PHYSICAL REVIEW D **82,** 085030 (2010)

Nonsupersymmetric strong coupling background from the large N quantum mechanics of two matrices coupled via a Yang-Mills interaction

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(Received 5 August 2008; published 29 October 2010)

We derive a planar sector of the large N nonsupersymmetric background of the quantum mechanical Hamiltonian of two Hermitian matrices coupled via a Yang-Mills interaction, in terms of the density of eigenvalues of one of the matrices. This background satisfies an implicit nonlinear integral equation, with a perturbative small coupling expansion and a solvable large coupling solution, which is obtained. The energy of system and the expectation value of several correlators are obtained in this strong coupling limit. They are free of infrared divergences.

DOI: 10.1103/PhysRevD.82.085030 PACS numbers: 11.15.Pg, 02.10.Yn, 11.25.Tq, 11.25.Yb

I. INTRODUCTION

The study of multimatrix models,[1] and particularly their large N limit [1], is of great interest. It is well known, for instance, that the large N limit of their description of D0 branes [2] has been conjectured to provide a definition of M theory [3]. In the context of the AdS/CFT duality [4–6], due to supersymmetry and conformal invariance, correlators of supergravity and $1/2$ Bogomol'nyi-Prasad-Sommerfeld monopoles (BPS) states reduce to calculation of free matrix model overlaps [7,8] or consideration of related matrix Hamiltonians [9]. For stringy states, in the context of the Berenstein-Maldacena-Nastase (BMN) limit [10] and $\mathcal{N} = 4$ Super Yang-Mills (SYM), similar considerations apply [11–13]. A plane-wave matrix theory [14] is related to the $\mathcal{N} = 4$ SYM dilatation operator [15]. Recently, multimatrix, multitrace operators with diagonal-free two point functions have been identified [16,17]. In earlier works [18–20] it has been argued that QCD can be reduced to a finite number of matrices with quenched momenta.

In this communication we will consider the quantum mechanics of two Hermitian matrices with harmonic potentials, interacting through a standard Yang-Mills quartic potential.

We will use the approach, first developed in [21], of treating one of the matrices, which generates the large N planar background, in a coordinate representation, and the other in a creation/annihilation basis. This was done in the context of $1/2$ BPS states and a dual- free harmonic Hamiltonian [8,9] with the matrix generating the background being the holomorphic component of a complex matrix. A precise phase space identification between the collective density description of the dynamics of this matrix [22], and the droplet description of the Lin-Lunin-Maldacena (LLM) [23] metric is obtained [21]. The generalization of this approach to include g_{YM} interactions was developed in [24]. By considering the planar background generated by of one of the two Hermitian matrices, further properties of the spectrum were established in [25].

In [21,24,25], a supersymmetric setting was always assumed, allowing one to consistently neglect normal ordering terms. As a result, the planar background is harmonic, and g_{YM} independent.

In this communication we explore the consequences of not requiring supersymmetry, and establish a g_{YM} dependent density description of a planar sector of the large N limit of the system. The communication is organized as follows: in Sec. II, the Hamiltonian of the two interacting harmonic matrices is introduced. It is shown how a Bogoliubov transformation brings the Hamiltonian to a form quadratic in the creation/annihilation oscillators of one of the matrices, with frequencies having a square root dependence on the eigenvalues of the other matrix [24].[2] The sector of the Hamiltonian contributing to the planar large N background is identified, and it corresponds to the ground state of the creation/annihilation sector. Several important subtleties are pointed out and addressed, and an Hamiltonian which depends only on degrees of freedom of the first matrix is derived. In Sec. III, the planar background for this Hamiltonian, in terms of its density of eigenvalues, is obtained implicitly through a nonlinear integral equation. In Sec. IV, the strong coupling limit of the background is obtained explicitly. In Sec. V, the relevance of this strong coupling limit to the system of two massless Hermitian matrices interacting through the Yang-Mills potential is highlighted. This background is free of infrared divergences. Section VI is reserved for a summary and discussions.

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[1]By matrix models we mean integrals over matrices or the quantum mechanics of matrix valued degrees of freedom.

[2]See also [26]

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II. MODEL HAMILTONIAN AND PLANAR LARGE N SECTOR

We consider the quantum mechanical Hamiltonian

$$\hat{H} \equiv \frac{1}{2}\,\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\,\mathrm{Tr}(X_1^2) + \frac{1}{2}\,Tr(P_2^2) + \frac{w^2}{2}\,\mathrm{Tr}(X_2^2) - g_{\mathrm{YM}}^2\mathrm{Tr}([X_1, X_2][X_1, X_2]), \tag{1}$$

where X_1 and X_2 are two $N \times N$ Hermitian matrices, and P_1 and P_2 their conjugate momenta, respectively.

We will consider the system to be gauged, i.e., we restrict ourselves to (residual) gauge invariant states.

One can think of (1) as associated with two of the six Higgs scalars of the bosonic sector of $\mathcal{N} = 4$ SYM, in the leading Kaluza-Klein compactification on $S^3 \times R$. The harmonic potential results from the coupling to the curvature of the manifold. It should be borne in mind that we do not require supersymmetry. Alternatively, compactification of QCD_{2+1} on a sphere results in a similar Hamiltonian.

We will follow the approach first suggested in [21] of treating one of the matrices, X_1, in coordinate space and exactly (in the large N limit), and the other, X_2, in a creation/annihilation basis. Letting

$$X_2 \equiv \frac{1}{\sqrt{2w}}(A_2 + A_2^\dagger) \qquad P_2 = -i\sqrt{\frac{w}{2}}(A_2 - A_2^\dagger), \tag{2}$$

the Hamiltonian (1) takes the form

$$\hat{H} = \frac{1}{2}\,\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\,\mathrm{Tr}(X_1^2) + w\mathrm{Tr}(A_2^\dagger A_2) + N^2\frac{w}{2} - \frac{g_{\mathrm{YM}}^2}{2w}\,\mathrm{Tr}(2[X_1, A_2^\dagger][X_1, A_2] + [X_1, A_2]^2 + [X_1, A_2^\dagger]^2) + \frac{g_{YM}^2 N}{w}\,\mathrm{Tr}(X_1^2) - \frac{g_{YM}^2}{w}(\mathrm{Tr}(X_1))^2. \tag{3}$$

As the interaction is quadratic in the oscillators, one can perform a Bogoliubov transformation

$$(V^\dagger A_2 V)_{ij} = \cosh(\phi_{ij})B_{ij} - \sinh(\phi_{ij})B_{ij}^\dagger \tag{4}$$

with

$$\tanh(2\phi_{ij}) = \frac{\frac{g_{\mathrm{YM}}^2}{w}(\lambda_i - \lambda_j)^2}{w + \frac{g_{\mathrm{YM}}^2}{w}(\lambda_i - \lambda_j)^2}, \tag{5}$$

where the λ_i's are the eigenvalues of the matrix X_1 and V is the unitary matrix that diagonalizes X_1. Then (3) takes the form

$$\hat{H} = \frac{1}{2}\,\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\,\mathrm{Tr}(X_1^2) + \sum_{i,j=1}^{N}\sqrt{w^2 + 2g_{\mathrm{YM}}^2(\lambda_i - \lambda_j)^2} \times\left(B_{ij}^\dagger B_{ji} + \frac{1}{2}\right). \tag{6}$$

We are interested in the leading large N configuration of the system and, in particular, for the purposes of this communication, in the sector with no B quanta. This comes from the zero point energies of the B, $B^\dagger$ oscillators, and we are therefore lead to the Hamiltonian

$$\hat{H}_0 = \frac{1}{2}\,\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\,\mathrm{Tr}(X_1^2) + \frac{1}{2}\sum_{i,j=1}^{N}\sqrt{w^2 + 2g_{YM}^2(\lambda_i - \lambda_j)^2}. \tag{7}$$

This seemingly simple Hamiltonian, however, hides at least two facts: first, the oscillators B, $B^\dagger$ depend on degrees of freedom of X_1. As a result, for instance, P_1 does not commute with B or $B^\dagger$. Secondly, the state with no B impurities also contains dependence on the eigenvalues λ_i of X_1.

The Hamiltonian (7) therefore requires careful treatment. We start, in the next subsection, by first deriving a canonical transformation that restores the standard commutators with appropriately shifted operators.

A. Canonical transformation

Rather than working in a creation/annihilation basis, we note that B_{ij}, $B_{ij}^\dagger$ are the creation and annihilation operators associated with the scalar fields

$$\bar{X}_2 \equiv V^\dagger X_2 V \qquad \bar{P}_2 \equiv V^\dagger P_2 V.$$

We find it easier to work with $(\bar{P}_2)_{ij}$ and $(\bar{X}_2)_{ij}$, instead of B, $B^\dagger$. The commutator of ${(P_1)_A}^3$ with these scalar fields is nonzero, and has the form

$$[(P_1)_A, (\bar{X}_2)^B] = -iF_A^{BC}(\bar{X}_2)_C, \tag{8}$$

$$[(P_1)_A, (\bar{P}_2)^B] = -iF_A^{BC}(\bar{P}_2)_C. \tag{9}$$

Therefore, we perform a canonical transformation of P_1 to get its commutator with $\bar{P}_2$ and $\bar{X}_2$ equal to zero. This canonical transformation is given as

$$(\bar{P}_1)_A = (P_1)_A + {F_A}^{BC}(\bar{X}_2)_B(\bar{P}_2)_C. \tag{10}$$

It can be shown that the commutator of $(\bar{P}_1)_A$ with $(\bar{P}_2)^B$ and $(\bar{X}_2)^B$ is indeed zero, provided ${F_A}^{BC} = -{F_A}^{CB}$. The explicit form of ${F_A}^{BC}$ follows from (8), and it takes the form

$$\begin{aligned} {F_{cd}}^{pq,lm} &= F_{cd,qp,ml} \\ &= \frac{V_{md}^\dagger V_{cp}}{\lambda_p - \lambda_m}(1 - \delta_{pm})\delta_{ql} \\ &\quad + \frac{V_{qd}^\dagger V_{cl}}{\lambda_q - \lambda_l}(1 - \delta_{ql})\delta_{pm}. \end{aligned} \tag{11}$$

[3] We have introduced a double index notation and metric, i.e., if $A = (ij)$, $T_A = T_{ij}$ and $T^A = T_{ji}$

This is the expression for ${F_A}^{BC}$, with $A = (cd)$, $B = (pq)$ and $C = (lm)$. It can be easily verified that ${F_A}^{BC} = -{F_A}^{CB}$.

B. Effective X_1 Hamiltonian

For the $\bar{P}_2$, $\bar{X}_2$ sector of (7), the ground state configuration with "no X_2 impurities" is given by

$$\Psi_0 = \exp\left(\frac{1}{4}\sum_A \ln\omega_A - \frac{1}{2}\omega_A(\bar{X}_{2A})({\bar{X}_2}^A)\right) \tag{12}$$

with

$$\omega_A = \omega_{ij} = \sum_{i,j=1}^{N}\sqrt{w^2 + 2g_{\mathrm{YM}}^2(\lambda_i - \lambda_j)^2}. \tag{13}$$

Given the dependence of Ψ_0 on λ_i, $P_1\Psi_0 \neq 0$. This will result in a shift of the kinetic term in the "1" sector of the Hamiltonian, in addition to the shift resulting from the canonical transformation of P_1 discussed in the previous subsection.

We are then led to introduce the effective X_1 Hamiltonian defined as

$$\hat{H}_1 = \int d\bar{X}_2 \Psi_0(\lambda, \bar{X}_2)\hat{H}_0\Psi_0(\lambda, \bar{X}_2) \tag{14}$$

which now acts on functions of X_1 only.

As a result of the canonical transformation obtained in the previous subsection, one has

$$\begin{aligned}\frac{1}{2}P_1^A P_{1A}\Psi_0(\lambda, X_2) &\to \frac{1}{2}(P_1^A - F^{ABC}X_{2B}P_{2C}) \\ &\quad\times(P_{1A} - {F_A}^{B'C'}X_{2B'}P_{2C'})\Psi_0(\lambda, X_2) \\ &= \frac{1}{2}(P_1^A P_{1A} - [P_1^A, {F_A}^{BC}X_{2B}P_{2C}] - 2{F_A}^{BC}X_{2B}P_{2C}P_1^A \\ &\quad + F^{ABC}X_{2B}P_{2C}{F_A}^{B'C'}X_{2B'}P_{2C'})\Psi_0(\lambda, X_2)\end{aligned} \tag{15}$$

where, for simplicity of notation, we have dropped the bar sign.

Details of the computation of the expectation values required by (14) are given in the Appendix. The result is

$$\frac{1}{2}P_1^A P_{1A} \to \frac{1}{2}P_1^A P_{1A} - \frac{1}{8}\sum_{ABC}{F_A}^{BC}{F^A}_{BC}\left(1 - \frac{\omega_C}{\omega_B}\right) + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B. \tag{16}$$

The above equation can be simplified using the explicit expression for ${F_A}^{BC}$ given in (11), yielding

$$\sum_A {F_A}^{BC}{F^A}_{BC} = \frac{(1-\delta_{pm})\delta_{ql}}{(\lambda_p - \lambda_m)^2} + \frac{(1-\delta_{ql})\delta_{pm}}{(\lambda_q - \lambda_l)^2}.$$

Therefore,

$$-\frac{1}{8}\sum_{ABC}{F_A}^{BC}{F^A}_{BC}\left(1 - \frac{\omega_C}{\omega_B}\right) = -\frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p - \lambda_m)^2}\times\left(1 - \frac{\omega_{lm}}{\omega_{lp}}\right).$$

Thus, from (16), we finally obtain

$$\frac{1}{2}{P_1}^A P_{1A} \to \frac{1}{2}P_1^A P_{1A} - \frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p - \lambda_m)^2}\left(1 - \frac{\omega_{lm}}{\omega_{lp}}\right) + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B \tag{17}$$

and we arrive at the following expression for $\hat{H}_1$:

$$\hat{H}_1 = \frac{1}{2}P_1^A P_{1A} - \frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p - \lambda_m)^2}\left(1 - \frac{\omega_{lm}}{\omega_{lp}}\right) + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B + \frac{1}{2}\sum_A \omega_A. \tag{18}$$

It is important to note that the effective potential depends only on the eigenvalues of X_1, and therefore one can use standard collective field theory techniques to obtain an effective Hamiltonian in terms of the density of these eigenvalues.

III. A SELF CONSISTENT NONLINEAR INTEGRAL EQUATION FOR THE LARGE N BACKGROUND

Considering the effective Hamiltonian (18), written as

$$\begin{aligned}\hat{H}_1 &= \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{\omega^2}{2}\mathrm{Tr}(X_1^2) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{N}\sqrt{\omega^2 + 2g_{\mathrm{YM}}^2(\lambda_i - \lambda_j)^2} \\ &\quad - \frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p - \lambda_m)^2}\left(1 - \frac{\omega_{lm}}{\omega_{lp}}\right) \\ &\quad + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B,\end{aligned} \tag{19}$$

we note that

$$\frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B = \frac{1}{8}\sum_{bc}\frac{1}{\omega_{bc}^4}(4g_{\mathrm{YM}}^2)^2(\lambda_b - \lambda_c)^2.$$

Hence,

$$\hat{H}_1 = \frac{1}{2}\,\mathrm{Tr}(P_1^2) + \frac{\omega^2}{2}\,\mathrm{Tr}(X_1^2) + \frac{1}{2}\sum_{i,j=1}^{N}\sqrt{\omega^2 + 2g_{\mathrm{YM}}^2(\lambda_i - \lambda_j)^2} - \frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p - \lambda_m)^2}\left(1 - \frac{\omega_{lm}}{\omega_{lp}}\right) + \frac{1}{8}\sum_{bc}\frac{1}{\omega_{bc}^4}(4g_{ym}^2)^2(\lambda_b - \lambda_c)^2. \tag{20}$$

Recall that ω_{ij} is defined in (13).

This Hamiltonian describes the dynamics of a single Hermitian matrix, and the large N background can be described in terms of the density of eigenvalues

$$\phi(x) = \sum_i \delta(x - \lambda_i),$$

as the minimum of the cubic field effective potential [22]

$$V_{\mathrm{eff}} = \frac{\pi^2}{6}\int dx\phi^3(x) + \frac{\omega^2}{2}\int dx\phi(x)x^2 - \mu\left(\int dx\phi(x) - N\right) + \frac{1}{2}\int dx\int dy\sqrt{\omega^2 + 2g_{YM}^2(x-y)^2} \times \phi(x)\phi(y) - \frac{1}{4}\int dx\int dy\int dz\phi(x)\phi(y)\phi(z) \times \frac{1}{(x-y)^2}\left(1 - \frac{\omega(z,y)}{\omega(z,x)}\right) + \frac{1}{8}\int dx\int dy\phi(x)\phi(y)\frac{1}{\omega_{xy}^4}(4g_{ym}^2)^2(x-y)^2, \tag{21}$$

where the Lagrange multiplier μ enforces the constraint $\int dx\phi(x) = N$. To exhibit explicitly the N dependence, we rescale

$$x \to \sqrt{N}x \qquad \phi(x) \to \sqrt{N}\phi(x) \qquad \mu \to N\mu. \tag{22}$$

Under the above rescaling, we see that the last term in (21) is of order N, and therefore is subleading. Thus, we obtain

$$V_{\mathrm{eff}} = N^2\left[\frac{\pi^2}{6}\int dx\phi^3(x) + \frac{\omega^2}{2}\int dx\phi(x)x^2 - \mu\left(\int dx\phi(x) - 1\right) + \frac{1}{2}\int dx\int dy\sqrt{\omega^2 + 2\lambda(x-y)^2}\phi(x)\phi(y) - \frac{1}{4}\int dx\int dy\int dz\phi(x)\phi(y)\phi(z)\frac{1}{(x-y)^2} \times\left(1 - \frac{\omega(z,y)}{\omega(z,x)}\right)\right] \tag{23}$$

where $\lambda = g_{\mathrm{YM}}^2 N$ is the usual 't Hooft's coupling.

As $N \to \infty$, the large N background configuration minimizes (23), yielding the following self-consistent integral equation for the density

$$\pi^2\phi_0^2(x) = 2\mu - \omega^2x^2 - 2\int dy\sqrt{\omega^2 + 2\lambda(x-y)^2}\phi_0(y) + \int dy\int dz\phi(y)\phi(z)\frac{1}{(x-y)^2} \times\left(1 - \frac{1}{2}\frac{\omega(z,y)}{\omega(z,x)} - \frac{1}{2}\frac{\omega(z,x)}{\omega(z,y)}\right) + \frac{1}{2}\int dz\phi(y)\phi(z)\frac{1}{(y-z)^2}\left(1 - \frac{\omega(x,y)}{\omega(x,z)}\right). \tag{24}$$

IV. STRONG COUPLING SOLUTION

A perturbative expansion of (24) can be carried out [27]. One should not expect moments of the type $\int dxx^2\phi_0(x)$ to agree perturbatively with corresponding large N correlators $<\mathrm{Tr}X_1^2>_{N\to\infty}$, as we project onto the $\bar{X}_2$ sector ground state. However, one expects the additional terms that shift the kinetic energy as a result of the canonical transformation and of the dependence on X_1 degrees of freedom of the $\bar{X}_2$ sector eigenstates to be universal. As evident from the last lines of Eqs. (24) and (23), these additional terms are subleading in a strong coupling expansion.[4]

It is therefore of great interest to investigate the $\lambda \to \infty$ of (24) and (23). In this limit

$$\pi^2\phi_0^2(x) = 2\mu - 2\sqrt{2\lambda}\int dy|x-y|\phi_0(y), \tag{25}$$

$$E_0 = N^2\left[\frac{\pi^2}{6}\int dx\phi_0^3(x) + \frac{\sqrt{2\lambda}}{2}\int dx\int dy|x - y|\phi_0(x)\phi_0(y)\right]. \tag{26}$$

We find it useful to introduce

$$f(x) = \sqrt{2\lambda}\int dy|x-y|\phi_0(y), \qquad \pi^2\phi_0^2(x) = 2(\mu - f(x)) \tag{27}$$

which satisfies

$$f(x) = \frac{\sqrt{2\lambda}}{\pi}\int dy|x-y|\sqrt{2(\mu - f(y))}. \tag{28}$$

[4]We have presented an argument in terms of (23), which results from an effective Hamiltonian. Alternatively, we can redefine $\bar{X}_{2A} \to \sqrt{\omega_A}\bar{X}_{2A}$ to exhibit explicitly all ω_A dependence in the Hamiltonian, and the Hamiltonian terms associated with the kinetic energy shift are indeed observed to be subleading in λ.

As it was the case in perturbation theory, we assume that $\phi_0(x)$ remains an even, single cut function defined in the interval $[-x_0, x_0]$. To show that this is a consistent ansatz, we note that then

$$f(x) = \sqrt{2\lambda}\bigg(|x|\int_{-|x|}^{|x|}\phi_0(y)dy + 2\int_{|x|}^{x_0}\phi_0(y)ydy\bigg). \quad (29)$$

Hence $f(x)$ is also even, establishing the consistency of the ansatz.

Using

$$\partial_x^2|x-y| = 2\delta(x-y),$$

Eq. (28) becomes[5]

$$\partial_x^2 f(x) = \frac{4\sqrt{\lambda}}{\pi}\sqrt{\mu - f(x)}. \quad (30)$$

This can be integrated in the usual way, to yield

$$\frac{1}{2}(\partial_x f)^2 + \frac{8\sqrt{\lambda}}{3\pi}(\mu - f(x))^{3/2} = e. \quad (31)$$

The "energy" constant is fixed by the condition $\partial_x f(0) = 0$. Denoting $f(0) \equiv f_0$ one obtains

$$\frac{df}{dx} = \frac{4\lambda^{1/4}}{\sqrt{3\pi}}\sqrt{(\mu - f_0)^{3/2} - (\mu - f(x))^{3/2}}. \quad (32)$$

The normalization condition

$$1 = \int_{-x_0}^{x_0} dx\phi_0(x) = 2\int_0^{x_0} dx\phi_0(x) = 2\int_{f_0}^{\mu} df\frac{\phi_0(f)}{\frac{df}{dx}}$$

fixes

$$(\mu - f_0)^{3/2} = \bigg(\frac{3\pi}{8}\bigg)\lambda^{1/2},$$

and hence (32) takes the form

$$\frac{df}{dx} = \sqrt{2\lambda}\sqrt{1 - \bigg(\frac{\mu - f(x)}{\mu - f_0}\bigg)^{3/2}}. \quad (33)$$

We will not need to invert (33) and obtain $f(x)$ explicitly, as all results presented in this paper will be expressed in terms of known definite integrals.

Of particular interest is the large N ground state energy. From (26) and (27) this can be written as

$$E_0 = N^2\bigg[\frac{\pi^2}{6}\int dx\phi_0^3(x) + \frac{1}{2}\int dxf(x)\phi_0(x)\bigg]$$
$$= N^2\bigg[\frac{\mu}{2} - \frac{\pi^2}{12}\int dx\phi_0^3(x)\bigg]. \quad (34)$$

[5] ϕ_0^2 satisfies a very similar equation.

One needs to know μ, or f_0, independently. From (29), one obtain

$$f_0 = \sqrt{2\lambda}x_0 - (\mu - f_0) \qquad \mu = \sqrt{2\lambda}x_0.$$

From (33) one obtains

$$\sqrt{2\lambda}x_0 = (\mu - f_0)\int_0^1 \frac{dz}{\sqrt{1-(1-z)^{3/2}}}$$
$$= 2(\mu - f_0)\int_0^1 \frac{tdt}{\sqrt{1-t^3}}.$$

Also,

$$\frac{\pi^2}{12}\int dx\phi_0^3(x) = \frac{1}{6}(\mu - f_0)\int_0^1 dz\sqrt{1-(1-z)^{3/2}}$$
$$= \frac{1}{3}(\mu - f_0)\int_0^1 tdt\sqrt{1-t^3}.$$

These integrals are tabulated [28], and are finite. Therefore

$$E_0 = N^2\bigg[\frac{6}{7}\bigg(\frac{3\pi}{8}\bigg)^{2/3}\int_0^1\frac{tdt}{\sqrt{1-t^3}}\lambda^{1/3}\bigg]$$
$$= N^2\bigg[\frac{9}{14}\bigg(\frac{\sqrt{3}}{4\pi}\bigg)^{1/3}\bigg(\Gamma\bigg(\frac{2}{3}\bigg)\bigg)^3\lambda^{1/3}\bigg] \quad (35)$$

we now consider the correlator

$$\langle \mathrm{Tr}X_1^2\rangle = N^2\int dxx^2\phi_0 = 2N^2\int_{f_0}^{\mu}x^2(f)\frac{\phi_0(f)}{\frac{df}{dx}}df.$$

By a sequence of integrations by parts, we obtain

$$<\mathrm{Tr}X_1^2> = N^2\bigg[-\frac{\mu^2}{2\lambda} + \frac{2}{\sqrt{2\lambda}}\int_{f_0}^{\mu}\frac{f}{\frac{df}{dx}}df\bigg]$$
$$= 2N^2\bigg(\frac{3\pi}{8}\bigg)^{4/3}\lambda^{-(1/3)}\bigg[\bigg(\int_0^1\frac{tdt}{\sqrt{1-t^3}}\bigg)^2 - \frac{2}{5}\int_0^1\frac{dt}{\sqrt{1-t^3}}\bigg]$$
$$= \frac{N^2}{\pi 2^{1/3}\sqrt{3}}\bigg(\frac{3\pi}{8}\bigg)^{4/3}\lambda^{-(1/3)} \times\bigg[\frac{3\sqrt{3}}{\pi}\bigg(\Gamma\bigg(\frac{2}{3}\bigg)\bigg)^6 - \frac{2}{5}\bigg(\Gamma\bigg(\frac{1}{3}\bigg)\bigg)^3\bigg]. \quad (36)$$

V. INTERACTING MASSLESS MATRICES

The results of the previous section could have also been obtained by taking the limit $w \to 0$ of the planar Hamiltonian (7). It is therefore relevant to discuss the relevance of these results for the system

$$\hat{H} \equiv \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{1}{2}\mathrm{Tr}(P_2^2) - g_{YM}^2\mathrm{Tr}([X_1, X_2][X_1, X_2]), \quad (37)$$

which results from the dimensional reduction of massless Higgs or non-Abelian vector potentials. The Hamiltonian (37) has a single dimensionful parameter, $\lambda = g_{YM}^2 N$, and therefore all observable quantities should depend only on well defined powers of λ. However, as is well known, perturbation theory is plagued with infrared divergences. In this context, w can be thought of as a standard "mass" regulator.

The results of the previous section are therefore remarkable, as they are finite and free of any infrared divergences, and depend only on the appropriate power of λ which is expected from dimension considerations.

It has already been pointed out the expression (7) has a smooth $w \to 0$ limit. The only place where this limit is potentially ill defined is in the transformation (4), where w has to be kept finite, if small, to ensure that the transformation is canonical. However, once the Bogoliubov transformation is implemented and the Hamiltonian (1) is recast in the form (6), its $w \to 0$ limit should provide a correct description of the system (37).

The results of the previous section provide further support for the expectation that strong coupling dynamics, appropriately resummed through the large N limit, is free of infrared divergences.

VI. SUMMARY

In this communication, we obtained a sector of the large N planar background of two Hermitian matrices, in an harmonic potential, interacting through a Yang-Mills potential, in terms of the density of eigenvalues of one of the matrices. This background is shown to satisfy a self-consistent g_{YM} dependent integral equation. Of particular physical interest is the strong coupling limit of this background. The self-consistent integral equation is solved explicitly in this limit, and the ground state energy and examples of correlators were to be finite and free of infrared divergences. We argue that this is related to the planar solution of the Hamiltonian of two "massless" matrices (i.e., without the harmonic potential, or in the zero curvature limit) with a Yang-Mills interaction.

ACKNOWLEDGMENTS

One of us (J. P. R.) would like to thank the High Energy Theory Group of Brown University for their hospitality during several research visits over the last two years, which allowed for concentrated effort leading to some of the ideas presented in this paper. We thank Antal Jevicki and Robert de Mello Koch for reading an earlier version of the manuscript and their comments.

APPENDIX

The detailed calculation of Eq. (15) proceeds as follows. Recall

$$\begin{aligned}\frac{1}{2}P_1^A P_{1A}\Psi_0(\lambda,X_2) \to{}& \frac{1}{2}(P_1^A - F^{ABC}X_{2B}P_{2C})\\ &\times(P_{1A} - F_A{}^{B'C'}X_{2B'}P_{2C'})\Psi_0(\lambda,X_2)\\ ={}& \frac{1}{2}(\underbrace{P_1^A P_{1A}}_{\text{Term1}} - \underbrace{[P_1^A, F_A{}^{BC}X_{2B}P_{2C}]}_{\text{Term2}}\\ &-\underbrace{2F_A{}^{BC}X_{2B}P_{2C}P_1^A}_{\text{Term3}}\\ &+\underbrace{F^{ABC}X_{2B}P_{2C}F_A{}^{B'C'}X_{2B'}P_{2C'}}_{\text{Term4}})\\ &\times\Psi_0(\lambda,X_2).\end{aligned} \tag{A1}$$

Term 1 of the above expression will have an additional shift coming from the fact that $P_1\psi_0 \neq 0$. We find that

$$P_{1A}\Psi_0(\lambda, X_2) = \Psi_0(\lambda, X_2)(P_{1A} - i(\Delta Y)_A) \tag{A2}$$

where, the additional shift is given by

$$(\Delta Y)_A = \frac{1}{4}\sum_D \partial_A \ln\omega_D(\delta_D^D - 2\omega_D X_2{}^D X_{2D}). \tag{A3}$$

Then Term 1 in (A1) is given as

$$\begin{aligned}\frac{1}{2}P_1{}^A P_{1A}\Psi_0(\lambda,X_2) ={}& \frac{1}{2}\Psi_0(P_1{}^A - i(\Delta Y)^A)(P_{1A} - i(\Delta Y)_A)\\ ={}& \frac{1}{2}\Psi_0(P_1{}^A P_{1A} - i[P_1{}^A, \Delta Y_A]\\ &- i2(\Delta Y)_A P_1{}^A - (\Delta Y)^A(\Delta Y)_A).\end{aligned} \tag{A4}$$

Taking the ground state expectation value[6] of (A4) gives

$$\frac{1}{2}\bigg(P_1{}^A P_{1A} - \overbrace{2i\langle(\Delta Y)_A\rangle P_1{}^A}^{A}\bigg) - \frac{1}{2}\overbrace{\langle(\Delta Y)^A(\Delta Y)_A\rangle}^{B} - \frac{1}{2}\overbrace{i\langle[P_1{}^A, \Delta Y_A]\rangle}^{C}. \tag{A5}$$

Term A of the above expression, multiplying P_{1A} linearly, is equal to zero, as is required by the consistency of this method. Using (A3), terms B and C can be calculated, with results

$$\begin{aligned}\text{Term B} ={}& -\frac{1}{2}\langle(\Delta Y)^A(\Delta Y)_A\rangle\\ ={}& -\frac{1}{2}\bigg\langle\bigg(\frac{1}{4}\partial^A \ln\omega_B(\delta_B^B - 2\omega_B X_2^B X_{2B})\bigg)\\ &\times\bigg(\frac{1}{4}\partial_A \ln\omega_B(\delta_B^B - 2\omega_B X_2^B X_{2B})\bigg)\bigg\rangle\\ ={}& -\frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B,\end{aligned} \tag{A6}$$

[6]The ground state expectation value of any operator O is given as $\langle\hat{O}(X_2)\rangle \equiv \int dX_2 \Psi_0^\star(X_2, \lambda)\hat{O}(X_2)\Psi_0(X_2, \lambda)$. In particular, $\langle X_{2A}X_2^B\rangle = \frac{\delta_A^B}{2\omega_B}$.

$$\begin{aligned}\text{Term C} &= -i\frac{1}{2}\langle[P_1{}^A, \Delta Y_A]\rangle \\ &= -i\frac{1}{2}(-i)\partial_1^A\left(\frac{1}{4}\sum_D \partial_A \ln\omega_D(\delta_D^D - 2\omega_D X_2{}^D X_{2D})\right) \\ &= \frac{1}{8}\partial_A \ln\omega_D \partial^A \ln\omega_D. \end{aligned} \tag{A7}$$

Using these in (A5) allows us to finally obtain term 1 in (A1).

$$\text{Term 1} = \frac{1}{2}P_1{}^A P_{1A} + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B. \tag{A8}$$

In order to calculate term 2 in (A1), we note that

$$P_{2C}\Psi_0 = i\omega_C X_{2C}\Psi_0 \tag{A9}$$

and hence,

$$\begin{aligned} & i\Psi_0(P_1^A F_A{}^{BC} X_{2B}\omega_C X_{2C}) \\ &\quad = i\Psi_0((P_1^A F_A{}^{BC})\omega_C \\ &\qquad < X_{2B}X_{2C} > + F_A{}^{BC}(P_1^A\omega_C) < X_{2B}X_{2C}>) \\ &\quad = i\Psi_0\left((P_1^A F_A{}^{BC})\omega_C \frac{g_{BC}}{2\omega_B} + F_A{}^{BC}(P_1^A\omega_C)\frac{g_{BC}}{2\omega_B}\right) \\ &\quad = 0 \quad (\text{Since } F_A{}^{BB} = 0). \end{aligned} \tag{A10}$$

In order to calculate Term 3, we use (A2) and (A9) to give

$$\begin{aligned} -(F_A{}^{BC} X_{2B} P_{2C} P_1^A)\Psi_0 &= \Psi_0(-iF_A{}^{BC} X_{2B}\omega_C X_{2C} \\ &\quad \times (P_1{}^A - i(\Delta Y)^A)) \\ &= \Psi_0(-iF_A{}^{BC}\omega_C < X_{2B}X_{2C} > P_1{}^A \\ &\quad - F_A{}^{BC}\omega_C < X_{2B}X_{2C}(\Delta Y)^A>) \\ &= 0. \end{aligned} \tag{A11}$$

Now we are left to calculate term 4 in (A1) which is given as

$$\begin{aligned} & \frac{1}{2}F^{ABC} X_{2B} P_{2C} F_A{}^{DE} X_{2D} P_{2E}\Psi_0(\lambda, X_2) \\ &\quad = \frac{1}{2}F^{ABC}F_A{}^{DE} X_{2B}P_{2C}(X_{2D} i\omega_E X_{2E})\Psi_0 \\ &\quad = \frac{1}{2}(iF^{ABC}F_A{}^{DE}\omega_E X_{2B}((-i\delta_{CD})X_{2E} \\ &\quad + X_{2D}(-i\delta_{CE})) + iF^{ABC}F_A{}^{DE}\omega_E X_{2B}X_{2D}X_{2E} \\ &\quad \times (i\omega_C X_{2C}))\Psi_0 \frac{1}{2}(F^{ABC}F_A{}^{CE}\omega_E < X_{2B}X_{2E}> \\ &\quad + F^{ABC}F_A{}^{DC}\omega_C < X_{2B}X_{2D}> \\ &\quad - F_A{}^{BC}F^{ADE}\omega_C\omega_E < X_{2B}X_{2C}X_{2D}X_{2E}>\Big) \\ &\quad = \frac{1}{2}(F^{ABC}F_A{}^{CE}\omega_E\frac{g_{BE}}{2\omega_B} + F^{ABC}F_A{}^{DC}\omega_C\frac{g_{BD}}{2\omega_B} \\ &\quad - F_A{}^{BC}F^{ADE}\omega_C\omega_E\Big(\frac{g_{BC}}{2\omega_B}\frac{g_{DE}}{2\omega_E} + \frac{g_{BD}}{2\omega_B}\frac{g_{CE}}{2\omega_C} \\ &\quad + \frac{g_{BE}}{2\omega_B}\frac{g_{CD}}{2\omega_C}\Big)) \\ &\quad = -\frac{1}{8}F^{ABC}F_{ABC}\left(1 - \frac{\omega_C}{\omega_B}\right). \end{aligned} \tag{A12}$$

Using (A8) and (A10)–(A12) for terms 1, 2, 3 and in (A1) we obtain the final expression for the shifted kinetic operator:

$$\begin{aligned} \frac{1}{2}P_1^A P_{1A} \rightarrow \frac{1}{2}P_1^A P_{1A} - \frac{1}{8}\sum_{ABC} F_A{}^{BC}F_{BC}^A\left(1 - \frac{\omega_C}{\omega_B}\right) \\ + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B. \end{aligned} \tag{A13}$$

[1] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[2] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995).
[3] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[4] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[5] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[6] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[7] S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, Adv. Theor. Math. Phys. **2**, 697 (1998).
[8] S. Corley, A. Jevicki, and S. Ramgoolam, Adv. Theor. Math. Phys. **5**, 809 (2002).
[9] D. Berenstein, J. High Energy Phys. 07 (2004) 018.
[10] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, J. High Energy Phys. 04 (2002) 013.
[11] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov, and W. Skiba, J. High Energy Phys. 07 (2002) 017.
[12] N. Beisert, C. Kristjansen, J. Plefka, and M. Staudacher, Phys. Lett. B **558**, 229 (2003).
[13] R. de Mello Koch, A. Donos, A. Jevicki, and J. P. Rodrigues, Phys. Rev. D **68**, 065012 (2003).
[14] N. w. Kim, T. Klose, and J. Plefka, Nucl. Phys. **B671**, 359 (2003); T. Fischbacher, T. Klose, and J. Plefka, J. High Energy Phys. 02 (2005) 039.
[15] N. Beisert, Phys. Rep. **405**, 1 (2005).
[16] T. W. Brown, P. J. Heslop, and S. Ramgoolam, J. High Energy Phys. 02 (2008) 030.
[17] R. Bhattacharyya, S. Collins, and R. d. M. Koch, J. High Energy Phys. 03 (2008) 044.
[18] T. Eguchi and H. Kawai, Phys. Rev. Lett. **48**, 1063 (1982).

[19] G. Bhanot, U. M. Heller, and H. Neuberger, Phys. Lett. B **113**, 47 (1982).

[20] D. J. Gross and Y. Kitazawa, Nucl. Phys. **B206**, 440 (1982).

[21] A. Donos, A. Jevicki, and J. P. Rodrigues, Phys. Rev. D **72**, 125009 (2005).

[22] A. Jevicki and B. Sakita, Nucl. Phys. **B165**, 511 (1980); **B185**, 89 (1981).

[23] H. Lin, O. Lunin, and J. Maldacena, J. High Energy Phys. 10 (2004) 025.

[24] J. P. Rodrigues, J. High Energy Phys. 12 (2005) 043.

[25] M. N. H. Cook and J. P. Rodrigues, Phys. Rev. D **78**,065024 2008.

[26] D. Berenstein, D. H. Correa, and S. E. Vazquez, J. High Energy Phys. 02 (2006) 048.

[27] A. Zaidi, Ph.D. thesis, University of the Witwatersrand, 2010.

[28] L. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 2000), 6th ed.