Aspects of self-consistency in the Dyson-Schwinger approach to QED and $\lambda(\phi^\star\phi)^2$ theories

Roberto Casalbuoni,[1,2] Massimo Ladisa,[3,4,5] and Valerio Olevano[3,4,5]

[1]*Department of Physics, University of Florence & INFN, I-50019 Sesto F., Firenze, Italy*
[2]*Galileo Galilei Institute, Largo Enrico Fermi 2, I-50125 Firenze, Italy*
[3]*Institut Néel, CNRS and UJF, 25 rue des Martyrs, BP 166, F-38042 Grenoble cedex 9, France*
[4]*European Theoretical Spectroscopy Facility (ETSF), 25 rue des Martyrs, BP 166, F-38042 Grenoble cedex 9, France*
[5]*Istituto di Cristallografia, CNR, Via G. Amendola 122/O, I-70126 Bari, Italy*

We investigate some aspects of the self-consistency in the Dyson-Schwinger approach to both the QED and the self-interacting scalar field theories. We prove that the set of the Dyson-Schwinger equations, together with the Green-Ward-Takahashi identity, is equivalent to the analogous set of integral equations studied in condensed matter, namely, many-body perturbation theory, where it is solved self-consistently and iteratively. In this framework, we compute the nonperturbative solution of the gap equation for the self-interacting scalar field theory.

I. INTRODUCTION

Nonperturbative approaches to quantum field theory (QFT) allow for a better understanding of some general properties of the exact scattering amplitudes. They historically pursued after the formulation of quantum electrodynamics (QED) to analyze the asymptotic behavior of renormalized operators at short distance (see, for instance, Ref. [1]).

Beyond that, they give considerable informations in detailed studies of the structure of higher order approximations in theories where the perturbation expansion, along with a strong coupling, fails in analyzing the short-distance behavior of the relevant theory operators. This approach (referred to as Dyson-Schwinger) has been historically introduced by a number of authors [2–6].

Since then, it has inspired plenty of papers on the subject (see, for instance, Refs. [7–15]). Among them we quote those dealing with nonabelian theories [10–15], such as quantum chromodynamics (QCD), where both the aymptotic freedom (large coupling values at low energy) and the confinement of the Lagrangian fields within the asymptotic ones intrinsically require a phenomenological analysis beyond the naive perturbative expansion. In these processes the short-distance contributions can be computed, to some extent, by using the factorization approximation for the local operators in the effective Hamiltonian, as argued by Bjorken on the basis of color-transparency [16].

Large coupling and field confinement also characterize solid state physics phenomenology, where a screened many-body interaction occurs at about the Fermi energy among a large number of electrons embedded in the crystal lattice. In condensed matter singularities appear in the perturbation series of the correlation energy of a fully degenerate Fermi-Dirac system with Coulomb interaction, otherwise named homogeneous electron gas (HEG); in Ref. [17] the large divergent logarithms are resummed according to the general scheme introduced by Feynman in QFT [6]. In spite of its simplicity, HEG is fundamental for computing the correlation energy of a wide class of complex systems within the so-called local-density approximation [18].

Also within this context, a Green's function theory approach, called (improperly) many-body perturbation theory (MBPT), based on a formalism of second quantization of operators, has been considered [19,20]. The fundamental degree of freedom is the Green's function or propagator, which represents the probability amplitude for the propagation of an electron. As in any other QFT, the many-body system can be expanded in perturbation theory, with the coupling being the many-body interaction term. The Green's function (as well as any other quantity of the theory, such as the self-energy or the polarization) can be calculated at a given order of perturbation theory. A Feynman diagrammatic analysis is hence possible. The theory at the first order is equivalent to the Hartree-Fock theory. However the coupling is not small (compared, for example, to the electron-ion interaction) and the expansion does not converge. The second order is not necessarily smaller than the first. Hence one needs to resort to more complicated methods to solve the theory. Beyond partial resummations of diagrams to all orders, iterative methods have been preferred.

Historically a Dyson-Schwinger approach has been introduced to account for such a nonperturbative phenomenology and to compute optical and electronic properties of complex systems by means of iterative schemes. The alternative formulation in terms of functional derivatives reduces the many-body problem to the solution of a coupled set of nonlinear integral equations, whose characteristic feature is that, besides the single-particle Green's function, a whole hierarchy of equations involving higher order Green's functions is generated. A truncation of Feynman diagrammatics corresponds to the custom of replacing this hierarchy of equations by another coupled set of nonlinear

equations connecting the single-particle Green's function to the mass, polarization and vertex operators, often referred to as the Hedin's equations in condensed matter [21], to be solved iteratively until self-consistency is achieved.[1] Here the higher order Green's functions dependence of the relevant quantities is recast within the functional derivative of the mass operator with respect to the fermion single-particle Green's function. The latter quantity is proved to equate the Bethe-Salpeter kernel for the two-body into two-body rescattering. Both the diagrammatic truncation and the nonlinear coupled set of equations correspond to a nonperturbative approximation to the solution of the many-body problem.

So far, nobody has solved the Hedin's equations for a real system, since the nonlinearity of the equation involving the Bethe-Salpeter kernel is computationally demanding. Approximations are required to simplify the problem. Among the most widely used computation schemes it is worth mentioning the so-called *GW approximation,*[2] where the vertex operator is simplified in the self-energy evaluation at the beginning, and the Bethe-Salpeter equation accounts for the vertex corrections within this approach (see, for instance, the reviews in [18,22]).

The analogies between these two fields, spanning ultrahigh and ultralow energies, have been cross fertilizing methodologies and approaches. Along with that, this paper aims at spotting: i) the equivalence between the Hedin's equations set and the Dyson-Schwinger approach to QFT, where the role played by the underlying gauge symmetry is crucial in relating the Bethe-Salpeter kernel to the functional derivative of the mass operator with respect to the fermion Green's function (throughout the Green-Ward-Takahashi identity [23–25]); ii) the iterative scheme solution of the Dyson-Schwinger equations, in the spirit of the condensed matter methods, by means of a S-matrix unitarity inspired ansatz on the mass operator, accounting for the nonlinearity of the present approach. Both points should be regarded as aspects of the self-consistency of the Dyson-Schwinger approach to the many-body quantum field theory, with regard to the actual theory investigated, in the context of the aforementioned S-matrix unitarity inspired picture. Indeed, in QFT the S matrix satisfies the unitarity condition, actually more fundamental than the concept of Hamiltonian and wave functions (see, for instance, Refs. [1,26–28] and references therein).

In that respect, we point out two examples (QED and the self-interacting scalar field theory), different on the

physical basis but akin as to the solution scheme. The QED example is intended to formally bridge the condensed matter scenario to its underlying fundamental theory: throughout the paper, a QFT analysis to the topics has been preferred to the functional one, in order to spot this idea. A functional approach is, of course, viable. On the other hand, the gap equation arising in the self-interacting scalar field theory is probably the simplest example to show how to implement the iterative scheme to the numerical solution of the Dyson-Schwinger equations.

The plan of the paper is as follows. In the next section we discuss the general formalism of Dyson-Schwinger approach in QED. In Sec. III, the Green-Ward-Takahashi identity is proved to be equivalent to the definition of the Bethe-Salpeter kernel as the functional derivative of the mass operator with respect to the Green's function. Sec. IV is devoted to the calculation of the Dyson-Schwinger equations for the self-interacting scalar field theory. Finally in Sec. V we compute the nonperturbative solution to the gap equation arising for the latter case and draw our conclusions.

II. THE SET OF DYSON-SCHWINGER EQUATIONS IN QED

It is worth to recall the Dyson-Schwinger equations for QED. Hereafter, Latin letters shall refer to noninteracting quantities, while calligraphic symbols represent interacting ones. Thus, ψ represents the free fermionic field, while Ψ is the exact fermionic field; analogously for the photonic field A ($\mathcal{A}$). Accordingly, G ($\mathcal{G}$) is the free (exact) fermionic Green's function, while D ($\mathcal{D}$) corresponds to the free (exact) photon propagator. The electromagnetic current j is defined according to Ref. [29], i.e. $j^\mu(x) \overset{\text{def}}{=} \bar{\Psi}(x)\gamma^\mu\Psi(x)$. They read

$$G^\alpha_\beta(x, y) \overset{\text{def}}{=} -i\langle 0|T(\psi^\alpha(x)\bar{\psi}_\beta(y))|0\rangle,$$
$$(i\not{\partial}_x - mI)\psi(x) = 0, \tag{1}$$

$$D^{\mu\nu}(x, y) \overset{\text{def}}{=} +i\langle 0|T(A^\mu(x)A^{\dagger\nu}(y))|0\rangle,$$
$$(\partial^\mu\partial^\nu - \partial_\alpha\partial^\alpha g^{\mu\nu})A_\nu(x) = 0, \tag{2}$$

$$\mathcal{G}^\alpha_\beta(x, y) \overset{\text{def}}{=} -i\langle 0|T(\Psi^\alpha(x)\bar{\Psi}_\beta(y))|0\rangle,$$
$$(i\not{\partial}_x - mI)\Psi(x) = e\gamma_\mu\mathcal{A}^\mu(x)\Psi(x), \tag{3}$$

$$\mathcal{D}^{\mu\nu}(x, y) \overset{\text{def}}{=} +i\langle 0|T(\mathcal{A}^\mu(x)\mathcal{A}^{\dagger\nu}(y))|0\rangle,$$
$$(\partial^\mu\partial^\nu - \partial_\alpha\partial^\alpha g^{\mu\nu})\mathcal{A}_\nu(x) = -4\pi e j^\mu(x), \tag{4}$$

$$\langle 0|T(\mathcal{A}^\mu(x)\Psi^\alpha(y)\bar{\Psi}_\beta(z))|0\rangle \overset{\text{def}}{=}$$
$$\times e \iiint d^4x' d^4y' d^4z' \{\mathcal{G}^{\beta'}_\beta(z, z')\Gamma^{\mu'\alpha'}_{\beta'}(z', y', x')$$
$$\times \mathcal{G}^\alpha_{\alpha'}(y', y)\mathcal{D}^\mu_{\mu'}(x', x)\}, \tag{5}$$

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[1]The approach described here is akin to the *truncation* procedure of the infinite tower of Dyson-Schwinger equations in particle physics (see, for instance, Ref. [7]).

[2]The name *GW* stands for the *product* of the *G* Green's function and the *W* dressed Coulomb interaction while computing the mass operator ΓGW, being Γ the vertex operator as it appears in the Lagrangian ($\Gamma \approx \mathbf{I}$ when magnetic and relativistic effects are neglected).

$$\mathcal{M}_\beta^\alpha(x, y) \overset{\text{def}}{=} G_\beta^{-1\alpha}(x, y) - \mathcal{G}_\beta^{-1\alpha}(x, y),$$

$$\mathcal{P}^{\mu\nu}(x, y) \overset{\text{def}}{=} D^{-1\mu\nu}(x, y) - \mathcal{D}^{-1\mu\nu}(x, y), \qquad (6)$$

where a sum over repeated indices is understood. The relevant equation for $\mathcal{M}$ operator is obtained by applying $G_\beta^{-1\alpha}(x, y) \left(\overset{\text{def}}{=} \delta^4(x - y)(i\slashed{\partial}_y - mI)_\beta^\alpha\right)$ to $\mathcal{G}$ operator defined in Eq. (3) and by using Eq. (5) for the vertex Γ.[3] It reads

$$\mathcal{M}_\beta^\alpha(x, y) = -ie^2 \iint d^4x' d^4y' \gamma_{\alpha'\mu}^\alpha \mathcal{G}_{\beta'}^{\alpha'}(x, x')$$
$$\times \Gamma_{\beta\nu}^{\beta'}(x', y, y') \mathcal{D}^{\nu\mu}(y', x); \qquad (7)$$

analogously for $\mathcal{P}$ operator one gets

$$\mathcal{P}^{\mu\nu}(x, y) = 4\pi ie^2 \iint d^4x' d^4y' \gamma_{\alpha'}^{\alpha\mu} \mathcal{G}_{\beta'}^{\alpha'}(x, x')$$
$$\times \Gamma_{\delta'}^{\beta'\nu}(x', y', y) \mathcal{G}_\alpha^{\delta'}(y', x). \qquad (8)$$

Beside these, another equation is needed for the vertex Γ:

$$i\Gamma_\beta^{\alpha\mu}(x, y, z) \overset{\text{def}}{=} i\gamma_\beta^{\alpha\mu}\delta^4(x - z)\delta^4(y - z)$$
$$+ i \iiiint d^4x' d^4y' d^4x'' d^4y'' \mathcal{G}_{\alpha'}^{\alpha''}(x'', x')\Gamma_{\beta'}^{\alpha'\mu}(x', y', z)$$
$$\times \mathcal{G}_{\beta''}^{\beta'}(y', y'')\mathbf{K}_{\alpha''\beta}^{\beta''\alpha}(y, y'', x, x''), \qquad (9)$$

where $\mathbf{K}$, the Bethe-Salpeter kernel, accounts for all the possible contributions coming from the fermion-fermion rescattering, except for those already embodied within the exact propagators $\mathcal{G}$, $\mathcal{D}$ and vertex Γ.[4]

Equations (6)–(9) are also reported in a diagrammatic representation:

$$\boxed{\mathcal{M}} = \qquad \Gamma \qquad\qquad \boxed{\mathcal{P}} = \qquad \Gamma \qquad (1)$$

$$\boxed{\mathcal{M}} \overset{def.}{=} \left[\ \rule{1cm}{0pt}\ \right]^{-1} - \left[\ \rule{1cm}{0pt}\ \right]^{-1} \qquad \boxed{\mathcal{P}} \overset{def.}{=} \left[\ \rule{1cm}{0pt}\ \right]^{-1} - \left[\ \rule{1cm}{0pt}\ \right]^{-1} \qquad (2)$$

$$\Gamma \overset{def.}{=} \quad \bullet \quad + \quad \boxed{\mathbf{K}} \qquad (3)$$

As a matter of fact, the set of five integral equations depicted above depends on six unknowns, i.e. $\mathcal{G}$, $\mathcal{D}$, $\mathcal{P}$, $\mathcal{M}$, Γ and $\mathbf{K}$. Thus, an additional equation is needed to close the system and to approach a solution. Two schemes have been assessed to accomplish with such an issue. In condensed matter the Bethe-Salpeter kernel is written as the functional derivative of the mass operator with respect

to the Green's function: $\mathbf{K} = \delta\mathcal{M}/\delta\mathcal{G}$ [18,21,22]. In QFT, the same definition formally holds. Indeed, in the framework of Cornwall-Jackiw-Tomboulis approach [30], it is known that the second functional derivative of the action with respect to the (bilocal) field equals the Bethe-Salpeter kernel [31]. On the other hand, in Ref. [11] such a second functional derivative is proved to be the functional derivative of the mass operator with respect to the fermion Green's function. While such a relation is crucial for the self-consistency of the coupled set of nonlinear equations, in order to avoid a hierarchy of higher order Green's functions, a functional derivative is computationally demanding and a diagrammatic, although nonperturbative, expansion of the Bethe-Salpeter kernel is usually preferred[5] [7–10,12–15].

III. THE GREEN-WARD-TAKAHASHI IDENTITY

Under a small phase (gauge) shift on the Ψ operators, $\delta\chi$, the $\mathcal{G}(x, x')$ operator is shifted by $ie\mathcal{G}(x, x')[\delta\chi(x) - \delta\chi(x')]$, according to the definition of Eq. (3). On the other hand, $\delta\mathcal{G}(x, x')$ can be directly computed as the coupling of the (small) gauge field $-\partial_\mu\delta\chi$ to the fermionic current.[6] By equating the former and the latter quantities, one gets the Green's equation [24] either in its integral formulation, i.e.

$$ie\mathcal{G}_\beta^\alpha(x, x')[\delta\chi(x) - \delta\chi(x')]$$
$$= e \iiint d^4y d^4y' d^4z \mathcal{G}_{\alpha'}^\alpha(x, y)\partial_z^\mu\Gamma_{\beta'\mu}^{\alpha'}(y, y', z)$$
$$\times \mathcal{G}_\beta^{\beta'}(y', x')\delta\chi(z), \qquad (10)$$

or in its differential one $-ie\mathcal{G}^{-1}(x, x')[\delta\chi(x) - \delta\chi(x')] = e \int d^4z \partial_z^\mu \Gamma_\mu(x, x', z)\delta\chi(z)$. We will need both forms in the sequel. The Ward-Takahashi identity corresponds to the soft photon limit in momentum space [23,25].

To accomplish with the task of completing the set of Dyson-Schwinger equations, we replace $\mathcal{G}^{-1}$ in the differential form of Green-Ward-Takahashi equation, according to the mass operator definition of Eq. (6), and we notice that the noninteracting propagator G is gauge invariant, therefore

$$iG^{-1}(x, x')[\delta\chi(x) - \delta\chi(x')]$$
$$= \int d^4z \delta\chi(z)\partial_z^\mu[\gamma_\mu\delta^4(x - z)\delta^4(x' - z)] = 0. \quad (11)$$

Thus,

$$ie\mathcal{M}(x,x')[\delta\chi(x)-\delta\chi(x')]$$

$$=e\int d^4z\delta\chi(z)\partial_z^\mu[-\gamma_\mu\delta^4(x-z)\delta^4(x'-z)+\Gamma_\mu(x,x',z)]$$

$$=e\int d^4z\delta\chi(z)\iiiint d^4\hat{x}'d^4\hat{y}'d^4\hat{x}''d^4\hat{x}''\mathcal{G}(\hat{x}',\hat{x}'')\partial_z^\mu$$

$$\Gamma_\mu(\hat{x}'',\hat{y}'',z)\mathcal{G}(\hat{y}'',\hat{y}')\mathbf{K}(\hat{y}',x',\hat{x}',x)$$

$$=\iint d^4\hat{x}'d^4\hat{y}'\delta\mathcal{G}(\hat{x}',\hat{y}')\mathbf{K}(\hat{y}',x',\hat{x}',x), \qquad (12)$$

where in the latter equations we used the vertex definition of Eq. (9) and the integral form of Green-Ward-Takahashi relation of Eq. (10). It can be easily seen that the l.h.s. of Eq. (12) is the shift of the mass operator under a gauge transform,[7] i.e. $\delta\mathcal{M}(x,x')$.

In conclusion

$$\delta\mathcal{M}(x,x')=\iint d^4\hat{x}'d^4\hat{y}'\delta\mathcal{G}(\hat{x}',\hat{y}')\mathbf{K}(\hat{y}',x',\hat{x}',x). \quad (13)$$

The meaning of this equation is twofold. It states that: i) the gauge shift on the mass operator is linearly dependent on the gauge shift of the fermion propagator throughout the Bethe-Salpeter kernel; ii) for the same reason, the Bethe-Salpeter kernel is gauge invariant. Moreover, the equation written above completes the set of Dyson-Schwinger equations and makes finding a solution a viable problem.

IV. THE SET OF DYSON-SCHWINGER EQUATIONS IN THE $\lambda(\phi^\star\phi)^2$ THEORY

The set of Dyson-Schwinger equations for self-interacting scalar field theory is simpler than the analogous in QED. Hereafter φ (ϕ) represents the free (exact) bosonic field and, accordingly, D ($\mathcal{D}$) is the free (exact) boson Green's function. It reads

$$D(x,y)\overset{\text{def}}{=}+i\langle0|T(\varphi(x)\varphi^\star(y))|0\rangle,$$
$$(\Box_x+m^2)\varphi(x)=0, \qquad (14)$$

$$\mathcal{D}(x,y)\overset{\text{def.}}{=}+i\langle0|T(\phi(x)\phi^\star(y))|0\rangle,$$
$$(\Box_x+m^2)\phi(x)=\frac{\lambda}{2!}\phi^\star(x)\phi^2(x), \qquad (15)$$

$$\langle0|T(\phi(x)\phi(y)\phi^\star(z)\phi^\star(w))|0\rangle\overset{\text{def}}{=}-i\lambda$$
$$\times\iiiint d^4x'd^4y'd^4z'd^4w'\{\mathcal{D}(w,w')\mathcal{D}(z,z')$$
$$\times\Pi(w',z',y',x')\mathcal{D}(y',y)\mathcal{D}(x',x)\}, \qquad (16)$$

$$\mathcal{M}(x,y)\overset{\text{def.}}{=}D^{-1}(x,y)-\mathcal{D}^{-1}(x,y), \qquad (17)$$

being $\Box_x\overset{\text{def}}{=}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}$. The latter and the former sets are akin as to the derivation. Let us consider, for instance, the mass operator equation. By applying D^{-1} to $\mathcal{D}$ (Eqs. (14) and (15)) one gets

$$\int d^4yD^{-1}(x,y)\mathcal{D}(y,x')=\delta^4(x-x')$$
$$+i\frac{\lambda}{2!}\langle0|T(\phi(x)^2\phi^\star(x)\phi^\star(x'))|0\rangle. \qquad (18)$$

Finally, by using Eq. (16), the mass operator is computed:

$$\mathcal{M}(x,x')=\frac{\lambda^2}{2!}\iiint d^4\hat{y}'d^4\hat{y}''d^4\hat{x}''\mathcal{D}(x,\hat{y}'')$$
$$\times\mathcal{D}(x,\hat{x}'')\Pi(\hat{y}'',\hat{x}'',\hat{y}',x')\mathcal{D}(\hat{y}',x). \qquad (19)$$

The vertex equation introduces a Bethe-Salpeter kernel for the two-body rescattering:

$$\Pi(y',x',y,x)=\delta^4(y'-x')\delta^4(y'-y)\delta^4(y'-x)$$
$$+\iiiint d^4\hat{x}'d^4\hat{x}''d^4\hat{y}'d^4\hat{y}''\Pi(y',x',\hat{y}'',\hat{x}'')\mathcal{D}(\hat{y}'',\hat{y}')$$
$$\times\mathcal{D}(\hat{x}'',\hat{x}')\mathbf{K}(\hat{y}',\hat{x}',y,x). \qquad (20)$$

Although the self-interacting scalar field theory is not a gauge theory, the gauge shifts of mass operator and propagator (under a phase shift of ϕ operator) are related in the same fashion of QED case[8]

$$\delta\mathcal{M}(x,x')=\iint d^4\hat{x}'d^4\hat{y}'\delta\mathcal{D}(\hat{x}',\hat{y}')\mathbf{K}(\hat{y}',x',\hat{x}',x). \quad (21)$$

The latter equation completes the set of Dyson-Schwinger ones, Eqs. (17), (19), and (20) for the self-interacting scalar field theory. They can be recapitulated in the following diagrammatics where $\delta[\cdots]$ means the gauge shift of the quantity within brackets:

$$(1)$$

[7]Indeed under a gauge shift: $\delta\{\int d^4y\mathcal{G}(x,y)\mathcal{G}^{-1}(y,x')\}=0$. By expanding the latter gauge variation and by noticing that the noninteracting propagator G (unlike $\mathcal{G}$ and $\mathcal{M}$) is gauge invariant, we achieve the result.

[8]For a strictly neutral ϕ field, i.e. $\phi^\star=\phi$, an additional (Schwinger) term arises. The absence of a global gauge symmetry for this case has been also pointed out in [32].

V. RESULTS AND DISCUSSION

Hereafter we shall assume that the space-time is homogeneous. Therefore the energy-momentum conservation law holds in momentum space, where the relevant quantities defined in the previous section simplify accordingly.[9] Thus the set of Dyson-Schwinger equations reduces to

$$\tilde{\mathcal{M}}(p) \overset{\text{def}}{=} \tilde{D}^{-1}(p) - \tilde{\mathcal{D}}^{-1}(p),$$

$$\tilde{\mathcal{M}}(p) = \frac{\lambda^2}{2!} \iint \frac{d^4\sigma}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} \tilde{\mathcal{D}}(\sigma)\tilde{\mathcal{D}}(\eta)\tilde{\Pi}(-\sigma, -\eta, p - \sigma - \eta, -p)\tilde{\mathcal{D}}(\sigma + \eta - p),$$

$$\tilde{\Pi}(p + \sigma, q - \sigma, p, q) \overset{\text{def}}{=} \mathbf{I} + \int \frac{d^4k}{(2\pi)^4} \tilde{\Pi}(p + \sigma, q - \sigma, p + k, q - k)\tilde{\mathcal{D}}(q - k)\tilde{\mathcal{D}}(p + k)\tilde{\mathbf{K}}(p + k, q - k, p, q),$$

$$\tilde{\mathcal{M}}(p + q) - \tilde{\mathcal{M}}(p) = \int \frac{d^4k}{(2\pi)^4} [\tilde{\mathcal{D}}(k) - \tilde{\mathcal{D}}(k - q)]\tilde{\mathbf{K}}(k - q, p + q, k, p). \quad (22)$$

For the sake of clarity, we prove the last equation in (22). We start from the last equation in (12) for the self-interacting scalar field theory:

$$\delta\mathcal{M}(x, x') = \iint d^4\hat{x}'d^4\hat{y}'\delta\mathcal{D}(\hat{x}', \hat{y}')\mathbf{K}(\hat{y}', x', \hat{x}', x) \Longleftrightarrow (\text{momentum space}) \Longleftrightarrow$$

$$\delta\tilde{\mathcal{M}}(k, k') = \iint \frac{d^4\hat{k}}{(2\pi)^4} \frac{d^4\hat{k}'}{(2\pi)^4} \delta\tilde{\mathcal{D}}(\hat{k}, \hat{k}')\tilde{\mathbf{K}}(\hat{k}', k', \hat{k}, k) \Longleftrightarrow (\delta\tilde{\chi} \text{ is the gauge shift}) \Longleftrightarrow$$

$$ie \int d^4q/(2\pi)^4 \delta\tilde{\chi}(q)\left[\tilde{\mathcal{M}}(k + q, k') - \tilde{\mathcal{M}}(k, k' - q) \right] = ie \int d^4q/(2\pi)^4 \delta\tilde{\chi}(q)$$

$$\times \iint \frac{d^4\hat{k}}{(2\pi)^4} \frac{d^4\hat{k}'}{(2\pi)^4}\left[\tilde{\mathcal{D}}(\hat{k} + q, \hat{k}') - \tilde{\mathcal{D}}(\hat{k}, \hat{k}' - q) \right]\tilde{\mathbf{K}}(\hat{k}', k', \hat{k}, k) \Longleftrightarrow (\text{homogeneous space-time}^9) \Longleftrightarrow$$

$$ie\delta\tilde{\chi}(k' - k)\left[\tilde{\mathcal{M}}(k') - \tilde{\mathcal{M}}(k) \right] = ie\delta\tilde{\chi}(k - k') \int \frac{d^4\hat{k}}{(2\pi)^4}\left[\tilde{\mathcal{D}}(k + \hat{k} - k') - \tilde{\mathcal{D}}(\hat{k}) \right]\tilde{\mathbf{K}}(k + \hat{k} - k', k', \hat{k}, k); \quad (23)$$

after a variable shift $(k' \to p + q, k \to p, \hat{k} \to k)$, and accounting for the $\delta\tilde{\chi}$ oddness,[10] the proof is completed.

The S-matrix unitarity naturally leads to physical conditions for the propagator structure, i.e. a pole term *times* a multiplicative constant (renormalization residue) [29]. Within this context, although not needed,[11] we shall introduce the following ansatz on the mass operator: $\tilde{\mathcal{M}}(p^2) = -\alpha\tilde{D}^{-1}(p^2) + \mu^2$, being α a (real) constant to be determined, while μ is a mass scale (renormalization point). By this choice, the interacting propagator reads

$$\tilde{\mathcal{D}}^{-1}(p^2) = (1 + \alpha)\tilde{D}^{-1}(p^2) - \mu^2 = \frac{1+\alpha}{\alpha}\left[\frac{\mu^2}{1+\alpha} - \tilde{\mathcal{M}}(p^2)\right],$$

and the pole is located at $\tilde{D}^{-1}(p_{\text{pole}}^2) \equiv \frac{\mu^2}{1+\alpha}$. Moreover, by

using this S-matrix unitarity inspired ansatz, the Bethe-Salpeter kernel $\tilde{\mathbf{K}}$ is promptly evaluated by means of the last equation in (22)

$$\tilde{\mathbf{K}}(k - q, p + q, k, p) = \frac{\alpha}{1 + \alpha} \tilde{\mathcal{D}}^{-1}(k - q)\tilde{\mathcal{D}}^{-1}(k)$$
$$\times (2\pi)^4\delta^4(p - k + q), \quad (24)$$

by which the vertex $\tilde{\Pi}$ is shortened to $(1 + \alpha)\mathbf{I}$ (vertex renormalization) by direct substitution in the third equation of (22).

Finally, the mass operator equation in (22) accounts for the self-consistency of the Dyson-Schwinger approach:

$$\tilde{\mathcal{M}}(p^2) = (1 + \alpha)\frac{\lambda^2}{2!} \int d^4x \exp(i\,px)[\mathcal{D}(x)]^3, \quad (25)$$

where $\mathcal{D}(x)$ is the interacting propagator in space-time (configuration space), while the implicit dependence upon α in both $\tilde{\mathcal{M}}$, $\mathcal{D}$ is understood.[12] Equation (25) describes a *setting sun* Feynman diagram; it provides either

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[9] For instance, the propagator in the momentum space reads $\tilde{\mathcal{D}}(k, k')$; it reduces to $(2\pi)^4\tilde{\mathcal{D}}(k)\delta^4(k - k')$. On the same footing, the gauge shift $\delta\tilde{\mathcal{D}}(p, p') = ie \int d^4q/(2\pi)^4 \delta\tilde{\chi}(q)[\tilde{\mathcal{D}}(p + q, p') - \tilde{\mathcal{D}}(p, p' - q)]$ is shortened to $\delta\tilde{\chi}(p' - p)[\tilde{\mathcal{D}}(p') - \tilde{\mathcal{D}}(p)]$, analogously for the mass operator. The Bethe-Salpeter kernel is also simplified: $\tilde{\mathbf{K}}(p, q, p', q') = (2\pi)^4\delta^4(p + q - p' - q')\tilde{\mathbf{K}}(p, q, p', q')$.

[10] The phase (gauge) shift oddness stems from the Green's function definition: $\mathcal{D}(x', x) = -\mathcal{D}(x, x')^\star$.

[11] In principle, the set of Eqs. (22) accounts for an even number of unknowns ($\tilde{\mathcal{M}}$, $\tilde{\mathcal{D}}$, $\tilde{\Pi}$ and $\tilde{\mathbf{K}}$): it can be numerically solved self-consistently.

[12] In Euclidean four-dimensional space the interacting propagator, together with the mass operator ansatz, reads $\mathcal{D}(x, m) = \frac{mx}{1+\alpha}\frac{K_1(mx)}{(2\pi)^2x^2}$, $m = \frac{\mu}{\sqrt{1+\alpha}}$, $x = \sqrt{x_\mu x^\mu}$, being K_1 the first order modified Bessel function of second kind, while $\tilde{\mathcal{M}}(p^2) = \alpha p^2 + \mu^2$.

the broken symmetry solution or the symmetric one. To accomplish with the task, the mass operator of Eq. (25) satisfies a dispersion relation (DR) with one subtraction [33–35] (in p variable)[13]: $\tilde{\mathcal{M}}_{\text{sub}}(p^2) = p^2 \int_0^{+\infty} \frac{dt}{\pi} \frac{\text{Im}(\tilde{M}(t))}{t(t-p^2)}$. We evaluate the subtracted version of Eq. (25) on the mass pole, resembling the mass/gap equation of [36,37]. In Euclidean space it reads [38]:

$$m^2 = m^2 \frac{\lambda_R^2}{2!(2\pi)^4} \int_0^{+\infty} dx\, m\left[\frac{1}{2} - \frac{J_1(mx)}{mx}\right] K_1(mx)^3 \mathcal{F}(x,\Lambda) \tag{26}$$

where J_1 is the first order Bessel function of first kind, $\lambda_R = \lambda/(1+\alpha)$ and $\mathcal{F}(x,\Lambda)$ is an ultraviolet cutoff factor [36]. The integral of Eq. (26) has to be regularized at some point $(1/\Lambda)$ since it exhibits a (logarithmic) divergence for small x ($\Lambda \to \infty$) (the variable x behaves like the inverse four-momentum p, thus at the origin $x \sim 1/\Lambda$).

After rescaling the variable $x \to \xi/m$ and normalizing the parameter m to the cutoff Λ ($m \to \eta\Lambda$), Eq. (26) reduces to[14]

$$\eta^2 = \eta^2 \frac{\lambda_R^2}{2!(2\pi)^4} \int_\eta^{+\infty} d\xi\left[\frac{1}{2} - \frac{J_1(\xi)}{\xi}\right] K_1(\xi)^3. \tag{27}$$

For vanishing η the mild divergence of the integral on the r.h.s. of Eq. (27) is compensated by the η^2 term and the trivial solution ($\eta = 0$) is achieved.

The importance of Eq. (27) is twofold. On one hand, a mapping between λ_R and η can be obtained beyond the perturbative regime of the coupling constant, helpful while investigating the cohexistence of the triviality (namely $\lambda_R \to 0$ when $\Lambda \to \infty$) and the spontaneous symmetry breaking (SSB) of the theory (crucial, for instance, for the self-consistency of the standard model in particle physics [39]).

On the other hand, a renormalization group equation (RGE) for the η-evolution of the coupling constant λ_R can be promptly assessed from Eq. (27):

$$\lambda_R^2(\eta) = \frac{\lambda_R^2(\eta_0)}{1 + \frac{\lambda_R^2(\eta_0)}{2!(2\pi)^4} \int_\eta^{\eta_0} dx\,\Omega(x)}$$

$$\overset{\eta \to 0,\ \eta_0/\eta\ \text{fixed}}{\approx} \frac{\lambda_R^2(\eta_0)}{1 + \frac{\lambda_R^2(\eta_0)}{2!(4\pi)^4} \ln(\frac{\eta_0}{\eta})}, \tag{28}$$

being $\Omega(x)$ the integrand function of Eq. (27). While the latter equation corresponds to the perturbative region for the coupling λ_R, the former one holds in nonperturbative regime too.

Beyond the trivial solution ($\eta = 0$), corresponding to the unbroken symmetry [40], Eq. (27) gives the broken

[13]Otherwise $\tilde{\mathcal{M}}_{\text{sub}}(p^2) = \tilde{\mathcal{M}}(p^2) - \tilde{\mathcal{M}}(0)$, being $\tilde{\mathcal{M}}(p^2) = \int_{\beta_4}^{+\infty} \frac{dt}{\pi} \frac{\text{Im}(\tilde{M}(t))}{t-p^2}$ the mass unsubtracted DR.
[14]Hereafter we implement $\mathcal{F}(x,\Lambda) = \theta[x - 1/\Lambda]$, being θ the Heaviside function.

symmetry solution, once the renormalized coupling λ_R is replaced according to the SSB mechanism. Indeed both m^2 and λ_R depend upon the renormalization constant α in the same fashion $\left(\propto \frac{1}{1+\alpha}\right)$; thus, by eliminating it, we get the formula $m^2 = \frac{\mu^2}{\lambda}\lambda_R$, accounting for the not vanishing vacuum expectation value of the field which minimizes the Hamiltonian $\left(\langle 0|\phi^\star\phi|0\rangle \overset{\text{def}}{=} v^2 \neq 0, \frac{\mu^2}{\lambda} = \frac{v^2}{2}\right)$. It reads a typical fixed point problem for the parameter $\hat{m}$ $\left(\hat{m} \overset{\text{def}}{=} \frac{m}{v}, \hat{\Lambda} \overset{\text{def}}{=} \frac{\Lambda}{v}\right)$ [41,42]:

$$\hat{m} = 4\pi \sqrt[4]{\frac{\rho}{2\ln(\hat{m})}}, \tag{29}$$

being $\rho \overset{\text{def}}{=} \left[\frac{\lambda_R(\hat{m}/\hat{\Lambda})}{\lambda_R(1/\hat{\Lambda})}\right]^2$. Here the parameter ρ can be computed by means of Eq. (27). In the limit $\hat{\Lambda} \to \infty$ it approaches the unity regardless of the actual value for $\hat{m}$: for instance, we find $\rho \in [1.019, 1.072]$ for $\hat{m} \in [2, 12]$ (at $\hat{\Lambda} \simeq 4 \times 10^{18}$).

Implementing Eq. (29) is straightforward and computationally not demanding. In spite of its simplicity, it clearly exhibits the iterative solution scheme, following the spirit of the typical condensed matter approach to the solution of the Dyson-Schwinger equations original sets (22).

We find $\hat{m} = 8.88 \pm 0.10$, where the error comes from the parameter ρ ranging in the aforementioned interval. Our finding is in agreement with the value of $\hat{m} = 2\sqrt{2}\pi \simeq 8.88577$ predicted by the classically scale invariant (CSI) theory and computed on the lattice [43]. Indeed, in CSI theory an effective potential is computed starting from the Euclidean action of a massless self-interacting scalar field and integrating out the field fluctuations around its vacuum expectation value (VEV). Different schemes are possible: in Ref. [43], for instance, two different renormalization constants are introduced for the field VEV and for the fluctuations. Another possibility is to add a mass term in the Euclidean action and to describe the broken-to-symmetric phase as a phase transition. In order to account for different phenomenological approaches, the effective potential depends on a parameter ranging between 1 (CSI case) and 3 (classical quartic potential), while the value 2 is a phase transition signature. The effective potential is then expanded up to the third power around its minimum and the three coefficients of the expansion, related to the aforementioned parameter, are fitted by the lattice simulation data. The lattice data fitted central value is 1.14, although the resulting uncertainty is large (see Ref. [43] and references therein for details). While the latter statement signals the need for more statistics in lattice calculations, the former result points at the CSI theory as the preferred scenario.

In conclusion, we believe that the Dyson-Schwinger equations in QED, together with the Green-Ward-Takahashi identity, we investigate in this paper, are equivalent to the analogous set of integral equations studied in condensed matter, often referred to as the Hedin's equations. They account for the self-consistency of the method, corresponding to the truncation of the perturbative/diagrammatic expansion in particle physics. Within the scheme proposed, the approach seems to be capable of producing the nonperturbative solution of the self-interacting scalar field theory, as suggested in the CSI theory, within theoretical errors. This method could be straightforwardly applied to other abelian (as well as non-abelian) theories, for instance the self-interacting fermion field theory, as an effective theory for describing the gap opening (mass generation) mechanism in condensed matter. We only mention here a result for the graphene, a system composed by sp_2-hybridized carbon atoms placed on a plane. Such a system is interesting both on experimental and theoretical side, due to its peculiar electronic properties (see, for instance, Ref. [44] and references therein). While from the former point of view the gap opening is still under investigation, it is not theoretically understood yet whether or not its spatial dimensionality is two (as assumed in all the theoretical works) or three (as suggested by the role played by the off plane p_z orbitals). Both issues (gap and dimensions) could be addressed by a Dyson-Schwinger approach together with a similar mass operator ansatz. Indeed the mass/gap equation, as it appears in the Eq. (25), actually depends on the space-time dimension [38]. Therefore, if a gap opens, it will depend on the actual spatial dimension (probably between two and three). The details of this work will be given elsewhere together with the results.

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