Applications of two-body Dirac equations to the meson spectrum with three versus two covariant interactions, SU(3) mixing, and comparison to a quasipotential approach

Horace W. Crater* and James Schiermeyer

The University of Tennessee Space Institute B. H. Goethert Parkway,Tullahoma, Tennessee 37388, USA

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In a previous paper, Crater and Van Alstine applied the two-body Dirac equations of constraint dynamics to quark-antiquark bound states using a relativistic extention of the Adler-Piran potential and compared their spectral results to those from other approaches which also considered meson spectroscopy as a whole and not in parts. In this paper, we explore in more detail the differences and similarities in an important subset of those approaches, the quasipotential approach. In the earlier paper, the transformation properties of the quark-antiquark potentials were limited to a scalar and an electromagnetic-like four-vector, with the former accounting for the confining aspects of the overall potential, and the latter the short range portion. The static Adler-Piran potential was first given an invariant form and then apportioned between those two different types of potentials. Here, we make a change in this apportionment that leads to a substantial improvement in the resultant spectroscopy by including a timelike confining vector potential over and above the scalar confining one and the electromagnetic-like vector potential. Our fit includes 19 more mesons than the earlier results and we modify the scalar portion of the potential in such a way that allows this formalism to account for the isoscalar mesons η and η' not included in the previous work. Continuing the comparisons of formalisms and spectral results made in the previous paper with other approaches to meson spectroscopy, we examine in this paper the quasipotential approach of Ebert, Faustov, and Galkin.

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I. INTRODUCTION

There are a number of strategies in computational treatments of quantum chromodynamics that emerge in the study of meson spectroscopy. One is to set up a discrete lattice analog of the full quantum field theory. A second is to first make analytic approximations which replace the quantum field theoretic problem by a classical variational problem involving an effective Lagrange function and action. The latter approach has been exploited by Adler and Piran [1], and in a previous paper Crater and Van Alstine gave a detailed account of applications of the two-body Dirac equations (TBDE) of constraint dynamics to the meson quark-antiquark bound states [2] using a relativistic extension of the Adler-Piran potential. That paper also included a comparison of this approach to others [3–6] who, like ours, considered the whole spectrum instead of just selected parts.

Here we update the results presented in [2] in four ways. First, we include 19 more mesons not included in the previous work. Second, we obtain a substantial improvement in our fit to most all of the mesons by allowing the confining interaction, pure scalar in [2], to take on a timelike vector portion. We still include the electromagnetic-like vector potential used previously. Third, we extend the relativistic Schrödinger-like form of the TBDE to include isoscalar mixing, thus incorporating the isoscalar mesons η and η'. And finally, we critically examine, by comparison with the TBDE, aspects of quasipotential approaches including a recent one presented in [7] as well as in [6].

In Sec. II we give a short review of the relativistic two-body constraint formalism, distinguishing between our new approach used for confining given in this paper and the one presented in [2] and including a discussion of the closely related quasipotential approach. In Sec. III we review the static Adler-Piran potential and how we apportion it between the three invariant potential functions $A(r)$, $S(r)$, and $V(r)$ used in our TBDE. In Sec. IV we present our main new results on meson spectroscopy. In Sec. V we include our treatment of SU(3) mixing, and in Sec. VI we discuss the meson spectral results of [7] including the advantages and shortcomings of their quasipotentials bound state formalism.

II. REVIEW OF RELATIVISTIC TWO-BODY FORMALISMS

A. Two-body constraint approach

When the interaction and the masses are known, a common starting point in describing the relativistic two-body bound state problem is the Bethe-Salpeter equation [8]. The Bethe-Salpeter equation is, however, usually not considered in its full four-dimensional form due to the difficulty of treating the relative time coordinate [9]. Numerous truncations of the Bethe-Salpeter equation have been proposed for the relativistic two-body problem [10,11]. Some of these types of approximate methods have previously been applied with considerable success to the $q\bar{q}$ meson spectrum

*hcrater@utsi.edu

[2,12–18]. The ladder approximation and the instantaneous approximation of the Salpeter equation have been widely used. It should be noted, however, that the simple ladder approximation and the Salpeter equation do not lead to the correct one-body limit [19], and do not respect gauge invariance [20]. Crossed-ladder diagrams must be included to insure gauge-invariant scattering amplitudes.

The two-body Dirac equations of constraint dynamics provide a covariant three-dimensional truncation of the Bethe-Salpeter equation. Sazdjian [21] has shown that the Bethe-Salpeter equation can be algebraically transformed into two independent equations. The first yields a covariant three-dimensional eigenvalue equation which for spinless particles takes the form

$$(\mathcal{H}_{10} + \mathcal{H}_{20} + 2\Phi_w)\Psi(x_1, x_2) = 0, \tag{1}$$

where $\mathcal{H}_{i0} = p_i^2 + m_i^2$. The quasipotential Φ_w is a modified geometric series in the Bethe-Salpeter kernel K such that in lowest order in K

$$\Phi_w = \pi i w \delta(P \cdot p) K, \tag{2}$$

where $P = p_1 + p_2$ is the total momentum, $p = \eta_2 p_1 - \eta_1 p_2$ is the relative momentum, w is the invariant total center of momentum (c.m.) energy with $P^2 = -w^2$. The η_i must be chosen so that the relative coordinate $x = x_1 - x_2$ and p are canonically conjugate, i.e. $\eta_1 + \eta_2 = 1$. The second equation overcomes the difficulty of treating the relative time in the center of momentum system by setting an invariant condition on the relative momentum p,

$$(\mathcal{H}_{10} - \mathcal{H}_{20})\Psi(x_1, x_2) = 0 = 2P \cdot p\Psi(x_1, x_2). \tag{3}$$

Note that this implies $p^\mu \Psi = p_\perp^\mu \Psi \equiv (\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu) p_\nu \Psi$ in which $\hat{P}^\mu = P^\mu / w$ is a timelike unit vector ($\hat{P}^2 = -1$) in the direction of the total momentum.

One can further combine the sum and the difference of Eqs. (1) and (3) to obtain a set of two relativistic equations one for each particle with each equation specifying two generalized mass-shell constraints

$$\mathcal{H}_i \Psi(x_1, x_2) = (p_i^2 + m_i^2 + \Phi_w)\Psi(x_1, x_2) = 0, \quad i = 1, 2, \tag{4}$$

including the interaction with the other particle. These constraint equations are just those of Dirac's Hamiltonian constraint dynamics[1] [22,23]. In order for the two simultaneous wave equations of (4) to have solutions other than zero, Dirac's constraint dynamics stipulate that these two constraints must be compatible among themselves, $[\mathcal{H}_1, \mathcal{H}_2]\Psi = 0$, that is, they must be first class. With no external potentials, the coordinate dependence of the quasipotential Φ_w would be through x and the compatibility condition becomes $[p_1^2 - p_2^2, \Phi_w]\Psi = P^\mu \partial \Phi_w / \partial x^\mu = 0$. In order for this to be true in general, Φ_w must depend on the relative coordinate x only through its component, $x_\perp$, perpendicular to P,

$$x_\perp^\mu = (\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu)(x_1 - x_2)_\nu. \tag{5}$$

Since the total momentum is conserved, the single component wave function Ψ in coordinate space is a product of a plane wave eigenstate of P and an internal part ψ [24], depending on this $x_\perp$.[2]

We find a plausible structure for the quasipotential Φ_w by observing that the one-body Klein-Gordon equation $(p^2 + m^2)\psi = (\mathbf{p}^2 - \varepsilon^2 + m^2)\psi = 0$ takes the form $(\mathbf{p}^2 - \varepsilon^2 + m^2 + 2mS + S^2 + 2\varepsilon A - A^2)\psi = 0$ when one introduces a scalar interaction and timelike vector interaction via $m \to m + S$ and $\varepsilon \to \varepsilon - A$. In the two-body case, separate classical [25] and quantum field theory [26] arguments show that when one includes world scalar and vector interactions, then Φ_w depends on two underlying invariant functions $S(r)$ and $A(r)$ through the two-body Klein-Gordon-like potential form with the same general structure, that is

$$\Phi_w = 2m_w S + S^2 + 2\varepsilon_w A - A^2. \tag{6}$$

Those field theories further yield the c.m. energy dependent forms

$$m_w = m_1 m_2 / w, \tag{7}$$

and

$$\varepsilon_w = (w^2 - m_1^2 - m_2^2)/2w, \tag{8}$$

ones that Tododov [23] introduced as the relativistic reduced mass and effective particle energy for the two-body meson system. Similar to what happens in the nonrelativistic two-body problem, in the relativistic case we have the motion of this effective particle taking place as if it were in an external field (here generated by S and A). The two kinematical variables (7) and (8) are related to one another by the Einstein condition

$$\varepsilon_w^2 - m_w^2 = b^2(w), \tag{9}$$

where the invariant

$$b^2(w) \equiv (w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2)/4w^2, \tag{10}$$

is the c.m. value of the square of the relative momentum expressed as a function of w. One also has

[1] These equations were originally proposed in the form of classical generalized mass-shell first class constraints $\mathcal{H}_i = (p_i^2 + m_i^2 + \Phi_i) \approx 0$, and their quantization $\mathcal{H}_i \Psi = 0$ without reference to a quantum field theory. For the classical $\mathcal{H}_i$ to be compatible, their Poisson bracket with one another must either vanish strongly or depend on the constraints themselves, $\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0$. The simplest solution of this equation is $\Phi_1 = \Phi_2$, a kind of relativistic third law condition, together with their common transverse coordinate dependence $\Phi_w(x_\perp)$, just as with its quantum version.

[2] We use the same symbol P for the eigenvalue so that the w dependence in Eq. (6) is regarded as an eigenvalue dependence. The wave function Ψ can be viewed either as a relativistic 2-body wave function (similar in interpretation to the Dirac wave function) or, if a close connection to field theory is required, related directly to the Bethe-Salpeter wave function χ by [21] $\Psi = -\pi i \delta(P \cdot p)\mathcal{H}_{10}\chi = -\pi i \delta(P \cdot p)\mathcal{H}_{20}\chi$.

$$b^2(w) = \varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2, \qquad (11)$$

in which ε_1 and ε_2 are the invariant c.m. energies of the individual particles satisfying

$$\varepsilon_1 + \varepsilon_2 = w, \qquad \varepsilon_1 - \varepsilon_2 = (m_1^2 - m_2^2)/w. \qquad (12)$$

In terms of these invariants, the relative momentum appearing in Eq. (2) and (3) is given by

$$p^\mu = (\varepsilon_2 p_1^\mu - \varepsilon_1 p_2^\mu)/w, \qquad (13)$$

so that $\eta_1 + \eta_2 = (\varepsilon_1 + \varepsilon_2)/w = 1$. In [27] the forms for these two-body and effective particle variables are given sound justifications based solely on relativistic kinematics, supplementing the dynamical arguments of [25,26].

Originally, the two-body Dirac equations of constraint dynamics arose from a supersymmetric treatment of two pseudoclassical constraints (with Grassmann variables in place of gamma matrices) which were then quantized [28–31]. Sazdjian later derived [21] different forms of these same equations, just as with their spinless counterparts above, as covariant and three-dimensional truncation of the Bethe-Salpeter equation. The forms of the equations are varied (see Appendix A), but the one that is the most familiar is the "external potential" form similar in structure to the ordinary Dirac equation. For two particles interacting through world scalar and vector interactions they are

$$\begin{aligned}\mathcal{S}_1\psi &\equiv \gamma_{51}(\gamma_1\cdot(p_1 - \tilde{A}_1) + m_1 + \tilde{S}_1)\Psi = 0,\\ \mathcal{S}_2\psi &\equiv \gamma_{52}(\gamma_2\cdot(p_2 - \tilde{A}_2) + m_2 + \tilde{S}_2)\Psi = 0.\end{aligned} \qquad (14)$$

Here Ψ is a 16 component wave function consisting of an external plane wave part that is an eigenstate of P and an internal part $\psi = \psi(x_\perp)$. The vector potential $\tilde{A}_i^\mu$ was taken to be an electromagnetic-like four-vector potential with the time and spacelike portions both arising from a single invariant function A.[3] The tilde on these four-vector potentials, as well as on the scalar ones $\tilde{S}_i$, indicates that they are not only position dependent but also spin-dependent by way of the gamma matrices. In this paper, we allow for the presence of a timelike portion arising from an independent invariant function V.[4] In either case, the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ must commute, or at the very least $[\mathcal{S}_1, \mathcal{S}_2]\psi = 0$, since they operate on the same wave function.[5] This compatibility condition gives restrictions on the spin dependence which the vector and scalar potentials

$$\begin{aligned}\tilde{A}_i^\mu &= \tilde{A}_i^\mu(A(r), V(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2),\\ \tilde{S}_i &= \tilde{S}_i(S(r), A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2).\end{aligned} \qquad (15)$$

are allowed to have[6] in addition to requiring that they depend on the invariant separation $r \equiv \sqrt{x_\perp^2}$ through the invariants $A(r)$, $V(r)$, and $S(r)$. The covariant constraint (3) can also be shown to follow from Eq. (14). We give the explicit connections between $\tilde{A}_i^\mu$, $\tilde{S}_i$ and the invariants $A(r)$, $V(r)$, and $S(r)$ in Appendix A. The Pauli reduction of these coupled Dirac equations lead to a covariant Schrödinger-like equation for the relative motion with an explicit spin-dependent potential Φ_w,[7]

$$\begin{aligned}&(p_\perp^2 + \Phi_w(A(r), V(r), S(r), p_\perp, \hat{P}, w, \sigma_1, \sigma_2))\psi_+\\ &\quad = b^2(w)\psi_+,\end{aligned} \qquad (16)$$

with $b^2(w)$ playing the role of the eigenvalue.[8] This eigenvalue equation can then be solved for the four-component effective particle spinor wave function ψ_+ related to the 16 component spinor $\psi(x_\perp)$ in Appendix A.

The set of Eq. (14) and the equivalent Schrödinger-like Eq. (16) possesses a number of important and desirable features. First, they reduce to the correct one-body Dirac form when one of the two constituents becomes very massive. (The Salpeter equation does not have this important property.) Second, the generalized three-dimensional Schrödinger Eq. (16) is quite similar to the nonrelativistic Schrödinger equation and it indeed goes over to the correct nonrelativistic Schrödinger equation in the limit of weak binding. One can thus employ familiar techniques to obtain its solutions. Third, Eq. (16) can be solved nonperturbatively for both QED bound states (e.g. positronium and muonium) and QCD bound states (i.e. bound states obtained from two-body relativistic potential models for mesons) since every term in Φ_w is nonsingular in the sense that they are less attractive than $-1/4r^2$ (no delta functions or attractive $1/r^3$ potentials, for example). Thus, unlike with the $1/m$ nonrelativistic and semirelativistic expansions, the covariant Dirac formalism itself introduces natural cutoff factors that smooth out singular spin-dependent interactions, there being no need to introduce them by hand (- see [28,29,32,33] and Sec. VI D) as in other approaches.

[3]In a perturbative context, i.e. for weak potentials, that would mean that this aspect of $\tilde{A}_i^\mu$ is regarded as arising from a Feynman gauge vertex coupling of a form proportional to $\gamma_1^\mu\gamma_{2\mu}A$ (see Appendix A).

[4]In a perturbative or weak potential context, that would mean that this aspect of $\tilde{A}_i^\mu$ is regarded as arising from an additional vertex coupling proportional to $-\gamma_1\cdot\hat{P}\gamma_2\cdot\hat{P}V$. Similarly, in a perturbative or weak potential context, $\tilde{S}_i$ is regarded as arising from a vertex coupling proportional to $1_1 1_2 S$. (See Appendix A).

[5]The γ_5 matrices for each of the two particles are designated by γ_{5i} $i = 1, 2$. The reason for putting these matrices in front of the whole expression is that including them facilitates the proof of the compatibility condition, see [24,28].

[6]The dependence of the scalar potentials $\tilde{S}_i$ on the invariant $A(r)$ responsible for the electromagnetic-like potential is seen in [24,26] to result from the way the scalar and vector fields combine. That combination without the presence of the independent timelike portion leads to a two-body Klein-Gordon-like potential portion of Φ_w to be of the form given in Eq. (6).

[7]In the presence of an additional and independent timelike vector interaction V, we assume the scalar and vector fields combine in such a way that leads to a two-body Klein-Gordon-like potential portion of Φ_w of the form $2m_wS + S^2 + 2\varepsilon_wA - A^2 + 2\varepsilon_wV - V^2$ instead of that given in Eq. (6).

[8]Because of the dependence of Φ_w on w, this is a nonlinear eigenvalue equation.

Fourth, the relativistic potentials appearing in these equations are directly related through Eq. (2) to the interactions of perturbative quantum field theory, while for QCD bound states they may be introduced semiphenomenologically through $A(r)$ and $S(r)$ (and in this paper $V(r)$). Fifth, these equations have been tested analytically [34] and numerically [29] against the known perturbative fine and hyperfine structures of QED bound states and related field theoretic bound states. The (nonperturbative) successes with these QED bound states provide strong motivation for applying the constrained Dirac formalism to meson bound states as in [2]. Sixth, these equations provide a covariant three-dimensional framework in which the local potential approximation consistently fulfills the requirements of gauge invariance in QED [35]. Finally, the same general structures of the Darwin, spin-orbit, spin-spin and tensor terms in $\Phi_w(A, V, S)$ of Eqs. (16) and (35), responsible for the accurate hyperfine structures of QED bound states arising naturally from the TBDE formalism when $A = -\alpha/r$ and $S = V = 0$, are used with the only alteration in its application to the QCD bound states being that A, V, and S are apportioned appropriately from the Adler-Piran potential.

We emphasize the importance of the nonperturbative QED bound state test. (See Appendix C for a review of the application of the TBDE to QED.) Many of the wave equations used in the standard approaches to QED bound states have been modified to include QCD inspired potentials and then applied to nonperturbative numerical calculations of QCD bound states without first testing those approaches nonperturbatively in QED. By this we mean that the accepted perturbative results of those equations (QED spectral results correct through order α^4) have not been replicated using numerical methods. Sommerer *et al.*. [4] have shown that the Blankenbecler-Sugar equation and the Gross equations fail this test. This indicates a danger in applying such three-dimensional truncations of the Bethe-Salpeter equation: if failure occurs in their applications to QED bound states this brings into question the spectral results of similar nonperturbative (i.e. numerical) approaches based on the same truncations when applied to QCD bound states. This would be true especially when the only difference between the vector portions of the QED and QCD potentials would be the replacing of the QED $-\alpha/r$ by a similar $A(r)$ from QCD.

In [2] we presented details of the application of this formalism to meson spectroscopy using a covariant version of the Adler-Piran static quark potential. Note especially that the equations used there displayed a single $\Phi(A(r), S(r), p_\perp, \hat{P}, w, \sigma_1, \sigma_2,)$ in Eq. (16). It depends on the quark masses through factors such as those that appear in Eq. (6). However, its dependence is the same for all quark mass ratios—hence a single structure for all the $Q\bar{Q}$, $q\bar{Q}$, and $q\bar{q}$ mesons in a single overall fit. We found that the fit provided by the TBDE for the entire meson spectrum (from the pion to the excited bottomonium states) competes with the best fits to partial spectra provided by other approaches and does so with the smallest number of interaction functions (just $A(r)$ and $S(r)$) without additional cutoff parameters necessary to make those approaches numerically tractable. We also found that the pion bound state displays some characteristics of a Goldstone boson. That is, as the quark mass tends to zero, the pion mass (unlike the ρ and the excited π) vanishes, in contrast to almost every other relativistic potential model. (For more discussion on this see footnote [23] below).

B. Two-body quasipotential approaches

Also presented in [2] was a detailed comparison between the meson spectroscopy results of our model and those of several other approaches: one based on the Breit equation [5], two on truncated versions of the Bethe-Salpeter equation [3,4], and one on a quasipotential approach [6]. We explore in this section, in more detail, the differences and similarities between our approach and the quasipotential approach. The quasipotential equation was first introduced by Logunov and Tavkhelidze [10]. In its homogeneous form, that equation describes a two-particle relativistic composite system with its c.m. momentum space form (in the notation used here $\mathbf{p}$ is the relative momentum given in Eq. (13)) for spinless particles given by

$$(w - \sqrt{\mathbf{p}^2 + m_1^2} - \sqrt{\mathbf{p}^2 + m_2^2})\Psi_w(\mathbf{p}) = \int V(\mathbf{p}, \mathbf{q}, w)\Psi_w(\mathbf{q})\frac{d^3q}{(2\pi)^3}, \tag{17}$$

where $\Psi_w(\mathbf{p})$ is the quasipotential wave function projected onto positive-frequency states and $V(\mathbf{p}, \mathbf{q}, w)$ is the quasipotential calculated by means of the off-energy shell scattering amplitude (so that the respective c.m. energies of the two particles are not given by the above square roots but by Eq. (12)). The corresponding inhomogeneous quasipotential equation is of the general form

$$T(\mathbf{p}, \mathbf{q}, w) + V(\mathbf{p}, \mathbf{q}, w) + \int V(\mathbf{p}, \mathbf{k}, w)G_w(\mathbf{k})V(\mathbf{k}, \mathbf{q}, w) = 0, \tag{18}$$

a linear integral equation of the Lippmann-Schwinger type relating the quasipotential to the off-energy shell extrapolation of the Feynman scattering amplitude. The choice of this equation and the accompanying homogeneous equation is not unique [11]. For example, the Green function $G_w(\mathbf{k})$ has only its imaginary part determined by requiring the condition of elastic unitarity on Eq. (18).[9] Todorov [10] took advantage of this nonuniqueness to write down a local version of the corresponding homogeneous equation of the form

[9]For Hermitian potentials, that condition has the symbolic form of $T - T^\dagger = T^\dagger(G - G^\dagger)T$.

$$(\mathbf{p}^2 - b^2)\phi(\mathbf{p}) + \frac{2\varepsilon_1\varepsilon_2}{w}\int V_w(\mathbf{p},\mathbf{k})\phi(\mathbf{k})\frac{d^3k}{(2\pi)^3} = 0, \quad (19)$$

with $\phi(\mathbf{p})$ the wave function in momentum space. In [32], Crater and Van Alstine showed that the spinless version of Eq. (16) in the case of QED ($V = S = 0$) has, in the c.m. frame, the form (see Eq. (66b) and discussion below Eq. (73b) of that paper)

$$(\mathbf{p}^2 - (\varepsilon_w - A)^2 + m_w^2 + \frac{1}{2}\nabla^2\mathcal{G} + \frac{1}{4}(\nabla\mathcal{G})^2)\psi = 0, \quad (20)$$

where

$$\mathcal{G} = \ln G, \qquad G = \frac{1}{(1-2A/w)^{1/2}}. \quad (21)$$

As was pointed out in that paper, for $A = -\alpha/r$, this reduces for weak potentials to the minimal or gauge structure form postulated by Todorov,

$$[(\mathbf{p} - \mathbf{A})^2 - (\varepsilon_w - A^0)^2 + m_w^2]\psi = 0. \quad (22)$$

Although not noticed at the time, Eq. (20) does in fact additionally have this minimal structure not only for arbitrary strength couplings but also for potentials not restricted to Coulomb potentials, provided just that

$$A^0 = A, \qquad \mathbf{A} = -\frac{i}{2}\nabla\mathcal{G}I_s, \quad (23)$$

where I_s is the space reflection operator satisfying

$$I_s f(\mathbf{r}) = f(-\mathbf{r}). \quad (24)$$

Aneva, Karchev, and Rizov [36] developed the weak potential version of this two-body Klein-Gordon equation for two spin combinations: for one spin-zero and one spin-one half particle and for a spin-one-half particle-antiparticle pair. For the latter it has the form,

$$(\mathbf{p}^2 - b^2)\phi_{\lambda_1\lambda_2}(\mathbf{p}) + \frac{2\varepsilon_1\varepsilon_2}{w}\int \bar{u}_{\lambda_1'}(\mathbf{p})\bar{v}_{\lambda_2}(-\mathbf{k})\mathbb{V}_w(\mathbf{p},\mathbf{k}) \times v_{\lambda_2'}(-\mathbf{p})u_{\lambda_1}(\mathbf{k})\phi_{\lambda_1'\lambda_2'}(\mathbf{k})\frac{d^3k}{(2\pi)^3} = 0. \quad (25)$$

Expressing the on shell free four-component Dirac spinors in terms of two-component Pauli spinors and assuming the local quasipotentials

$$\begin{aligned}\mathbb{V}_w(\mathbf{p},\mathbf{k}) &= \mathbb{V}_w(\mathbf{p}-\mathbf{k}) \\ &= \mathcal{A}(\mathbf{p}-\mathbf{k})\gamma_1^\mu\gamma_{2\mu} + \mathcal{V}(\mathbf{p}-\mathbf{k})\beta_1\beta_2 \\ &\quad + \mathcal{S}(\mathbf{p}-\mathbf{k})1_1 1_2, \end{aligned} \quad (26)$$

Equation (25) can be brought to a four-component wave equation form superficially similar to Eq. (16) in the c.m. frame. We write it as

$$(\mathbf{p}^2 + \mathcal{V}_w(A(r), V(r), S(r), \mathbf{p}, w, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2))\phi(\mathbf{r}) = b^2(w)\phi(\mathbf{r}). \quad (27)$$

However, there are distinct differences. First of all, the spin structure of Φ_w is not identical to that of $\mathcal{V}_w$ even if the functions $A(r)$, $V(r)$, and $S(r)$ are the same. The reason is that the spin dependence of the vector and scalar potentials $\tilde{A}_i^\mu$ and $\tilde{S}_i$ and, in particular, the minimal type of context in which they appear in Eqs. (14) arise from (nonlinear) hyperbolic functions (see Appendix A and [37,38]) of matrices such as appear in Eq. (26). Without that hyperbolic structure the external potential forms in which the minimal structures appear would be absent. To reproduce the effects of those nonlinear functions in Eq. (27) one would have to supplement Eq. (26) with types of invariants other than just scalar and vector [26,39,40] (a pseudovector invariant, for example). Second, the desirable minimal scalar structures as appear in the first line on the right hand side of Eq. (35) below would not appear in $\mathcal{V}_w$ without including higher order diagrams that would again require invariants other than just scalar and vector. These minimal scalar structures are not only desirable, they arise naturally and strictly from $O(1/c^2)$ expansions of classical and quantum field theoretic potentials [25,26] and from gauge invariance considerations (see Todorov in [10], and also [24,36,41]). In a later section we discuss the recent work of [7], which uses a quasipotential equation similar to Eq. (27) in meson spectroscopy calculations.

III. THE ADLER-PIRAN POTENTIAL FOR THE TWO-BODY DIRAC EQUATIONS

In this section, we use the relativistic Schrödinger-like Eq. (16) to construct a relativistic naive quark model by choosing the three invariant functions A, V, and S to incorporate the Adler-Piran static quark potential [1]. This potential was originally obtained from the QCD field theory through a nonlinear effective action model for heavy quark statics. Adler and Piran used the renormalization group approximation to obtain both total flux confinement and a linear static potential at large distances. Their model uses nonlinear electrostatics with displacement and electric fields related through a nonlinear constitutive equation with the effective dielectric constant given by a leading log-log model which fixes all parameters in their model apart from a mass scale Λ. Their static potential also contains an unknown "integration constant" U_0 in the final form of their potential (hereafter called $V_{AP}(r)$). We insert into Eq. (16) invariants A, V, and S with forms determined so that the sum $A + V + S$ appearing as the potential in the nonrelativistic limit of our equations becomes the Adler-Piran nonrelativistic $Q\bar{Q}$ potential (which depends on two parameters Λ and U_0) plus the Coulomb interaction between the quark and antiquark. That is,

$$V_{AP}(r) + V_{\rm coul} = \Lambda(U(\Lambda r) + U_0) + \frac{e_1 e_2}{r} = A + V + S. \quad (28)$$

As determined by Adler and Piran, the short and long distance behaviors of $U(\Lambda r)$ generate known lattice and continuum results through the explicit appearance of an effective running coupling constant in coordinate space. That is, the Adler-Piran potential incorporates asymptotic freedom through

$$\Lambda U(\Lambda r \ll 1) \sim 1/(r \ln \Lambda r), \tag{29}$$

and linear confinement through

$$\Lambda U(\Lambda r \gg 1) \sim \Lambda^2 r. \tag{30}$$

In addition to obtaining these leading behavior analytic forms for short and long distances, they converted the numerically obtained values of the potential at all distances (short, intermediate, and long distances) to compact analytic expressions. The explicit closed form expressions [1] for $U(\Lambda r)$ are different for each of the four regions, and are linked continuously. Letting $x = \Lambda r$,

$$\begin{aligned}
U(x) &= -(16\pi/27)(1 + a_1 x^{a_2}) f(w_p)/w_p, \\
&\quad 0 < x \leq 0.0125, \\
w_p &= 1/(a_3 x)^2, \\
U(x) &= K + \alpha(x/0.125)^E, \qquad 0.0125 \leq x \leq 0.125, \\
E &= \beta + \gamma \ln(1/x) + \delta(\ln(1/x))^2 + \varepsilon(\ln(1.x))^3 \\
U(x) &= K' + \alpha' \ln x + \beta'(\ln x)^2 + \gamma'(\ln x)^3 + \delta'(\ln x)^4 \\
&\quad + \varepsilon'(\ln x)^4, \qquad 0.125 \leq \Lambda r \leq 2, \\
U(x) &= \Lambda\left(c_1 x + c_2 \ln x + \frac{c_3}{\sqrt{x}} + \frac{c_4}{x} + c_5\right), \\
&\quad 2 \leq \Lambda r < \infty.
\end{aligned} \tag{31}$$

The function f is defined by $w_p = f(\ln f + \zeta \ln \ln f)$; $\zeta = 2(51 - 19n_f/3)/(11 - 2n_f/3)^2 = 64/81$ for $n_f = 3$. The various constants a_1 to a_3, K, α, β, γ, δ, ε, K', α', β', γ', δ', ε', and c_1 to c_5 are given by the Adler-Piran leading log-log model [1] and are not adjustable parameters. We modify these closed forms so that the connections between different regions are continuous in second derivatives. The nonrelativistic analysis used by Adler and Piran, however, does not determine the relativistic transformation properties of the potential. How this potential is apportioned between vector and scalar is therefore somewhat, although not completely, arbitrary.

In earlier work [13], we divided the potential in the following way among three relativistic invariants A, V, and S for all $x = \Lambda r$. (In our former construction, the additional invariant V was responsible for a possible independent timelike vector interaction.)

$$\begin{aligned}
S &= \eta\left(\Lambda\left(c_1 x + c_2 \ln(x) + \frac{c_3}{\sqrt{x}} + c_5 + U_0\right)\right), \\
V &= (1 - \eta)\Lambda\left(c_1 x + c_2 \ln(x) + \frac{c_3}{\sqrt{x}} + c_5 + U_0\right), \\
A &= U(x) - \Lambda\left(c_1 x + c_2 \ln(x) + \frac{c_3}{\sqrt{x}} + c_5\right),
\end{aligned} \tag{32}$$

in which $\eta = \frac{1}{2}$. That is, we assumed that (with the exception of the Coulomb-like term (c_4/x)) the long distance part was equally divided between scalar and a proposed timelike vector.

In the present investigation, we compute the best fit meson spectrum for the following apportionment of the Adler-Piran potential:

$$\begin{aligned}
A &= \exp(-\beta\Lambda r)\left[V_{\mathrm{AP}} - \frac{c_4}{r}\right] + \frac{c_4}{r} + \frac{e_1 e_2}{r}, \\
V + S &= V_{\mathrm{AP}} + \frac{e_1 e_2}{r} - A \\
&= \left(V_{\mathrm{AP}} - \frac{c_4}{r}\right)(1 - \exp(-\beta\Lambda r)) \equiv \mathcal{U},
\end{aligned} \tag{33}$$

In order to covariantly incorporate the Adler-Piran potential into our equations, we treat the short distance portion as purely electromagnetic-like (in the sense of the transformation properties of the potential). The attractive ($c_4 = -0.58$) QCD-Coulomb-like portion (not to be confused with the electrostatic $V_{\mathrm{coul}} = e_1 e_2/r$) is assigned completely to the electromagnetic-like part A. That is, the constant portion of the running coupling constant corresponding to the exchange diagram is expected to be electromagnetic-like ($\sim \gamma_{1\mu}\gamma_2^{\mu}$). Through the additional parameter β, the exponential factor gradually turns off the electromagnetic-like contribution (i.e. A) to the potential at long distance except for the $1/r$ portion mentioned above, while the scalar and timelike portions (i.e. S and V) gradually turn on, becoming fully responsible for the linear confining and subdominant terms at long distance. We choose not to consider an apportionment function with a large number of parameters as the simple exponential gives a single length scale for the turning of the potential from electromagnetic-like to scalar and timelike. Altogether our three invariant potential functions depend on three parameters: Λ, U_0, and β. We furthermore let a free parameter ξ divide the relative portions of $\mathcal{U}$ as follows

$$\begin{aligned}
S &= \xi\mathcal{U} = \xi\left(V_{AP} - \frac{c_4}{r}\right)(1 - \exp(-\beta\Lambda r)), \\
V &= \mathcal{U} - \mathcal{S} = (1 - \xi)\left(V_{AP} - \frac{c_4}{r}\right)(1 - \exp(-\beta\Lambda r)).
\end{aligned} \tag{34}$$

This differs from the division in the earlier work [13]. Also, the earlier work did not include the effects of the tensor interaction or spin-orbit difference terms or the $u - d$ quark mass differences[10] (see Eq. (35) below). In [2],

[10] In the present treatment, we treat the entire interaction present in our equations, thereby keeping each of these effects. In our former treatment [13], we also performed a decoupling between the upper-upper and lower-lower components of the wave functions for spin-triplet states which turned out to be defective but which we subsequently corrected in our numerical test of our formalism for QED [29].

Crater and Van Alstine chose $\xi = 1$ and thus assumed that the scalar interaction is solely responsible for the long distance confining terms.

When inserted into the constraint equations, V, S, and A become relativistic invariant functions of the invariant separation $r = \sqrt{x_\perp^2}$. The covariant structures of the constraint formalism then automatically determine the exact forms by which the central static potential is supplemented with accompanying relativistic spin-dependent and recoil terms.

IV. MESON SPECTROSCOPY FROM THE SCHRÖDINGER-LIKE FORM OF THE TWO-BODY DIRAC EQUATIONS

A. Center of momentum form of covariant Pauli-Schrödinger reduction of the two-body Dirac equations

In Appendix A we outline the steps needed to obtain the explicit c.m. form of Eq. (16). That form is [2,40,42],

$$\begin{aligned}\{\mathbf{p}^2 + \Phi(\mathbf{r}, m_1, m_2, w, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)\}\psi_+ = \{&\mathbf{p}^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2 + 2\varepsilon_w V - V^2 + \Phi_{\mathrm{D}} \\ &+ \mathbf{L}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{\mathrm{SO}} + \boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\mathbf{L}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\Phi_{\mathrm{SOT}} + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\Phi_{\mathrm{SS}} \\ &+ (3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}} - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Phi_{\mathrm{T}} + \mathbf{L}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\Phi_{\mathrm{SOD}} + i\mathbf{L}\cdot\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\Phi_{\mathrm{SOX}}\}\psi_+ \\ = b^2\psi_+. \end{aligned} \quad (35)$$

The detailed forms of the separate quasipotentials Φ_i are given in Appendix A together with their forms for weak potential and in the static limit. In Appendix B we give the radial forms of Eq. (35). The subscripts of most of the quasipotentials are self explanatory.[11] After the eigenvalue b^2 of (35) is obtained, the invariant mass of the composite two-body system w can then be obtained by inverting Eq. (10). It is given explicitly by

$$w = \sqrt{b^2 + m_1^2} + \sqrt{b^2 + m_2^2}. \quad (36)$$

The structure of the linear and quadratic terms in Eq. (35), as well as the Darwin and spin-orbit terms, are plausible in light of the discussion given above Eq. (6), and in light of the static limit Dirac structures that come about from the Pauli reduction of the Dirac equation (see Eq. (75) below for the two-component Pauli reduction of the Dirac equation). Their appearances as well as that of the remaining spin structures are direct outcomes of the Pauli reductions of the simultaneous TBDE Eq. (14).

B. Spectral results

Theory 1 (abbreviated by Th1) has two invariant interaction functions: $A(r)$ for the short distance behavior and $S(r)$ for scalar confinement. Theory 2 (abbreviated Th2) has three invariant interaction functions including the previous two plus $V(r)$ for timelike vector confinement. In Appendix A 1 we outline how these invariant interaction functions are related to what we call vertex invariants ($\mathcal{J}(r)$, $\mathcal{G}(r)$, $\mathcal{L}(r)$), which define the covariant gamma matrix interaction structures which enter into the hyperbolic form of the TBDE as seen in Eqs. (A2), (A3), and (A7)–(A12), and the related energy and mass potentials $E_{1,2}$ and $M_{1,2}$, which characterize the external potential forms of the TBDE as seen in Eqs. (A13)–(A16). The distinction between Th1 and Th2 includes more than the partial cancellation $S^2 - V^2 = \mathcal{U}^2(2\xi - 1)$ between the quadratic scalar and timelike vector interactions. It also includes their partial cancellations for the spin-orbit and Darwin terms (see Appendix A 4). In this section, we present spectral results with (Th2) and without (Th1) the added timelike invariant function $V(r)$.

We display our results in Tables I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII, XIII, XIV, XV, and XVI. The first table lists the best fit values for the quark masses and the potential parameters Λ, ΛU_0, β for Th1 (scalar only confinement) and Th2. The ratio that optimized the fit for Th2 is $\xi = 0.704$, (see Eq. (34)). In the first two columns

TABLE I. Parameters for Theory 1 and Theory 2. The 4 potential parameters Λ, ΛU_0, $1/(\beta\Lambda)$, and ξ are, respectively, the QCD scale factor, the Adler-Piran integration constant, the vector-scalar transition distance, and the $S/(S+V)$ ratio.

Parameter	Th1	Th2
m_b	4.917 GeV	4.953 GeV
m_c	1.546 GeV	1.585 GeV
m_s	0.2874 GeV	0.3079 GeV
m_u	0.0713 GeV	0.0985 GeV
m_d	0.0771 GeV	0.1045 GeV
Λ	0.2213 GeV	0.2255 GeV
ΛU_0	1.815 GeV	1.770 GeV
β	1.502	4.408
ξ	1	0.704

[11]The subscript on quasipotential Φ_{D} refers to Darwin. It consist of what are called Darwin terms, those that are the two-body analogue of terms that accompany the spin-orbit term in the one-body Pauli reduction of the ordinary one-body Dirac equation, and ones related by canonical transformations to Darwin interactions [25,43], momentum dependent terms arising from retardation effects. The subscripts on the other quasipotentials refer, respectively, to SO (spin-orbit), SOD (spin-orbit difference), SOX (spin-orbit cross terms), SS (spin-spin), T (tensor), SOT (spin-orbit-tensor)

TABLE II. $u\bar{d}$ Mesons, Theory 1 and Theory 2—In this table and the ones below, the meson masses are in units of GeV, with experimental errors given parenthetically in units of MeV.

$u\bar{d}$ mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
π: $u\bar{d}1^1S_0$	0.140(0.0)	0.141	0.134	−0.002	0.006	0.0	0.3
ρ: $u\bar{d}1^3S_1$	0.775(0.4)	0.790	0.781	−0.015	−0.005	1.9	0.2
b_1: $u\bar{d}1^1P_1$	1.230(3.2)	1.283	1.243	−0.053	−0.014	2.5	0.2
a_1: $u\bar{d}1^3P_1$	1.230(40.)	1.425	1.320	−0.195	−0.090	0.2	0.1
π: $u\bar{d}2^1S_0$	1.300(100)	1.493	1.435	−0.193	−0.135	0.0	0.0
a_2: $u\bar{d}1^3P_2$	1.318(0.6)	1.276	1.310	0.042	0.008	12.9	0.5
ρ: $u\bar{d}2^3S_1$	1.465(25.)	1.745	1.684	−0.280	−0.219	1.3	0.8
a_0: $u\bar{d}1^3P_0$	1.474(19.)	1.165	1.024	0.309	0.450	2.6	5.6
b_2: $u\bar{d}1^1D_2$	1.672(3.2)	1.815	1.763	−0.143	−0.090	18.2	7.2
a_3: $u\bar{d}1^3D_3$	1.689(2.1)	1.663	1.718	0.026	−0.029	1.2	1.5
a_1: $u\bar{d}1^3D_1$	1.720(20.)	1.944	1.847	−0.224	−0.127	1.2	0.4
a_2: $u\bar{d}2^3P_2$	1.732(16.)	2.025	2.009	−0.293	−0.277	3.3	3.0
π: $u\bar{d}3^1S_0$	1.816(14.)	2.090	2.037	−0.274	−0.221	3.8	2.5
b_2: $u\bar{d}2^1D_2$	1.895(16.)	2.300	2.267	−0.405	−0.372	6.4	5.4
a_4: $u\bar{d}1^3F_4$	2.011(12.)	1.984	2.057	0.027	−0.046	0.1	0.1
b_2: $u\bar{d}3^1D_2$	2.090(29.)	2.704	2.700	−0.614	−0.610	4.5	4.4
ρ: $u\bar{d}3^3S_1$	2.149(17.)	2.281	2.326	−0.132	−0.177	0.6	1.1
a_3: $u\bar{d}2^3D_3$	2.250(45.)	2.275	2.290	−0.025	−0.040	0.0	0.0
a_5: $u\bar{d}1^3G_5$	2.330(35.0)	2.258	2.349	0.072	−0.019	0.0	0.0
a_6: $u\bar{d}1^3H_6$	2.450(130)	2.500	2.609	−0.050	−0.159	0.0	0.0

of Tables II, III, IV, V, VI, VII, VIII, and IX, we list quantum numbers and experimental rest mass values (in GeV) and experimental errors listed parenthetically (in MeV) for 105 known mesons. We include all well known and plausible candidates listed in the standard reference ([44]). We omit only those mesons with substantial flavor mixing, like the η and η' mesons. In the tables, the quantum numbers listed are those of the ψ_+ part of the

TABLE III. $s\bar{u}$ and $s\bar{d}$ Mesons, Theory 1 and Theory 2.

$s\bar{u}$, $s\bar{d}$ Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
K^-: $s\bar{u}1^1S_0$	0.494(0.0)	0.485	0.519	0.008	−0.025	0.7	6.4
K^0: $s\bar{d}1^1S_0$	0.498(0.0)	0.488	0.520	0.010	−0.022	1.0	5.0
K^{*-}: $s\bar{u}1^3S_1$	0.892(0.3)	0.917	0.896	−0.025	−0.004	6.0	0.2
K^{*-}: $s\bar{d}1^3S_1$	0.896(0.3)	0.919	0.897	−0.023	−0.001	4.8	0.0
K^-: $s\bar{u}1^1P_1$	1.272(7.0)	1.356	1.339	−0.084	−0.067	1.4	0.9
K^{*-}: $s\bar{u}1^3P_1$	1.403(7.0)	1.419	1.359	−0.016	0.044	0.1	0.4
K^{*-}: $s\bar{u}2^3S_1$	1.414(15.)	1.759	1.706	−0.345	−0.292	5.3	3.8
K^{*-}: $s\bar{u}1^3P_0$	1.425(50.)	1.132	1.079	0.293	0.346	0.3	0.5
K^{*-}: $s\bar{u}1^3P_2$	1.426(50.)	1.379	1.404	0.047	0.022	6.8	1.4
K^{*-}: $s\bar{d}1^3P_2$	1.432(1.3)	1.380	1.405	0.052	0.027	10.2	2.8
K^-: $s\bar{u}2^1S_0$	1.460(40.)	1.523	1.476	−0.063	−0.016	0.0	0.0
K^{*-}: $s\bar{u}1^3D_1$	1.717(27.)	1.922	1.837	−0.205	−0.120	0.6	0.2
K^-: $s\bar{u}1^1D_2$	1.773(8.0)	1.835	1.803	−0.062	−0.030	0.6	0.1
K^{*-}: $s\bar{u}1^3D_3$	1.776(7.0)	1.740	1.792	0.036	−0.016	0.3	0.0
K^{*-}: $s\bar{u}1^3D_2$	1.816(13.)	1.824	1.795	−0.008	0.021	0.0	0.0
K^-: $s\bar{u}3^1S_0$	1.830(13.)	2.115	2.081	−0.285	−0.251	0.5	0.4
K^{*-}: $s\bar{u}2^3P_2$	1.973(33.)	2.078	2.060	−0.105	−0.087	0.1	0.1
K^{*-}: $s\bar{u}1^3F_4$	2.045(9.0)	2.045	2.117	0.000	−0.072	0.0	0.6
K^{*-}: $s\bar{u}2^3D_2$	2.247(17.)	2.326	2.313	−0.079	−0.066	0.2	0.1
K^{*-}: $s\bar{u}2^3F_3$	2.324(24.)	2.642	2.600	−0.318	−0.276	1.7	1.3
K^{*-}: $s\bar{u}1^3G_5$	2.382(14.)	2.309	2.401	0.073	−0.019	0.3	0.0
K^{*-}: $s\bar{u}2^3F_4$	2.490(20.)	2.555	2.600	−0.065	−0.110	0.1	0.3

TABLE IV. $s\bar{s}$ Mesons, Theory 1 and Theory 2.

$s\bar{s}$ Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
ϕ: $s\bar{s}1^3S_1$	1.019(0.0)	1.050	1.013	−0.031	0.006	9.3	0.4
ϕ: $s\bar{s}1^3P_0$	1.370(100)	1.211	1.175	0.159	0.195	0.0	0.0
ϕ: $s\bar{s}1^3P_1$	1.518(5.0)	1.480	1.437	0.038	0.081	0.5	2.5
ϕ: $s\bar{s}1^3P_2$	1.525(5.0)	1.496	1.506	0.029	0.019	0.3	0.1
ϕ: $s\bar{s}2^3S_1$	1.680(20.)	1.811	1.875	−0.131	−0.195	0.4	0.9
ϕ: $s\bar{s}1^3D_3$	1.854(7.0)	1.839	1.879	0.015	−0.025	0.0	0.1
ϕ: $s\bar{s}2^3P_2$	2.011(70)	2.149	2.128	−0.138	−0.117	0.0	0.0
ϕ: $s\bar{s}3^3P_2$	2.297(28.)	2.612	2.603	−0.315	−0.306	1.3	1.2

TABLE V. $c\bar{u}$ and $c\bar{d}$ Mesons, Theory 1 and Theory 2.

$c\bar{u}$, $c\bar{d}$ Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
D^0: $c\bar{u}1^1S_0$	1.865(0.2)	1.865	1.876	0.000	−0.011	0.0	1.1
D^+: $c\bar{d}1^1S_0$	1.870(0.2)	1.872	1.883	−0.003	−0.013	0.1	1.7
D^{*0}: $c\bar{u}1^3S_1$	2.007(0.2)	2.013	2.007	−0.006	0.000	0.4	0.0
D^{*+}: $c\bar{d}1^3S_1$	2.010(0.2)	2.019	2.013	−0.008	−0.002	0.7	0.1
D^{*0}: $c\bar{u}1^3P_0$	2.352(50.)	2.224	2.221	0.128	0.131	0.1	0.1
D^{*+}: $c\bar{d}1^3P_0$	2.403(14.)	2.232	2.230	0.171	0.173	1.5	1.5
D^+: $c\bar{d}1^3P_2$	2.460(3.0)	2.398	2.414	0.062	0.046	3.9	2.1
D^{*0}: $c\bar{u}1^3P_2$	2.461(1.6)	2.393	2.409	0.069	0.052	13.2	7.7

TABLE VI. $c\bar{s}$ Mesons, Theory 1 and Theory 2.

$c\bar{s}$ Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
D_s: $c\bar{s}1^1S_0$	1.968(0.3)	1.972	1.974	−0.004	−0.006	0.1	0.3
D_s^*: $c\bar{s}1^3S_1$	2.112(0.5)	2.138	2.119	−0.026	−0.007	5.4	0.4
D_s^*: $c\bar{s}1^3P_0$	2.318(0.6)	2.348	2.340	−0.031	−0.022	6.9	3.5
D_s: $c\bar{s}1^1P_1$	2.535(0.3)	2.505	2.499	0.030	0.036	8.3	11.6
D_s^*: $c\bar{s}1^3P_2$	2.573(0.9)	2.534	2.532	0.039	0.040	8.4	8.9
D_s^*: $c\bar{s}2^3S_1$	2.690(7.0)	2.714	2.702	−0.024	−0.012	0.1	0.0

TABLE VII. $c\bar{c}$ Mesons, Theory 1 and Theory 2.

$c\bar{c}$ Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
η_c: $c\bar{c}1^1S_0$	2.980(1.2)	2.965	2.973	0.015	0.007	1.0	0.2
$J/\psi(1S)$: $c\bar{c}1^3S_1$	3.097(0.0)	3.131	3.128	−0.034	−0.031	11.4	9.7
χ_0: $c\bar{c}1^3P_0$	3.415(0.3)	3.395	3.397	0.020	0.018	3.7	3.0
χ_1: $c\bar{c}1^3P_1$	3.511(0.1)	3.506	3.505	0.005	0.006	0.2	0.4
h_1: $c\bar{c}1^1P_1$	3.526(0.3)	3.522	3.523	0.004	0.003	0.2	0.1
χ_2: $c\bar{c}1^3P_2$	3.556(0.1))	3.562	3.557	−0.006	−0.001	0.4	0.0
η_c: $c\bar{c}2^1S_0$	3.637(4.0)	3.606	3.602	0.031	0.035	0.6	0.7
$\psi(2S)$: $c\bar{c}2^3S_1$	3.686(0.0)	3.688	3.689	−0.002	−0.002	0.0	0.1
$\psi(1D)$: $c\bar{c}1^3D_1$	3.773(0.4)	3.807	3.807	−0.034	−0.034	0.9	0.9
χ_2: $c\bar{c}2^3P_2$	3.929(5.0)	3.980	3.983	−0.051	−0.054	1.0	1.1
$\psi(3S)$: $c\bar{c}3^3S_1$	4.039(10.)	4.086	4.092	−0.047	−0.053	0.2	0.3
$\psi(2D)$: $c\bar{c}2^3D_1$	4.153(3.0)	4.164	4.169	−0.011	−0.016	0.1	0.3
$\psi(4S)$: $c\bar{c}4^3S_1$	4.421(4.0)	4.410	4.426	0.011	−0.005	0.1	0.0
$\psi(3D)$: $c\bar{c}3^3D_1$	4.421(4.0)	4.467	4.483	−0.046	−0.062	1.2	2.3
$\psi(5S)$: $c\bar{c}5^3S_1$	4.800(100)	4.690	4.719	0.110	0.081	0.0	0.0
$\psi(4D)$: $c\bar{c}4^3D_1$	4.880(100)	4.735	4.764	0.145	0.116	0.0	0.0
$\psi(6S)$: $c\bar{c}6^3S_1$	5.180(100)	4.940	4.983	0.203	0.197	0.0	0.0
$\psi(5D)$: $c\bar{c}5^3D_1$	5.290(100)	4.977	5.020	0.350	0.270	0.1	0.1

TABLE VIII. $b\bar{u}$, $b\bar{d}$ and $b\bar{s}$ Mesons, Theory 1 and Theory 2.

$b\bar{u}$, $b\bar{d}$ $b\bar{s}$ Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
B^-: $b\bar{u}1^1S_0$	5.279(0.3)	5.281	5.283	−0.002	−0.004	0.0	0.2
B^0: $b\bar{d}1^1S_0$	5.280(0.3)	5.282	5.284	−0.003	−0.005	0.1	0.2
B^{*-}: $b\bar{u}1^3S_1$	5.325(0.5)	5.335	5.333	−0.010	−0.008	0.8	0.5
B^{*-}: $b\bar{u}1^3P_2$	5.747(2.9)	5.671	5.687	0.076	0.059	6.2	3.8
B_s^0: $b\bar{s}1^1S_0$	5.366(0.6)	5.373	5.367	−0.007	−0.001	0.3	0.0
B_s^{*0}: $b\bar{s}1^3S_1$	5.413(1.3)	5.441	5.430	−0.029	−0.017	3.0	1.0
B_s^{*0}: $b\bar{s}1^3P_1$	5.829(0.7)	5.789	5.792	0.040	0.037	10.9	9.4
B_s^{*0}: $b\bar{s}1^3P_2$	5.840(0.6)	5.805	5.805	0.035	0.035	8.9	9.0
B_c^-: $b\bar{c}1^1S_0$	6.276(21.)	6.249	6.251	0.027	0.025	0.4	0.4

TABLE IX. $b\bar{b}$ Mesons, Theory 1 and Theory 2.

$b\bar{b}$ Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
η_b: $b\bar{b}1^1S_0$	9.389(4.0)	9.337	9.330	0.052	0.059	1.6	2.0
$\Upsilon(1S)$: $b\bar{b}1^3S_1$	9.460(0.3)	9.444	9.444	0.016	0.017	2.4	2.6
χ_{b0}: $b\bar{b}1^3P_0$	9.859(0.4)	9.836	9.834	0.023	0.026	4.6	5.6
χ_{b1}: $b\bar{b}1^3P_1$	9.893(0.3)	9.886	9.886	0.007	0.007	0.4	0.4
χ_{b2}: $b\bar{b}1^3P_2$	9.912(0.3)	9.922	9.920	−0.010	−0.008	0.9	0.6
$\Upsilon(2S)$: $b\bar{b}2^3S_1$	10.023(0.3)	10.022	10.022	0.001	0.002	0.0	0.0
$\Upsilon(D)$: $b\bar{b}2^3D_2$	10.161(0.6)	10.178	10.179	−0.017	−0.018	2.1	2.3
χ_{b0}: $b\bar{b}2^3P_0$	10.232(0.4)	10.230	10.229	0.002	0.003	0.1	0.1
χ_{b1}: $b\bar{b}2^3P_1$	10.255(0.5)	10.261	10.262	−0.006	−0.007	0.3	0.4
χ_{b2}: $b\bar{b}2^3P_2$	10.269(0.4)	10.284	10.286	−0.015	−0.017	1.9	2.5
$\Upsilon(3S)$: $b\bar{b}3^3S_1$	10.355(0.6)	10.366	10.368	−0.011	−0.013	0.8	1.2
$\Upsilon(4S)$: $b\bar{b}4^3S_1$	10.579(1.2)	10.626	10.633	−0.046	−0.053	8.8	11.7
$\Upsilon(5S)$: $b\bar{b}5^3S_1$	10.865(8.0)	10.844	10.857	0.021	0.008	0.1	0.0
$\Upsilon(6S)$: $b\bar{b}6^3S_1$	11.019(8.0)	11.036	11.055	−0.017	−0.036	0.0	0.2

16-component wave function. In the third and fourth columns are the theoretical values, with Th1 referring to the results without the timelike vector interaction and Th2 with the timelike vector interaction. In the fifth and sixth columns, we give the differences between our theoretical results and the experimental, and in the last two columns the contributions of each theoretically computed value to the total χ^2 of 237 for Th1 and 173 for Th2.[12]

To generate the fits, in addition to varying the five quark masses we vary the parameters Λ, U_0, β, and ξ in the apportioned static Adler-Piran potential in A, V, and S. Those invariants are put into our relativistic wave equations just as we have inserted the invariant Coulomb potential $A = -\alpha/r$ (but with $V = S = 0$) to obtain the results of QED bound states [29,34]. Note especially that we use a single $\Phi(A, S)$ for Th1 and a single $\Phi(A, V, S)$ for Th2 for all quark mass ratios. Hence in each theory, we use a single structure for all the $Q\bar{Q}$, $q\bar{Q}$, *and* $q\bar{q}$ mesons in a single overall fit. The entire confining part of the potential

TABLE X. χ^2 by Family for Theory 1 and Theory 2.

	χ^2			Average χ^2	
Meson Family	Th1	Th2	# Mesons	Th1	Th2
$u\bar{d}$	62.1	30.5	20	3.1	1.5
$s\bar{u}$, $s\bar{d}$	37.7	26.5	22	1.8	1.3
$s\bar{s}$	11.0	4.5	8	1.4	0.6
$c\bar{u}$, $c\bar{d}$	20.9	14.6	8	2.6	1.8
$c\bar{s}$	29.4	26.6	6	4.9	4.4
$c\bar{c}$	28.2	23.4	18	1.6	1.3
$b\bar{u}$, $b\bar{s}$, $b\bar{c}$	31.2	25.9	9	3.5	2.9
$b\bar{b}$	24.8	25.1	14	1.8	1.8
Total	245.3	177.1	105	2.4	1.7

[12]The reader of [2] may notice that the total χ^2 of the model there of 101 (corresponding to Th1 here) was substantially lower than the 237 that we found here. There are several reasons for this difference. The main reason is that in this paper we do not include a 5% addition to the calculational uncertainty based on the total meson widths. Second, there are 19 more mesons in the present model, many of which were difficult to fit. Third, the experimental errors changed. Fourth, many of the newer mesons (for example the η_b) not only added more to the χ^2 from their own fit, but also indirectly to the older ones (e.g. the 1^3S_1 Υ meson). There is no difference in the parameter sets and potentials in Th1 and those used in [2] although the values are different.

TABLE XI. Ground State Singlet-Triplet Splittings (MeV).

Family	Exp.	Th1.	Th2.
$u\bar{d}$	635	649	647
$s\bar{u}$	398	432	377
$s\bar{d}$	398	431	377
$c\bar{u}$	142	148	131
$c\bar{d}$	140	147	130
$c\bar{s}$	144	166	145
$c\bar{c}$	117	166	155
$b\bar{u}$	46	54	50
$s\bar{s}$	47	68	63
$b\bar{b}$	71	107	114

TABLE XII. Spin-Orbit Splitting R Ratios.

Family	Exp.	Th1.	Th2.
$u\bar{d}$	−0.36	−0.57	−0.03
$s\bar{u}$	−1.05	−0.14	0.16
$s\bar{s}$	0.05	0.06	0.26
$c\bar{c}$	0.47	0.50	0.48
$b\bar{b}$ $(1^3P_{2,1,0})$	0.56	0.72	0.65
$b\bar{b}$ $(2^3P_{2,1,0})$	0.61	0.74	0.73

TABLE XIII. Splitting Between $^1P_1$ and weighted triplet states (MeV).

Family	Exp.	Th1.	Th2.
$u\bar{d}$	76	30	36
$s\bar{u}$	146	9	14
$c\bar{c}$	−1	3	−1

TABLE XIV. Tensor Term Mixing Between Orbital D and Radial S Excitations of the Spin-Triplet Ground States (in MeV).

Family	Exp.	Th1.	Th2.
$u\bar{d}$ $(1^3D_1 - 2^3S_1)$	255	199	163
$s\bar{u}$ $(1^3D_1 - 2^3S_1)$	303	163	131
$c\bar{c}$ $(1^3D_1 - 2^3S_1)$	87	119	118
$c\bar{c}$ $(2^3D_1 - 3^3S_1$	114	78	77

transforms as a world scalar for Th1 and combined time-like and scalar for Th2. Since $\xi > 0.5$ in our equations, this structure leads in both models to linear confinement at long distances and quadratic confinement at extremely long distances (where the quadratic contribution S^2 outweighs the linear term $2m_wS$ in Th1 and $S^2 - V^2$ outweighing the linear terms $2m_wS + 2\varepsilon_wV$ in Th2). At distances at which $\exp(-\beta\Lambda r) \ll 1$, the corresponding fine and hyperfine structures producing spin-orbit, Thomas, Darwin, spin-spin, and tensor terms (the last two are relatively small in that domain) are dominated by the confining interaction, while at short distances $(\exp(-\beta\Lambda r) \sim 1)$ the electromagnetic-like portion of the interaction gives the dominant contribution to the fine and hyperfine structures. Furthermore because the signs of each of the spin-orbit and Darwin terms in the Pauli form of our TBDE are opposite for the scalar and vector interactions (see Appendix A 4), the spin-orbit contributions of those parts of the interaction produce opposite effects with degrees of cancellation depending on the size of the quarkonia atom. Another point to make is that because of the various sizes of the quarkonia atoms and the c.m. energy dependence the behavior of Φ_w is sharply different for the light mesons compared with the heavy ones. This may possibly account for the ability of our formalism to obtain good fits for the light meson hyperfine splittings while at the same time giving good overall fits to the heavy mesons.

We obtain the meson masses given in columns three and four as the result of a least squares fit using the known experimental masses and errors from the Particle Data Group (PDG) tables [44] and an assumed calculational error of 1.0 MeV. We employ the calculational error not to represent the uncertainty of our algorithm but more to prevent the mesons that are stable with respect to the strong interaction from being weighted too heavily. Our χ^2 is per datum (105) minus parameters (8 or 9). In Table I, the value of β for Th1 implies that (in the best fit) as the quark separation increases, our apportioned Adler-Piran potential switches from primarily vector to scalar at about $(\beta\Lambda)^{-1} \sim 0.60$ fermi. This shift is a relativistic effect since the effective nonrelativistic limit of the potential $(\mathcal{A} + S)$ exhibits no such shift (i.e., by construction β drops out). For Th2, this distance is substantially less, $(\beta\Lambda)^{-1} \sim 0.22$ fermi.

TABLE XV. Radial Excitations (MeV).

Family	Exp. Difference	Th1. Difference	Th2. Difference
$u\bar{d}$, π: 1, 2, 3^1S_0	1160, 516	1352, 597	1301, 602
$u\bar{d}$ ρ: 1, 2, 3^3S_1	690, 684	955, 536	903, 642
$s\bar{u}$ K^-: 1, 2, 3^1S_0	966, 370	1038, 592	957, 605
$s\bar{u}$ K^{*-}: 1, 2^2S_1	522	842	810
$s\bar{s}$ ϕ: 1, 2^2S_1	661	761	862
$c\bar{c}$ η_c: 1, 2^1S_0	657	641	629
$c\bar{c}$ ψ: 1, 2, 3^3S_1	589, 353	557, 398	561, 403
$b\bar{b}$ Υ: 1, 2, 3, 4, $5{,}6^3S_1$	563, 332, 224, 286, 154	578, 344, 260, 218, 192	578, 346, 265, 224 198

TABLE XVI. Isospin Splitting (MeV).

Family	Exp	Theory 1	Theory 2
$s\bar{d} - s\bar{u}$: 1^1S_0	4	3	0
$s\bar{d} - s\bar{u}$: 1^3S_1	4	2	1
$s\bar{d} - s\bar{u}$: 1^3P_2	6	1	1
$c\bar{d} - c\bar{u}$: 1^1S_0	5	7	7
$c\bar{d} - c\bar{u}$: 1^3S_1	3	6	6
$c\bar{d} - c\bar{u}$: 1^3P_0	51	8	9
$c\bar{d} - c\bar{u}$: 1^3P_2	1	−5	−5
$b\bar{d} - b\bar{u}$: 1^1S_0	1	2	1
$d - u$ mass		5.8	6.0

Table X lists the 8 meson families, their respective χ^2 contributions and their averages. The most striking feature is that as the quark masses increase from the lightest to the heaviest, the differences of the respective χ^2 shifts from about a factor of 2 to almost even. The heaviest mesons are also the smallest in mean radius. This means that they are less likely to experience the effects of the S^2 and $-V^2$ portions of the confining interactions. As the mesons become large, they experience more of the effects of these parts of the potentials. The most dramatic improvement from the inclusion of the timelike vector confining potential $V(r)$ (Th2) is with the light quark $u\bar{d}$ family. Referring now to Tables II, III, IV, V, VI, VII, VIII, and IX most of the improvement comes from that of the fits to the a_2, b_2 and b_1 mesons. For the $s\bar{u}$, $s\bar{d}$ family the largest improvement comes from the lowest lying $^3P_2$ mesons. The ground state singlet-triplet splitting changes from an overestimation to an underestimation. For the $s\bar{s}$ family the most significant improvement is in the ground state, although somewhat off-balanced by a worse fit for the lowest lying $^3P_1$ state. In the case of the $c\bar{u}$, $c\bar{d}$ family the main improvement is from the $^3P_2$ mesons. For the $c\bar{s}$ mesons there is a slight overall improvement for Th2 with offsetting changes for the $^3P_0$ and $^1P_1$ mesons. There is only a very slight improvement for the $c\bar{c}$ mesons. For both Th1 and Th2 the worst fit is to the J/ψ meson with a mass too large by about 30 MeV. It cannot be adjusted downward by lowering the charm mass due to the fact that other mesons in this family would be pushed further from the data. With the heavy-light family, a single b quark, there again is not much overall change and even less in the $b\bar{b}$ family although another significant improvement is in the $^3P_2$ $b\bar{u}$ state. An oddity with the $b\bar{b}$ is the sudden increase in χ^2 at the 4^3S_1 meson, the worst fit of all the mesons in terms of the incremental χ^2. Since that meson is closest to threshold, its mass will be most affected by it, whereas our theoretical model does not take threshold effects into account.

A possible explanation of why most improvements come for the $^3P_2$ states is that the effect on the spin-orbit coupling due to the Thomas terms is opposite in sign for the timelike vector and scalar mesons. Without the balancing effect of the timelike vector confining interaction, the scalar interaction enforces an inversion of the spin-orbit splittings of the light mesons that are far too distorted for the $u\bar{d}$ and $s\bar{u}$ multiplets.[13] Also, the long range scalar parts contribute oppositely in sign from the short range vector part attributable to the $A(r)$ potential.

We now examine another important feature of our method: the goodness with which our equations account for spin-dependent effects (both fine- and hyperfine- splittings). Table XI shows the best fit vs experimental ground state singlet-triplet splittings and six vs four of the ten hyperfine splittings are improved using Th2 over Th1. Both give good fits for all hyperfine ground state splittings except for the $\eta_c - \psi$ system and $\eta_b - \Upsilon$ system which for the latter over estimate the splittings by about 50%.[14] One problem with the fit for the $c\bar{c}$ system of mesons may be due to the fact that the $D^{*3}P_2$, $^1P_1$ and $D_s^{*3}P_2$ fits are significantly low while the J/ψ fit is significantly high. Lowering the c quark mass corrects the J/ψ mass while raising the D^*, D_s^* P state masses would require raising the c quark mass. Reducing one discrepancy would worsen the other, at least in our three invariant function approach.

For the spin-orbit splittings Table XII gives the R ratios $(^3P_2 - {}^3P_1)/(^3P_1 - {}^3P_0)$. Both sets of fits are very poor for the two lightest multiplets. The fact that Th1 has the same sign for the $u\bar{d}$ as the experiment values is not an indication that it gives reasonable results since the negative sign originates from the numerator instead of the denominator. Of the four remaining multiplets, Th2 gives a better fit on 3. It must be said, however, that none of the better fits are very good except for the $c\bar{c}$. From the experimental point of view the poor R value for the $u\bar{d}$ and $u\bar{s}$ may be the uncertain status of the $^3P_0$ light quark meson bound states, or, theoretically, our low $^3P_0$ theoretical meson masses. Also, the lack of any mechanism in our model to account for the effects of decay rates on level shifts undoubtedly has an effect. Another likely cause for the poorer performance of Th1 as one goes from heavier to light mesons is that the radial size of the meson grows so that the long distance interactions, in which the scalar interaction becomes

[13] Another possible source of the strange multiplet inversion, is that the observed $^3P_0$ states of 1450 for the $u\bar{d}$ and 1430 for the $u\bar{s}$ systems are, in fact not zero node states, but rather one node excited states. This may, in our formalism, give room to an interpretation of the $u\bar{d}(980)$ and the $\kappa(700\text{–}900)$ mesons as possible candidates for the zero node states. Our tables support that more than the identification of a zero node 1474 for the $u\bar{d}$ and 1425 for the $u\bar{s}$ systems. However, using the parameters on our model, we would obtain the 2^3P_0 values of 1800 and 1850 for those one node states, well above the 1474 and 1425 experimental values. So this does not appear to be a plausible alternative for either Th1 or Th2.

[14] We point out, however, that our model does much better in simultaneously working with heavy and light $q\bar{q}$ hyperfine splittings than that obtained by typical constituent (nonchiral) quark models (a major exception discussed here in detail is the quasipotential model of [7])

dominant, play a more important role. This effect is blunted by Th2 as seen in the $u\bar{d}$ numbers. In Table II The spin-orbit terms due to scalar interactions are opposite in sign and tend (at long distance) to dominate the spin-orbit terms due to vector interactions for Th1, but less so in Th2. This results in partial to full multiplet inversions as we proceed from the $s\bar{s}$ to the $u\bar{d}$ mesons. This inversion mechanism is less for Th2 than for Th1 because of the value of ξ.

The hyperfine structure of our equations also influences the splitting between the $^1P_1$ and the weighted sum $[5(^3P_2) + 3(^3P_1) + 1(^3P_0)]/9$ of bound states. Table XIII indicates the agreement of the theoretical and experimental mass differences is excellent for the $c\bar{c}$ system, too small but of the right sign for the $u\bar{s}$ and $u\bar{d}$ systems. The agreement, however, for the light systems is nevertheless considerably better than that in the case of the fine structure splitting R ratios. Note that in the case of unequal mass P states, our calculations of the two values incorporate the effects of $\mathbf{L}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$ and $\mathbf{L}\cdot\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2$ which mix spin.

Next consider the mixing due to the tensor term between orbital D and radial S excitations of the spin-triplet ground states. This mixing occurs most notably in the $c\bar{c}$, $u\bar{s}$ and $u\bar{d}$ systems. Table XIV shows that Th1 is better than Th2 although both are pretty far off the mark. For the charmonium system, the lower doublet results are high, whereas the higher doublet results are low.

Next we consider the effects of the change from Th1 to Th2 on the radial excitations. Table XV shows that, for the most part, Th1 gives slightly better results for the radially excited states, although where both theories are furthest off (the $u\bar{d}$ states) Th2 gives better results. The radially excited $u\bar{d}$ mesons have a larger mean radius than for the heavier meson and thus the temporizing effects of the $-V^2$ term tends to counteract more the increased confining potential for large r from linear to quadratic due to the S^2 terms.

Finally we comment on the isospin splittings shown in Table XVI. There are two effects we must consider here: the positive $d-u$ mass differences of about 6 MeV for both theories and the Coulomb interaction between the quarks on the order of $\alpha \times 197$ MeV or less depending on the meson sizes. The Coulomb interaction is counter to the $d-u$ mass difference for the $s\bar{d} - s\bar{u}$ and $b\bar{d} - b\bar{u}$ splittings while enhancing the $d-u$ mass difference for the $c\bar{d} - c\bar{u}$ splittings. These alternatively competing and enhancing effects are seen in the sizes of the splittings for both theories as you read down the table from the $s\bar{d} - s\bar{u}$ through the $c\bar{d} - c\bar{u}$ to the $b\bar{d} - b\bar{u}$ splitting. For the $K - K^*$ family the values for the isospin splittings are 3 and 0 MeV for Th1 and 2 vs the experimental value 4 MeV for the singlet ground states while for the triplet the isospin splittings are 2 and 1 MeV vs the experimental value 4 MeV. The experimental splitting grows for the orbital excitation (K_2^*) to 6 MeV. The probable reason for the increase is that at the larger distances, the influence of the Coulomb differences becomes small so that only the $d-u$ mass difference influences the result. Our theories do not show a similar increase for the orbital excitations. In the case of the $D^+ - D^0$ splitting our mass differences for Th1 and Th2 are 7 and 7 MeV, respectively, vs the experimental mass difference of just 5 MeV. Here we see the opposite overall effect between the combined effects of the $d-u$ mass difference and the slightly increased electromagnetic binding present in the case of the D^0 and the slightly decreased binding in the case of the D^+. Whereas in the kaon system the results are too small, for the D the results are too large. This can be partially understood since the Coulomb and $d-u$ mass differences work in concert with the Coulomb potential for these doublets. These effects work in the same way for the spin-triplet splitting resulting in the theoretical values of 6 and 6 MeV for the two theories compared with the experimental value 3 MeV. For the $^3P_2$ isodoublet we obtain -5 and -5 MeV vs about 1 for the experimental value again showing the expected opposite trend from that of the kaon system. The experimental splitting between the $^3P_0$ isodoublet of 51 MeV appears incomprehensibly large. Our two values are 8 and 9 MeV. The isospin splittings that we obtain for the spin singlet B meson system are 2 and 1 MeV for Th1 and Th2 vs 1 MeV. Here the competing effects cancel as in the kaon system only more so since the mesons are smaller and thus the Coulomb parts play a stronger role than for the kaon.

V. THE EFFECTIVE RELATIVISTIC SCHRÖDINGER EQUATION WITH FLAVOR MIXING FOR SPIN-ZERO ISOSCALAR MESONS

Consider the general eigenvalue Eq. (35) for an isoscalar meson, one with quark structure $q\bar{q}$. As seen in Appendix A the mass dependence appearing in Φ_w directly or indirectly through m_w, ε_w, ε_1, ε_2, is of four types: $m_1 m_2$, m_1^2 and m_2^2, $m_1^2 + m_2^2$ and $m_1^2 - m_2^2$. The actual isoscalar mesons consist of mixtures of three equal mass quark-antiquark pairs. We write the three separate equal mass versions of (35), using Eq. (10), together in shorthand as

$$[\mathbf{p}^2 + \Phi_w(\mathbf{r}, m_1 = m_2, w, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)]\begin{bmatrix}\psi_{u\bar{u}}\\ \psi_{d\bar{d}}\\ \psi_{s\bar{s}}\end{bmatrix} \equiv [\mathbf{p}^2 + \Phi_w(\mathbf{r}, \mathbb{M})]\begin{bmatrix}\psi_{u\bar{u}}\\ \psi_{d\bar{d}}\\ \psi_{s\bar{s}}\end{bmatrix} = \frac{1}{4}(w^2 - 4\mathbb{M}^2)\begin{bmatrix}\psi_{u\bar{u}}\\ \psi_{d\bar{d}}\\ \psi_{s\bar{s}}\end{bmatrix}. \quad (37)$$

in which

$$\mathbb{M} = \begin{bmatrix} m_u & 0 & 0\\ 0 & m_d & 0\\ 0 & 0 & m_s\end{bmatrix}. \quad (38)$$

Equation (37) does not include mixing between the pairs. Motivated by ideas presented by Brayshaw[15] [5], we model the effects of $q_i\bar{q}_i \rightarrow q_j\bar{q}_j$ via two-gluon annihilation and creation as an effective scalar potential by postulating a symmetric matrix $\mathbb{M}$ that is not diagonal,

$$\begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix} \rightarrow \mathbb{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix} + \begin{bmatrix} \delta m_u & \sqrt{\delta m_u \delta m_d} & \sqrt{\delta m_u \delta m_s} \\ \sqrt{\delta m_u \delta m_d} & \delta m_d & \sqrt{\delta m_d \delta m_s} \\ \sqrt{\delta m_u \delta m_s} & \sqrt{\delta m_d \delta m_s} & \delta m_s \end{bmatrix}. \tag{39}$$

Suppose that an orthogonal matrix $\mathbb{R}$ diagonalizes $\mathbb{M}$[16]

$$\mathbb{R}\mathbb{M}\mathbb{R}^{-1} = \mathbb{M}_D. \tag{40}$$

Then Eq. (37) becomes

$$[\mathbf{p}^2 + \Phi_w(\mathbf{r}, \mathbb{M}_D)]\psi = \frac{1}{4}(w^2 - 4\mathbb{M}_D^2)\psi.$$

This gives us, in essence, three new effective families of equal quark-anti-quark mesons, like ones that contain b, c, s, u, d except that mixtures are involved. In this paper we see if this idea is successful for the ground state pseudoscalar isoscalar family of mesons alone. With the three parameters, one obtains three different effective quark masses, one for each isoscalar family. The three δm_i are adjusted to give the best fit to the correct π^0, η, η' masses.[17] Table XVII gives the values of δm_u, δm_d, δm_s together with the three effective quark masses, the eigenvalues of $\mathbb{M}$ which we call $m_{q(\pi^0)}$, $m_{q(\eta)}$, $m_{q(\eta')}$.

[15]Brayshaw considered the modification of the meson mass by $w\mathbf{1} \rightarrow w\mathbf{1} + |G\rangle Z\langle G|$ with Z a fixed parameter and $|G\rangle$ in the ns subspace. This is equivalent to the mixing matrix

$$\mathbb{T} = Z\begin{bmatrix} \langle n\bar{n}|G\rangle\langle G|n\bar{n}\rangle & \langle n\bar{n}|G\rangle\langle G|s\bar{s}\rangle \\ \langle s\bar{s}|G\rangle\langle G|n\bar{n}\rangle & \langle s\bar{s}|G\rangle\langle G|s\bar{s}\rangle \end{bmatrix} \equiv \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix},$$

with the property that (for real matrix elements)

$$t_{11}t_{22} = t_{12}t_{21} = |t_{12}|^2 = t_{12}^2,$$

just as with the choice of Eq. (39).

[16]We ignore here the coupling that would result from this diagonalization that would be brought on by the Coulomb interactions between the equal mass $q\bar{q}$ pairs.

[17]These fits are simultaneous with the fits of the earlier 105 mesons, with the same quark masses and potential parameters used in the generation of Tables (II, III, IV, V, VI, VII, VIII, and IX). Oddly, a precise fit to the π^0 was not possible in either theory, in spite of the three extra parameters δm_u, δm_d, and δm_s available.

TABLE XVII. Mixing Parameters for Theory 1 and Theory 2 (GeV).

Parameter	Th1	Th2
δm_u	0.1004	0.1070
δm_d	0.1378	0.1055
δm_s	0.0468	0.0578
$m_q(\pi^0)$	0.0737	0.1015
$m_q(\eta)$	0.2175	0.2261
$m_q(\eta')$	0.4297	0.4536

Paralleling the earlier Tables II, III, IV, V, VI, VII, VIII, and IX, Table XVIII gives the best fit values for the π^0, η, and η' mesons. The predicted quark content becomes a further test of our model.

The corresponding eigenvectors are quite close to the mixtures

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \equiv |\pi_3\rangle,$$
$$|\eta\rangle = \frac{\cos\theta}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \frac{\sin\theta}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \equiv \cos\theta|\eta_8\rangle - \sin\theta|\eta_1\rangle,$$
$$|\eta'\rangle = \frac{\sin\theta}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{\cos\theta}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \equiv \sin\theta|\eta_8\rangle + \cos\theta|\eta_1\rangle, \tag{41}$$

With the three eigenvectors in matrix form we find

$$[|\pi^0\ \ |\eta\rangle\ \ |\eta'\rangle] = \begin{bmatrix} 0.770 & 0.470 & 0.431 \\ -0.638 & 0.574 & 0.514 \\ -0.006 & -0.671 & 0.742 \end{bmatrix}, \tag{42}$$

corresponding to $\theta = -12.6$ degrees for Th1 and

$$[|\pi^0\ \ |\eta\rangle\ \ |\eta'\rangle] = \begin{bmatrix} 0.717 & 0.542 & 0.438 \\ -0.697 & 0.565 & 0.442 \\ -0.008 & -0.622 & 0.783 \end{bmatrix}, \tag{43}$$

corresponding to $\theta = -16.3$ degrees for Th2. The Th2 value is consistent with chiral perturbation theory results corresponding to using the formula [44].

$$\tan^2\theta_{[\text{quad}]} = \frac{4m_K^2 - m_\pi^2 - 3m_\eta^2}{m_\pi^2 + 3m_{\eta'}^2 - 4m_K^2}. \tag{44}$$

With the meson masses listed in Tables II and III for Th2 we obtain $\theta_{[\text{quad}]} = -17.2$ degrees, reasonably close to our

TABLE XVIII. $c\bar{s}$ Mesons, Theory 1 and Theory 2 (GeV).

Mesons	Exp.	Th1.	Th2.	Exp.-Th1.	Exp.-Th2.	χ^2 -Th1.	χ^2 -Th2.
π^0: 1^1S_0	0.135(0.0)	0.139	0.134	−0.004	0.001	0.2	0.0
η: 1^1S_0	0.548(0.0)	0.548	0.548	0.000	0.000	0.0	0.0
η': 1^1S_0	0.958(0.2)	0.958	0.958	0.000	0.000	0.0	0.0

Th2 result of −16.3 degrees. On the other hand, using the values listed there for Th1 gives $\theta_{[\mathrm{quad}]} = -7.3$ degrees which is significantly different from −12.6 degrees. Using the experimental masses gives $\theta_{[\mathrm{quad}]} = -11.5$ degrees. Radiative vector meson decays give an angle between −10 and −20 degrees while fits to tensor decay widths give −17 degrees. Note that even though the chiral symmetry and its breaking are not built into our model as in, for example, it is in [45], it is of interest that the use of Eq. (44) does produce a consistent result by use of our theoretically computed masses.[18]

VI. THE QUASIPOTENTIAL EQUATION OF EBERT, FAUSTOV AND GALKIN

A. The model and comparisons with TBDE

The model which we critically examine here gives excellent fits to the meson spectrum as well as numerous meson decay rates. The quasipotential approach of Ebert, Faustov and Galkin (EFG)[7] is a local one very similar to that of Todorov [10] and Aneva, Karchev, and Rizov [36], discussed in Sec. II (see Eq. (25)). Insofar as our discussion given in that section is concerned, the main difference between the TBDE and EFG approach is the replacement of the timelike vector confining interaction in Eq. (26) with a different confining vector interaction:

$$\mathcal{V}(\mathbf{p}-\mathbf{k})\beta_1\beta_2 \to \mathcal{V}(\mathbf{p}-\mathbf{k})\Gamma_{1\mu}\Gamma_2^{\mu},$$
$$\Gamma_{i\mu} = \gamma_{i\mu} - \frac{i\kappa(p-k)_\nu\sigma_{i\mu\nu}}{2m_i}, \qquad i = 1, 2. \tag{45}$$

They include as do we a scalar confining potential. In coordinate space their choice is

$$V(r) = (1-\varepsilon)(Ar+B), \qquad S(r) = \varepsilon(Ar+B). \tag{46}$$

For their electromagnetic-like vector interaction they use the Coulomb gauge (instead of the Feynman gauge used in the TBDE).

$$\frac{4}{3}\alpha_s D_{\mu\nu}(\mathbf{p}-\mathbf{k})\gamma_1^{\mu}\gamma_2^{\nu}.$$

Its momentum space form is

$$D^{00}(\mathbf{p}-\mathbf{k}) = -\frac{4\pi}{(\mathbf{p}-\mathbf{k})^2},$$
$$D^{ij} = -\frac{4\pi}{(p-k)^2}\left(\delta^{ij} - \frac{(p-k)^i(p-k)^j}{(\mathbf{p}-\mathbf{k})^2}\right). \tag{47}$$

The addition of the Pauli term with their value $\kappa = -1$ has the effect of cancelling the lowest order spin-dependent contributions in each factor of $\Gamma_{i\mu}$ when sandwiched between on energy shell spinors. In the nonrelativistic limit their scalar and vector confining interactions combine to $(Ar+B)$ with $A = 0.18\ \mathrm{GeV}^2$, $B = -0.3$ GeV. In essence, their approach embodies a modified version of the Cornell potential into the local quasipotential approach. Their choice of κ was fixed by an analysis of the fine structure splitting of heavy quarkonia ^3P_J states [46] and their choice of $\varepsilon = -1$ is determined from considerations of charmonium radiative decay.

We compare in Tables XIX and XX their results to our Th2 by listing the deviations from the experimental results and respective χ^2. Of the 86 common mesons fit to the respective models, the collected results of quasipotential approach of [7,46–50] (EFG) are more accurate in 69 of the fits (61 when comparing χ^2).[19] This includes the difficult singlet-triplet splittings for the ground and excited states of charmonium and the ground state of bottomonium, as well as good fits to most of the radial and orbital excitations of the ground states of the light mesons. Particularly interesting examples for the $\pi-\rho$ family are the $^3P_1 - {}^1P_1$ splitting and the radial excitation of the singlet and triplet S states, and for the K-K^* families the $^3P_1 - {}^1P_1$ splitting and the $^3P_0$ mass. These three areas of their spectrum are noteworthy improvements over the TBDE approach. It is hard, however, to give an even theoretical comparison between

[18]There are at least three other additional hints of a possible emergent chiral symmetry and its breaking in the TBDE model. a) the relatively small quark u, d, masses ~60–100 MeV compared with ~300 MeV with most other quark models b) Our π mass decreases toward zero as $m_q \to 0$, c) the matrix element of the divergence of the axial vector current being proportional to the quark mass [13,39].

[19]Because of their large descrepancy on the (controversial) $^3P_0$ $D_s(2370)$ meson, their χ^2 of 389 is actually larger than our 255 (our fit here is altered to include less mesons and $m_u = m_d$ so there are slight differences from the previous tables). If one eliminates that meson from the fit our χ^2 reduces to 251 while their χ^2 reduces to 69. It should be pointed out that their fit did not appear to be a least χ^2 one like our fit. It also was not an overall fit like ours.

TABLE XIX. Comparison of Theory 2 (A, S, V) Fits with Ebert *et al.* [7], [46–50].

Old Mesons	Exp.	TBDE	EFG	Exp.- TBDE	Exp.—EFG.	χ^2 (TBDE)	χ^2 (EFG).
π: $u\bar{d}1^1S_0$	0.140	0.134	0.154	0.005	−0.014	0.4	2.5
ρ: $u\bar{d}1^3S_1$	0.775	0.779	0.776	−0.003	−0.001	0.1	0.0
b_1: $u\bar{d}1^1P_1$	1.230	1.237	1.258	−0.007	−0.028	0.1	0.9
a_1: $u\bar{d}1^3P_1$	1.230	1.311	1.254	−0.081	−0.024	0.0	0.0
π: $u\bar{d}2^1S_0$	1.300	1.426	1.292	−0.126	0.008	0.0	0.0
a_2: $u\bar{d}1^3P_2$	1.318	1.303	1.317	0.015	0.001	2.0	0.0
ρ: $u\bar{d}2^3S_1$	1.465	1.674	1.486	−0.209	−0.021	0.8	0.0
a_0: $u\bar{d}1^3P_0$	1.474	1.015	1.176	0.459	0.298	7.0	3.0
b_2: $u\bar{d}1^1D_2$	1.672	1.752	1.643	−0.080	0.029	6.8	0.9
a_3: $u\bar{d}1^3D_3$	1.689	1.706	1.714	−0.017	−0.025	0.7	1.4
a_1: $u\bar{d}1^3D_1$	1.720	1.836	1.742	−0.116	−0.022	0.4	0.0
a_2: $u\bar{d}2^3P_2$	1.732	1.997	1.779	−0.265	−0.047	3.3	0.1
π: $u\bar{d}3^1S_0$	1.816	2.022	1.788	−0.206	0.028	2.6	0.0
b_2: $u\bar{d}2^1D_2$	1.895	2.252	1.960	−0.357	−0.065	6.0	0.2
a_4: $u\bar{d}1^3F_4$	2.011	2.042	2.018	−0.031	−0.007	0.1	0.0
b_2: $u\bar{d}3^1D_2$	2.090	2.682	2.216	−0.592	−0.126	5.0	0.2
ρ: $u\bar{d}3^3S_1$	2.149	2.309	1.921	−0.160	0.228	1.1	2.2
a_6: $u\bar{d}1^3H_6$	2.450	2.590	2.475	−0.140	−0.025	0.0	0.0
K^-: $s\bar{u}1^1S_0$	0.494	0.528	0.482	−0.034	0.012	14.2	1.6
K^{*-}: $s\bar{u}1^3S_1$	0.892	0.898	0.897	−0.007	−0.005	0.5	0.3
K^-: $s\bar{u}1^1P_1$	1.272	1.336	1.294	−0.064	−0.022	1.0	0.1
K^{*-}: $s\bar{u}1^3P_1$	1.403	1.354	1.412	0.049	−0.009	0.6	0.0
K^{*-}: $s\bar{u}2^3S_1$	1.414	1.698	1.675	−0.284	−0.261	4.3	3.6
K^{*-}: $s\bar{u}1^3P_0$	1.425	1.075	1.362	0.350	0.063	0.6	0.0
K^{*-}: $s\bar{u}1^3P_2$	1.426	1.401	1.424	0.025	0.002	0.0	0.0
K^-: $s\bar{u}2^1S_0$	1.460	1.414	1.538	0.046	−0.078	0.0	0.0
K^{*-}: $s\bar{u}1^3D_1$	1.717	1.828	1.699	−0.111	0.018	0.2	0.0
K^-: $s\bar{u}1^1D_2$	1.773	1.795	1.709	−0.022	0.064	0.1	0.8
K^{*-}: $s\bar{u}1^3D_3$	1.776	1.784	1.789	−0.008	−0.013	0.0	0.0
K^{*-}: $s\bar{u}1^3D_2$	1.816	1.787	1.824	0.029	−0.008	0.1	0.0
K^-: $s\bar{u}3^1S_0$	1.830	2.069	2.065	−0.239	−0.235	4.1	3.9
K^{*-}: $s\bar{u}2^3P_2$	1.973	2.050	1.896	−0.077	0.077	0.1	0.1
K^{*-}: $s\bar{u}1^3F_4$	2.045	2.106	2.096	−0.061	−0.051	0.5	0.4
K^{*-}: $s\bar{u}2^3D_2$	2.247	2.301	2.163	−0.054	0.084	0.1	0.3
K^{*-}: $s\bar{u}2^3F_3$	2.324	2.585	2.348	−0.261	−0.024	1.4	0.0
K^{*-}: $s\bar{u}1^3G_5$	2.382	2.387	2.356	−0.005	0.026	0.0	0.0
K^{*-}: $s\bar{u}2^3F_4$	2.490	2.585	2.436	−0.095	0.054	0.3	0.1
ϕ: $s\bar{s}1^3S_1$	1.019	1.017	1.038	0.002	−0.019	0.1	4.1
ϕ: $s\bar{s}1^3P_0$	1.370	1.175	1.420	0.195	−0.050	0.0	0.0
ϕ: $s\bar{s}1^3P_1$	1.518	1.436	1.464	0.082	0.054	3.1	1.4
ϕ: $s\bar{s}1^3P_2$	1.525	1.505	1.529	0.020	−0.004	0.2	0.0
ϕ: $s\bar{s}2^3S_1$	1.680	1.868	1.698	−0.188	−0.018	1.1	0.0
ϕ: $s\bar{s}1^3D_3$	1.854	1.874	1.950	−0.020	−0.096	0.1	2.2
ϕ: $s\bar{s}2^3P_2$	2.011	2.120	2.030	−0.109	−0.019	0.0	0.0
ϕ: $s\bar{s}3^3P_2$	2.297	2.590	2.412	−0.293	−0.115	1.3	0.2

the two approaches for a number of different reasons. It is worthwhile, however, to point out the differences in the two approaches, summarized in Table XXI. First of all, in our approach, we give an overall fit to the entire spectrum. The approach of EFG to the spectrum is spread over several papers and it is not clear that a uniform parametrization would yield the same results as given in the tables (summarized here) from their separate papers. They do use the same values for the constants A, B, κ, and ε as well as the quark masses in the various papers. However, the Λ parameter they

TABLE XX. Comparison of Th2(A, S, V) Fits with Ebert *et al.* [7], [46–50].

New Mesons	Exp.	TBDE.	EFG	Exp.-TBDE	Exp.-EFG	χ^2 -TBDE	χ^2 -EFG
D^+: $c\bar{d}1^1S_0$	1.870	1.881	1.871	−0.012	−0.001	1.5	0.0
D^{*+}: $c\bar{d}1^3S_1$	2.010	2.010	2.010	0.000	0.000	0.0	0.0
D^{*+}: $c\bar{d}1^3P_0$	2.403	2.224	2.406	0.179	−0.003	2.0	0.0
D^{*0}: $c\bar{u}1^3P_2$	2.460	2.408	2.460	0.052	0.000	3.3	0.0
D_s: $c\bar{s}1^1S_0$	1.968	1.976	1.969	−0.007	−0.001	0.5	0.0
D_s^*: $c\bar{s}1^3S_1$	2.112	2.120	2.111	−0.008	0.001	0.7	0.0
D_s^*: $c\bar{s}1^3P_0$	2.318	2.338	2.509	−0.020	−0.191	3.7	320
D_s: $c\bar{s}1^1P_1$	2.535	2.498	2.536	0.038	−0.001	15.4	0.0
D_s^*: $c\bar{s}1^3P_2$	2.573	2.531	2.571	0.042	0.002	11.8	0.0
D_s^*: $c\bar{s}2^3S_1$	2.690	2.698	2.731	−0.008	−0.041	0.0	0.4
η_c: $c\bar{c}1^1S_0$	2.980	2.972	2.978	0.008	0.002	0.3	0.0
$J/\psi(1S)$: $c\bar{c}1^3S_1$	3.097	3.126	3.097	−0.029	0.000	10.3	0.0
χ_0: $c\bar{c}1^3P_0$	3.415	3.393	3.423	0.022	−0.008	5.2	0.7
χ_1: $c\bar{c}1^3P_1$	3.511	3.500	3.509	0.010	0.002	1.3	0.0
h_1: $c\bar{c}1^1P_1$	3.526	3.519	3.525	0.007	0.001	0.6	0.0
χ_2: $c\bar{c}1^3P_2$	3.556	3.553	3.556	0.004	0.000	0.1	0.0
η_c: $c\bar{c}2^1S_0$	3.637	3.597	3.663	0.040	−0.026	1.1	0.5
$\psi(2S)$: $c\bar{c}2^3S_1$	3.686	3.683	3.684	0.004	0.002	0.1	0.1
$\psi(1D)$: $c\bar{c}1^3D_1$	3.773	3.801	3.795	−0.028	−0.022	8.4	5.2
χ_2: $c\bar{c}2^3P_2$	3.929	3.975	3.972	−0.046	−0.043	1.0	0.9
$\psi(3S)$: $c\bar{c}3^3S_1$	4.039	4.083	4.088	−0.044	−0.049	0.2	0.3
$\psi(2D)$: $c\bar{c}2^3D_1$	4.153	4.160	4.194	−0.007	−0.041	0.1	2.0
B^-: $b\bar{u}1^1S_0$	5.279	5.285	5.280	−0.006	−0.001	0.4	0.0
B^0: $b\bar{d}1^1S_0$	0.000	0.000	0.000	0.000	0.000	0.0	0.0
B^{*-}: $b\bar{u}1^3S_1$	5.325	5.334	5.326	−0.009	−0.001	0.8	0.0
B^{*-}: $b\bar{u}1^3P_2$	5.747	5.687	5.741	0.060	0.006	4.7	0.0
B_s^0: $b\bar{s}1^1S_0$	5.366	5.370	5.372	−0.004	−0.006	0.1	0.3
B_s^{*0}: $b\bar{s}1^3S_1$	5.413	5.432	5.414	−0.019	−0.001	1.7	0.0
B_s^{*0}: $b\bar{s}1^3P_1$	5.829	5.793	5.831	0.037	−0.002	11.0	0.0
B_s^{*0}: $b\bar{s}1^3P_2$	5.840	5.805	5.842	0.034	−0.002	10.5	0.0
η_b: $b\bar{b}1^1S_0$	9.389	9.334	9.400	0.055	−0.011	2.1	0.1
$\Upsilon(1S)$: $b\bar{b}1^3S_1$	9.460	9.447	9.460	0.014	0.000	2.1	0.0
χ_{b0}: $b\bar{b}1^3P_0$	9.859	9.835	9.864	0.024	−0.005	6.1	0.2
χ_{b1}: $b\bar{b}1^3P_1$	9.893	9.887	9.892	0.006	0.001	0.4	0.0
χ_{b2}: $b\bar{b}1^3P_2$	9.912	9.921	9.912	−0.009	0.000	0.8	0.0
$\Upsilon(2S)$: $b\bar{b}2^3S_1$	10.023	10.021	10.020	0.002	0.003	0.0	0.1
$\Upsilon(1D)$: $b\bar{b}1^3D_2$	10.161	10.178	10.157	−0.017	0.004	2.4	0.1
χ_{b0}: $b\bar{b}2^3P_0$	10.232	10.228	10.232	0.005	0.001	0.2	0.0
χ_{b1}: $b\bar{b}2^3P_1$	10.255	10.261	10.253	−0.005	0.002	0.3	0.1
χ_{b2}: $b\bar{b}2^3P_2$	10.269	10.284	10.267	−0.015	0.002	2.4	0.0
$\Upsilon(3S)$: $b\bar{b}3^3S_1$	10.355	10.366	10.355	−0.011	0.000	1.0	0.0
$\Upsilon(4S)$: $b\bar{b}4^3S_1$	10.579	10.628	10.604	−0.049	−0.025	11.7	3.0

TABLE XXI. Comparison of TBDE with EFG.

Properties	TBDE	Quasipotential Approach of EFG
Invariant Interactions	3- A (EM-like vector), S (scalar), V (timelike vector)	3- A (EM-like vector), S (scalar), V (Pauli-modified vector)
QCD Coupling	Coordinate Space Dependent	Quark Mass Dependent
Meson Fits	Overall Spectrum, Same Parametrizations	Spectrum in Parts with Different Parametrizations
Spin- Dependence	Fixed by the TBDE given A, S, V	Fixed by the Quasipotential and A, S, V
Kinematics	Exact	Exact
Singular potentials	Avoided by Dirac Equation Formalism	Avoided by *Ad hoc* Substitutions
Numerical evaluations	Yes	Not for Heavy Mesons
Chiral Symmetry	Zero Pion Mass for $m_q \to 0$	Not Tested
QED Spectral Tests	Both Perturbative and Nonperturbative	Perturbative Only
Static Limit ($m_2 \to \infty$)	Reduces to One-Body Dirac Eq.	Does Not Reduce to One-Body Dirac Eq.

use in the parametrization of the short distance QCD-Coulomb part of the potential is different.[20] Another difference is that the static QCD potential in the Adler-Piran model displays explicitly the asymptotic freedom behavior by its radial dependence [21] whereas in the Coulomb potential used by EFG, the asymptotic freedom behavior is displayed indirectly in their α_s by the quark mass dependence as seen in footnote [20]. On the other hand, for their heavy quark bound states a radial modification displaying short distance QCD asymptotic freedom corrections was used but was not for the light quark bound states. Both the quasipotential approach of EFG and the TBDE used here, extending earlier work of Crater and Van Alstine, have three invariant interaction functions. Both use electromagnetic-like four-vector interactions, in the Feynman gauge for the TBDE and the Coulomb gauge for the quasipotential approach. In the EFG quasipotential approach, the third interaction is a Pauli-modified vector interaction whereas in our approach it is a timelike vector interaction. The potentials used in each of the three parts are of course different in the two approaches. The spin dependence, although similar for the most part have distinctly different origins. In the 16 component TBDE the kinematics and dynamics and spinors are tied together in one wave equation (see. e.g. Eq. (16)). Its off-shell dependence is fixed by the wave equation and the spinors are all interacting. In the quasipotential approach the potential is constructed in part from the actions of free particle spinors. This leaves substantial leeway in how the off-shell behavior is fixed.

Both approaches have exact relativistic kinematics (from the use of $b^2(w)$) and do not use either v/c or $1/m_q$ expansions. Both approaches lead to nonlinear eigenvalue equations and give good values of the pion and kaon masses. Both approaches avoid singular effective potentials that would otherwise prevent nonperturbative spectral calculations. In the quasipotential approach of EFG, those singularities are avoided in an *ad hoc* though plausible fashion (see Eq. (12) in [7]). In the case of the TBDE, natural smoothing mechanisms appear in the Dirac formalism allowing one to avoid these *ad hoc* assumptions [28,29].[22] In addition, in the approach of the TBDE, strictly nonperturbative (i.e. numerical techniques) are used for the spectral evaluations. This does not appear to be the case for the work of EFG, particularly for the heavy mesons, where use of perturbation theory is required because of singular potentials. It may very well be that their *ad hoc* substitutions used in the later paper [7] will render the use of perturbation theory unnecessary for those mesons. In that case, clear tests must ensure that not only do the *ad hoc* substitutions give the same results as the perturbative treatments, but that this holds in the sensitive testing grounds of QED [4,29] ground states and those of related field theories. Both approaches display chiral symmetry breaking through the appearance of quark masses. It has been demonstrated in the TBDE, however, that the pion mass vanishes in the limit in which the quark mass vanishes [2]. That is not demonstrated in the EFG formalism nor for any other potential model formalism that we know of for the mesons (In an exception, Sazdjian

[20]In their earlier papers where the fits to the heavier mesons are given they use $\alpha_s = 4\pi/(\beta_0 \ln(\mu^2/\Lambda^2))$ where μ is the reduced mass and in the recent papers where the fits to lighter mesons are given they use $\alpha_s = 4\pi/(\beta_0 \ln((\mu^2 + M_B^2)/\Lambda^2))$. In the former papers they use $\Lambda = 169$ or 178 MeV and in the recent ones $\Lambda = 413$ MeV. It is not clear how using just one form for all the mesons would affect the overall fit.

[21]In [33], the Adler-Piran potential was replaced by a form that displays asymptotic freedom in the QCD coupling via $(8\pi/27)/\ln(K + B/(\Lambda r)^2)$. Although the model used there does not give as good a fit to the meson spectrum as the Adler-Piran model it does display asymptotic freedom in a simpler form.

[22]In [33], Crater, Yoon, and Wong described some unusual singularity structures of the effective potentials and wave functions that show up in Eq. (17) for singlet and triplet states, in both QED and QCD. In these cases, the TBDE lead to effective potentials and wave functions that are nevertheless not singular. The most noteworthy case was for the coupled ${}^3S_1$-${}^3D_1$ triplet system, when the tensor coupling is properly taken into account. There it was shown that including the tensor coupling is essential in order that the effective potentials and wave functions are well behaved at short distances, with the S state and D state wave functions becoming simply proportional to each other at short distance (see Appendix C).

has demonstrated this using pseudoscalar interactions for the TBDE [39]).[23]

The final point we want to make about the differences is that the wave equation arising from the TBDE has been tested in perturbative QED and related field theories. Todorov, and others [10,36,51] showed using perturbative methods how their local version of the quasipotential equation displays the accepted QED spectral results through order α^4 for two oppositely charged particles with arbitrary mass ratios. The works by Crater and Van Alstine [34] and others [29] go beyond this and show that not only do the TBDE display the correct fine and hyperfine spectral results when treated perturbatively, but those same results can be recovered when the equations are treated nonperturbatively. In [52], a local quasipotential equation closely related to that used by EFG in the meson spectrum was shown to also reproduce perturbatively the spectral results through order α^4 of QED for two oppositely charged particles with arbitrary mass ratios. However, the important nonperturbative tests as done in [29] of the bound state formalism for QED have not been carried out with the local quasipotential equation of EFG.

It is our contention that any relativistic potential model that includes four-vector interactions should, when the vector interaction is replaced by its QED counterpart, and confining potentials are set $= 0$, reproduce the standard QED spectral results. There are two models for which we carry out this test. We limit our test to the singlet positronium ground state. The fundamental question we ask is, do these two approaches, [6,7], which are used quite successfully for meson spectroscopy, give the correct spectral results when restricted to QED. The first one we examine is that of Ebert, Faustov and Galkin [7].

B. Positronium ground state spectral test of the quasipotential equation of Ebert, Faustov and Galkin

The effective Schrödinger equation of this approach, restricted to an equal mass bound system for vector interactions, is given in Eqs. (1–4) and (13–22) of [7]. For that restriction we have, in the notation of the present paper,

$$\begin{aligned} \left(\frac{\mathbf{p}^2}{2\mu_R}+V(r)\right)\Psi_w &= \frac{b^2(w)}{2\mu_R}\Psi_w, \\ \mu_R &= \frac{\varepsilon_1\varepsilon_2}{w}=\frac{w}{4}, \\ b^2(w) &= \frac{1}{4}(w^2-4m^2), \\ V(r) &= A(r)\left[1+\left(\frac{w-2m}{w}\right)^2\right] \\ &\quad +\left(\frac{2}{w(w+2m)}+\frac{8}{3w^2}\mathbf{S}_1\cdot\mathbf{S}_2\right)\nabla^2 A, \\ A &= -\frac{\alpha}{r}. \end{aligned} \tag{48}$$

The unperturbed effective Hamiltonian is

$$H_0=\frac{\mathbf{p}^2}{2\mu_R}-\frac{\alpha}{r}, \tag{49}$$

and the perturbation is for singlet states

$$H_1=-\left(\frac{w-2m}{w}\right)^2\frac{\alpha}{r}-\left(\frac{8\pi\alpha}{w(w+2m)}-\frac{8\pi\alpha}{w^2}\right)\delta^3(\mathbf{r}). \tag{50}$$

Comparing the nonrelativistic hydrogenic Schrödinger equation

$$\left(\frac{\mathbf{p}^2}{2\mu}-\frac{\alpha}{r}\right)\psi=\mathcal{E}\psi_w, \tag{51}$$

with ground state energy

$$\mathcal{E}=-\frac{\mu\alpha^2}{2}, \tag{52}$$

we see that for the eigenvalue Eq. (48), the total c.m. invariant energy from H_0 is determined by analogy with Eqs. (51) and (52) from

$$\frac{b^2(w)}{2\mu_R}=-\frac{\mu_R\alpha^2}{2}. \tag{53}$$

Thus, with

$$w^2-4m^2=-4\mu_R^2\alpha^2=-\frac{w^2}{4}\alpha^2 \tag{54}$$

we find that

$$w=2m-\frac{m\alpha^2}{4}+\frac{3m\alpha^4}{64}. \tag{55}$$

Substituting this into Eq. (50) at the appropriate order gives

$$H_1=-\left(\frac{m\alpha^2}{8m}\right)^2\frac{\alpha}{r}+\frac{\pi\alpha}{m^2}\delta^3(\mathbf{r})\to\frac{\pi\alpha}{m^2}\delta^3(\mathbf{r}), \tag{56}$$

with their spin-spin terms partially canceling their spin-independent contact (Darwin) term while the first term is

[23] H. Sazdjian has considered chiral symmetry and its breaking in the context of a closely related version of the TBDE. This was later discussed by Crater and Van Alstine, [13]. In particular, it is found that the matrix element of the divergence of the axial vector current is proportional to the quark mass. This demonstrates that the quark masses in the TBDE play the same role as the quark masses in QFT. In addition, Sazdjian shows that the pion decay constant in the context of the TBDE does not vanish in the limit of $m_q\to 0$. Sazdjian shows analytically, in the context of a pseudoscalar confining potential, the existence of a massless pseudoscalar meson when $m_q\to 0$. These are two of the main effects of the spontaneous breakdown of chiral symmetry. In our earlier work with Van Alstine [13], we showed numerically for the case of scalar confinining intereractions that the calculated pion mass tends to zero as $m_q\to 0$. In a later work ([2]) we showed, however, that the behavior is not of the square-root relation ($m_\pi\sim\sqrt{m_q}/F_\pi$). The same behavior appears to hold with the present calculations.

of higher order. The ground state unperturbed wave function is

$$\Psi_w = \frac{\exp(-r/a_{\rm eff})}{(\pi a_{\rm eff}^3)^{1/2}}, \qquad a_{\rm eff} = \frac{1}{\mu_R \alpha} \to \frac{2}{m\alpha}. \tag{57}$$

The expectation value is

$$\langle H_1 \rangle = \frac{\alpha}{m^2(\frac{2}{m\alpha})^3} = \frac{m\alpha^4}{8}, \tag{58}$$

and so Eq. (53), including $\langle H_1 \rangle$ then becomes, to the appropriate order

$$\begin{aligned} \frac{b^2(w)}{2\mu_R} &= -\frac{\mu_R \alpha^2}{2} + \frac{m\alpha^4}{8}, \\ w^2 - 4m^2 &= -\frac{w^2}{4}\alpha^2 + \frac{m^2\alpha^4}{2}, \end{aligned} \tag{59}$$

so that

$$\begin{aligned} w &= 2m - \frac{m\alpha^2}{4} + \frac{3m\alpha^4}{64} + \frac{m\alpha^4}{8} \\ &= 2m - \frac{m\alpha^2}{4} + \frac{11m\alpha^4}{64}. \end{aligned} \tag{60}$$

This result is in disagreement with the accepted fine structure result of

$$w = 2m - \frac{m\alpha^2}{4} - \frac{21m\alpha^4}{64}. \tag{61}$$

In the two-body Dirac equation, this spectrum results from an exact solution of the Schrödinger-like form [29,34,41][24]

$$(\mathbf{p}^2 + 2\varepsilon_w A - A^2)\psi = b^2\psi, \tag{62}$$

which yields

$$\begin{aligned} w &= m\sqrt{2 + 2\Big/\sqrt{1 + \frac{\alpha^2}{(1 + \sqrt{\frac{1}{4} - \alpha^2} - \frac{1}{2})^2}}} \\ &= 2m - \frac{m\alpha^2}{4} - \frac{21m\alpha^4}{64} + \ldots, \end{aligned} \tag{63}$$

obtained by steps similar to those outlined in Eq. (60).

The local quasipotential approach of [7] does not include the term $-A^2 = -\alpha^2/r^2$ which gauge invariance considerations would demand. It is of interest that if their approach includes this term, then the added contribution from this potential is

$$\frac{1}{2\mu}\left\langle -\frac{\alpha^2}{r^2}\right\rangle = \frac{1}{m}\left\langle -\frac{\alpha^2}{r^2}\right\rangle = -\frac{m\alpha^4}{2}. \tag{64}$$

[24]This equation and its gauge structure can also be seen to result from the equal mass singlet equation version of (35) for $S = V = 0$, $A = -\alpha/r$ or its radial version given by Eq. (B1). (It is noted that under these conditions, $\Phi_D = 3\Phi_{SS}$, and $\Phi_{SOD} = \Phi_{SOX} = 0$).

This, together with the fact that combining this with the earlier results gives the correct added $O(\alpha^4)$ correction,

$$\frac{11m\alpha^4}{64} - \frac{m\alpha^4}{2} = -\frac{21m\alpha^4}{64}, \tag{65}$$

points strongly to the lack of this term as being the cause of the incorrect QED spectral prediction in this approach.

We should emphasize that the closely related formalism of [52] *does* produce the correct $-\frac{21m\alpha^4}{64}$ relativistic correction. The difference between the formalism of EFG and that of [52] is that the former does not include two loop and iterated Born diagrams[25] contained in the latter. Those combined diagrams to lowest order in α do produce the $-A^2 = -\alpha^2/r^2$ term which would account for the spectral difference seen in Eq. (65) (see also [51]). Since that $-A^2$ term is not included in the EFG meson spectral formalism, it is likely that possibly important relativistic corrections for their meson spectrum will be missing. In a private communication, Faustov stated that the reason they did not include the contributions of two and more gluon exchange diagrams within QCD in calculations of the meson spectra, is that the effects of these diagrams would be contained in the confining, long range potential, the origin of which is not known and which is thus added phenomenologically. However, the $-A^2$ contribution from those two-gluon exchange diagrams due to the Coulomb-like potential A is short range and therefore would not by itself contribute to the confining potential. In other words, we claim that since its effects are short range, it should be considered apart from the phenomenologically added confining interaction.

C. Positronium test of the approach of Godfrey and Isgur

Although Crater and Van Alstine carried out an earlier comparison [2] with this approach [6], in light of the problem with the above quasipotential approach it is instructive to include a parallel perturbative treatment on their different quasipotential equation. Their relativistic Schrödinger equation (see their Eqs. (1–4)) relevant for the case considered here has the Hamiltonian which includes the spin-spin term in addition to the modified Coulomb term (see their Eq. (A15)[26]). Their equation was of the quasipotential type given in Eq. (18) extended

[25]The authors are grateful to Professor R. N. Faustov for pointing out to us the results of [52] and for their reason that the bound state equation used in [7] did not include the $-A^2$ term.

[26]In their Appendix A, Godfrey and Isgur modify the Coulomb and contact spin-spin term used here with smearing functions and extra nonlocal parts in order to account for the off mass-shell effects not present in the on shell scattering amplitudes from which they extract their potentials. We do not include the effects of the Gaussian smoothing factors in the determination of the modification of the Coulomb term from their Eq. (A15).

to include spin. Here we consider its semirelativistic expansion.

$$H = 2\sqrt{\mathbf{p}^2 + m^2} - \frac{\alpha}{r} + \frac{2}{3m^2}\mathbf{S}_1 \cdot \mathbf{S}_2 \nabla^2 A - \frac{\alpha}{2m^2}\left\{\mathbf{p}^2, \frac{1}{r}\right\}$$
$$\to 2m + H_0 + H_1,$$
$$H_0 = \frac{\mathbf{p}^2}{m} - \frac{\alpha}{r},$$
$$H_1 = -\frac{(\mathbf{p}^2)^2}{4m^3} + \frac{2\pi\alpha}{m^2}\delta^3(\mathbf{r}) - \frac{\alpha}{2m^2}\left\{\mathbf{p}^2, \frac{1}{r}\right\}. \quad (66)$$

The ground state unperturbed wave function is

$$\Psi = \frac{\exp(-r/a)}{(\pi a^3)^{1/2}}, \qquad a = \frac{2}{m\alpha}. \quad (67)$$

We find that

$$\langle H_1 \rangle = -\frac{1}{4m}\left\langle \Psi \frac{\mathbf{p}^2}{m}\frac{\mathbf{p}^2}{m}\Psi \right\rangle + \left\langle \Psi \frac{2\pi\alpha}{m^2}\delta^3(\mathbf{r})\Psi \right\rangle - \frac{\alpha}{2m^2}\left\langle \Psi \left\{\mathbf{p}^2, \frac{1}{r}\right\}\Psi \right\rangle \quad (68)$$

$$= -\frac{1}{4m}\left\langle \Psi \left(\mathcal{E} + \frac{\alpha}{r}\right)^2 \Psi \right\rangle + \frac{m\alpha^4}{4} - \frac{\alpha}{2m^2}\left\langle \Psi \left\{\mathbf{p}^2, \frac{1}{r}\right\}\Psi \right\rangle, \quad (69)$$

and with $\mathcal{E} = -\frac{m\alpha^2}{4}$ we have

$$-\frac{1}{4m}\mathcal{E}^2 = -\frac{m\alpha^4}{64},$$
$$-\frac{\mathcal{E}}{2m}\left\langle \Psi \frac{\alpha}{r}\Psi \right\rangle = \frac{\alpha^2}{8}(-2\mathcal{E}) = \frac{m\alpha^4}{16},$$
$$-\frac{1}{4m}\left\langle \Psi \frac{\alpha^2}{r^2}\Psi \right\rangle = -\frac{1}{4m}\frac{\alpha^2}{\pi a^3}4\pi\int_0^\infty dr \exp(-2r/a)$$
$$= -\frac{m\alpha^4}{8}, \quad (70)$$

and using

$$-\frac{\alpha}{2m^2}\left\langle \Psi \left\{\mathbf{p}^2, \frac{1}{r}\right\}\Psi \right\rangle = -\frac{2(2\mu)\alpha}{2m^2}\left\langle \Psi \frac{1}{r}\left(\mathcal{E} + \frac{\alpha}{r}\right)\Psi \right\rangle$$
$$= -\frac{\mathcal{E}}{m}\left\langle \Psi \frac{\alpha}{r}\Psi \right\rangle - \frac{1}{m}\left\langle \Psi \frac{\alpha^2}{r^2}\Psi \right\rangle$$
$$= \frac{m\alpha^4}{8} - \frac{m\alpha^4}{2} \quad (71)$$

and find that

$$\langle H_1 \rangle = \frac{m\alpha^4}{4}\left(-\frac{1}{16} + \frac{1}{4} - \frac{1}{2} + 1 + \frac{1}{2} - 2\right), \quad (72)$$

and so

$$w = 2m + \langle H_0 \rangle + \langle H_1 \rangle = 2m - \frac{m\alpha^2}{4} - \frac{13m\alpha^4}{64}. \quad (73)$$

This also does not agree with the accepted result of Eq. (61). Since the addition of the $-\alpha^2/r^2$ term would drive this to the other side of the accepted value it is not clear where the error is in this approach.

D. Comparison of two approaches to Dirac equation in the static limit

The dynamics of the heavy-light $q\bar{Q}$ bound states, particularly the $u\bar{b}$ and $d\bar{b}$, should be well approximated by the ordinary one-body Dirac equation. In the limit when, say, $m_2 \to \infty$, details outlined in Appendix A 5 show that our TBDE reduce to the single particle Dirac equation for a spin-one-half particle in an external scalar and vector potential,

$$(\boldsymbol{\gamma} \cdot \mathbf{p} - \beta(\varepsilon - A) + m + S)\psi = 0, \quad (74)$$

in which ε is the total energy of the single particle of mass m. In this same limit, Eq. (35) (see Appendix A 5) becomes

$$\left(\mathbf{p}^2 + 2mS + S^2 + 2\varepsilon A - A^2 + \frac{1}{2}\frac{\nabla^2 A - \nabla^2 S}{m + S + \varepsilon - A} + \frac{3}{4}\left(\frac{S' - A'}{m + S + \varepsilon - A}\right)^2 + \frac{A' - S'}{m + S + \varepsilon - A}\frac{\mathbf{L} \cdot \boldsymbol{\sigma}_1}{r}\right)\psi_+ = (\varepsilon^2 - m^2)\psi_+, \quad (75)$$

which agrees with the Pauli reduction of the Dirac Eq. (74) for a single particle in an external scalar and vector potential when the first order momentum terms are scaled away (see for example [33]). The wave function ψ_+ is the upper two-component spinor. From the point of view of the single particle Dirac equation the quadratic S^2 and $-A^2$ terms above are not put in by hand but arise naturally from the Pauli reduction.

We compare Eq. (75) with the corresponding equations from the two quasipotential approaches, including scalar and vector interactions.[27] Referring to Eqs. (1–4) and (13–22) of [7], we have the $m_2 \to \infty$ limit of

$$\left(\mathbf{p}^2 + 2\varepsilon(A + S) + \frac{1}{2}\frac{\nabla^2 A}{m + \varepsilon} + \frac{A' - S'}{m + \varepsilon}\frac{\mathbf{L} \cdot \boldsymbol{\sigma}_1}{r}\right)\psi = (\varepsilon^2 - m^2)\psi, \quad (76)$$

in which we have used $\mu_R \to \varepsilon$. It is evident that for the vector interaction alone ($S = 0$), this equation will not yield a spectrum perturbatively equivalent to the Dirac spectrum for $A = -\alpha/r$ and for the same reason as with positronium, that is, the lacking of the $-A^2$ term. This

[27] In [53], the static limit Dirac equation was recovered from a two-body quasipotential equation by techniques with some similarity to the Gross equation [54]. In that equation the relative energy is constrained by restricting one of the spin-one half particles to its positive energy mass-shell. This differs from the TBDE which treats both particles off-shell but yet constraining the relative energy covariantly through $P \cdot p\psi = 0$. The EFG equation also has this constraint, but unlike the equation derived in [53], the Gross equation and the TBDE, it does not have the Dirac equation as a static limit as seen by a comparison of Eq. (76) with Eq. (75).

would result in incorrect fine structure for hydrogenlike atoms (see discussions and footnotes below Eq. (65) for the reason for this omission). Also, there are three parts for the scalar potential interaction that differ from the Dirac equation: the appearance of $2\varepsilon S$ instead of the expected $2mS$, the lacking of the S^2 term, and the absence of any scalar Darwin term. The appearance of ε instead of m can be traced to the use of a common reduced mass for multiplying both vector and scalar interactions. Beyond that is the absence of the potential energy terms in the denominators of the spin-orbit and Darwin terms. The authors correct this by hand, but their correction does not match the forms in the Dirac equation. On the other hand those potential energy terms in the denominators of the Darwin and spin-orbit terms of Eq. (75) provide a natural smoothing mechanism that eliminates such singular potentials as delta functions and $1/r^3$ potentials. For example take the case of $S = 0$, $A = -\alpha/r$. The Laplacian term would produce $4\pi\delta^3(\mathbf{r})$. However, the A term in the denominator would then be evaluated at the origin and completely cancel the effects of the delta function. Its perturbative effects are reproduced by the adjacent $3/4$ term. Similarly, the $1/r^3$ behavior of the spin-orbit weak potential form in which the A in the denominator is ignored is modified to very near the origin to a less singular $1/r^2$ potential by the effect of the A term in the denominator as well as the $3/4$ term. Similar smoothing mechanics naturally built in to the Pauli structure of the Dirac equation occur in the Pauli reduction of the TBDE (see [28,29,32,33]).

Referring to A-15, 16 of [6], we have the $m_2 \to \infty$ limit of

$$\left(\sqrt{\mathbf{p}^2 + m^2} + A + S + \frac{A' - S'}{4m^2}\frac{\mathbf{L}\cdot\boldsymbol{\sigma}_1}{r}\right)\psi = w\psi. \quad (77)$$

This Hamiltonian form is missing Darwin terms for not only scalar interactions, but also for vector interactions as well. Those are as important for spectral studies as the spin-orbit terms and their lack is a serious defect in both these equations. The lack of the vector Darwin term would result in incorrect fine structure for $L = 0$ hydrogen levels.

VII. CONCLUSION AND FUTURE DIRECTIONS

The application of Dirac's constraint dynamics applied to the relativistic two-body problem leads quite naturally to the two-body Dirac equations of constraint dynamics when both particles have spin-one-half. This paper follows many earlier ones analyzing the structures and applications of those equations. It has several sets of aims and results. First, we showed that when the interaction structure used in these equations is extended from two invariant functions (generated by what we have called $A(r)$ and $S(r)$) to three (not only the two that generate an electromagnetic-like interaction and a confining world scalar interaction but also the $V(r)$ that generates a confining timelike vector interaction), that the fit to the meson spectrum is improved substantially. However, there is still a considerable amount of improvement that is needed, primarily in the radial and orbital excitations of the singlet and triplet ground states. Work in progress seeks to extend invariant functions to ones that generate covariant pseudoscalar, pseudovector, and tensor interactions. Second, this paper also included 19 mesons not included in earlier work [2] where only two invariant functions were used. Among those 19 were the isoscalar η and η' mesons. Here, we developed an approach motivated by some work of Brayshaw [5] which introduces a constant symmetric but nondiagonal mass matrix that couples isoscalar $q\bar{q}$ channels. The three parameters introduced are adjusted (when possible) to fit the π^0, η, and η' meson masses and then used to predict accurately, at least in Th2, the $SU(3)$ pseudoscalar mixing angle. Missing is an attempt to connect this mass matrix to the compatibility condition between the constraints $\mathcal{S}_1$ and $\mathcal{S}_2$.

A glance at Ref. [11] shows that there are no shortages of attempts to stake claims of which relativistic two-body truncation of the Bethe-Salpeter equation is most successful. One of the purposes of [2] was to clarify some guidelines and important benchmarks that such equations should have when applied to any relativistic two-body problem be it for QED bound states, QCD bound states, or nucleon-nucleon scattering. This clarification continues in this paper with a fairly detailed analysis of the local quasipotential approach of Ebert, Faustov, and Galikin. This approach was chosen among numerous others for two reasons: (a) the close connection between the minimal dynamical structure of the constraint approach and the early work done on the local quasipotential approach by Todorov and his coauthors and (b) the extensive phenomenological studies by the local quasipotential approach of EFG in meson spectral and decay studies.

The two quasipotential methods that we discussed have three weaknesses. Neither method uses a wave equation to uncover their respective spin-dependent corrections. Rather, they use an on shell version of the scattering amplitude and the quasipotential equation for the potential. However, as discussed in both papers, they each must make assumptions that allow them to include expected off-shell effects. In contrast, the TBDE of constraint dynamics include automatically by their mathematically consistent construction, off mass-shell effects. The second weakness, is that both quasipotential approaches, when applied to QED, do not produce the correct hyperfine structure. This, in our opinion, is a serious but easily correctable drawback in both approaches. Let us be precise here about our concerns. In both of these two quasipotential approaches to the relativistic QCD potential model, if one turns off the confining interaction and replaces the nonconfining vector potential by the Coulomb potential their resultant QED spectrum will be incorrect whether computed perturbatively or numerically. This calls into question their QCD spectral results since the resultant relativistic corrections which the omitted term $-A^2$ would have contributed is of the same order as the spin-dependent corrections (dependent on A' and $\nabla^2 A$) which, of course, are not omitted. Even if that correction is made, the wave equation should be shown to have (just as can be shown with the one-body Dirac equation) the same spectral results whether treated perturbatively or nonperturbatively. In addition, their modeling of the scalar interactions does not conform with the approach of the classical and quantum field theories used by Crater, Van Alstine, Yang, Sazdjian and Jolluli which supports our choice of vector interaction structures.[28] The third weakness is the lack of agreement in the $m_2 \to \infty$ limit with the Pauli form of the one-body Dirac equation. In particular, the lack of scalar Darwin terms in both approaches and vector Darwin terms in the approach of [6] is a serious weakness.

The constraint approach has been tested against both classical and quantum field theories for both scalar and vector interactions. In our construction of the vector potential [24,29,32], three primary guideposts were used beyond that of a minimal structure. The first is the use of just the barest (lowest order) input from field theory, the nonrelativistic Coulomb potential in QED.

The second is the gaugelike minimal coupling structure in Eq. (22) of the potential which Todorov postulated and was later confirmed in three independent ways: (a) Rizov, Todorov, and Aneva [51] demonstrated how the gauge structure (particularly the form $(\varepsilon_w - A)^2 - m_w^2$) arises in perturbation theory at a higher order than the Born approximation,[29] (b) by a comparison of the Fokker-Tetrode classical field theory $O(1/c^2)$ expansion for the Hamiltonian with the general quasipotential structure of $\mathbf{p}^2 + \Phi(\mathbf{r}, w) = (\varepsilon_w^2 - m_w^2)$ [25],[30] and (c) Jollouli and Sazdjian [26] who found a similar structure from nonperturbative quantum field theoretic arguments both for scalar and vector interactions. These three arguments demonstrate that the Born approximation structures (particularly the first term of (26) used to model the QCD potentials in [6]) cannot possibly yield that gauge structure and thus cannot yield the correct positronium spectrum when applied to QED. The higher order structures in [26,51] as well as the nonlinear hyperbolic structures in Eqs. (A2), (A3), (A7), and (A22)–(A25) argue that the Born structure of the first term in (26) are insufficient and must be supplemented by other invariant couplings. The papers by Sazdjian and collaborators [26,39,40] demonstrate this explicitly, showing that pseudovector coupling is essential if done perturbatively in order to get the Dirac equation into an external field form in which the minimal structure can be demonstrated. The work in [37,38] as well as by Sazdjian and collaborators shows that the vector coupling (see Eq. (A10)) alone will, when placed in the nonlinear context of the hyperbolic parametrization given in Eqs. (A2) and (A3), yield that external field form in which the minimal structure can be demonstrated.

The third guidepost is the use of the relativistic reduced mass m_w and energy ε_w (see Eqs. (7) and (8)). In addition to the discussions given in [27], their appearance in the forms $2m_w S + 2\varepsilon_w A$ as part of the minimal structure is also dictated by the same field theory mechanisms discussed in item (b) and (c) above. Note that even though in Eq. (19) there is the appearance of another relativistic reduced mass $(\varepsilon_1 \varepsilon_2 / w)$, in working out the quasipotential using the spinors as in Eq. (25) that reduced mass is replaced by m_w in the scalar and ε_w in the vector case. This replacement does not appear in the work of [7].

So, while the works of Ebert, Faustov, and Galkin, and Godfrey and Isgur are quite impressive in terms of spectral and decay agreements, it would be of value for adherents of those approaches to consider the criticisms presented in this paper, related to the ability of those equations to reproduce the static limit Dirac equation structures (which would include the $-A^2$ term) and general short range structure related to the $-A^2$ gauge term. These criticisms are not about their choice of QCD inspired potentials, but rather about how their relativistic wave equations translate the physics of those potentials into spectral results. Thus, these concern the impact on their QCD meson spectral results those two approaches would have based on their field theory connections to QED bound states.

Finally, a few words about future direction lines of research related to this paper. The approach given in our paper views the meson as a two-body bound state in a first quantized formalism. In place of the nonrelativistic Schrödinger equation for two interacting particles, we use the TBDE of constraint dynamics. There are systematic corrections that should follow the completion of this first step which would end with the inclusion of covariant pseudoscalar, pseudovector, and tensor interactions, in addition to the scalar and vector interactions we have included in this paper. First is a second quantized version of the TBDE similar to what has been accomplished for nonrelativistic second quantized formalisms by the Cornell group [55], Tornqvist [56], and more recently by Barnes and Swanson [57]. The latter includes a microscopic theory [58] of the $^3P_0$ model which includes pair production. The aim, as in their recent paper, would be to gain a measure of the effects of two-body meson decays on

[28] In particular, their approach does not include the quadratic structure $2m_w S + S^2$ implied by the those authors' approaches to scalar field theories.

[29] In particular, the quasipotential equation Eq. (18) for the potential to be used in Eq. (19) must be iterated to second order ($V^{(2)} = T_1 G T_1 - T_2$). It is remarkable that this gaugelike structure postulate Eq. (22) anticipates the systematic inclusion of higher order terms by the quasipotential formalism.

[30] In this, not only does the gaugelike minimal structure $(\varepsilon_w - A)^2$ appear to be a natural outgrowth of classical $O(1/c^2)$ expansion to at least order $1/c^4$, but also a minimal scalar interaction structure appears so that combined they yield $(\varepsilon_w - A)^2 - (m_w + S)^2$, again, at least through order $1/c^4$.

the observed rest mass of the decaying meson. Beyond that would be many body formalisms which view the meson more generally as a linear combination of $q\bar{q} + q\bar{q}g + \ldots$.

The primary weakness in our results of this paper are: (1) the radial and orbital excitations of the old meson spectroscopy; (2) the less than good hyperfine splittings of the $c\bar{c}$ and $b\bar{b}$ families compared with the good splitting results we obtained with all the other families, including the lightest and most highly relativistic $u\bar{d}$; (3) the failure to account for the light ($^3P_0$) scalar mesons; (4) Failure to reproduce the square-root Gell-Mann-Oakes-Renner relation. It is too early to say whether the completion of the first quantization program will rectify any of these problems. We point out, however, that should the weakness of our model, relating to the radial excitations of the light $q\bar{q}$ bound states be substantially improved by including pseudoscalar, pseudovector, and tensor interactions, this would offer the opportunity of allowing the hypothesis discussed in footnote 13 to be actively considered. Whether including these other interactions leads to substantial improvements or not, it will be essential to follow in parallel with our relativistic formalism, the second quantized nonrelativistic formalisms developed earlier.

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APPENDIX A: RELATIVISTIC SCHRÖDINGER EQUATION DETAILS

1. Connections between the TBDE and Eq. (35) and forms for $\tilde{A}_i^\mu$, $\tilde{S}_i$ in terms of the invariants $A(r)$, $V(r)$, and $S(r)$

Here we present an outline of some details of Eq. (14) and its Pauli-Schrödinger reduction given in full elsewhere (see [24,37,38,42]). Each of the two Dirac equations in (14) has a form similar to a single particle Dirac equation in an external four-vector and scalar potential but here acting on a 16 component wave function Ψ which is the product of an external part (being a plane wave eigenstate of P) multiplying the internal wave function ψ

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}. \tag{A1}$$

The four ψ_i are each four-component spinor wave functions. To obtain the actual general spin-dependent forms of those $\tilde{A}_i^\mu$, $\tilde{S}_i$ potentials which were required by the compatibility condition $[\mathcal{S}_1, \mathcal{S}_2]\psi = 0$ was a most perplexing problem, involving the discovery of underlying supersymmetries in the case of scalar and timelike vector interactions [24,28]. Extending those external potential forms to more general covariant interactions necessitated an entirely different approach leading to what is called the hyperbolic form of the TBDE. The most general hyperbolic form for compatible TBDE is

$$\begin{aligned} \mathcal{S}_1\psi &= (\cosh(\Delta)\mathbf{S}_1 + \sinh(\Delta)\mathbf{S}_2)\psi = 0, \\ \mathcal{S}_2\psi &= (\cosh(\Delta)\mathbf{S}_2 + \sinh(\Delta)\mathbf{S}_1)\psi = 0, \end{aligned} \tag{A2}$$

where Δ represents any invariant interaction singly or in combination. It has a matrix structure in addition to coordinate dependence. Depending on that matrix structure we have either vector, scalar or more general tensor interactions [37]. The operators $\mathbf{S}_1$ and $\mathbf{S}_2$ are auxiliary constraints satisfying

$$\begin{aligned} \mathbf{S}_1\psi &\equiv (\mathcal{S}_{10}\cosh(\Delta) + \mathcal{S}_{20}\sinh(\Delta))\psi = 0, \\ \mathbf{S}_2\psi &\equiv (\mathcal{S}_{20}\cosh(\Delta) + \mathcal{S}_{10}\sinh(\Delta))\psi = 0, \end{aligned} \tag{A3}$$

in which the $\mathcal{S}_{i0}$ are the free Dirac operators

$$\mathcal{S}_{i0} = \frac{i}{\sqrt{2}}\gamma_{5i}(\gamma_i \cdot p_i + m_i). \tag{A4}$$

This, in turn leads to the two compatibility conditions [31,37,39]

$$[\mathcal{S}_1, \mathcal{S}_2]\psi = 0, \tag{A5}$$

and

$$[\mathbf{S}_1, \mathbf{S}_2]\psi = 0, \tag{A6}$$

provided that $\Delta(x) = \Delta(x_\perp)$. These compatibility conditions do not restrict the gamma matrix structure of Δ. That matrix structure is determined by the type of vertex-vertex structure we wish to incorporate in the interaction. The three types of invariant interactions Δ that we use in this paper are

$$\begin{aligned} &\Delta_{\mathcal{L}}(x_\perp) = -1_1 1_2 \frac{\mathcal{L}(x_\perp)}{2}\mathcal{O}_1, \\ &\qquad \mathcal{O}_1 = -\gamma_{51}\gamma_{52}, \quad \text{scalar}, \\ &\Delta_{\mathcal{J}}(x_\perp) = \beta_1\beta_2 \frac{\mathcal{J}(x_\perp)}{2}\mathcal{O}_1, \quad \text{time-like vector}, \\ &\Delta_{\mathcal{G}}(x_\perp) = \gamma_{1\perp}\cdot\gamma_{2\perp}\frac{\mathcal{G}(x_\perp)}{2}\mathcal{O}_1, \quad \text{space-like vector}, \end{aligned} \tag{A7}$$

where

$$\begin{aligned} &\gamma^\mu_{i\perp} = (\eta^{\mu\nu} + \hat{P}^\mu\hat{P}^\nu)\gamma_{\nu i}, \qquad \gamma_{5i} = \gamma_i^0\gamma_i^1\gamma_i^2\gamma_i^3, \\ &\beta_i = -\gamma_i\cdot\hat{P}, \qquad i = 1, 2. \end{aligned} \tag{A8}$$

For general independent scalar, timelike vector, and spacelike vector interactions we have

$$\Delta(x_\perp) = \Delta_{\mathcal{L}} + \Delta_{\mathcal{J}} + \Delta_{\mathcal{G}}. \tag{A9}$$

The special case of an electromagnetic-like interaction (in the Feynman gauge) corresponds to $\mathcal{J} = -\mathcal{G}$ or

$$\begin{aligned}\Delta_{\mathcal{J}} + \Delta_{\mathcal{G}} &\equiv \Delta_{\mathcal{EM}} \\ &= (-\gamma_1 \cdot \hat{P}\gamma_2 \cdot \hat{P} + \gamma_{1\perp} \cdot \gamma_{2\perp})\frac{\mathcal{G}(x_\perp)}{2}\mathcal{O}_1 \\ &= \gamma_1 \cdot \gamma_2 \frac{\mathcal{G}(x_\perp)}{2}\mathcal{O}_1. \end{aligned} \tag{A10}$$

Our Th1 corresponds to a scalar and electromagnetic interaction,

$$\Delta(x_\perp) = \Delta_{\mathcal{L}} + \Delta_{\mathcal{EM}}. \tag{A11}$$

Our Th2 corresponds to a modification of the timelike portion of $\Delta_{\mathcal{EM}}$ to

$$\begin{aligned}\Delta(x_\perp) &= \Delta_{\mathcal{L}} + \Delta_{\mathcal{J}} + \Delta_{\mathcal{G}} \\ &= (-1_1 1_2 \mathcal{L}(x_\perp) + \beta_1\beta_2\mathcal{J}(x_\perp) \\ &\quad + \gamma_{1\perp} \cdot \gamma_{2\perp}\mathcal{G}(x_\perp))\frac{\mathcal{O}_1}{2}, \\ \mathcal{J} &\neq -\mathcal{G}. \end{aligned} \tag{A12}$$

This leads to[31] [37,38] ($\partial_\mu = \partial/\partial x^\mu$)

$$\begin{aligned}\mathcal{S}_1\psi &= (-G\beta_1\Sigma_1 \cdot \mathcal{P}_2 + E_1\beta_1\gamma_{51} + M_1\gamma_{51} \\ &\quad - G\frac{i}{2}\Sigma_2 \cdot \partial(\mathcal{L}\beta_2 - \mathcal{J}\beta_1)\gamma_{51}\gamma_{52})\psi = 0, \\ \mathcal{S}_2\psi &= (G\beta_2\Sigma_2 \cdot \mathcal{P}_1 + E_2\beta_2\gamma_{52} + M_2\gamma_{52} \\ &\quad + G\frac{i}{2}\Sigma_1 \cdot \partial(\mathcal{L}\beta_1 - \mathcal{J}\beta_2)\gamma_{51}\gamma_{52})\psi = 0, \end{aligned} \tag{A13}$$

with

$$G = \exp\mathcal{G}, \qquad \mathcal{P}_i \equiv p_\perp - \frac{i}{2}\Sigma_i \cdot \partial\mathcal{G}\Sigma_i. \tag{A14}$$

The connections between what we call the vertex invariants, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{G}$, and the mass and energy potentials, M_i, E_i, are found to be

$$\begin{aligned}M_1 &= m_1\cosh\mathcal{L} + m_2\sinh\mathcal{L}, \\ M_2 &= m_2\cosh\mathcal{L} + m_1\sinh\mathcal{L}, \\ E_1 &= \varepsilon_1\cosh\mathcal{J} + \varepsilon_2\sinh\mathcal{J}, \\ E_2 &= \varepsilon_2\cosh\mathcal{J} + \varepsilon_1\sinh\mathcal{J}. \end{aligned} \tag{A15}$$

Equation (A13) depends on the standard Pauli-Dirac representation of gamma matrices in block forms (see Eq. (2.28) in [2] for their explicit forms) and where[32]

$$\Sigma_i = \gamma_{5i}\beta_i\gamma_{\perp i}. \tag{A16}$$

Comparing Eq. (A13) with Eq. (14) we find that the spin-dependent vector interactions of Eq. (14) are [24,29]

$$\begin{aligned}\tilde{A}_1^\mu &= ((\varepsilon_1 - E_1) - i\frac{G}{2}(\gamma_2 \cdot \partial\mathcal{J})\gamma_2 \cdot \hat{P})\hat{P}^\mu + (1 - G)p_\perp^\mu \\ &\quad - \frac{i}{2}\partial G \cdot \gamma_2\gamma_{2\perp}^\mu, \\ A_2^\mu &= ((\varepsilon_2 - E_2) + i\frac{G}{2}(\gamma_1 \cdot \partial\mathcal{J}))\gamma_1 \cdot \hat{P})\hat{P}^\mu - (1 - G)p_\perp^\mu \\ &\quad + \frac{i}{2}\partial G \cdot \gamma_1\gamma_{1\perp}^\mu. \end{aligned} \tag{A17}$$

Note that the first portion of the vector potentials is timelike (parallel to $\hat{P}^\mu$) while the next portion is spacelike (perpendicular to $\hat{P}^\mu$). The spin-dependent scalar potentials $\tilde{S}_i$ are

$$\begin{aligned}\tilde{S}_1 &= M_1 - m_1 - \frac{i}{2}G\gamma_2 \cdot \partial\mathcal{L}, \\ \tilde{S}_2 &= M_2 - m_2 + \frac{i}{2}G\gamma_1 \cdot \partial\mathcal{L}. \end{aligned} \tag{A18}$$

Equation (A17) simplifies to

$$\begin{aligned}\tilde{A}_1^\mu &= ((\varepsilon_1 - E_1))\hat{P}^\mu + (1 - G)p_\perp^\mu - \frac{i}{2}\partial G \cdot \gamma_2\gamma_2^\mu, \\ A_2^\mu &= ((\varepsilon_2 - E_2))\hat{P}^\mu - (1 - G)p_\perp^\mu + \frac{i}{2}\partial G \cdot \gamma_1\gamma_1^\mu, \end{aligned} \tag{A19}$$

for electromagnetic-like interactions.

We have chosen a parametrization for $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{G}$ that takes advantage of the Todorov effective external potential forms and at the same time will display the correct static limit form for the Pauli reduction (see Eq. (75)). The choice for these parametrizations is fixed due to the fact that for classical [25] or quantum field theories [26] for separate scalar and vector interactions the

[31] In short, one inserts Eq. (A3) into (A2) and brings the free Dirac operator (A4) to the right of the matrix hyperbolic functions. Using commutators and $\cosh^2\Delta - \sinh^2\Delta = 1$ one arrives at Eq. (A13). The structure of these equations are very much the same as that of a Dirac equation for each of the two particles, with M_i and E_i playing the roles that $m + S$ and $\varepsilon - A$ do in the single particle Dirac Eq. (74). Over and above the usual kinetic part, the spin-dependent modifications involving $G\mathcal{P}_i$ and the last set of derivative terms are two-body recoil effects essential for the compatibility (consistency) of the two equations.

[32] Just as x^μ is a four-vector, so are γ^μ (in the sense of Dirac) and P^μ. Thus, the matrix structures of the timelike and spacelike interactions in Eq. (A7) are $\gamma_1^0\gamma_2^0$ and $\boldsymbol{\gamma}_1 \cdot \boldsymbol{\gamma}_2$ only in the c.m. system due to the fact that from Eq. (A8), $\beta_i = \gamma_i^0$ only in the c.m. frame. Likewise, $\Sigma_i^\mu = (0, \boldsymbol{\Sigma})$ only in the c.m. frame just as is $x_\perp^\mu = (0, \mathbf{r})$ in that frame only.

spin-independent part of the quasipotential Φ_w involves the difference of squares of the invariant mass and energy potentials (M_i and E_i respectively)

$$M_i^2 = m_i^2 + 2m_w S + S^2; \quad E_i^2 = \varepsilon_i^2 - 2\varepsilon_w V + V^2, \tag{A20}$$

so that

$$M_i^2 - E_i^2 = 2m_w S + S^2 + 2\varepsilon_w V - V^2 - b^2(w). \tag{A21}$$

Strictly speaking, the forms in Eqs. (A20) and (A21) are for scalar and timelike vector interactions. Equations (A14) and (A13) involve combined scalar, electromagnetic-like, and separate timelike vector interactions. Without the separate timelike interactions this amounts to working in the Feynman gauge with the simplest relation between spacelike and timelike parts, (see Eqs. (A10) and (A11), and [2,13]). In the general case the mass and energy potentials in place of Eq. (A20) are respectively

$$M_i^2 = m_i^2 + \exp(2\mathcal{G})(2m_w S + S^2), \tag{A22}$$

$$E_i^2 = \exp(2\mathcal{G}(\mathcal{A}))((\varepsilon_i - A)^2 - 2\varepsilon_w V + V^2), \tag{A23}$$

so that from Eq. (A15),

$$\begin{aligned} \exp(\mathcal{L}) = \exp(\mathcal{L}(S, A)) &= \frac{\sqrt{m_1^2 + \exp(2\mathcal{G})(2m_w S + S^2)} + \sqrt{m_2^2 + \exp(2\mathcal{G})(2m_w S + S^2)}}{m_1 + m_2}, \\ \exp(\mathcal{J}) = \exp(\mathcal{J}(V, A)) &= \exp(\mathcal{G})\frac{\sqrt{(\varepsilon_1 - A)^2 - 2\varepsilon_w V + V^2} + \sqrt{(\varepsilon_2 - A)^2 - 2\varepsilon_w V + V^2}}{\varepsilon_1 + \varepsilon_2}, \end{aligned} \tag{A24}$$

with

$$\exp(2\mathcal{G}(A)) = \frac{1}{(1 - 2A/w)} \equiv G^2. \tag{A25}$$

Below we present, the consequent connections to the invariant interaction functions A, V, and S.

(a) (a) In the case of electromagnetic interactions ($V = 0$) with scalar confinement (Th1), we have

$$\begin{aligned} \mathcal{J} = -\mathcal{G} &= \frac{1}{2}\log(1 - 2A/w) = \log\frac{E_1 + E_2}{w}, \\ E_i^2 &= \exp(2\mathcal{G})(\varepsilon_i - A)^2, \\ M_i^2 &= m_i^2 + \exp(2\mathcal{G})(2m_w S + S^2), \end{aligned} \tag{A26}$$

and the spin-independent (SI) minimal coupling comes from

$$\begin{aligned} \exp(2\mathcal{G})p^2 + M_i^2 - E_i^2 &\to p^2 + \exp(-2\mathcal{G})(M_i^2 - E_i^2) \\ &= p^2 + \Phi_{\rm SI} - b^2 \end{aligned} \tag{A27}$$

and appears as

$$\begin{aligned} \Phi_{SI} - b^2 &= 2m_w S + S^2 + m_i^2(1 - 2A/w) - (\varepsilon_i - A)^2 \\ &= 2m_w S + S^2 + 2\varepsilon_w A - A^2 - b^2. \end{aligned} \tag{A28}$$

(b) (b) In the case of pure timelike vector interactions and scalar interactions (with no electromagnetic-like interactions, this model is not appropriate for meson spectroscopy) we have

$$\begin{aligned} \mathcal{J} &= \mathcal{J}_0 \equiv \log\frac{E_{10} + E_{20}}{w}, \\ E_{i0}^2 &\equiv \varepsilon_i^2 - 2\varepsilon_w V + V^2, \qquad \mathcal{G} = 0, \end{aligned} \tag{A29}$$

and the spin-independent minimal coupling appears as

$$\Phi_{\rm SI} = 2m_w S + S^2 + 2\varepsilon_w V - V^2. \tag{A30}$$

(c) (c) When we include independent timelike and electromagnetic-like simultaneously together with scalar interactions (Th2) then we have

$$\begin{aligned} -\mathcal{G} &= \frac{1}{2}\log(1 - 2A/w), \\ \mathcal{J} &= \log\frac{E_1 + E_2}{w}, \\ E_i^2 &= \exp(2\mathcal{G})((\varepsilon_i - A)^2 - 2\varepsilon_w V + V^2) \\ &= \exp(2\mathcal{G})(E_{i0}^2 - 2\varepsilon_i A + A^2) \end{aligned} \tag{A31}$$

and the spin-independent minimal coupling appears like

$$\Phi_{\rm SI} = 2m_w S + S^2 + 2\varepsilon_w A - A^2 + 2\varepsilon_w V - V^2. \tag{A32}$$

2. Details on Eq. (35)

The Klein-Gordon like potential energy terms appearing at the beginning of the Pauli form (35) arise from

$$M_i^2 - E_i^2 = \exp(2\mathcal{G})[2m_w S + S^2 + 2\varepsilon_w A - A^2 + 2\varepsilon_w V - V^2 - b^2(w)].$$

To obtain the symbolic Pauli form of Eq. (16) and the subsequent detailed form in Eq. (35) involves steps similar to those used in the Pauli reduction of the single particle Dirac equation [33] but with the combinations $\phi_\pm = \psi_1 \pm \psi_4$ and $\chi_\pm = \psi_2 \pm \psi_3$ instead of the individual single particle wave function. This reduces the Pauli forms to 4 uncoupled four-component relativistic Schrödinger equations [2,16,38,40,42]. We work in the c.m. frame in which $\hat{P} = (1, \mathbf{0})$ and $\hat{r} = (0, \hat{\mathbf{r}})$. The final four-component wave functions $\psi_\pm$, $\eta_\pm$ that appear in Eq. (35) are defined by [42]

$$\begin{aligned}\phi_\pm &= \exp(\mathcal{F} + \mathcal{K}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\psi_\pm\\ &= (\exp\mathcal{F})(\cosh\mathcal{K} + \sinh\mathcal{K}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\psi_\pm,\\ \chi_\pm &= \exp(\mathcal{F} + \mathcal{K}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\eta_\pm\\ &= (\exp\mathcal{F})(\cosh\mathcal{K} + \sinh\mathcal{K}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\eta_\pm,\end{aligned} \tag{A33}$$

in which

$$\begin{aligned}\mathcal{F} &= \frac{1}{2}\log\frac{\mathcal{D}}{\varepsilon_2 m_1 + \varepsilon_1 m_2} - \mathcal{G},\\ \mathcal{D} &= E_2M_1 + E_1M_2,\\ \mathcal{K} &= \frac{(\mathcal{L} - \mathcal{J})}{2}.\end{aligned} \tag{A34}$$

In analogy to what occurs in the decoupled form of the Schrödinger equation for the individual single particle wave function, this substitution has the convenient property that in the resultant bound state equation, the coefficients of the first order relative momentum terms vanish.

Using the results in [33,42], we obtain for the general case of unequal masses the relativistic Schrödinger Eq. (35) that is a detailed c.m. form of Eq. (16). In that equation we have introduced the abbreviations

$$\begin{aligned}\Phi_{\mathrm{D}} &= -\frac{2(\mathcal{F}' + 1/r)(\cosh 2\mathcal{K} - 1)}{r} + \mathcal{F}'^2 + \mathcal{K}'^2 + \frac{2\mathcal{K}'\sinh 2\mathcal{K}}{r} - \boldsymbol{\nabla}^2\mathcal{F} + m(r),\\ \Phi_{\mathrm{SO}} &= -\frac{\mathcal{F}'}{r} - \frac{(\mathcal{F}' + 1/r)(\cosh 2\mathcal{K} - 1)}{r} + \frac{\mathcal{K}'\sinh 2\mathcal{K}}{r},\\ \Phi_{\mathrm{SOD}} &= (l'\cosh 2\mathcal{K} - q'\sinh 2\mathcal{K}),\\ \Phi_{\mathrm{SOX}} &= (q'\cosh 2\mathcal{K} - l'\sinh 2\mathcal{K}),\\ \Phi_{\mathrm{SS}} &= k(r) + \frac{2\mathcal{K}'\sinh 2\mathcal{K}}{3r} - \frac{2(\mathcal{F}' + 1/r)(\cosh 2\mathcal{K} - 1)}{3r} + \frac{2\mathcal{F}'\mathcal{K}'}{3} - \frac{\boldsymbol{\nabla}^2\mathcal{K}}{3},\\ \Phi_{\mathrm{T}} &= \frac{1}{3}\left[n(r) + \frac{(3\mathcal{F}' - \mathcal{K}' + 3/r)\sinh 2\mathcal{K}}{r} + \frac{(\mathcal{F}' - 3\mathcal{K}' + 1/r)(\cosh 2\mathcal{K} - 1)}{r} + 2\mathcal{F}'\mathcal{K}' - \boldsymbol{\nabla}^2\mathcal{K}\right],\\ \Phi_{\mathrm{SOT}} &= -\mathcal{K}'\frac{\cosh 2\mathcal{K} - 1}{r} - \frac{\mathcal{K}'}{r} + \frac{(\mathcal{F}' + 1/r)\sinh 2\mathcal{K}}{r},\end{aligned} \tag{A35}$$

where

$$\begin{aligned}k(r) &= \frac{1}{3}\nabla^2(\mathcal{K} + \mathcal{G}) - \frac{2\mathcal{F}'(\mathcal{G}' + \mathcal{K}')}{3} - \frac{1}{2}\mathcal{G}'^2,\\ n(r) &= \frac{1}{3}\left[\nabla^2\mathcal{K} - \frac{1}{2}\nabla^2\mathcal{G} + \frac{3(\mathcal{G}' - 2\mathcal{K}')}{2r} + \mathcal{F}'(\mathcal{G}' - 2\mathcal{K}')\right],\\ m(r) &= -\frac{1}{2}\nabla^2\mathcal{G} + \frac{3}{4}\mathcal{G}'^2 + \mathcal{G}'\mathcal{F}' - \mathcal{K}'^2,\end{aligned} \tag{A36}$$

and

$$\begin{aligned}l'(r) &= -\frac{1}{2r}\frac{E_2M_2 - E_1M_1}{E_2M_1 + E_1M_2}(\mathcal{L}' + \mathcal{J}'),\\ q'(r) &= \frac{1}{2r}\frac{E_2M_1 - E_1M_2}{E_2M_1 + E_1M_2}(\mathcal{L}' + \mathcal{J}').\end{aligned} \tag{A37}$$

(The prime symbol stands for d/dr, and the explicit forms of the derivatives are given in Eq. (A38) below). For $L = J$ states, the hyperbolic terms cancel and the spin-orbit difference terms in general produce spin mixing except for equal masses or $J = 0$. For ease of use we have listed in Appendix A 3 the explicit forms that appear in the above Φ's in Eqs. (A35) and (A36) in terms of the general invariant potentials $A(r)$, $V(r)$, and $S(r)$. The radial components of Eq. (35) are given in Appendix B.

3. Explicit expressions for terms in the relativistic Schrödinger Eq. (35) from $A(r)$, $V(r)$ and $S(r)$

Given the functions $A(r)$, $V(r)$, and $S(r)$ for the interaction, users of the relativistic Schrödinger Eq. (35) will find it convenient to have an explicit expression in an order that would be useful for programing the terms in the associated Eqs. (A35)–(A37). We use the definitions above given in Eqs. (A22)–(A25) and (A34). In order that the terms in Eq. (A35) be reduced to expressions involving just $A(r)$, $V(r)$, $S(r)$ and their derivatives, we list the following formulae

$$\mathcal{F}' = \frac{(\mathcal{L}' + \mathcal{J}')(E_2M_2 + E_1M_1)}{2(E_2M_1 + E_1M_2)} - \mathcal{G}',$$
$$\mathcal{G}' = \frac{A'}{w - 2A},$$
$$\mathcal{L}' = \frac{M_1'}{M_2} = \frac{M_2'}{M_1} = \frac{w}{M_1M_2}\left(\frac{S'(m_w + S)}{w - 2A} + \frac{(2m_wS + S^2)A'}{(w - 2A)^2}\right),$$
$$\mathcal{J}' = \frac{E_1'}{E_2} = \frac{E_2'}{E_1} = -\frac{(\mathcal{G}'[(\varepsilon_1 - A)(\varepsilon_2 - A) + 2\varepsilon_w V - V^2] + (\varepsilon_w - V)V')}{E_1E_2(w - 2A)/w},$$
$$\mathcal{K}' = \frac{(\mathcal{L}' - \mathcal{J}')}{2}. \tag{A38}$$

Also needed are

$$\cosh 2\mathcal{K} = \frac{1}{2}\left(\frac{(\varepsilon_1 + \varepsilon_2)(M_1 + M_2)}{(m_1 + m_2)(E_1 + E_2)} + \frac{(m_1 + m_2)(E_1 + E_2)}{(\varepsilon_1 + \varepsilon_2)(M_1 + M_2)}\right),$$
$$\sinh 2\mathcal{K} = \frac{1}{2}\left(\frac{(\varepsilon_1 + \varepsilon_2)(M_1 + M_2)}{(m_1 + m_2)(E_1 + E_2)} - \frac{(m_1 + m_2)(E_1 + E_2)}{(\varepsilon_1 + \varepsilon_2)(M_1 + M_2)}\right), \tag{A39}$$

and

$$\nabla^2\mathcal{F} = \frac{(\nabla^2\mathcal{L} + \nabla^2\mathcal{J})(E_2M_2 + E_1M_1)}{2(E_2M_1 + E_1M_2)} - (\mathcal{L}' + \mathcal{J}')^2\frac{(m_1^2 - m_2^2)^2}{2(E_2M_1 + E_1M_2)^2} - \nabla^2\mathcal{G},$$
$$\nabla^2\mathcal{L} = \frac{-\mathcal{L}'^2(M_1^2 + M_2^2)}{M_1M_2} + \frac{w}{M_1M_2}\left(\frac{\nabla^2S(m_w + S) + S'^2}{w - 2A} + \frac{4S'(m_w + S)A' + (2m_wS + S^2)\nabla^2A}{(w - 2A)^2} + \frac{4(2m_wS + S^2)A'^2}{(w - 2A)^3}\right),$$
$$\nabla^2\mathcal{J} = -\left[\left(\frac{E_1^2 + E_2^2}{E_1E_2}\right)\mathcal{J}' - 2\mathcal{G}'\right]\mathcal{J}' - \frac{\exp(2\mathcal{G})}{E_1E_2}\{\nabla^2\mathcal{G}[(\varepsilon_1 - A)(\varepsilon_2 - A) + 2\varepsilon_wV - V^2] + (\varepsilon_w - V)\nabla^2V$$
$$- \mathcal{G}'^2(w - 2A)^2 - V'^2 + 2V'\mathcal{G}'(\varepsilon_w - V)\}$$
$$\nabla^2\mathcal{G} = \frac{\nabla^2A}{w - 2A} + 2\mathcal{G}'^2. \tag{A40}$$

The expressions for $k(r)$, $m(r)$, and $n(r)$ that appear in Eq. (A35)) are given in Eq. (A36). They can be evaluated using the above expressions plus

$$\nabla^2\mathcal{K} = \frac{\nabla^2\mathcal{L} - \nabla^2\mathcal{J}}{2}. \tag{A41}$$

The only remaining parts of Eq. (A35) that need expressing are those for l' and q' given in Eq. (A37) Using Eq. (A34) they can be obtained in terms of the above formulae.

4. Weak potential limits of quasipotentials

The weak potential forms of the quasipotentials are needed for working out perturbative spectral corrections. For weak potentials, we take $\varepsilon_i = m_i$ wherever they appear in the potentials and assume $|A|, |V|, |S| \ll m_i$. Thus

$$\mathcal{L} \rightarrow \frac{m_wS}{m_1m_2} \rightarrow \frac{S}{m_1 + m_2},$$
$$\mathcal{J} \rightarrow -\frac{A}{w} - \frac{\varepsilon_wV}{\varepsilon_1\varepsilon_2} \rightarrow -\frac{A + V}{m_1 + m_2}, \tag{A42}$$
$$\mathcal{G} \rightarrow \frac{A}{w} \rightarrow \frac{A}{m_1 + m_2}.$$

Among other results, this will allow us to see how the scalar interaction contributes oppositely to the spin-orbit and Darwin terms from both vector interactions to lowest order. In that same limit, the dominant portions of Φ's in Eq. (35) are

$$\Phi_{\rm D} \to -\nabla^2\mathcal{F} \to -\frac{1}{4r(m_1+m_2)}\left(\frac{\nabla^2(\mathcal{S}-A-V)(m_2^2+m_1^2)}{m_2m_1} - \nabla^2 A\right),$$
$$\Phi_{\rm SO} \to -\frac{\mathcal{F}'}{r} \to -\frac{1}{4r(m_1+m_2)}\left(\frac{(\mathcal{S}'-A'-V')(m_2^2+m_1^2)}{m_2m_1} - A'\right)$$
$$\Phi_{\rm SOD} \to l' \to -\frac{1}{4r(m_1+m_2)}\frac{(\mathcal{S}'-A'-V')(m_2^2-m_1^2)}{m_1m_2}$$
$$\Phi_{\rm SOX} \to q' \to 0 \tag{A43}$$
$$\Phi_{\rm SS} \to \frac{1}{3}\nabla^2\mathcal{G} \to \frac{1}{6(m_1+m_2)}\nabla^2 A,$$
$$\Phi_{\rm T} \to -\boldsymbol{\nabla}^2\frac{(S+A+V)}{9(m_1+m_2)},$$
$$\Phi_{\rm SOT} \to -\frac{\mathcal{K}'}{r} \to -\frac{(S'+A'+V')}{2r(m_1+m_2)}.$$

Note how in the Darwin and spin-orbit terms the $\mathcal{S}' - V'$ dependence shows how the scalar and timelike confining effects tend to cancel for $\xi < 1$. As anticipated, only the Darwin and spin-orbit terms survive in the static limit when one of the two particles becomes very massive. In that limit (say $m_2 \to \infty$), the two spin-orbit terms $\Phi_{\rm SO}$ and $\Phi_{\rm SOD}$ combine to

$$-\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\frac{\mathcal{S}'-A'-V'}{4r} - \mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2) \times \frac{\mathcal{S}'-A'-V'}{4r} = -\mathbf{L}\cdot\boldsymbol{\sigma}_1\frac{\mathcal{S}'-A'-V'}{2r}. \tag{A44}$$

5. Single particle limit ($m_2 \to \infty$) of the TBDE

In addition to using $p_2 \to (m_2, \mathbf{0})$, $p_1 = (\varepsilon_1, \mathbf{p})$, the single particle limit of the TBDE is obtained by substituting $m_2 \to \infty$ in the expressions for the various potentials listed in Appendices A1, A2, and A3. To determine that limit, we use that the total c.m. energy $w = \varepsilon_2 + \varepsilon_1 \to m_2 + \varepsilon_1$. In that case using the expressions for m_w and ε_w in Eqs. (7) and (8) we find

$$m_w \to m_1 \equiv m, \qquad \varepsilon_w \to \varepsilon_1 \equiv \varepsilon,$$
$$G = (1 - 2A/(m_2+\varepsilon))^{-1/2} \to 1, \tag{A45}$$

and thus for the quantities related to the mass potentials, we have

$$M_1 \to \sqrt{(m+S)^2} = m + S,$$
$$M_2 - m_2 \to \sqrt{m_2^2 + 2mS + S^2} - m_2 \to 0, \tag{A46}$$
$$\exp(\mathcal{L}) = \exp\left(\frac{M_1+M_2}{m_1+m_2}\right) \to 1, \qquad \tilde{S}_1 \to S,$$
$$\tilde{S}_2 \to 0.$$

For the energy potentials we have, since $\hat{P} \to (1, \mathbf{0})$ that

$$E_1 \to \sqrt{\varepsilon^2 - 2\varepsilon(A+V) + A^2 + V^2} \equiv \varepsilon - U,$$
$$\varepsilon_2 - E_2 \to \varepsilon_2 - \sqrt{\varepsilon_2^2 - 2\varepsilon_2 A - 2\varepsilon V + A^2 + V^2}$$
$$\to \varepsilon_2 - \varepsilon_2\sqrt{1 - 2A/\varepsilon_2} \to A,$$
$$\exp(\mathcal{J}) = \exp\left(\frac{E_1+E_2}{\varepsilon_1+\varepsilon_2}\right) \to 1,$$
$$\tilde{A}_1^{\mu} \to U(1, \mathbf{0}),$$
$$\tilde{A}_2^{\mu} \to A(1, \mathbf{0}). \tag{A47}$$

The TBDE (14) thus reduce to

$$(\boldsymbol{\gamma}_1\cdot\mathbf{p} - \beta_1(\varepsilon - U) + m + S)\psi = 0,$$
$$(-\beta_2(m_2 - A) + m_2)\psi \to -m_2(\beta_2 - 1)\psi = 0, \tag{A48}$$

effectively to a single ordinary one-body Dirac Eq. (74) for a particle in combined external scalar and time only component vector potential. Note that $U \neq A + V$.[33] Doing the usual Pauli reduction yields for the upper component wave

[33]Strictly speaking, in the static limit, the square roots forms obtained from Eq. (A22) imply $M_1 \to |m+S| > 0$. For for large distances, $A \to 0$, and so Eq. (A23) implies $E_1 \to |\varepsilon - V|$ and for short distances $V \to 0$, $E_1 \to |\varepsilon - A|$. These could possibly be in opposition to the forms with indefinite sign of $m + S$ and $\varepsilon - V$ or $\varepsilon - A$ that would appear in the one-body Dirac equation. In our applications to meson spectroscopy, S is always positive so we need not worry about the use of the square-root form of M_i in the computation of M_i. Since V is positive and increasing with distance, it is possible that at large enough distances $\varepsilon - V < 0$. In that case, use of the positive root for the square root would not agree with the sign of $\varepsilon - V$. We found that for the $^1S_0$, $b\bar{u}$, and $b\bar{d}$ mesons, that does occur near the very end of the integration cutoff distance of about 1.92 Fermis. That is so close to the end, however, that use of the positive square root is unlikely to have any effect on the spectral results (for the $^3P_2$, $b\bar{u}$, and $b\bar{d}$ mesons, $\varepsilon - V$ remained positive throughout the integration region.) Future work may address the theoretical problem of how to make the approach to the static limit of the square-root forms smooth and exact.

function the Schrödinger-like form which is the same as the $m_2 \to \infty$ limit of Eq. (35). That limit can be readily seen from Eqs. (A34) and (A45)–(A47) which shows

$$\begin{aligned}
\mathcal{F} &\to \frac{1}{2}\log\frac{m+S+\varepsilon-U}{m+\varepsilon},\\
\mathcal{F}' &= \frac{1}{2}\frac{S'-U'}{m+S+\varepsilon-A},\\
\nabla^2\mathcal{F} &= \frac{1}{2}\frac{\nabla^2 S-\nabla^2 U}{m+S+\varepsilon-U}-\frac{1}{2}\left(\frac{S'-U'}{m+S+\varepsilon-U}\right)^2,\\
\mathcal{L}', \mathcal{J}', \mathcal{G}', \mathcal{K}' &\to 0,\\
\Phi_D &\to \mathcal{F}'^2-\boldsymbol{\nabla}^2\mathcal{F},\\
\Phi_{SO} &\to -\frac{\mathcal{F}'}{r},\\
\Phi_{SOD} &\to l' = -\frac{1}{2r}\frac{E_2M_2-E_1M_1}{E_2M_1+E_1M_2}(\mathcal{L}'+\mathcal{J}')\\
&\to -\frac{1}{2r}\frac{E_2M_2}{E_2M_1+E_1M_2}(\mathcal{L}'+\mathcal{J}') \to -\frac{\mathcal{F}'}{r}.
\end{aligned} \tag{A49}$$

From the last line we obtain

$$\begin{aligned}
&\Phi_{SO}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+\Phi_{SOD}\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\\
&\quad = -\frac{2\mathcal{F}'}{r}\mathbf{L}\cdot\boldsymbol{\sigma}_1 = -\frac{S'-U'}{m+S+\varepsilon-U}\mathbf{L}\cdot\boldsymbol{\sigma}_1.
\end{aligned} \tag{A50}$$

and hence Eq. (35) reduces to Eq. (75). In Sec. VI D we display our static limit equations for Th1.

APPENDIX B: RADIAL EQUATIONS

The following are radial eigenvalue equations, [33,42], corresponding to Eq. (35). For a general singlet ${}^1J_J$ wave function $v_{LSJ} = v_{J0J} \equiv v_0$ coupled to a general triplet ${}^3J_J$ wave function $v_{J1J} \equiv v_1$, the wave equation

$$\begin{aligned}
&\left\{-\frac{d^2}{dr^2}+\frac{J(J+1)}{r^2}+2m_wS+S^2+2\varepsilon_wA-A^2+2\varepsilon_wV\right.\\
&\left.\quad -V^2+\Phi_D-3\Phi_{SS}\right\}v_0+2\sqrt{J(J+1)}(\Phi_{SOD}-\Phi_{SOX})v_1\\
&= b^2v_0,
\end{aligned} \tag{B1}$$

is coupled to

$$\begin{aligned}
&\left\{-\frac{d^2}{dr^2}+\frac{J(J+1)}{r^2}+2m_wS+S^2+2\varepsilon_wA-A^2+2\varepsilon_wV\right.\\
&\left.\quad -V^2+\Phi_D-2\Phi_{SO}+\Phi_{SS}+2\Phi_T-2\Phi_{SOT}\right\}v_1\\
&\quad +2\sqrt{J(J+1)}(\Phi_{SOD}+\Phi_{SOX})v_0\\
&= b^2v_1.
\end{aligned} \tag{B2}$$

For a general $S=1$, $J=L+1$ wave function $v_{J-11J} \equiv v_+$ coupled to a general $S=1$, $J=L-1$ wave function $v_{J+11J} \equiv v_-$ the equation

$$\begin{aligned}
&\left\{-\frac{d^2}{dr^2}+\frac{J(J-1)}{r^2}+2m_wS+S^2+2\varepsilon_wA-A^2+2\varepsilon_wV\right.\\
&\quad -V^2+\Phi_D+2(J-1)\Phi_{SO}+\Phi_{SS}\\
&\left.\quad +\frac{2(J-1)}{2J+1}(\Phi_{SOT}-\Phi_T)\right\}v_+\\
&\quad +\frac{2\sqrt{J(J+1)}}{2J+1}\{3\Phi_T-2(J+2)\Phi_{SOT}\}v_-\\
&= b^2v_+,
\end{aligned} \tag{B3}$$

is coupled to

$$\begin{aligned}
&\left\{-\frac{d^2}{dr^2}+\frac{(J+1)(J+2)}{r^2}+2m_wS+S^2+2\varepsilon_wA-A^2\right.\\
&\quad +2\varepsilon_wV-V^2+\Phi_D-2(J+2)\Phi_{SO}+\Phi_{SS}\\
&\left.\quad +\frac{2(J+2)}{2J+1}(\Phi_{SOT}-\Phi_T)\right\}v_-\\
&\quad +\frac{2\sqrt{J(J+1)}}{2J+1}\{3\Phi_T+2(J-1)\Phi_{SOT}\}v_+\\
&= b^2v_-.
\end{aligned} \tag{B4}$$

For the uncoupled ${}^3P_0$ states the single equation is

$$\begin{aligned}
&\left\{-\frac{d^2}{dr^2}+\frac{2}{r^2}+2m_wS+S^2+2\varepsilon_wA-A^2+2\varepsilon_wV-V^2\right.\\
&\left.\quad +\Phi_D-4\Phi_{SO}+\Phi_{SS}+4(\Phi_{SOT}-\Phi_T)\right\}v_-\\
&= b^2v_-.
\end{aligned} \tag{B5}$$

APPENDIX C: TWO-BODY DIRAC EQUATIONS FOR QED

The Schrödinger-like form Eq. (35) of the TBDE given in Eq. (14) can be used for QCD bound state (meson spectroscopy) and for QED bound states (positronium, muonium, and hydrogenlike systems). In our meson spectroscopy work presented in this paper we use the three invariant functions $S(r)$, $A(r)$, and $V(r)$ specified in Sec. III. Once these are specified then so are the three vertex invariants $\mathcal{L}(r)$, $\mathcal{G}(r)$, and $\mathcal{J}(r)$. They, in turn, fix the quasipotentials given in Eq. (A35) and below to (A41) that appear in Eq. (35) and its radial forms of Appendix B. To make the transition from QCD meson bound states to QED bound states we simply take $S(r) = V(r) = 0$ and $A(r) = -\alpha/r$ (we remind the reader that only in the c.m. system is the invariant $r = \sqrt{x_\perp^2}$ equal to $|\mathbf{r}|$). Our QED spectral results then follow from solving nonperturbatively (i.e. numerically or analytically) the radial eigenvalue equations of Appendix B. For example, since $\Phi_D = 3\Phi_{SS}$ for $S = V = 0$, the equal mass spin singlet Eq. (B1) collapses to

$$\left\{-\frac{d^2}{dr^2}+\frac{J(J+1)}{r^2}-\frac{2\varepsilon_w\alpha}{r}-\frac{\alpha^2}{r^2}\right\}v_0 = b^2 v_0, \qquad \text{(C1)}$$

which has the analytic spectral solution given in Eq. (63) for $J = 0$. As long as $J(J+1) - \alpha^2 > -1/4$, since the above equation takes the limiting form of

$$\left\{-\frac{d^2}{dr^2}+\frac{J(J+1)}{r^2}-\frac{\alpha^2}{r^2}\right\}v_0 = 0, \qquad \text{(C2)}$$

it is clear that the effective potential is nonsingular, thus allowing a well defined eigenvalue solution.[34] The paper [33] demonstrated that the corresponding short distance behavior of the potentials in the other radial equations of Appendix B for the other QED bound states also allow well defined eigenvalue solutions.[35] Beyond that, numerical solutions of these eigenvalue equations yield spectra (not limited just to the singlet ground states) in agreement with standard perturbative $O(\alpha^4)$ results [29]. In electron volts the numerical binding energy for the singlet ground state of positronium from Eq. (B1) is -6.8033256279 vs the perturbative result of Eq. (61) or $m(-\alpha^2/4 - 21\alpha^4/64) = -6.8033256719$. The difference in units of $m\alpha^4/2$ is $-6.08\mathrm{E}-05$ which is on the order of α^2, so that, as expected from (61) the difference is on the order of $m\alpha^6$. The corresponding numerical binding energy in electron volts for the triplet ground state from the coupled equations Eqs. (B3) and (B4) is -6.8028426132 vs the perturbative result [29] of $m(-\alpha^2/4 + \alpha^4/192) = -6.8028426636$. These results do not include the effects of the annihilation diagram. The difference in units of $m\alpha^4/2$ is $6.97\mathrm{E}-05$ which is on the order of α^2, so that the difference is also on the order of $m\alpha^6$. These results (from a very extensive list of numerically computed spectral results in [29]) taken together, represent crucial tests of this formalism, ones that have not been demonstrated in any other relativistic bound state formalism. In fact, the authors of [4] have found a particular quasipotential formalism that does give such agreement, but only for the ground state. They also demonstrate that several well know two-body relativistic bound state formalisms (including the Blankenbecler-Sugar formalism and the formalism of Gross) fail this test. Let us be explicit about the implications of either the failure of this test or the lack of performing this test. When one proposes a new bound state formalism such as Dirac did in 1928, it was essential that at the very least it reproduce the nonrelativistic hydrogenic spectrum. Beyond that it provides two other remarkable results. First of all, by way of an order $1/c^2$ expansion, it gave an order α^4 perturbative correction to the nonrelativistic spectral results. Remarkably, an exact solution of the same equation later by Darwin gave spectral results which when truncated to order α^4 agreed precisely with the perturbative treatment, and, at the time, with experimental fine structure measurements. Breit, in the development of his two-body equation gave us an equation with interactions beyond the Coulomb potential that ultimately reproduced, when treated perturbatively, spectral results for two-body systems that agreed through terms of order α^4 with experiment. Unlike the one-body Dirac equation which has an exact spectral solution which agrees, at order α^4, with its perturbative solution, the Breit equation has no exact or for that matter numerical solution which agrees, at order α^4, with its perturbative solution. The same could be said for most all other two-body relativistic treatments proposed since then, with the two exceptions ([4,29]) noted above. One's two-body formalism having an agreement of its nonperturbative treatment with its perturbative treatment of the spectra in a well established field theory such as QED, in our opinion, should be regarded as a necessary hurdle to pass before going on to apply these formalisms to potential models for meson spectroscopy.

[34] In particular, the short distance radial behavior is $v_{J0J} \to r^{\sqrt{J(J+1)-\alpha^2}}$.

[35] The short distance radial dependencies of the wave functions that arise from Eqs. (B3) and (B4) are the well behaved forms $v_{J-11J}(r) = r^{(1/2+\sqrt{J(J+1)-\alpha^2})}$ and $v_{J+11J}(r) = \frac{J}{\sqrt{J(J+1)}} v_{J-11J}(r)$ [33]. Such behaviors as represented here and in the previous footnote arise because the effective potentials that appear in our bound state equations are well defined (no delta function potentials, for example). We refer the interested reader to Eq. (42–43) of section V of [33] for the intriguing details of the cancellation of otherwise singular potentials due to the presence of the tensor coupling terms.

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