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Yukawa terms in noncommutative SO(10) and E_6 GUTs

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We propose a method for constructing Yukawa terms for noncommutative SO(10) and E_6 GUTs when these GUTs are formulated within the enveloping-algebra formalism. The most general noncommutative Yukawa term that we propose contains, at first order in $\theta^{\mu\nu}$, the most general Becchi-Rouet-Stora invariant Yukawa contribution whose only dimensionful parameter is the noncommutativity parameter. This noncommutative Yukawa interaction is thus renormalizable at first order in $\theta^{\mu\nu}$.

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I. INTRODUCTION

The SO(10) and E_6 GUTs, which were introduced [1–3] in the mid 1970's, are the most popular GUTs in four dimensional space-time. They incorporate right-handed neutrinos in the fermionic multiplets and realize the idea of family unification—each standard model family snugly fits into an irreducible multiplet, in addition to gauge coupling unification. These theories can be made supersymmetric to achieve gauge coupling unification after crossing the desert [4,5] but may also –at least in the SO(10) case—lead to nonsupersymmetric unification, if intermediate symmetry breaking scales (oases are thus created in the desert) are introduced between the electroweak scale and the GUT scale [5,6]. In view of all the results obtained so far, and reviewed in [4,5], that GUTs may be relevant in the understanding of the data which will come out of the LHC is a thought that one cannot be rid of easily. This is a thought that is also prompted by the fact that SO(10) and E_6 GUTs arise naturally F theory [7].

More than a decade [8,9] has gone by since it became clear that field theories on noncommutative space-time—which are named noncommutative field theories—are to be considered in earnest. The formulation of noncommutative gauge theories, which are deformations of ordinary theories with simple gauge groups in arbitrary representations, demanded the introduction of the enveloping-algebra formalism [10–12]—a formalism which may find stringy accommodation in F theory [13]. The main feature of this formalism—see Ref. [14] for a review—is that both noncommutative gauge fields and infinitesimal noncommutative gauge transformations take values on the universal enveloping algebra of the corresponding Lie algebra and are functions of the ordinary gauge fields, these functions defining the corresponding Seiberg-Witten maps. The formulation of a noncommutative generalization—called the noncommutative standard model—of the standard model demands the use of the enveloping-algebra formalism, if no new particles are introduced—for noncommutative generalisations of the standard model outside the enveloping-algebra formalism see Refs. [15–17]. The noncommutative standard model was put forward in Ref. [18], and a fair amount of phenomenological consequences—which might be tested against the data from the LHC—have been drawn from it, Refs. [19–23] to quote only a few—the reader may wish to find further information in Ref. [24]. Renormalizability [25–29], anomaly freedom [30,31], and existence of classical solutions [32–34] are other issues which have been studied for noncommutative gauge theories formulated within the enveloping-algebra formalism.

The general procedure to construct the noncommutative counterpart of the ordinary SO(10) GUT within the enveloping-algebra formalism was laid down in Ref. [35]—see also Ref. [36]. However, the relevance in its phenomenological applications—footprints of a noncommutative space-time may be found at the LHC—of the Yukawa and Higgs sectors of this theory demands that a detailed analysis and construction of these sectors be carried out. At this point, we would like to stress that, against all odds, theories which contain the fermionic and gauge sectors—but have no Higgses—of the noncommutative SO(10) and E_6 GUTs are one-loop renormalizable at first order in the noncommutativity parameter—see Ref. [37]. So, it is a pressing issue to carry out a detailed construction of the first-order-in-θ Yukawa and Higgs sectors of these theories, if the renormalizability properties of phenomenological relevant noncommutative GUTs are to be studied. In this paper, we shall remedy this state of affairs and propose a new strategy to construct the noncommutative counterparts of the ordinary SO(10) andE_6 Yukawa terms that are renormalizable at first order in the noncommutativity parameter. The ideas introduced here will be certainly of help in the construction of the Higgs potential of noncommutative SO(10) and E_6 GUTs, but its construction will not be tackled here, since it is very involved and surely deserves to be carried out separately.

The layout of this paper is as follows. In Sec. II, we put forward our procedure to construct noncommutative Yukawa terms for SO(10) and E_6 GUTs. In Sec. III, we work out our noncommutative Yukawa terms at first order in the noncommutativity parameter taking into account the symmetry properties, under the exchange of the fermionic multiplets, of the invariant tensor that occur in the ordinary

Yukawa terms. Section IV is devoted to the discussion of redundant Yukawa terms. In Sec. V, we state our conclusions.

II. NONCOMMUTATIVE YUKAWA TERMS FOR SO(10) AND E_6

In ordinary SO(10) and E_6 GUTs the fermionic degrees of freedom are given by three fermionic field multiplets $\psi_{\alpha Af}$–$f = 1, 2, 3$, labels the three fermionic families of the GUT. For each "A" and "f," $\psi_{\alpha Af}$, $\alpha = 1$, and 2, denote, respectively, the components of a left-handed Weyl spinor, here we follow the conventions of Ref. [38]; whereas for each "α" and f the index A labels the components of the fermionic multiplet carrying certain—the 16 for SO(10) and the 27 for E_6—irreducible representations of the GUT gauge group. The ordinary Becchi-Rouet-Stora (BRS) transformations of $\psi_{\alpha Af}$ are defined as follows:

$$\begin{aligned} s\psi_{\alpha Af} &= i\lambda^{(\psi)}_{AB}\psi_{\alpha Bf},\\ s\lambda^{(\psi)}_{AB} &= i\lambda^{(\psi)}_{AC}\lambda^{(\psi)}_{CB},\\ \lambda^{(\psi)}_{AB} &= \lambda^a\Sigma^a_{AB}, \end{aligned}\tag{2.1}$$

where Σ^a_{AB} stands for a generic generator of the gauge group in the representation furnished by the fermionic multiplet of each family. We shall denote by ϕ_i the components of a generic Higgs multiplet which couples in the Yukawa terms to the fermions of our theory. We shall assume that this multiplet carries an irreducible representation of the GUT gauge group. The BRS transformation of ϕ_i is given by

$$\begin{aligned} s\phi_i &= i\lambda^{(\phi)}_{ij}\phi_j,\\ s\lambda^{(\phi)}_{ij} &= i\lambda^{(\phi)}_{ik}\lambda^{(\phi)}_{kj},\\ \lambda^{(\phi)}_{ij} &= \lambda^a M^a_{ij}, \end{aligned}\tag{2.2}$$

where M^a_{ij} denotes a generic generator of the GUT gauge group in the irreducible representation supplied by the Higgs multiplet. As is well known, for SO(10), ϕ_i will transform under either the 10 or the 120 or the $\overline{126}$, whereas the 27, the $351'$, and the 351 are the representations that may carry the Higgs multiplets in a Yukawa term of the E_6 GUT.

The ordinary Yukawa $\mathcal{Y}^{(\mathrm{ord})}$ term for the gauge groups SO(10) and E_6 reads

$$\mathcal{Y}^{(\mathrm{ord})} = \int d^4x\,\mathcal{Y}_{ff'}\mathcal{C}_{AiB}\tilde{\psi}^{\alpha}_{Af}\psi_{\alpha Bf'}\phi_i, \tag{2.3}$$

where $\mathcal{Y}_{ff'}$ denotes the Yukawa couplings and $\mathcal{C}_{AiB}$ is a group invariant three-index tensor, i.e.,

$$\tilde{\Sigma}^a_{AC}\mathcal{C}_{CiB} + \mathcal{C}_{AjB}M^a_{ji} + \mathcal{C}_{AjC}\Sigma^a_{CB} = 0, \tag{2.4}$$

where $\tilde{\Sigma}^a_{AC} \equiv \Sigma^a_{CA}$. For later convenience, we have expressed $\mathcal{Y}^{(\mathrm{ord})}$ in terms of the A component of the transpose of the fermionic multiplet ψ^{α}_f: $\tilde{\psi}^{\alpha}_f = (\psi^{\alpha}_f)^{\top}$—of course, $\tilde{\psi}^{\alpha}_{Af} = \psi^{\alpha}_{Af}$. The ordinary gauge transformations act on $\tilde{\psi}^{\alpha}_f$ on the right by means of the transpose matrix. Hence, the BRS variation of $\tilde{\psi}^{\alpha}_{Af}$ reads

$$\begin{aligned} s\tilde{\psi}_{\alpha Af} &= i\tilde{\psi}_{\alpha Bf}\tilde{\lambda}^{(\psi)}_{BA},\\ s\tilde{\lambda}^{(\psi)}_{BA} &= -i\tilde{\lambda}^{(\psi)}_{BC}\tilde{\lambda}^{(\psi)}_{CA},\\ \tilde{\lambda}^{(\psi)}_{BA} &= \lambda^a\tilde{\Sigma}^a_{BA},\\ \tilde{\Sigma}^a &= (\Sigma^a)^{\top}. \end{aligned}\tag{2.5}$$

Let us now introduce the following fields: ϕ_{AB}, $\tilde{\psi}^{\alpha}_{iBf}$, and $\psi_{\alpha Aif'}$, which are defined as follows

$$\begin{aligned} \phi_{AB} &= \mathcal{C}_{AiB}\phi_i,\\ \tilde{\psi}^{\alpha}_{iBf} &= \tilde{\psi}^{\alpha}_{Af}\mathcal{C}_{AiB},\\ \psi_{\alpha Aif'} &= \mathcal{C}_{AiB}\psi_{\alpha Bif'}. \end{aligned}\tag{2.6}$$

To construct noncommutative versions of $\mathcal{Y}^{(\mathrm{ord})}$ in Eq. (2.3), we shall find it useful to have $\mathcal{Y}^{(\mathrm{ord})}$ expressed in terms of the fields ϕ_{AB}, $\tilde{\psi}^{\alpha}_{iBf}$, and $\psi_{\alpha Aif'}$:

$$\begin{aligned} \mathcal{Y}^{(\mathrm{ord})}_1 &\equiv \mathcal{Y}^{(\mathrm{ord})} = \int d^4x\,\mathcal{Y}_{ff'}\tilde{\psi}^{\alpha}_{Af}\phi_{AB}\psi_{\alpha Bf'},\\ \mathcal{Y}^{(\mathrm{ord})}_2 &\equiv \mathcal{Y}^{(\mathrm{ord})} = \int d^4x\,\mathcal{Y}_{ff'}\tilde{\phi}_i\tilde{\psi}^{\alpha}_{iBf}\psi_{\alpha Bf'},\\ \mathcal{Y}^{(\mathrm{ord})}_3 &\equiv \mathcal{Y}^{(\mathrm{ord})} = \int d^4x\,\mathcal{Y}_{ff'}\tilde{\psi}^{\alpha}_{Af}\psi_{\alpha Aif'}\phi_i, \end{aligned}\tag{2.7}$$

where, for later convenience, we have introduced $\tilde{\phi}_i$, which is the "i"component of the transpose of the Higgs multiplet: $\tilde{\phi} = (\phi)^{\top}$. The fields ϕ_{AB}, $\tilde{\psi}^{\alpha}_{iBf}$, and $\psi_{\alpha Aif'}$ do not carry irreducible representations of the GUT gauge group, but they carry the very same number of physical degrees of freedom as do ϕ_i, $\tilde{\psi}^{\alpha}_{Bf}$, and $\psi_{\alpha Af'}$, respectively. The BRS transformations of ϕ_{AB}, $\tilde{\psi}^{\alpha}_{iBf}$, and $\psi_{\alpha Aif'}$ are

$$\begin{aligned} s\phi_{AB} &= -i\tilde{\lambda}^{(\psi)}_{AC}\phi_{CB} - i\phi_{AC}\lambda^{(\psi)}_{CB},\\ s\tilde{\psi}^{\alpha}_{iBf} &= -i\tilde{\lambda}^{(\phi)}_{ij}\tilde{\psi}^{\alpha}_{jBf} - i\tilde{\psi}^{\alpha}_{iCf}\lambda^{(\psi)}_{CB},\\ s\psi_{\alpha Aif'} &= -i\tilde{\lambda}^{(\psi)}_{AC}\psi_{\alpha Cif'} - i\psi_{\alpha Ajf'}\lambda^{(\phi)}_{ji}. \end{aligned}\tag{2.8}$$

In our notation, $\tilde{\lambda}^{(\phi)}_{ij} = \lambda^{(\phi)}_{ji}$. The BRS transformations in the previous Eq. are a by-product of the BRS transformations in Eqs. (2.2), (2.5), and (2.1) and of $\mathcal{C}_{AiB}$ being, as shown in Eq. (2.4), a group invariant tensor.

It can be seen [35] that the naive noncommutative version of $\mathcal{Y}^{(\mathrm{ord})}$ as defined in Eq. (2.3) would not do, since, on the one hand, the $\star$ product is noncommutative and, on the other hand, the fact that the noncommutative gauge transformations are valued on the universal enveloping algebra of the Lie algebra yields the conclusion that Eq. (2.4) only leads to gauge invariance at zero order in the noncommutative parameter. By the naive noncommutative version of $\mathcal{Y}^{(\mathrm{ord})}$, we mean the expression

$$\int d^4x \mathcal{Y}_{ff'} \mathcal{C}_{AiB} \tilde{\Psi}^{\alpha}_{Af} \star \Psi_{\alpha Bf'} \star \Phi_i,$$

where $\tilde{\Psi}^{\alpha}_{Af}$, $\Psi_{\alpha Bf'}$, and Φ_i are defined in terms of the ordinary fields by means of the standard—see Eq. (3.3) in Ref. [12]—Seiberg-Witten maps. However, if we include in our formalism the notion of hybrid Seiberg-Witten map introduced in Ref. [39], one can naturally associate a noncommutative Yukawa term to each $\mathcal{Y}_n^{(\mathrm{ord})}$, $n = 1, 2, 3$, in Eq. (2.7). We shall see that the three noncommutative Yukawa terms so obtained are not equal to one another, so our most general noncommutative Yukawa term will be the sum of them all.

To obtain the noncommutative version of $\mathcal{Y}_1^{(\mathrm{ord})}$ in Eq. (2.7), one first introduces three noncommutative fields $\tilde{\Psi}^{\alpha}_{Af}$, Φ_{AB}, and $\Psi_{\alpha Bf'}$, which are, respectively, the noncommutative counterparts of the ordinary fields $\tilde{\psi}^{\alpha}_{Af}$, ϕ_{AB}, and $\psi_{\alpha Bf'}$ in $\mathcal{Y}_1^{(\mathrm{ord})}$. The noncommutative fields are functions of the ordinary fields and $\theta^{\mu\nu}$ that solve the Seiberg-Witten map equations and go to its ordinary counterpart as $\theta^{\mu\nu} \to 0$. To define the Seiberg-Witten map equations, one first introduces the noncommutative BRS transformations of $\tilde{\Psi}^{\alpha}_{Af}$, Φ_{AB}, and $\Psi_{\alpha Bf'}$:

$$\begin{aligned} &s_{\mathrm{nc}}\tilde{\Psi}^{\alpha}_{Af} = i\tilde{\Psi}^{\alpha}_{Bf} \star \tilde{\Lambda}^{(\psi)}_{BA}, \qquad s_{\mathrm{nc}}\Psi_{\alpha Bf'} = i\Lambda^{(\psi)}_{BC} \star \Psi_{\alpha Cf'},\\ &s_{\mathrm{nc}}\Phi_{AB} = -i\tilde{\Lambda}^{(\psi)}_{AC} \star \Phi_{CB} - i\Phi_{AC} \star \Lambda^{(\psi)}_{CB},\\ &s_{\mathrm{nc}}\tilde{\Lambda}^{(\psi)}_{BA} = -i\tilde{\Lambda}^{(\psi)}_{BC} \star \tilde{\Lambda}^{(\psi)}_{CA}, \qquad s_{\mathrm{nc}}\Lambda^{(\psi)}_{BC} = i\Lambda^{(\psi)}_{BD} \star \Lambda^{(\psi)}_{DC}. \end{aligned} \tag{2.9}$$

Let us stress that we have defined the noncommutative BRS transformation of $\tilde{\Psi}^{\alpha}_{Af}$ by acting, via the $\star$ product, with $\tilde{\Lambda}^{(\psi)}_{BA}$ on the right of $\tilde{\Psi}^{\alpha}_{Af}$. Hence, by definition, the noncommutative gauge transformations act on $\tilde{\Psi}^{\alpha}_{Af}$ on the right. We shall see below that this right action makes the noncommutative Yukawa term gauge invariant, and it is to be compared with the noncommutative BRS transformation of $\Psi_{\alpha Bf}$, which is defined by left action with the $\star$ product.

The Seiberg-Witten map equations, which give

$$\begin{aligned} &\tilde{\Psi}^{\ \alpha}_{Af}[\tilde{a}^{(\psi)}_{\mu}, \tilde{\psi}^{\alpha}_{Bf}, \theta^{\mu\nu}], \qquad \Phi_{AB}[\tilde{a}^{(\psi)}_{\mu}, a^{(\psi)}_{\mu}, \phi_{AB}, \theta^{\mu\nu}],\\ &\Psi_{\alpha Bf'}[a^{(\psi)}_{\mu}, \psi^{\ \alpha}_{\alpha Cf'}, \theta^{\mu\nu}],\\ &\tilde{\Lambda}^{(\psi)}_{BA}[\tilde{a}^{(\psi)}_{\mu}, \tilde{\lambda}^{(\psi)}, \theta^{\mu\nu}] \quad \text{and} \quad \Lambda^{(\psi)}_{BC}[a^{(\psi)}_{\mu}, \lambda^{(\psi)}, \theta^{\mu\nu}] \end{aligned}$$

as a function of their arguments, are the following:

$$\begin{aligned} &s_{\mathrm{nc}}\tilde{\Lambda}^{(\psi)}_{BA} = s\tilde{\Lambda}^{(\psi)}_{BA}, \qquad s_{\mathrm{nc}}\Lambda^{(\psi)}_{BA} = s\Lambda^{(\psi)}_{BA},\\ &s_{\mathrm{nc}}\tilde{\Psi}^{\alpha}_{Af} = s\tilde{\Psi}^{\alpha}_{Af}, \qquad s_{\mathrm{nc}}\Psi_{\alpha Bf'} = s\Psi_{\alpha Bf'},\\ &s_{\mathrm{nc}}\Phi_{AB} = s\Phi_{AB}. \end{aligned} \tag{2.10}$$

The symbol s denotes the ordinary BRS operator defined in Eqs. (2.1), (2.2), (2.5), and (2.8), along with

$$\begin{aligned} &s\tilde{a}^{(\psi)}_{\mu AB} = \partial_\mu \tilde{\lambda}^{(\psi)}_{AB} + i[\tilde{a}^{(\psi)}_{\mu}, \tilde{\lambda}^{(\psi)}]_{AB}, \qquad \tilde{a}^{(\psi)}_{\mu AB} = a^a_\mu \tilde{\Sigma}^a_{AB},\\ &sa^{(\psi)}_{\mu AB} = \partial_\mu \lambda^{(\psi)}_{AB} - i[a^{(\psi)}_{\mu}, \lambda^{(\psi)}]_{AB}, \qquad a^{(\psi)}_{\mu AB} = a^a_\mu \Sigma^a_{AB}. \end{aligned} \tag{2.11}$$

Recall that $\tilde{\Sigma}^a_{AB} = \Sigma^a_{BA}$.

Solutions to the Seiberg-Witten map equations in Eq. (2.10) can be obtained as formal powers series in $\theta^{\mu\nu}$. Up to first order, these solutions, which define the corresponding Seiberg-Witten maps, read

$$\begin{aligned} \tilde{\Lambda}^{(\psi)}_{BA} &= \tilde{\lambda}^{(\psi)}_{BA} + \frac{1}{4}\theta^{\mu\nu}\{\tilde{a}^{(\psi)}_{\mu}, \partial_\nu \tilde{\lambda}^{(\psi)}\}_{BA} + O(\theta^2),\\ \Lambda^{(\psi)}_{BC} &= \lambda^{(\psi)}_{BC} - \frac{1}{4}\theta^{\mu\nu}\{a^{(\psi)}_{\mu}, \partial_\nu \lambda^{(\psi)}\}_{BC} + O(\theta^2),\\ \tilde{\Psi}^{\alpha}_{Af} &= \tilde{\psi}^{\alpha}_{Af} - \frac{1}{2}\theta^{\mu\nu}\partial_\mu \tilde{\psi}^{\alpha}_{Bf}\tilde{a}^{(\psi)}_{\nu BA}\\ &\quad + \frac{i}{4}\theta^{\mu\nu}\tilde{\psi}^{\alpha}_{Cf}\tilde{a}^{(\psi)}_{\mu CB}\tilde{a}^{(\psi)}_{\nu BA} + O(\theta^2),\\ \Phi_{AB} &= \phi_{AB} + \frac{1}{2}\theta^{\mu\nu}\tilde{a}^{(\psi)}_{\mu AC}\partial_\nu \phi_{CB} + \frac{i}{4}\theta^{\mu\nu}\tilde{a}^{(\psi)}_{\mu AC}\tilde{a}^{(\psi)}_{\nu CD}\phi_{DB}\\ &\quad + \frac{1}{2}\theta^{\mu\nu}\partial_\mu \phi_{AC}a^{(\psi)}_{\nu CB} + \frac{i}{4}\theta^{\mu\nu}\phi_{AC}a^{(\psi)}_{\mu CD}a^{(\psi)}_{\nu DB}\\ &\quad + \frac{i}{2}\theta^{\mu\nu}\tilde{a}^{(\psi)}_{\mu AC}\phi_{CD}a^{(\psi)}_{\nu DB} + O(\theta^2),\\ \Psi_{\alpha Bf'} &= \psi_{\alpha Bf'} - \frac{1}{2}\theta^{\mu\nu}a^{(\psi)}_{\mu BC}\partial_\mu \psi_{\alpha Cf'}\\ &\quad + \frac{i}{4}\theta^{\mu\nu}a^{(\psi)}_{\mu BC}a^{(\psi)}_{\nu CD}\psi^{\ \alpha}_{Df'} + O(\theta^2). \end{aligned} \tag{2.12}$$

Notice that Φ_{AB} is defined by a hybrid Seiberg-Witten map, a notion which was put forward in Ref. [39].

We are now in the position to introduce and—using Eq. (2.12)—compute up to first order in $\theta^{\mu\nu}$ the noncommutative counterpart, $\mathcal{Y}_1^{(\mathrm{nc})}$, of $\mathcal{Y}_1^{(\mathrm{ord})}$ in Eq. (2.7):

$$\begin{aligned}\mathcal{Y}_1^{(\mathrm{nc})} &= \int d^4x \mathcal{Y}^{(1)}_{ff'} \tilde{\Psi}^\alpha_{Af} \star \Phi_{AB} \star \Psi_{\alpha Bf'} \\ &= \int d^4x \mathcal{Y}^{(1)}_{ff'} \mathcal{C}_{AiB} \tilde{\psi}^\alpha_{Af} \phi_i \psi_{\alpha Bf'} \\ &\quad + \int d^4x \Big(-\frac{i}{2}\Big)\theta^{\mu\nu} \mathcal{Y}^{(1)}_{ff'} \mathcal{C}_{AiB} (D_\mu \tilde{\psi}^\alpha_f)_A \phi_i (D_\nu \psi_{\alpha f'})_B \\ &\quad + \int d^4x \Big(-\frac{1}{4}\Big)(\mathcal{Y}^{(1)}_{ff'}\mathcal{C}_{AiB} \\ &\quad - \mathcal{Y}^{(1)}_{f'f}\mathcal{C}_{BiA})\theta^{\mu\nu}\phi_i \tilde{\psi}^\alpha_{Af} f^{(\psi)}_{\mu\nu BC}\psi_{\alpha Cf'} + O(\theta^2),\end{aligned} \tag{2.13}$$

where $(D_\mu \tilde{\psi}^\alpha_f)_A = \partial_\mu \tilde{\psi}^\alpha_{Af} - i\tilde{\psi}^\alpha_{Bf}\tilde{a}^{(\psi)}_{\mu BA}$, $(D_\nu \psi_{\alpha f'})_B = \partial_\nu \psi_{\alpha Bf'} - i a^{(\psi)}_{\nu BC}\psi_{\alpha Cf'}$ and $f^{(\psi)}_{\mu\nu} = \partial_\mu a^{(\psi)}_\nu - \partial_\nu a^{(\psi)}_\mu - i[a^{(\psi)}_\mu, a^{(\psi)}_\nu]$. It is apparent that $\mathcal{Y}_1^{(\mathrm{nc})}$ is invariant under the noncommutative BRS variations defined in Eq. (2.9). Next, we define the noncommutative counterpart, $\mathcal{Y}_2^{(\mathrm{nc})}$, of $\mathcal{Y}_2^{(\mathrm{ord})}$ in Eq. (2.7):

$$\mathcal{Y}_2^{(\mathrm{nc})} = \int d^4x \mathcal{Y}^{(2)}_{ff'} \tilde{\Phi}_i \star \tilde{\Psi}^\alpha_{iBf} \star \Psi_{\alpha Bf'}, \tag{2.14}$$

where

$$\begin{aligned}\tilde{\Phi}_i &= \tilde{\phi}_i - \frac{1}{2}\theta^{\mu\nu}\partial_\mu \tilde{\phi}_j \tilde{a}^{(\phi)}_{\nu ji} + \frac{i}{4}\theta^{\mu\nu}\tilde{\phi}_j \tilde{a}^{(\phi)}_{\mu jk}\tilde{a}^{(\phi)}_{\nu ki} + O(\theta^2), \\ \tilde{\Psi}^\alpha_{iBf} &= \tilde{\psi}^\alpha_{iBf} + \frac{1}{2}\theta^{\mu\nu}\tilde{a}^{(\phi)}_{\mu ij}\partial_\nu \tilde{\psi}^\alpha_{jBf} + \frac{i}{4}\theta^{\mu\nu}\tilde{a}^{(\phi)}_{\mu ik}\tilde{a}^{(\phi)}_{\nu kj}\tilde{\psi}^\alpha_{jBf} \\ &\quad + \frac{1}{2}\theta^{\mu\nu}\partial_\mu \tilde{\psi}^\alpha_{iCf} a^{(\psi)}_{\nu CB} + \frac{i}{4}\theta^{\mu\nu}\tilde{\psi}^\alpha_{iDf} a^{(\psi)}_{\mu DC} a^{(\psi)}_{\nu CB} \\ &\quad + \frac{i}{2}\theta^{\mu\nu}\tilde{a}^{(\phi)}_{\mu ij}\tilde{\psi}^\alpha_{jCf} a^{(\psi)}_{\nu CB} + O(\theta^2), \\ \Psi_{\alpha Bf'} &= \psi_{\alpha Af} - \frac{1}{2}\theta^{\mu\nu}a^{(\psi)}_{\mu BC}\partial_\mu \psi_{\alpha Cf'} \\ &\quad + \frac{i}{4}\theta^{\mu\nu}a^{(\psi)}_{\mu BC}a^{(\psi)}_{\nu CD}\psi^\alpha_{Df'} + O(\theta^2),\end{aligned} \tag{2.15}$$

with $\tilde{a}^{(\phi)}_{\mu ij} = a^a_\mu \tilde{M}^a_{ij}$, $\tilde{M}^a_{ij} = M^a_{ji}$. The noncommutative fields in the previous equation are solutions to the following Seiberg-Witten map equations:

$$\begin{aligned}& -i\tilde{\Lambda}^{(\phi)}_{ij} \star \tilde{\Psi}^\alpha_{jBf} - i\tilde{\Psi}^\alpha_{iCf} \star \Lambda^{(\psi)}_{CB} \equiv s_{\mathrm{nc}}\tilde{\Psi}^\alpha_{iBf} = s\tilde{\Psi}^\alpha_{iBf}, \\ & i\Lambda^{(\psi)}_{BC} \star \Psi_{\alpha Cf'} \equiv s_{\mathrm{nc}}\Psi_{\alpha Bf'} = s\Psi_{\alpha Bf'}, \\ & i\tilde{\Phi}_j \star \tilde{\Lambda}^{(\phi)}_{ji} \equiv s_{\mathrm{nc}}\tilde{\Phi}_i = s\tilde{\Phi}_i, \\ & i\Lambda^{(\psi)}_{AC} \star \Lambda^{(\psi)}_{CB} \equiv s_{\mathrm{nc}}\Lambda^{(\psi)}_{AC} = s\Lambda^{(\psi)}_{AC}, \\ & -i\tilde{\Lambda}^{(\phi)}_{ik} \star \tilde{\Lambda}^{(\phi)}_{kj} \equiv s_{\mathrm{nc}}\tilde{\Lambda}^{(\phi)}_{ij} = s\tilde{\Lambda}^{(\phi)}_{ij},\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}\tilde{\Lambda}^{(\phi)}_{ij} &= \tilde{\lambda}^{(\phi)}_{ij} + \frac{1}{4}\theta^{\mu\nu}\{\tilde{a}^{(\phi)}_\mu, \partial_\nu \tilde{\lambda}^{(\phi)}\}_{ij} + O(\theta^2), \\ \Lambda^{(\psi)}_{BC} &= \lambda^{(\psi)}_{BC} - \frac{1}{4}\theta^{\mu\nu}\{a^{(\psi)}_\mu, \partial_\nu \lambda^{(\psi)}\}_{BC} + O(\theta^2),\end{aligned}$$

with $\tilde{\lambda}^{(\phi)}_{ij} = \tilde{\lambda}^a \tilde{M}^a_{ij}$. To check that the Seiberg-Witten maps in Eq. (2.15) are solutions to Eq. (2.16), one needs the following results:

$$\begin{aligned}s\tilde{a}^{(\phi)}_{\mu ij} &= \partial_\mu \tilde{\lambda}^{(\phi)}_{ij} + i[\tilde{a}^{(\phi)}_\mu, \tilde{\lambda}^{(\phi)}]_{ij}, \\ s a^{(\phi)}_{\mu ij} &= \partial_\mu \lambda^{(\phi)}_{ij} - i[a^{(\phi)}_\mu, \lambda^{(\phi)}]_{ij},\end{aligned} \tag{2.17}$$

where $a^{(\phi)}_{\mu ij} = a^a_{\mu ij}M^a_{ij}$.

By using the results in Eq. (2.15), one obtains the θ expansion of $\mathcal{Y}_2^{(\mathrm{nc})}$ in Eq. (2.14):

$$\begin{aligned}\mathcal{Y}_2^{(\mathrm{nc})} &= \int d^4x \mathcal{Y}^{(2)}_{ff'}\mathcal{C}_{AiB}\tilde{\psi}^\alpha_{Af}\phi_i \psi_{\alpha Bf'} \\ &\quad + \int d^4x \Big(\frac{i}{2}\Big)\theta^{\mu\nu}\mathcal{Y}^{(2)}_{ff'}\mathcal{C}_{AiB}(D_\mu \tilde{\psi}^\alpha_f)_A \phi_i (D_\nu \psi_{\alpha f'})_B \\ &\quad + \int d^4x \Big(-\frac{1}{4}\Big)(\mathcal{Y}^{(2)}_{ff'}\mathcal{C}_{AiB} \\ &\quad + \mathcal{Y}^{(2)}_{f'f}\mathcal{C}_{BiA})\theta^{\mu\nu}\phi_i\tilde{\psi}^\alpha_{Af} f^{(\psi)}_{\mu\nu BC}\psi_{\alpha Cf'} + O(\theta^2).\end{aligned} \tag{2.18}$$

In obtaining the previous result, the following equation is of much help:

$$\tilde{f}^{(\psi)}_{\mu\nu AC}\mathcal{C}_{CiB} + \mathcal{C}_{AjB}f^{(\phi)}_{\mu\nu ji} + \mathcal{C}_{AiC}f^{(\psi)}_{\mu\nu CB} = 0. \tag{2.19}$$

Notice that $\tilde{f}^{(\psi)}_{\mu\nu} = \partial_\mu \tilde{a}^{(\psi)}_\nu - \partial_\nu \tilde{a}^{(\psi)}_\mu + i[\tilde{a}^{(\psi)}_\mu, \tilde{a}^{(\psi)}_\nu]$ and $f^{(\phi)}_{\mu\nu} = \partial_\mu a^{(\phi)}_\nu - \partial_\nu a^{(\phi)}_\mu - i[a^{(\phi)}_\mu, a^{(\phi)}_\nu]$. Equation (2.19), and similar equations involving $a^{(\psi)}_\mu$ and $a^{(\phi)}_\mu$, follow from Eq. (2.4).

Finally, we shall introduce the noncommutative version $\mathcal{Y}_3^{(\mathrm{nc})}$ of $\mathcal{Y}_3^{(\mathrm{ord})}$ in Eq. (2.7)

$$\mathcal{Y}_3^{(\mathrm{nc})} = \int d^4x \mathcal{Y}^{(3)}_{ff'}\tilde{\Psi}^\alpha_{Af} \star \Psi_{\alpha Aif'} \star \Phi_i. \tag{2.20}$$

The fields in the previous equation are given, at first order in θ, by the following expressions:

$$\begin{aligned}\tilde{\Psi}^\alpha_{Af} &= \tilde{\psi}^\alpha_{Af} - \frac{1}{2}\theta^{\mu\nu}\partial_\mu \tilde{\psi}^\alpha_{Bf}\tilde{a}^{(\psi)}_{\nu BA} + \frac{i}{4}\theta^{\mu\nu}\tilde{\psi}^\alpha_{Cf}\tilde{a}^{(\psi)}_{\mu CB}\tilde{a}^{(\psi)}_{\nu BA} \\ &\quad + O(\theta^2), \\ \Psi_{\alpha Aif'} &= \psi_{\alpha Aif'} + \frac{1}{2}\theta^{\mu\nu}\tilde{a}^{(\psi)}_{\mu AB}\partial_\nu \psi_{\alpha Bif'} \\ &\quad + \frac{i}{4}\tilde{a}^{(\psi)}_{\mu AB}\tilde{a}^{(\psi)}_{\nu BC}\psi_{\alpha Cif'} + \frac{1}{2}\theta^{\mu\nu}\partial_\mu \psi_{\alpha Ajf'}a^{(\phi)}_{\nu ji} \\ &\quad + \frac{i}{4}\psi_{\alpha Akf'}a^{(\phi)}_{\mu kj}a^{(\phi)}_{\nu ji} + \frac{i}{2}\theta^{\mu\nu}\tilde{a}^{(\psi)}_{\mu AB}\psi_{\alpha Bjf'}a^{(\phi)}_{\nu ji} \\ &\quad + O(\theta^2), \\ \Phi_i &= \phi_i - \frac{1}{2}\theta^{\mu\nu}a^{(\phi)}_{\mu ij}\partial_\nu \phi_j + \frac{i}{4}\theta^{\mu\nu}a^{(\phi)}_{\mu ij}a^{(\phi)}_{\nu jk}\phi_k + O(\theta^2).\end{aligned} \tag{2.21}$$

The Seiberg-Witten maps in the previous set of equations are solutions to

$$\begin{aligned}
&i\tilde\Psi^\alpha_{Bf}\star\tilde\Lambda^{(\psi)}_{BA}\equiv s_{\rm nc}\tilde\Psi^\alpha_{Af}=s\tilde\Psi^\alpha_{Af},\\
&-i\tilde\Lambda^{(\psi)}_{AC}\star\Psi_{\alpha Cif'}-i\Psi_{\alpha Ajf'}\star\Lambda^{(\phi)}_{ji}\equiv s_{\rm nc}\Psi_{\alpha Aif'}\\
&\quad=s\Psi_{\alpha Aif'},\\
&i\Lambda^{(\phi)}_{ij}\star\Phi_j\equiv s_{\rm nc}\Phi_i=s\Phi_i,\\
&-i\tilde\Lambda^{(\psi)}_{AC}\star\tilde\Lambda^{(\psi)}_{CB}\equiv s_{\rm nc}\tilde\Lambda^{(\psi)}_{AB}=s\tilde\Lambda^{(\psi)}_{AB},\\
&i\Lambda^{(\phi)}_{ik}\star\Lambda^{(\phi)}_{kj}\equiv s_{\rm nc}\Lambda^{(\phi)}_{ij}=s\Lambda^{(\phi)}_{ij},
\end{aligned}\tag{2.22}$$

if

$$\begin{aligned}
\Lambda^{(\phi)}_{ij}&=\lambda^{(\phi)}_{ij}-\frac14\theta^{\mu\nu}\{a^{(\phi)}_\mu,\partial_\nu\lambda^{(\phi)}\}_{ij}+O(\theta^2),\\
\tilde\Lambda^{(\psi)}_{AB}&=\tilde\lambda^{(\psi)}_{AB}+\frac14\theta^{\mu\nu}\{\tilde a^{(\psi)}_\mu,\partial_\nu\tilde\lambda^{(\psi)}\}_{AB}+O(\theta^2).
\end{aligned}$$

Now, substituting the Seiberg-Witten maps in Eq. (2.21) in Eq. (2.20), one gets

$$\begin{aligned}
\mathcal{Y}^{(\rm nc)}_3=&\int d^4x\,\mathcal{Y}^{(3)}_{ff'}\mathcal{C}_{AiB}\tilde\psi^\alpha_{Af}\phi_i\psi_{\alpha Bf'}\\
&+\int d^4x\left(\frac{i}{2}\right)\theta^{\mu\nu}\mathcal{Y}^{(3)}_{ff'}\mathcal{C}_{AiB}(D_\mu\tilde\psi^\alpha_f)_A\phi_i(D_\nu\psi_{\alpha f'})_B\\
&+\int d^4x\left(\frac14\right)(\mathcal{Y}^{(3)}_{ff'}\mathcal{C}_{AiB}\\
&+\mathcal{Y}^{(3)}_{f'f}\mathcal{C}_{BiA})\theta^{\mu\nu}\phi_i\tilde\psi^\alpha_{Af}f^{(\psi)}_{\mu\nu BC}\psi_{\alpha Cf'}+O(\theta^2).
\end{aligned}\tag{2.23}$$

We have found no reason to discard any of the $\mathcal{Y}^{(\rm nc)}_n$, $n=1,2,3$, in Eqs. (2.13), (2.14), and (2.20), respectively, as a valid noncommutative Yukawa contribution; we then conclude that our noncommutative Yukawa term $\mathcal{Y}^{(\rm nc)}$ is the sum of the three of them:

$$\mathcal{Y}^{(\rm nc)}\equiv\mathcal{Y}^{(\rm nc)}_1+\mathcal{Y}^{(\rm nc)}_2+\mathcal{Y}^{(\rm nc)}_3.\tag{2.24}$$

Using the expansions in Eqs. (2.13), (2.18), and (2.23), one can show that the most general functional which is linear in $\theta^{\mu\nu}$, contains one ϕ_i and two $\psi_{\alpha Af}$, involves the derivatives of these fields, has no dimensionful parameter other than $\theta^{\mu\nu}$, and whose BRS variation vanishes, is given by the first order in θ contribution to $\mathcal{Y}^{(\rm nc)}$ above. Hence, the noncommutative Yukawa interaction introduced in Eq. (2.24) is renormalizable at first order in $\theta^{\mu\nu}$, a property not to be overlooked.

III. TAKING INTO ACCOUNT THE INDEX SYMMETRY PROPERTIES OF $\mathcal{C}_{AiB}$

Let ϕ_i in Eq. (2.3) carry an irreducible representation of SO(10), and let $\mathcal{C}_{AiB}$ be the invariant tensor also in Eq. (2.3). Then, the Clebsch-Gordan decomposition [40] of the $16\otimes16$ representation of SO(10) leads to the conclusion that $\mathcal{C}_{AiB}=\mathcal{C}_{BiA}$, if ϕ_i carries either the 10 or the $\overline{126}$ of SO(10), and that $\mathcal{C}_{AiB}=-\mathcal{C}_{BiA}$, if Φ_i transforms under the 120 of SO(10). Analogously [40], that for E_6 we have $27\otimes27=(\overline{27}\oplus\overline{351'})_{\rm s}\oplus\overline{351}_{\rm as}$ implies that $\mathcal{C}_{AiB}=\mathcal{C}_{BiA}$ when the Higgs field is in either the 27 or the 351′ of E_6 and $\mathcal{C}_{AiB}=-\mathcal{C}_{BiA}$ when ϕ_i carries the 351 of E_6.

That in our case $\mathcal{C}_{AiB}$ has well-defined symmetry properties under the exchange of A and "B" leads to simplified expressions for $\mathcal{Y}^{(\rm nc)}$ in Eq. (2.24). Indeed, if $\mathcal{C}_{AiB}=\mathcal{C}_{BiA}$, Eqs. (2.13), (2.18), (2.23), and (2.24) yield

$$\begin{aligned}
\mathcal{Y}^{(\rm nc)}=&\int d^4x\,\mathcal{Y}^{(\rm s)}_{ff'}\mathcal{C}_{AiB}\tilde\psi^\alpha_{Af}\phi_i\psi_{\alpha Bf'}+\int d^4x\left(\frac{i}{2}\right)\\
&\times(-\mathcal{Y}^{(1,\rm as)}_{ff'}+\mathcal{Y}^{(2,\rm as)}_{ff'}\\
&+\mathcal{Y}^{(3,\rm as)}_{ff'})\theta^{\mu\nu}\mathcal{C}_{AiB}(D_\mu\tilde\psi^\alpha_f)_A\phi_i(D_\nu\psi_{\alpha f'})_B\\
&+\int d^4x\left(-\frac12\right)(\mathcal{Y}^{(1,\rm as)}_{ff'}+\mathcal{Y}^{(2,\rm s)}_{ff'}\\
&-\mathcal{Y}^{(3,\rm s)}_{ff'})\theta^{\mu\nu}\mathcal{C}_{AiB}\phi_i\tilde\psi^\alpha_{Af}f^{(\psi)}_{\mu\nu BC}\psi_{\alpha Cf'}+O(\theta^2),
\end{aligned}$$

where $\mathcal{Y}^{(\rm s)}_{ff'}=\mathcal{Y}^{(1,\rm s)}_{ff'}+\mathcal{Y}^{(2,\rm s)}_{ff'}+\mathcal{Y}^{(3,\rm s)}_{ff'}$. $\mathcal{Y}^{(n,\rm s)}_{ff'}$ and $\mathcal{Y}^{(n,\rm as)}_{ff'}$ denote, respectively, the symmetric and antisymmetric parts of $\mathcal{Y}^{(n)}_{ff'}$ with regard to the indices f, f'. $\mathcal{Y}^{(n)}_{ff'}$, $n=1,2,3$ were introduced in Eqs. (2.13), (2.14), and (2.20). Similarly, when $\mathcal{C}_{AiB}=-\mathcal{C}_{BiA}$, Eq. (2.24) boils down to

$$\begin{aligned}
\mathcal{Y}^{(\rm nc)}=&\int d^4x\,\mathcal{Y}^{(\rm as)}_{ff'}\mathcal{C}_{AiB}\tilde\psi^\alpha_{Af}\phi_i\psi_{\alpha Bf'}+\int d^4x\left(\frac{i}{2}\right)\\
&\times(-\mathcal{Y}^{(1,\rm s)}_{ff'}+\mathcal{Y}^{(2,\rm s)}_{ff'}\\
&+\mathcal{Y}^{(3,\rm s)}_{ff'})\theta^{\mu\nu}\mathcal{C}_{AiB}(D_\mu\tilde\psi^\alpha_f)_A\phi_i(D_\nu\psi_{\alpha f'})_B\\
&+\int d^4x\left(-\frac12\right)(\mathcal{Y}^{(1,\rm s)}_{ff'}+\mathcal{Y}^{(2,\rm as)}_{ff'}\\
&-\mathcal{Y}^{(3,\rm as)}_{ff'})\theta^{\mu\nu}\mathcal{C}_{AiB}\phi_i\tilde\psi^\alpha_{Af}f^{(\psi)}_{\mu\nu BC}\psi_{\alpha Cf'}+O(\theta^2),
\end{aligned}$$

where $\mathcal{Y}^{(\rm as)}_{ff'}=\mathcal{Y}^{(1,\rm as)}_{ff'}+\mathcal{Y}^{(2,\rm as)}_{ff'}+\mathcal{Y}^{(3,\rm as)}_{ff'}$.

IV. REDUNDANT CHOICES

Recall that $\tilde\Psi^\alpha_{iBf}$ is the noncommutative counterpart of $\tilde\psi^\alpha_{iBf}$ in Eq. (2.6). The reader may rightly ask whether a new Yukawa term can be obtained by making the following choice—to be compared with the definition in Eq. (2.16)—for the noncommutative BRS transformations of $\tilde\Psi^\alpha_{iBf}$:

$$s_{\rm nc}\tilde\Psi^\alpha_{iBf}=-i\tilde\Psi^\alpha_{jBf}\star\tilde\Lambda^{(\phi)}_{ij}-i\Lambda^{(\psi)}_{CB}\star\tilde\Psi^\alpha_{iCf}.\tag{4.1}$$

Notice that this is a noncommutative generalization of the BRS transformations, in Eq. (2.8), of $\tilde\psi^\alpha_{iBf}$. Also notice that we go back to $s_{\rm nc}\tilde\Psi^\alpha_{iBf}$ in Eq. (2.16) when we change the order in which the Λ's and $\tilde{\Psi}^{\alpha}_{iBf}$ occur in Eq. (4.1). Since the way in which the contracted indices occur in Eq. (4.1) is a little odd, we shall rename the objects in that equation as follows:

$$\tilde{\Psi}^{\,\alpha}_{iBf} \equiv \Psi'^{\alpha}_{iBf}, \qquad \tilde{\Lambda}^{(\phi)}_{ij} \equiv \Lambda'^{(\phi)}_{ji}, \qquad \Lambda^{(\psi)}_{CB} \equiv \tilde{\Lambda}'^{(\psi)}_{BC}.$$

In terms of the fields we have just introduced, Eq. (4.1) reads

$$s_{\rm nc}\Psi'^{\alpha}_{Bif} = -i\tilde{\Psi}'^{\alpha}_{Bjf} \star \Lambda'^{(\phi)}_{ji} - i\tilde{\Lambda}'^{(\psi)}_{BC} \star \tilde{\Psi}'^{\alpha}_{Cif}. \qquad (4.2)$$

This equation is to be supplemented with

$$\begin{aligned} s_{\rm nc}\Lambda'^{(\phi)}_{ji} &= i\Lambda'^{(\phi)}_{jk} \star \Lambda'^{(\phi)}_{ki}, \\ s_{\rm nc}\tilde{\Lambda}'^{(\psi)}_{BC} &= -i\tilde{\Lambda}'^{(\psi)}_{BD} \star \tilde{\Lambda}'^{(\psi)}_{DC}, \end{aligned} \qquad (4.3)$$

if we want $s^2_{\rm nc} = 0$.

Let us next introduce Φ'_i and $\tilde{\Psi}'_{\alpha iBf'}$ as the new noncommutative counterparts of the ordinary ϕ_i and $\tilde{\psi}'_{\alpha Bf'} = \psi'_{\alpha Bf'}$, the latter entering the ordinary Yukawa term in Eq. (2.3). The BRS transformations of Φ'_i and $\tilde{\Psi}'_{\alpha iBf'}$ are defined as follows:

$$s_{\rm nc}\tilde{\Psi}'_{\alpha Af} \equiv i\tilde{\Psi}'_{\alpha Bf} \star \tilde{\Lambda}'_{BA}, \qquad s_{\rm nc}\Phi'_i \equiv i\Lambda'^{(\phi)}_{ij} \star \Phi'_j. \qquad (4.4)$$

Now, it is plain that

$$\mathcal{Y}^{(\rm nc)}_4 = \int d^4x\, \mathcal{Y}^{(4)}_{f'f}\tilde{\Psi}'^{\alpha}_{Af'} \star \Psi'_{\alpha Aif} \star \Phi'_i \qquad (4.5)$$

is invariant under noncommutative BRS transformations if the fields in it are solutions to the following Seiberg-Witten map equations:

$$\begin{aligned} &s_{\rm nc}\tilde{\Psi}'^{\alpha}_{Af'} = s\tilde{\Psi}'^{\alpha}_{Af'}, \qquad s_{\rm nc}\Psi'_{\alpha Bif} = s\Psi'_{\alpha Bif}, \\ &s_{\rm nc}\Phi'_i = s\Phi'_i, \qquad s_{\rm nc}\Lambda'^{(\phi)}_{ji} = s\Lambda'^{(\phi)}_{ji}, \\ &s_{\rm nc}\tilde{\Lambda}'^{(\psi)}_{BC} = s\tilde{\Lambda}'^{(\psi)}_{BC}, \end{aligned} \qquad (4.6)$$

where the action of the noncommutative BRS operator $s_{\rm nc}$ is defined in Eqs. (4.2), (4.3), and (4.4), and the ordinary BRS operator s is given in Eqs. (2.1), (2.2), (2.5), (2.8), (2.11), and (2.17). However, the Yukawa term in Eq. (4.5) is not a new Yukawa term, but it is the Yukawa term in Eq. (2.20). Indeed, notice that i) the Seiberg-Witten map equations in Eq. (4.6) are those in Eq. (2.22), and ii) that at $\theta^{\mu\nu} = 0$ the solutions to Eq. (4.6) must satisfy

$$\begin{aligned} &\tilde{\Psi}'^{\alpha}_{Af'}[\theta = 0] = \tilde{\psi}^{\alpha}_{Af'}, \\ &\Psi'_{\alpha Bif}[\theta = 0] = \tilde{\psi}_{\alpha iBf} \equiv \tilde{\psi}_{\alpha Af}\mathcal{C}_{AiB}, \\ &\Phi'_i[\theta = 0] = \phi_i, \qquad \Lambda'^{(\phi)}_{ji}[\theta = 0] = \lambda^{(\phi)}_{ji}, \\ &\tilde{\Lambda}'^{(\psi)}_{BC}[\theta = 0] = \tilde{\lambda}^{(\psi)}_{BC}. \end{aligned}$$

Then, the fact that $\mathcal{C}_{AiB} = \pm\mathcal{C}_{BiA}$—see previous section—leads to $\tilde{\psi}_{\alpha Af}\mathcal{C}_{AiB} = \pm\mathcal{C}_{BiA}\psi_{\alpha Af} \equiv \pm\psi_{\alpha Bif}$, which combined with i) and ii) above implies that

$$\tilde{\Psi}'^{\alpha}_{Af'} = \tilde{\Psi}^{\alpha}_{Af'}, \qquad \Psi'_{\alpha Bif} = \pm\Psi_{\alpha Bif}, \qquad \Phi'_i = \Phi_i, \qquad (4.7)$$

where $\tilde{\Psi}^{\alpha}_{Af'}$, $\Psi_{\alpha Bif}$, and Φ_i are the solutions to Eq. (2.22), whose first-order-in-θ expansions are displayed in Eq. (2.21). Finally, by substituting Eq. (4.7) in Eq. (4.5), one recovers Eq. (2.20). We thus conclude that the Yukawa term in Eq. (4.5) is redundant.

Analogously, if the fields $\Psi_{\alpha Aif'}$ and Φ_{AB}—which are, respectively, the noncommutative counterparts of the ordinary fields $\psi_{\alpha Aif'}$ and ϕ_{AB} in Eq. (2.6)—are defined so that their noncommutative BRS transformations are given by

$$\begin{aligned} s_{\rm nc}\Psi_{\alpha Aif'} &= -i\Psi_{\alpha Cif'} \star \tilde{\Lambda}^{(\psi)}_{AC} - i\Lambda^{(\phi)}_{ji} \star \Psi_{\alpha Ajf'}, \\ s_{\rm nc}\Phi_{AB} &= -i\Phi_{CB} \star \tilde{\Lambda}^{(\psi)}_{AC} - i\Lambda^{(\psi)}_{CB} \star \Phi_{AC}, \end{aligned} \qquad (4.8)$$

one may show that no new Yukawa terms arise out of them. Indeed, proceeding similarly as we did above, one may show that $\Psi_{\alpha Aif'}$ and Φ_{AB} transforming as in Eq. (4.8) yield $\mathcal{Y}^{(\rm nc)}_2$ and $\mathcal{Y}^{(\rm nc)}_1$, respectively. $\mathcal{Y}^{(\rm nc)}_2$ is given in Eq. (2.14), and $\mathcal{Y}^{(\rm nc)}_1$ was introduced in Eq. (2.13).

A last remark, the two Λ's in the noncommutative BRS transformations of Φ_{AB}, $\tilde{\Psi}^{\alpha}_{iBf}$, and $\Psi_{\alpha Aif'}$ cannot both occur in the BRS transformation on the same side of the corresponding field, for then $s^2_{\rm nc}$ will not vanish when acting on those fields, which in turn will render meaningless the Seiberg-Witten map equations for Φ_{AB}, $\tilde{\Psi}^{\alpha}_{iBf}$, and $\Psi_{\alpha Aif'}$—recall that $s^2 = 0$ if s is the ordinary BRS operator.

V. CONCLUSIONS

We have seen in this paper that noncommutative Yukawa GUT terms can be constructed in a natural way by applying the enveloping-algebra formalism to ordinary fields—ϕ_{AB}, $\tilde{\psi}^{\alpha}_{iBf}$, and $\psi_{\alpha Aif'}$ in Eq. (2.6), which transform under reducible representations of the gauge group, but which involve the very same number of physical degrees as the ordinary irreducible multiplets they are made out of. Let us stress that in the noncommutative case, in sharp contrast with ordinary case, Yukawa terms cannot be constructed in general—and in particular for SO(10) and E_6—by applying the Seiberg-Witten map to ordinary irreducible multiplets, so other procedures such as the one put forward in this paper are needed. Our procedure, which takes advantage of the notion of hybrid Seiberg-Witten map introduced in Ref. [39], yields a renormalizable Yukawa term at first order in θ, thus paving the way—in view of the results in Ref. [37]—to constructing renormalizable noncommutative SO(1O) andE_6 GUTs, at least at first order in $\theta^{\mu\nu}$. Of course, the next challenging issue is to define a noncommutative Higgs potential which deforms the already involved—see, e.g., Refs. [41,42]—ordinary GUT Higgs potential. This, although certainly feasible within the noncommutative GUT formalism of Ref. [35] with help from the ideas presented in this paper, is a much involved piece of research and deserves a separate study. Let us finally point out that Eqs. (2.13), (2.14), and (2.20) generalize naively to higher space-time dimensions, so the procedure introduced in this paper to construct Yukawa terms may be of help in formulating GUTs in higher dimensional noncommutative space-times [13,43,44].

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