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Massive scaling limit of the β-deformed matrix model of Selberg type

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We consider a series of massive scaling limits $m_1 \to \infty$, $q \to 0$, $\lim m_1 q = \Lambda_3$ followed by $m_4 \to \infty$, $\Lambda_3 \to 0$, $\lim m_4 \Lambda_3 = (\Lambda_2)^2$ of the β-deformed matrix model of Selberg type ($N_c = 2$, $N_f = 4$) which reduce the number of flavors to $N_f = 3$ and subsequently to $N_f = 2$. This keeps the other parameters of the model finite, which include $n = N_L$ and $N = n + N_R$, namely, the size of the matrix and the "filling fraction." Exploiting the method developed before, we generate instanton expansion with finite g_s, $\epsilon_{1,2}$ to check the Nekrasov coefficients ($N_f = 3$, 2 cases) to the lowest order. The limiting expressions provide integral representation of irregular conformal blocks which contains a $2d$ operator $\lim \frac{1}{C(q)} :\mathrm{e}^{(1/2)\alpha_1 \phi(0)}:(\int_0^q dz :\mathrm{e}^{b_E \phi(z)}:)^n:\mathrm{e}^{(1/2)\alpha_2 \phi(q)}:$ and is subsequently analytically continued.

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I. INTRODUCTION

There has already been an ample amount of literature on the Seiberg-Witten prepotential for the cases when massive flavors are present. Just restricting our attention to the $SU(2)$ case, a partial list includes [1–9]. In particular, Refs. [4,5] compute the prepotentials of lower flavors as decoupling limits of those of higher flavors.

In the recent intense activity on the conjectured equivalence between the (irregular) conformal block and the Nekrasov partition function [10–14] (for partial proof, see [15,16]), the discussions of these decoupling limits are further advanced and augmented to contain the parameters g_s, $\epsilon_{1,2}$ of genus expansion and quantization and are derived both from the $2d - 6d$ perspective of the Riemann surface [17] and from the Shapovalov form [18–20].

Somewhat separately, the relevance of the β deformation of the one-matrix model [21] and that of the more general quiver matrix model [21,22] to the above equivalence have been noted. At the planar level, both the spectral curve of the one-matrix model [21] and that of the quiver matrix model [22] are shown to be isomorphic to the corresponding Seiberg-Witten curve written in the Witten-Gaiotto form [23,24]. (To check the decoupling limits at the planar free energy, see [25].)

The connection between the conformal block and the β deformed matrix model becomes firmer through the Dotsenko-Fateev integral representation [26–29]. In particular, the first few Nekrasov coefficients are derived and the $0d - 4d$ dictionary has been established in [29]. See also [30–38]. This representation permits rigorous treatments for arbitrary values of g_s and $\epsilon_{1,2}$. In this paper, we will consider a series of massive scaling limits which reduce the number of flavors to $N_f = 3$ and subsequently to $N_f = 2$, using the technology established in [29]. The way in which these limits are taken is different from that considered on the basis of the Shapovalov form [18,20]. The limiting expressions provide integral representation of irregular conformal blocks which contains a $2d$ operator $\lim \frac{1}{C(q)} :\mathrm{e}^{(1/2)\alpha_1 \phi(0)}:(\int_0^q dz :\mathrm{e}^{b_E \phi(z)}:)^n:\mathrm{e}^{(1/2)\alpha_2 \phi(q)}:$ and is subsequently analytically continued.

A similar consideration at higher (say 5) point conformal block yields interesting chiral 3- and 4-point functions to study.

In the next section, we briefly recall [29]. In Sec. III, we take the massive scaling limit to the three flavor case and generate its first expansion coefficient. In Sec. IV, we subsequently take the limit to the two flavor case and generate its first expansion coefficient. In the Appendix, the contour deformations assumed in Secs. III and IV are justified.

II. REVIEW OF GENERIC 4-POINT CONFORMAL BLOCK REPRESENTED BY SELBERG TYPE MATRIX MODEL

The Dotsenko-Fateev multiple integral is an integral representation of the generic 4-point conformal block $\mathcal{F}(q|c; \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_I)$. In [29], we have managed to put this into the form of the perturbed double-Selberg matrix model. Renaming the same quantity as $\mathcal{F}$ as $Z_{\mathrm{pert}-(\mathrm{Selberg})^2}$, we have obtained

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$$Z_{\text{pert}-(\text{Selberg})^2} = q^{\sigma}(1-q)^{(1/2)\alpha_2\alpha_3}\left(\prod_{I=1}^{N_L}\int_0^1 dx_I\right) \times \prod_{I=1}^{N_L} x_I^{b_E\alpha_1}(1-x_I)^{b_E\alpha_2}(1-qx_I)^{b_E\alpha_3} \times \prod_{1\le I<J\le N_L}|x_I-x_J|^{2b_E^2}\left(\prod_{J=1}^{N_R}\int_0^1 dy_J\right) \times \prod_{J=1}^{N_R} y_J^{b_E\alpha_4}(1-y_J)^{b_E\alpha_3}(1-qy_J)^{b_E\alpha_2} \times \prod_{1\le I<J\le N_R}|y_I-y_J|^{2b_E^2}\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}(1-qx_Iy_J)^{2b_E^2}. \tag{2.1}$$

Here $n := N_L$ and $N_R := N - n$ are originally the number of the screening operators we put between 0 and q and between 1 and ∞, respectively. The remaining parameters are related to those of the original conformal block by $c = 1 - 6Q_E^2$, $Q_E = b_E - (1/b_E)$, $\Delta_i = (1/4)\alpha_i(\alpha_i - 2Q_E)$, $\Delta_I = (1/4)\alpha_I(\alpha_I - 2Q_E)$. Also,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(N_L + N_R)b_E = 2Q_E, \tag{2.2}$$

$$\sigma := \frac{1}{2}\alpha_1\alpha_2 + N_L + N_L b_E(\alpha_1+\alpha_2) + N_L(N_L-1)b_E^2. \tag{2.3}$$

The Selberg integral is denoted by

$$S_N(\beta_1,\beta_2,\gamma) = \left(\prod_{I=1}^{N}\int_0^1 dx_I\right) \times \prod_{I=1}^{N} x_I^{\beta_1-1}(1-x_I)^{\beta_2-1}\prod_{1\le I<J\le N}|x_I-x_J|^{2\gamma}. \tag{2.4}$$

The Selberg integral (2.7) is convergent [39] and equals

$$S_N(\beta_1,\beta_2,\gamma) = \prod_{j=1}^{N}\frac{\Gamma(1+j\gamma)\Gamma(\beta_1+(j-1)\gamma)\Gamma(\beta_2+(j-1)\gamma)}{\Gamma(1+\gamma)\Gamma(\beta_1+\beta_2+(N+j-2)\gamma)}, \tag{2.5}$$

when N is a positive integer and the complex parameters above obey

$$\mathrm{Re}\,\beta_1 > 0, \qquad \mathrm{Re}\beta_2 > 0, \qquad \mathrm{Re}\gamma > -\min\left\{\frac{1}{N}, \frac{\mathrm{Re}\beta_1}{N-1}, \frac{\mathrm{Re}\beta_2}{N-1}\right\}. \tag{2.6}$$

The perturbed double-Selberg model (2.1) has a well-defined q expansion if

$$\mathrm{Re}\,(b_E\alpha_i) > -1, \qquad (i = 1,2,3,4), \tag{2.7}$$

$$\mathrm{Re}(b_E^2) > -\min\left\{\frac{1}{N_L}, \frac{1}{N_R}, \frac{\mathrm{Re}(b_E\alpha_1)+1}{N_L-1}, \frac{\mathrm{Re}(b_E\alpha_2)+1}{N_L-1}, \frac{\mathrm{Re}(b_E\alpha_3)+1}{N_R-1}, \frac{\mathrm{Re}(b_E\alpha_4)+1}{N_R-1}\right\}, \tag{2.8}$$

and $|q| < 1$.

Let

$$Z_{(\text{Selberg})^2}(b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3) = Z_{\text{Selberg}}(b_E; N_L, \alpha_1, \alpha_2) Z_{\text{Selberg}}(b_E; N_R, \alpha_4, \alpha_3) := S_{N_L}(1 + b_E\alpha_1, 1 + b_E\alpha_2, b_E^2) \times S_{N_R}(1 + b_E\alpha_4, 1 + b_E\alpha_3, b_E^2). \tag{2.9}$$

Averaging with respect to $Z_{(\text{Selberg})^2}$, $Z_{\text{Selberg}}(N_L)$, and $Z_{\text{Selberg}}(N_R)$ is denoted by $\langle\langle\cdots\rangle\rangle_{N_L,L_R}$, $\langle\langle\cdots\rangle\rangle_{N_L}$ and $\langle\langle\cdots\rangle\rangle_{N_R}$, respectively.

The q expansion of the perturbed double-Selberg model is a special case of that of the more general perturbed Selberg model and is exactly calculable. Consider the following function:

$$Z_{\text{pert}-\text{Selberg}}(\beta_1,\beta_2,\gamma;\{g_i\}) := S_N(\beta_1,\beta_2,\gamma)\left\langle\left\langle\exp\left(\sum_{I=1}^{N}W(x_I;g)\right)\right\rangle\right\rangle_N, \tag{2.10}$$

where the averaging is with respect to the Selberg integral (2.4) and

$$W(x;g) = \sum_{i=0}^{\infty} g_i x^i. \tag{2.11}$$

Let us expand the exponential of the potential into the Jack polynomials

$$\exp\left(\sum_{I=1}^{N}W(x_I;\{g_i\})\right) = \sum_{\lambda} C_\lambda^{(\gamma)}(g) P_\lambda^{(1/\gamma)}(x). \tag{2.12}$$

Here $P_\lambda^{(1/\gamma)}(x)$ is a polynomial of $x = (x_1, \cdots, x_N)$, and $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a partition: $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$. Jack polynomials are the eigenstates of

$$\sum_{I=1}^{N}\left(x_I\frac{\partial}{\partial x_I}\right)^2 + \gamma\sum_{1\le I<J\le N}\left(\frac{x_I+x_J}{x_I-x_J}\right)\left(x_I\frac{\partial}{\partial x_I} - x_J\frac{\partial}{\partial x_J}\right), \tag{2.13}$$

with homogeneous degree $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ and are normalized such that for dominance ordering

$$P_\lambda^{(1/\gamma)}(x) = m_\lambda(x) + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu(x). \tag{2.14}$$

Here $m_\lambda(x)$ is the monomial symmetric polynomial.

Let λ' be the conjugate partition of λ, i.e., whose diagram of partition is the transpose of that of λ along the main diagonal. Then the Macdonald-Kadell integral [40–42] implies that

$$\langle\langle P_\lambda^{(1/\gamma)}(x)\rangle\rangle_N = \prod_{i\ge 1}\frac{(\beta_1+(N-i)\gamma)_{\lambda_i}((N+1-i)\gamma)_{\lambda_i}}{(\beta_1+\beta_2+(2N-1-i)\gamma)_{\lambda_i}} \times \prod_{(i,j)\in\lambda}\frac{1}{(\lambda_i - j + (\lambda'_j - i + 1)\gamma)}, \tag{2.15}$$

where $(a)_n$ is the Pochhammer symbol:

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1. \tag{2.16}$$

Computation of (2.10) reduces to that of the expansion coefficients $C_\lambda^{(\gamma)}(g)$.

An important relation established in [29] is the 0d-4d version of the relation proposed by Alday-Gaiotto-Tachikawa. It reads

$$b_E N_L = \frac{a-m_2}{g_s}, \quad b_E N_R = -\frac{a+m_3}{g_s},$$
$$\alpha_1 = \frac{1}{g_s}(m_2 - m_1 + \epsilon), \quad \alpha_2 = \frac{1}{g_s}(m_2 + m_1),$$
$$\alpha_3 = \frac{1}{g_s}(m_3 + m_4), \quad \alpha_4 = \frac{1}{g_s}(m_3 - m_4 + \epsilon). \tag{2.17}$$

Also, $b_E = \epsilon_1/g_s$ and $\epsilon = \epsilon_1 + \epsilon_2$, $(1/b_E) = -\epsilon_2/g_s$. These relations convert the seven parameters of the matrix model

$$b_E, \quad N_L, \quad \alpha_1, \quad \alpha_2, \quad N_R, \quad \alpha_4, \quad \alpha_3 \tag{2.18}$$

under the constraint (2.2) into the six unconstrained parameters of the $\mathcal{N}=2$ $SU(2)$ gauge theory with $N_f = 4$:

$$\frac{\epsilon_1}{g_s}, \quad \frac{a}{g_s}, \quad \frac{m_1}{g_s}, \quad \frac{m_2}{g_s}, \quad \frac{m_3}{g_s}, \quad \frac{m_4}{g_s}. \tag{2.19}$$

III. THE LIMIT $m_1 \to \infty$: FROM $N_f = 4$ TO $N_f = 3$

First, let us consider the limit $m_1 \to \infty$, $q \to 0$, keeping $\Lambda_3 := m_1 q$ finite. Because of the left-right reflection symmetry, this limit is equivalent to the limit $m_4 \to \infty$, $q \to 0$, with $m_4 q$ fixed. Without loss of generality, we therefore restrict ourselves to the former one. Let $q_3 := \Lambda_3/g_s$. Under this limit, the parameters α_3, α_4, b_E, N_L, N_R are unchanged and

$$\lim_{q\to 0} q\alpha_1 = -q_3, \qquad \lim_{q\to 0} q\alpha_2 = q_3. \tag{3.1}$$

Note that

$$\alpha_1 + \alpha_2 = \frac{2m_2 + \epsilon}{g_s} \tag{3.2}$$

remains finite in this limit. This is why we take the limit $m_1 \to \infty$ instead of $m_2 \to \infty$. But in the naive limit $m_1 \to \infty$ of $Z_{\text{pert}-(\text{Selberg})^2}$ (2.1) diverges since $\text{Re}(b_E\alpha_1) \to -\infty$ which is in the outside of the parameter region (2.7).[1] Hence, we should modify the multiple integral (2.1) before taking the limit.

[1]For simplicity, we assume that b_E is real and positive.

In order to examine this limit, we first rescale the integration variables x_I as $z_I = qx_I$:

$$Z_{\text{pert}-(\text{Selberg})^2} = q^{(1/2)\alpha_1\alpha_2}(1-q)^{(1/2)\alpha_2\alpha_3}\Big(\prod_{I=1}^{N_L}\int_0^q dz_I\Big) \times \Big(\prod_{J=1}^{N_R}\int_0^1 dy_J\Big)\prod_{I=1}^{N_L} z_I^{b_E(\alpha_1+\alpha_2)}\Big(\frac{q}{z_I}-1\Big)^{b_E\alpha_2}(1-z_I)^{b_E\alpha_3} \times \prod_{1\le I<J\le N_L}|z_I - z_J|^{2b_E^2} \times \prod_{J=1}^{N_R} y_J^{b_E\alpha_4 + 2b_E^2 N_L}(1-y_J)^{b_E\alpha_3}(1-qy_J)^{b_E\alpha_2} \times \prod_{1\le I<J\le N_R}|y_I - y_J|^{2b_E^2}\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\Big(\frac{1}{y_J} - z_I\Big)^{2b_E^2}. \tag{3.3}$$

The multiple integral part can be written as follows:

$$\Big(\prod_{J=1}^{N_R}\int_0^1 dy_J\Big)\Phi(y)\prod_{J=1}^{N_R} y_J^{b_E\alpha_4+2b_E^2N_L}(1-y_J)^{b_E\alpha_3}(1-qy_J)^{b_E\alpha_2} \times \prod_{1\le I<J\le N_R}|y_I - y_J|^{2b_E^2}, \tag{3.4}$$

where

$$\Phi(y) := \Big(\prod_{I=1}^{N_L}\int_0^q dz_I\Big)\prod_{I=1}^{N_L} z_I^{b_E(\alpha_1+\alpha_2)}\Big(\frac{q}{z_I}-1\Big)^{b_E\alpha_2}(1-z_I)^{b_E\alpha_3} \times \prod_{1\le I<J\le N_L}|z_I - z_J|^{2b_E^2}\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\Big(\frac{1}{y_J} - z_I\Big)^{2b_E^2}. \tag{3.5}$$

We assume that by using a certain contour integral, the integration path $[0, q]$ can be converted to some path $\mathcal{C}'_q$[2]:

$$\Phi(y) = C(q)\Big(\prod_{I=1}^{N_L}\int_{\mathcal{C}'_q} dz_I\Big)\cdots. \tag{3.6}$$

Here $C(q)$ is a constant which also depends on other parameters. Justification of this assumption is given in the Appendix.

In the limit $q \to 0$,

$$\lim_{q\to 0}\frac{1}{C(q)}\Phi(y) = \Big(\prod_{I=1}^{N_L}\int_{\mathcal{C}'_0} dz_I\Big) \times \prod_{I=1}^{N_L} z_I^{b_E(\alpha_1+\alpha_2)}\exp\Big(-\frac{b_E q_3}{z_I}\Big)(1-z_I)^{b_E\alpha_3} \times \prod_{1\le I<J\le N_L}(z_I - z_J)^{2b_E^2}\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\Big(\frac{1}{y_J} - z_I\Big)^{2b_E^2}. \tag{3.7}$$

[2]In the original variable x_I, this contour $\mathcal{C}'_q$ corresponds to the contour $\tilde{C}_\rho$ in the Appendix.

Therefore, we have

$$\begin{aligned} Z^{(3)} &:= \lim_{q\to 0} \frac{q^{-(1/2)\alpha_1\alpha_2}(1-q)^{-(1/2)\alpha_2\alpha_3}}{C(q)} Z_{\mathrm{pert}-(\mathrm{Selberg})^2} \\ &= \left(\prod_{I=1}^{N_L}\int_{\mathcal{C}_0'} dz_I\right)\left(\prod_{J=1}^{N_R}\int_0^1 dy_J\right)\prod_{I=1}^{N_L} z_I^{b_E(\alpha_1+\alpha_2)}\exp\left(-\frac{b_E q_3}{z_I}\right)(1-z_I)^{b_E\alpha_3}\prod_{1\le I<J\le N_L}(z_I-z_J)^{2b_E^2} \\ &\quad \times \prod_{J=1}^{N_R} y_J^{b_E\alpha_4}(1-y_J)^{b_E\alpha_3}\exp(-b_E q_3 y_J)\prod_{1\le I<J\le N_R}|y_I-y_J|^{2b_E^2}\times\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}(1-y_J z_I)^{2b_E^2} \\ &= (b_E q_3)^{\hat\sigma}\left(\prod_{I=1}^{N_L}\int_{\mathcal{C}} dw_I\right)\prod_{I=1}^{N_L} w_I^{b_E\hat\alpha_1}\mathrm{e}^{-w_I}\prod_{1\le I<J\le N_L}(w_I-w_J)^{2b_E^2}\left(\prod_{J=1}^{N_R}\int_0^1 dy_J\right)\prod_{J=1}^{N_R} y_J^{b_E\alpha_4}(1-y_J)^{b_E\alpha_3} \\ &\quad \times \prod_{1\le I<J\le N_R}|y_I-y_J|^{2b_E^2}\prod_{I=1}^{N_L}\left(1-\frac{b_E q_3}{w_I}\right)^{b_E\alpha_3}\prod_{J=1}^{N_R}\mathrm{e}^{-b_E q_3 y_J}\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{b_E q_3 y_J}{w_I}\right)^{2b_E^2}. \end{aligned} \tag{3.8}$$

Here we have changed $z_I = b_E q_3/w_I$ and

$$b_E\hat\alpha_1 := -2 - 2(N_L-1)b_E^2 - b_E(\alpha_1+\alpha_2), \tag{3.9}$$

$$\hat\sigma := N_L + b_E N_E(\alpha_1+\alpha_2) + N_L(N_L-1)b_E^2. \tag{3.10}$$

Using the $0d-4d$ dictionary, we find

$$\hat\alpha_1 = \frac{\epsilon - 2a}{g_s}. \tag{3.11}$$

The contour $\mathcal{C}$ for w_I is shown in Fig. 1.

The radius of the arc around the origin of this contour C is assumed to be greater than 1. Then, on the contour $\mathcal{C}$, it holds that $|w_I| > 1$. Hence, this multiple integral can serve as a well-defined generating function of the q_3 expansion.

Without specifying the integration contours, the large N limit of this type of ensemble average was studied in [25].

Let

$$T_N(\beta,\gamma) := \left(\prod_{I=1}^{N}\int_{\mathcal{C}} dw_I\right)\prod_{I=1}^{N} w_I^{\beta-1}\mathrm{e}^{-w_I}\prod_{1\le I<J\le N}(w_I-w_J)^{2\gamma}. \tag{3.12}$$

Now we have

$$\begin{aligned} Z^{(3)} &= (b_E q_3)^{\hat\sigma} T_{N_L}(1+b_E\hat\alpha_1, b_E^2) \\ &\quad\times S_{N_R}(1+b_E\alpha_4, 1+b_E\alpha_3, b_E^2)\left\langle\left\langle\prod_{I=1}^{N_L}\left(1-\frac{b_E q_3}{w_I}\right)^{b_E\alpha_3}\right.\right. \\ &\quad\times\left.\left.\prod_{J=1}^{N_R}\mathrm{e}^{-b_E q_3 y_J}\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{b_E q_3 y_J}{w_I}\right)^{2b_E^2}\right\rangle\right\rangle_{N_L', N_R}, \end{aligned} \tag{3.13}$$

where the averaging $\langle\langle\cdots\rangle\rangle_{N_L',N_R}$ is with respect to $T_{N_L}(1 + b_E\hat\alpha_1, b_E^2)S_{N_R}(1+b_E\alpha_4, 1+b_E\alpha_3, b_E^2)$.

Recall that $x_I = b_E q_3/(q w_I) = b_E m_1/(g_s w_I)$. By taking the limit of the Macdonald-Kadell formula, we have

$$\langle\langle P_\lambda^{(1/b_E^2)}(1/w)\rangle\rangle_{N_L'} = \left(\frac{g_s^2}{\epsilon_1}\right)^{|\lambda|}\prod_{(i,j)\in\lambda}\frac{(-a+m_2+\epsilon_1(i-1)+\epsilon_2(j-1))}{(2a-\epsilon-\epsilon_1(i-1)-\epsilon_2(j-1))(\epsilon_1(\lambda_j'-i+1)-\epsilon_2(\lambda_i-j))}. \tag{3.14}$$

Here $P_\lambda^{(1/b_E^2)}(1/w)$ is the Jack symmetric polynomials in $\{1/w_I\}_{1\le I\le N_L}$ and λ' is the conjugate partition of λ.

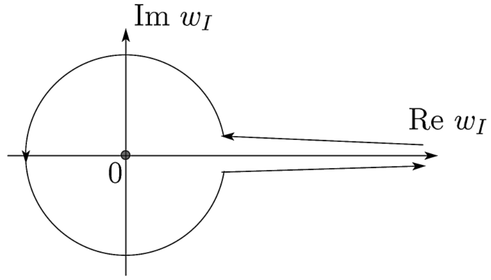

FIG. 1. Integration contour $\mathcal{C}$.

First expansion coefficient

Let us consider the following Λ_3 expansion:

$$\begin{aligned} \mathcal{A}^{(3)}(b_E q_3) &:= \left\langle\left\langle\prod_{I=1}^{N_L}\left(1-\frac{b_E q_3}{w_I}\right)^{b_E\alpha_3}\prod_{J=1}^{N_R}\mathrm{e}^{-b_E q_3 y_J}\right.\right. \\ &\quad\times\left.\left.\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{b_E q_3 y_J}{w_I}\right)^{2b_E^2}\right\rangle\right\rangle_{N_L',N_R} = 1 + \sum_{\ell=1}^{\infty}\Lambda_3^{\ell}\mathcal{A}_\ell^{(3)}. \end{aligned} \tag{3.15}$$

We have

$$\mathcal{A}_1^{(3)} = -\frac{\alpha_3}{g_s}\left\langle\left\langle\sum_{I=1}^{N_L}\frac{b_E^2}{w_I}\right\rangle\right\rangle_{N_L'} - \frac{1}{g_s}\left\langle\left\langle b_E\sum_{J=1}^{N_R} y_J\right\rangle\right\rangle_{N_R} - 2\frac{1}{g_s}\left\langle\left\langle\sum_{I=1}^{N_L}\frac{b_E^2}{w_I}\right\rangle\right\rangle_{N_L'}\left\langle\left\langle b_E\sum_{J=1}^{N_R} y_J\right\rangle\right\rangle_{N_R}. \quad (3.16)$$

By using

$$\left\langle\left\langle b_E^2\sum_{I=1}^{N_L}\frac{1}{w_I}\right\rangle\right\rangle_{N_L'} = \frac{b_E N_L}{\hat{\alpha}_1} = \frac{(a-m_2)}{(\epsilon-2a)}, \quad (3.17)$$

$$\left\langle\left\langle b_E\sum_{J=1}^{N_R} y_J\right\rangle\right\rangle_{N_R} = \frac{b_E N_R(b_E N_R - Q_E + \alpha_4)}{(\alpha_3+\alpha_4+2b_E N_R - 2Q_E)} = -\frac{(a+m_3)(a+m_4)}{g_s(2a+\epsilon)}, \quad (3.18)$$

we have

$$\mathcal{A}_1^{(3)} = \frac{(a+m_2)(a+m_3)(a+m_4)}{2a(2a+\epsilon)g_s^2} - \frac{(a-m_2)(a-m_3)(a-m_4)}{2a(2a-\epsilon)g_s^2}. \quad (3.19)$$

This is equivalent to the first Nekrasov function Z_1^{Nek} for $SU(2)$ with $N_f = 3$.

IV. THE LIMIT $m_4 \to \infty$: FROM $N_f = 3$ TO $N_f = 2$

Next, we consider the limit $m_4 \to \infty$, $\Lambda_3 \to 0$, keeping $(\Lambda_2)^2 := m_4\Lambda_3$ finite. Note that in this limit $q_3 = \Lambda_3/g_s \to 0$. Let $q_2 := \Lambda_2/g_s$. Under this limit,

$$\lim_{q_3\to 0}\alpha_3 q_3 = q_2^2, \qquad \lim_{q_3\to 0}\alpha_4 q_3 = -q_2^2, \quad (4.1)$$

and the following combination of the parameters

$$\alpha_1+\alpha_2 = \frac{2m_2+\epsilon}{g_s}, \qquad \alpha_3+\alpha_4 = \frac{2m_3+\epsilon}{g_s}, \quad (4.2)$$

remain finite and b_E, N_L, N_R are unchanged.

Recall that

$$(b_E q_3)^{-\hat{\sigma}} Z^{(3)} = \left(\prod_{I=1}^{N_L}\int_{\mathcal{C}} dw_I\right)\prod_{I=1}^{N_L} w_I^{b_E\hat{\alpha}_1}\mathrm{e}^{-w_I} \times \prod_{1\le I<J\le N_L}(w_I-w_J)^{2b_E^2}\left(\prod_{J=1}^{N_R}\int_0^1 dy_J\right) \times \prod_{J=1}^{N_R} y_J^{b_E(\alpha_3+\alpha_4)}\left(\frac{1}{y_J}-1\right)^{b_E\alpha_3}\prod_{1\le I<J\le N_R}|y_I-y_J|^{2b_E^2} \times \prod_{I=1}^{N_L}\left(1-\frac{b_E q_3}{w_I}\right)^{b_E\alpha_3}\prod_{J=1}^{N_R}\mathrm{e}^{-b_E q_3 y_J} \times \prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{b_E q_3 y_J}{w_I}\right)^{2b_E^2}. \quad (4.3)$$

By setting $y_J = b_E\alpha_3/u_J$, we have

$$(b_E q_3)^{-\hat{\sigma}} Z^{(3)} = (b_E\alpha_3)^{\hat{\sigma}'}\left(\prod_{I=1}^{N_L}\int_{\mathcal{C}} dw_I\right)\prod_{I=1}^{N_L} w_I^{b_E\hat{\alpha}_1}\mathrm{e}^{-w_I} \times \prod_{1\le I<J\le N_L}(w_I-w_J)^{2b_E^2}\left(\prod_{J=1}^{N_R}\int_{b_E\alpha_3}^{\infty} du_J\right) \times \prod_{J=1}^{N_R} u_J^{b_E\hat{\alpha}_4}\left(\frac{u_J}{b_E\alpha_3}-1\right)^{b_E\alpha_3}\prod_{1\le I<J\le N_R}(u_I-u_J)^{2b_E^2} \times \prod_{I=1}^{N_L}\left(1-\frac{b_E q_3}{w_I}\right)^{b_E\alpha_3}\prod_{J=1}^{N_R}\mathrm{e}^{-b_E q_3 y_J} \times \prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{b_E^2 q_3\alpha_3}{w_I u_J}\right)^{2b_E^2}, \quad (4.4)$$

where

$$b_E\hat{\alpha}_4 := -2 - b_E(\alpha_3+\alpha_4) - 2(N_R-1)b_E^2, \quad (4.5)$$

$$\hat{\sigma}' := N_R + b_E N_R(\alpha_3+\alpha_4) + N_R(N_R-1)b_E^2. \quad (4.6)$$

Note that

$$\hat{\alpha}_4 = \frac{\epsilon+2a}{g_s}. \quad (4.7)$$

As in the first limit (from $N_f = 4$ to 3), we assume that the integration over u_J in the path $[b_E\alpha_3, \infty]$ can be converted into a contour integral over a certain path $\mathcal{C}_{q_3}''$. When converting all u_J integrals, we denote an overall constant $C'(b_E q_3)$.

We also assume that

$$Z^{(2)} := \lim_{q_3\to 0}\frac{(b_E q_3)^{-\hat{\sigma}}(b_E\alpha_3)^{-\hat{\sigma}'}}{C'(b_E q_3)}Z^{(3)} = \left(\prod_{I=1}^{N_L}\int_{\mathcal{C}} dw_I\right)\prod_{I=1}^{N_L} w_I^{b_E\hat{\alpha}_1}\mathrm{e}^{-w_I}\prod_{1\le I<J\le N_L}(w_I-w_J)^{2b_E^2} \times \left(\prod_{J=1}^{N_R}\int_{\mathcal{C}} du_J\right)\prod_{J=1}^{N_R} u_J^{b_E\hat{\alpha}_4}\mathrm{e}^{-u_J}\prod_{1\le I<J\le N_R}(u_I-u_J)^{2b_E^2} \times \prod_{I=1}^{N_L}\exp\left(-\frac{(b_E q_2)^2}{w_I}\right)\prod_{J=1}^{N_R}\exp\left(-\frac{(b_E q_2)^2}{u_J}\right) \times \prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{(b_E q_2)^2}{w_I u_J}\right)^{2b_E^2}. \quad (4.8)$$

Hence,

$$Z^{(2)} = T_{N_L}(1+b_E\hat{\alpha}_1, b_E^2)T_{N_R}(1+b_E\hat{\alpha}_4, b_E^2) \times \left\langle\left\langle\prod_{I=1}^{N_L}\exp\left(-\frac{(b_E q_2)^2}{w_I}\right)\prod_{J=1}^{N_R}\exp\left(-\frac{(b_E q_2)^2}{u_J}\right) \times \prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{(b_E q_2)^2}{w_I u_J}\right)^{2b_E^2}\right\rangle\right\rangle_{N_L', N_R'}. \quad (4.9)$$

Here the averaging $\langle\langle\cdots\rangle\rangle_{N_L',N_R'}$ is with respect to $T_{N_L}(1+b_E\hat{\alpha}_1, b_E^2)T_{N_R}(1+b_E\hat{\alpha}_4, b_E^2)$.

Also. we have the formula for the average of the Jack symmetric polynomials $P_\lambda^{(1/b_E^2)}(1/u)$ of $\{1/u_J\}_{1\le J\le N_R}$:

$$\langle\langle P_\lambda^{(1/b_E^2)}(1/u)\rangle\rangle_{N_R'} = \left(-\frac{g_s^2}{\epsilon_1}\right)^{|\lambda|}\prod_{(i,j)\in\lambda}\frac{(a+m_3+\epsilon_1(i-1)+\epsilon_2(j-1))}{(2a+\epsilon+\epsilon_1(i-1)+\epsilon_2(j-1))(\epsilon_1(\lambda_j'-i+1)-\epsilon_2(\lambda_i-j))}. \tag{4.10}$$

First expansion coefficient

Let us consider the following Λ_2 expansion:

$$\mathcal{A}^{(2)}(b_Eq_2) := \left\langle\left\langle\prod_{I=1}^{N_L}\exp\left(-\frac{(b_Eq_2)^2}{w_I}\right)\times\prod_{J=1}^{N_R}\exp\left(-\frac{(b_Eq_2)^2}{u_J}\right)\times\prod_{I=1}^{N_L}\prod_{J=1}^{N_R}\left(1-\frac{(b_Eq_2)^2}{w_Iu_J}\right)^{2b_E^2}\right\rangle\right\rangle_{N_L',N_R'} = 1+\sum_{\ell=1}^{\infty}\Lambda_2^{2\ell}\mathcal{A}_\ell^{(2)}. \tag{4.11}$$

We have

$$\mathcal{A}_1^{(2)} = -\frac{1}{g_s^2}\left\langle\left\langle\sum_{I=1}^{N_L}\frac{b_E^2}{w_I}\right\rangle\right\rangle_{N_L'} - \frac{1}{g_s^2}\left\langle\left\langle\sum_{J=1}^{N_R}\frac{b_E^2}{u_J}\right\rangle\right\rangle_{N_R'} - \frac{2}{g_s^2}\left\langle\left\langle\sum_{I=1}^{N_L}\frac{b_E^2}{w_I}\right\rangle\right\rangle_{N_L'}\left\langle\left\langle\sum_{J=1}^{N_R}\frac{b_E^2}{u_J}\right\rangle\right\rangle_{N_R'}. \tag{4.12}$$

By using (3.17) and

$$\left\langle\left\langle\sum_{J=1}^{N_R}\frac{b_E^2}{u_J}\right\rangle\right\rangle_{N_R'} = \frac{b_EN_R}{\hat{\alpha}_4} = -\frac{(a+m_3)}{(\epsilon+2a)}, \tag{4.13}$$

we find

$$\mathcal{A}_1^{(2)} = \frac{(a+m_2)(a+m_3)}{2a(2a+\epsilon)g_s^2} + \frac{(a-m_2)(a-m_3)}{2a(2a-\epsilon)g_s^2}, \tag{4.14}$$

which is the first Nekrasov function for $SU(2)$ with $N_f = 2$.

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APPENDIX: GENERALIZATION OF INTEGRAL REPRESENTATION OF IGURI

In this Appendix, we obtain the analytic continuation of the Selberg average as multiple complex contour integrals where we can take our limits. We employ a method similar to that used in [43].

First, we introduce some definitions. In this Appendix, we study complex integrals along four paths specified at Figs. 2 and 3. The symbol $\mathcal{C}_\rho$ denotes the contour shown in Fig. 2 (left). The symbol $\mathcal{C}_{[0,1]}$ stands for a path composed of the segments connecting $x = 0 \pm 0\mathrm{i}$ and $x = 1 \pm 0\mathrm{i}$. See Fig. 2 (right). The symbol $\tilde{\mathcal{C}}_\rho$ is defined by subtracting $\mathcal{C}_{[0,1]}$ from $\mathcal{C}_\rho$: $\tilde{\mathcal{C}}_\rho := \mathcal{C}_\rho - \mathcal{C}_{[0,1]}$. See Fig. 3 (left). The symbol $\tilde{\mathcal{D}}_\rho$ stands for a tadpole-type path specified at Fig. 3 (right). Let us introduce a symbol representing an

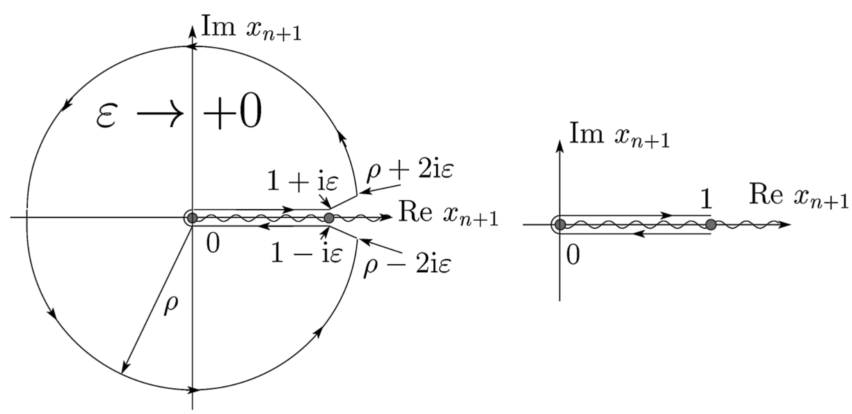

FIG. 2. x_{n+1} plane is illustrated. The wiggly lines are cuts of $\Phi_N^{(0)}$ with $N = 3$. The left figure shows a contour $\mathcal{C}_\rho$. Note that this contour does not get across the cuts of $\Phi_N^{(0)}$. We denote by $\mathcal{C}_{[0,1]}$ the path appearing in the right figure.

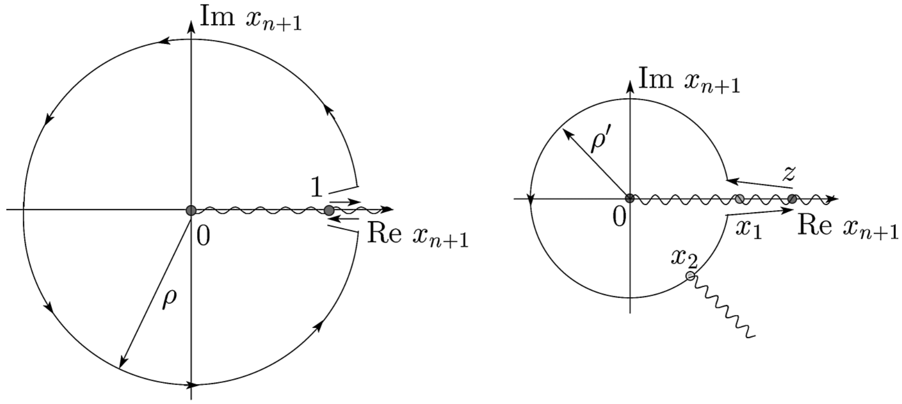

FIG. 3. x_{n+1} plane is illustrated. The wiggly lines are cuts of $\Phi_N^{(0)}$ with $N = 3$. A path $\tilde{\mathcal{C}}_\rho$ is illustrated in the left figure. This path is obtained by $\mathcal{C}_\rho - \mathcal{C}_{[0,1]}$. A path $\tilde{\mathcal{D}}_\rho$ is illustrated in the right figure. As a matter of convenience, we introduce parameter z. $z + 0\mathrm{i}$ and $z - 0\mathrm{i}$ are the starting point and the end point of $\tilde{\mathcal{D}}(\rho', z)$, respectively.

ordering along the path $\mathcal{C}_\rho$ by $\preceq$. We mean by $x \preceq y$ that y is ahead of x along the path. To complete the definition of $\preceq$, we regard $1 - 0 - 0\mathrm{i}$ and $1 + 0 - 0\mathrm{i}$ as the starting point and the end point of $\mathcal{C}_\rho$, respectively. The symbol $\Phi_N^{(n)}$ represents an integral kernel used in this Appendix:

$$\begin{aligned}\Phi_N^{(n)}(\beta_1, \beta_2, \gamma; x) := &\left(\prod_{I=1}^{n} |x_I|^{\beta_1 - 1} |1 - x_I|^{\beta_2 - 1}\right)\\ &\times\left(\prod_{\bar{I}=n+1}^{N} (-x_{\bar{I}})^{\beta_1 - 1}(1 - x_{\bar{I}})^{\beta_2 - 1}\right)\\ &\times\left(\prod_{1\le\hat{I}\le n<\hat{J}\le N}\left(1 - \frac{x_{\hat{I}}}{x_{\hat{J}}}\right)^{2\gamma}(-x_{\hat{J}})^{2\gamma}\right)\\ &\times\left(\prod_{n+1\le\tilde{I}<\tilde{J}\le N}\left(1 - \frac{x_{\tilde{I}}}{x_{\tilde{J}}}\right)^{2\gamma}(-x_{\tilde{J}})^{2\gamma}\right)\\ &\times\left(\prod_{1\le\check{I}<\check{J}\le n} |x_{\check{I}} - x_{\check{J}}|^{2\gamma}\right)\end{aligned}$$

$$\Phi_N(\beta_1, \beta_2, \gamma; x) := \Phi_N^{(N)}(\beta_1, \beta_2, \gamma; x). \tag{A1}$$

The reason why we employ the complicated term $(1 - \frac{x_I}{x_J})^{2\gamma}(-x_J)^{2\gamma}$ instead of $(x_I - x_J)^{2\gamma}$ is that this term has following good properties:

$$\left(1 - \frac{x}{y}\right)^{2\gamma}(-y)^{2\gamma} = \mathrm{e}^{-2\gamma\pi\mathrm{i}}\left(1 - \frac{y}{x}\right)^{2\gamma}(-x)^{2\gamma}, \quad (2\pi > \mathrm{Arg}x > \mathrm{Arg}y \ge 0), \tag{A2}$$

$$\left(1 - \frac{x}{y}\right)^{2\gamma}(-y)^{2\gamma} = (x - y)^{2\gamma}, \qquad (x \in \mathbb{R}^+). \tag{A3}$$

Let $f(x) = f(x_1, x_2, \cdots, x_N)$ be a holomorphic function in the region excluding $x \in (1, +\infty)$ and be invariant under the permutations of $x_1, x_2, \cdots, x_N$. For the sake of convenience, we introduce the symbol $\mathbf{s}(z)$ that stands for

$$\mathbf{s}(z) := \sin\pi z. \tag{A4}$$

Second, we show the following equations:

$$\begin{aligned}\left(\prod_{I=1}^{N}\int_{\tilde{\mathcal{C}}_\rho} dx_I\right)\Phi_N^{(0)}(x)f(x) = &\left(\prod_{I=1}^{N}\mathrm{e}^{\mathrm{i}\pi\gamma(I-N)}\frac{\mathbf{s}(\gamma I)}{\mathbf{s}(\gamma)}\right)\\ &\times\int_{1+0\mathrm{i}\preceq x_1\preceq\cdots\preceq x_N\preceq 1+0-0\mathrm{i}}\left(\prod_{I=1}^{N} dx_I\right)\Phi_N^{(0)}(x)f(x),\end{aligned} \tag{A5}$$

$$\begin{aligned}\left(\prod_{I=1}^{N}\int_{\mathcal{C}[0,1]} dx_I\right)\Phi_N^{(0)}(x)f(x) = &\left(\prod_{I=1}^{N}\mathrm{e}^{\mathrm{i}\pi\gamma(I-N)}\frac{\mathbf{s}(\gamma I)}{\mathbf{s}(\gamma)}\right)\\ &\times\int_{1+0\mathrm{i}\succeq x_1\succeq\cdots\succeq x_N\succeq 1-0-0\mathrm{i}}\left(\prod_{I=1}^{N} dx_I\right)\Phi_N^{(0)}(x)f(x).\end{aligned} \tag{A6}$$

Let $\mathcal{S}_N$ be the symmetric group of degree N. We define the action of $\sigma \in \mathcal{S}_N$ on a symmetric function $g(x)$ as follows:

$$\sigma \cdot g(x_1, x_2, \cdots, x_N) = g(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(N)}). \tag{A7}$$

Equation (A5) can be written as

$$\begin{aligned}\sum_{\sigma\in\mathcal{S}_N}\int_{1+0\mathrm{i}\preceq x_1\preceq\cdots\preceq x_N\preceq 1+0-0\mathrm{i}}&\left(\prod_{I=1}^{N} dx_I\right)\sigma\cdot\Phi_N^{(0)}(x)f(x)\\ = &\left(\prod_{I=1}^{N}\mathrm{e}^{\mathrm{i}\pi\gamma(I-N)}\frac{\mathbf{s}(\gamma I)}{\mathbf{s}(\gamma)}\right)\\ &\times\int_{1+0\mathrm{i}\preceq x_1\preceq\cdots\preceq x_N\preceq 1+0-0\mathrm{i}}\left(\prod_{I=1}^{N} dx_I\right)\Phi_N^{(0)}(x)f(x).\end{aligned} \tag{A8}$$

Let $\rho_{i,j}$ be a transposition and ϖ_n be a cyclic permutation defined as follows:

$$\rho_{i,j}(k) = \begin{cases} j & (k = i) \\ i & (k = j) \\ k & (k \neq i, j)\end{cases}, \tag{A9}$$

$$\varpi_n(k) = \begin{cases} k & (k < n) \\ N+1 & (k = n) \\ k-1 & (k > n) \end{cases}. \tag{A10}$$

Note that

$$\varpi_n = \rho_{N+1,N} \cdot \rho_{N,N-1} \cdot \cdots \cdot \rho_{n+1,n}. \tag{A11}$$

Equation (A2) implies

$$\rho_{i,i+1} \cdot \Phi_N^{(0)}(x_1, x_2, \cdots, x_N) = \mathrm{e}^{-2\pi\gamma\mathrm{i}} \Phi_N^{(0)}(x_1, x_2, \cdots, x_N)$$
$$(1 + 0\mathrm{i} \preceq x_1 \preceq \cdots \preceq x_N \preceq 1 + 0 - 0\mathrm{i}). \tag{A12}$$

Furthermore, introduce

$$\mathcal{S}_{N+1}^N := \{\sigma \in \mathcal{S}_{N+1} | \sigma(N+1) = N+1\}. \tag{A13}$$

Note that

$$\mathcal{S}_{N+1} = \bigcup_n (\mathcal{S}_{N+1}^N \cdot \varpi_n), \tag{A14}$$

$$\varnothing = (\mathcal{S}_{N+1}^N \cdot \varpi_i) \cap (\mathcal{S}_{N+1}^N \cdot \varpi_j) \qquad (i \neq j), \tag{A15}$$

where

$$\mathcal{S}_{N+1}^N \cdot \varpi_n = \{\sigma \cdot \varpi_n \in \mathcal{S}_{N+1} | \sigma \in \mathcal{S}_{N+1}^N\}. \tag{A16}$$

Now, we employ the mathematical induction on N to show (A5). If $N = 2$, (A5) is obvious. Suppose that if $N = k$, (A5) is true. For $N = k + 1$, we obtain

$$\begin{aligned}
&\sum_{\sigma \in \mathcal{S}_{k+1}} \int_{1+0\mathrm{i} \preceq x_1 \preceq \cdots \preceq x_{k+1} \preceq 1+0-0\mathrm{i}} \left(\prod_{I=1}^{k+1} dx_I\right) \sigma \cdot \Phi_{k+1}^{(0)}(x) f(x) \\
&= \sum_{n=1}^{k+1} \sum_{\sigma_k \in \mathcal{S}_{k+1}^k} \int_{1+0\mathrm{i} \preceq x_1 \preceq \cdots \preceq x_{k+1} \preceq 1+0-0\mathrm{i}} \left(\prod_{I=1}^{k+1} dx_I\right) \sigma_k \cdot \varpi_n \cdot \Phi_{k+1}^{(0)}(x) f(x) \\
&= \sum_{n=1}^{k+1} \mathrm{e}^{-2\pi\mathrm{i}\gamma(n-1)} \sum_{\sigma_k \in \mathcal{S}_{k+1}^k} \int_{1+0\mathrm{i} \preceq x_1 \preceq \cdots \preceq x_{k+1} \preceq 1+0-0\mathrm{i}} \left(\prod_{I=1}^{k+1} dx_I\right) \sigma_k \cdot \Phi_{k+1}^{(0)}(x) f(x) \\
&= \left(\prod_{I=1}^{k+1} \mathrm{e}^{\mathrm{i}\pi\gamma(I-k)} \frac{\mathbf{s}(\gamma I)}{\mathbf{s}(\gamma)}\right) \int_{1+0\mathrm{i} \preceq x_1 \preceq \cdots \preceq x_{k+1} \preceq 1+0-0\mathrm{i}} \left(\prod_{I=1}^{k+1} dx_I\right) \Phi_{k+1}^{(0)}(x) f(x).
\end{aligned} \tag{A17}$$

Thus, (A5) is proven. Equation (A6) is shown by the same proof.

Third, we show

$$\left(\prod_{I=1}^N \int_0^1 dx_I\right) \Phi_N^{(N)}(x) f(x) = \left(\frac{\mathrm{i}}{2}\right)^N \left(\prod_{I=1}^N \frac{\mathrm{e}^{\mathrm{i}\pi\gamma(I-N)}}{\mathbf{s}(\beta + (N-I)\gamma)}\right)$$
$$\times \left(\prod_{I=1}^N \int_{\mathcal{C}_\rho} dx_I\right) \Phi_N^{(0)}(x) f(x). \tag{A18}$$

Now, we consider the case

$$0 \leq x_1 \leq x_2 \cdots \leq x_n \leq 1, \tag{A19}$$

$$1 + 0\mathrm{i} \preceq x_{n+2} \preceq \cdots \preceq x_N \preceq 1 + 0 - 0\mathrm{i}. \tag{A20}$$

The n points $x_1, \cdots, x_n$ lie on the segments $\mathcal{C}_{[0,1]}$. Let us consider

$$\int_{\mathcal{C}_\rho} \Phi_N^{(n)}(\beta_1, \beta_2, \gamma; x) f(x) dx_{n+1}. \tag{A21}$$

This integral vanishes as there is no singularity in the region enclosed by this contour. Converting the integral over $\mathcal{C}_{[0,1]}$ into that over the segment (0, 1) on the real axis, we obtain

$$\begin{aligned}
0 = &-\Bigg[e^{-\mathrm{i}\pi\beta_1}\left(\int_0^{x_1} + e^{-2\mathrm{i}\pi\gamma}\int_{x_1}^{x_2} + e^{-4\mathrm{i}\pi\gamma}\int_{x_2}^{x_3} + \cdots + e^{-2n\mathrm{i}\pi\gamma}\int_{x_n}^1\right) \\
&- e^{\mathrm{i}\pi\beta_1}\left(\int_0^{x_1} + e^{2\mathrm{i}\pi\gamma}\int_{x_1}^{x_2} + e^{4\mathrm{i}\pi\gamma}\int_{x_2}^{x_3} + \cdots + e^{2n\mathrm{i}\pi\gamma}\int_{x_n}^1\right)\Bigg] \Phi_N^{(n+1)}(\beta_1, \beta_2, \gamma; x) f(x) dx_{n+1} \\
&+ \left(\int_{1+0\mathrm{i} \preceq x_{n+1} \preceq x_{n+2}} + \cdots + \int_{x_N \preceq x_{n+1} \preceq 1-0\mathrm{i}}\right) \Phi_N^{(n)}(\beta_1, \beta_2, \gamma; x) f(x) dx_{n+1}.
\end{aligned} \tag{A22}$$

Here, the phase factors appear due to the replacement $\Phi_N^{(n)} \to \Phi_N^{(n+1)}$.

Let us integrate the above expression over $x_1, \cdots, x_n, x_{n+2}, \cdots, x_N$, keeping the ordering (A19) and (A20). In this formula, the integrals over the real axis can be converted into those over the region $0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1$ and $1 + 0\mathrm{i} \preceq x_{n+2} \preceq \cdots \preceq x_N \preceq 1 + 0 - 0\mathrm{i}$ by the appropriate interchange of x_I's. The remaining integrals can also be converted into those over the region $0 \leq x_1 \leq \cdots \leq x_n \leq 1$ and $1 + 0\mathrm{i} \preceq x_{n+1} \preceq \cdots \preceq x_N \preceq 1 + 0 - 0\mathrm{i}$. We obtain

$$-\frac{2\mathrm{i}\mathbf{s}((n+1)\gamma)\mathbf{s}(\beta_1 + n\gamma)}{\mathbf{s}(\gamma)}\int_{c0\leq x_1\leq\cdots\leq x_{n+1}\leq 1 1+0\mathrm{i}\preceq x_{n+2}\preceq\cdots\preceq x_N\preceq 1-0\mathrm{i}}\left(\prod_{I=1}^{N} dx_I\right)\Phi_N^{(n+1)}(\beta_1,\beta_2,\gamma;x)f(x)$$
$$= \int_{c0\leq x_1\leq\cdots\leq x_n\leq 1 1+0\mathrm{i}\preceq x_{n+1}\preceq\cdots\preceq x_N\preceq 1-0\mathrm{i}}\left(\prod_{I=1}^{N} dx_I\right)\Phi_N^{(n)}(\beta_1,\beta_2,\gamma;x)f(x), \quad \text{(A23)}$$

where we use the following formula:

$$\sum_{j=0}^{n}\mathbf{s}(\beta_1 + 2j\gamma) = \frac{\mathbf{s}((n+1)\gamma)\mathbf{s}(\beta_1 + n\gamma)}{\mathbf{s}(\gamma)}. \quad \text{(A24)}$$

Using (A23) repeatedly and (A5), we obtain (A18).

Fourth, we show

$$\left(\prod_{I=1}^{N}\int_0^1 dx_I\right)\Phi_N^{(N)}(x)f(x) = \left(-\frac{\mathrm{i}}{2}\right)^N\left(\prod_{I=1}^{N}\frac{\mathrm{e}^{\mathrm{i}\pi\gamma(I-N)}}{\mathbf{s}(\beta+(N-I)\gamma)}\right) \times\left(\prod_{I=1}^{N}\int_{\mathcal{C}_{[0,1]}} dx_I\right)\Phi_N^{(0)}(x)f(x). \quad \text{(A25)}$$

Consider

$$\int_{\mathcal{C}_\rho}\Phi_N^{(0)}(\beta_1,\beta_2,\gamma;x)f(x)dx_{n+1}, \quad \text{(A26)}$$

with the ordering (A20) and

$$1 + 0\mathrm{i} \succeq x_1 \succeq \cdots \succeq x_N \succeq 1 - 0 - 0\mathrm{i}. \quad \text{(A27)}$$

We follow the proof of (A18) but this time not converting the integral over $\mathcal{C}_{[0,1]}$ into that over the segment (0, 1) on the real axis. By using (A5) and (A6), we obtain

$$\left(\prod_{I=1}^{N}\int_{\tilde{\mathcal{C}}_\rho} dx_I\right)\Phi_N^{(0)}(x)f(x) = (-1)^N\left(\prod_{I=1}^{N}\int_{\mathcal{C}_{[0,1]}} dx_I\right)\Phi_N^{(0)}(x)f(x). \quad \text{(A28)}$$

Now, we derive (A25) from (A18).

Fifth, we prove

$$\left(\prod_{I=1}^{N}\int_0^1 dx_I)\Phi_N(\beta_1,\beta_2,\gamma;x)f(x\right) = \left(\frac{\mathrm{i}}{2}\right)^N\left(\prod_{I=1}^{N}\frac{\mathrm{e}^{\mathrm{i}\pi\gamma(I-N)}}{\mathbf{s}(\beta+(N-I)\gamma)}\right)\left(\prod_{I=1}^{N}\int_{\tilde{\mathcal{D}}(1/\rho,1)} dx_I\right) \times\Phi_N^{(0)}(1-\beta_1-\beta_2-2(N-1)\gamma,\beta_2,\gamma;x)f\left(\frac{1}{x}\right). \quad \text{(A29)}$$

By the transformation $x \to 1/x$, we obtain

$$\left(\prod_{I=1}^{N}\int_{\tilde{\mathcal{C}}_\rho} dx_I\right)\Phi_N(\beta_1,\beta_2,\gamma;x)f(x) = \left(\prod_{I=1}^{N}\int_{\tilde{\mathcal{D}}(1/\rho,1)} dx_I\right) \times\Phi_N(1-\beta_1-\beta_2-2(N-1)\gamma,\beta_2,\gamma;x)f\left(\frac{1}{x}\right), \quad \text{(A30)}$$

where $f(\frac{1}{x})$ is obtained by replacing x_I by $1/x_I$ in $f(x)$ and $\tilde{\mathcal{D}}_{1/\rho,z}$ is defined by Fig. 3. In obtaining the above equation, we have used (A3) with $x = 1$ and convert the variable x_I to x_{N+1-I}. By using (A18), we have (A29). Notice that this formula holds independently of the radius parameter ρ.

Let us consider the limit of (A29) as $1/\rho \to 0$. Then, $\tilde{\mathcal{D}}(1/\rho, 1)$ reduce to $-\mathcal{C}_{[0,1]}$. Suppose that $f(1/x)$ is holomorphic function at $x \in (0, 1)$. Then, we can apply (A25) to (A29) and obtain

$$\left(\prod_{I=1}^{N}\int_0^1 dx_I\right)\Phi_N(\beta_1,\beta_2,\gamma;x)f(x) = C_N(\beta_1,\beta_2,\gamma)\left(\prod_{I=1}^{N}\int_0^1 dx_I\right) \times\Phi_N^{(0)}(1-\beta_1-\beta_2-2(N-1)\gamma,\beta_2,\gamma;x)f\left(\frac{1}{x}\right), \quad \text{(A31)}$$

where

$$C_N(\beta_1,\beta_2,\gamma) := \prod_{I=1}^{N}\frac{\mathbf{s}(\beta_1+\beta_2+(2N-I-1)\gamma)}{\mathbf{s}(\beta_1+(N-I)\gamma)}. \quad \text{(A32)}$$

The above formula has been shown by Iguri [43].

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