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Ginsparg-Wilson relation on a fuzzy 2-sphere for adjoint matter

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We formulate a Ginsparg-Wilson relation on a fuzzy 2-sphere for matter in the adjoint representation of the gauge group. Because of the Ginsparg-Wilson relation, an index theorem is satisfied. Our formulation is applicable to topologically nontrivial configurations as monopoles. It gives a solid basis for obtaining chiral fermions, which are an important ingredient of the standard model, from matrix model formulations of the superstring theory, such as the IIB matrix model, by considering topological configurations in the extra dimensions. We finally discuss whether this mechanism really works.

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I. INTRODUCTION

Matrix models are a promising candidate to formulate the superstring theory nonperturbatively [1,2], and they indeed include quantum gravity and gauge theory. One of the important subjects in such studies is to connect these models to phenomenology. Spacetime structures can be analyzed dynamically in the IIB matrix model [3], and four dimensionality seems to be preferred [3,4]. Assuming four-dimensional spacetime is obtained, we next want to show the standard model of particle physics on it. A crucial issue for it is to realize chiral fermions, which also ensures the existence of massless fermions. Without chiral symmetries, quantum corrections would induce mass of order of the Planck scale in general.

A way to obtain chiral spectrum in our spacetime is to consider topologically nontrivial configurations in the extra dimensions.[1] Owing to the index theorem [8], topological charge of the background provides the index of the Dirac operator, i.e., the difference of the numbers of chiral zero modes, which then produce massless chiral fermions in our spacetime. Generalizations of the index theorem to matrix models or noncommutative spaces are, however, mostly formulated in spaces with an infinite size, and it is widely believed that topological charges cannot be defined in a system with finite degrees of freedom.

The situation is similar to the lattice gauge theories, where the theory is defined on a finite number of lattice points. There a problem to properly define the chiral symmetry and the index theorem arises due to the doubling problem [9]. The problem has been solved successfully by introducing Dirac operators satisfying a Ginsparg-Wilson (GW) relation [10]. While all the gauge field configurations are continuously connected and there seems to be no room for defining separate topological sectors, the configuration space becomes disconnected by introducing the admissibility condition and the various topological sectors can then be realized [11].

The ideas of using the GW relation were applied to matrix models or noncommutative geometries. In Ref. [12], we have provided a general prescription to construct a GW Dirac operator with coupling to background gauge fields. As a concrete example, a GW Dirac operator on a fuzzy 2-sphere [13] was given.[2] As topologically nontrivial configurations, 't Hooft–Polyakov (TP) monopole configurations were introduced [15,16], and an index theorem for those backgrounds was formulated by introducing a projection operator [17]. This index theorem was further extended to general configurations, which enabled us to define all of the topological sectors in a single theory [17,18].

While our formulation has been given so far to fermionic fields with the fundamental representation of the gauge group, the matrix models of superstrings, such as the IIB matrix model, have fermions with the adjoint representation. It is then desirable to provide formulations for the adjoint matter. Since it is a highly delicate problem to formulate GW relations in each concrete case, we will study it in this paper. We further extend our formulation to configurations where the $U(\sum_p k_p)$ gauge symmetry is broken down to $\prod_p U(k_p)$, which seem phenomenologically interesting.

The formulations using the GW relation provide a firm foundation for studying the above mentioned mechanism of obtaining chiral fermions by embedding topological configurations in the extra dimensions. Indeed, the GW relation ensures the existence of chiral zero modes against any perturbations since the index is a topological quantity. However, one should study carefully whether the chiral zero modes in the extra dimensions really give chiral spectrum in our spacetime. By considering TP monopole-type configurations, where the gauge symmetry is broken

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[1]Having this mechanism in mind, we analyzed dynamics of a model on a fuzzy 2-sphere and showed that topologically nontrivial configurations are indeed realized [5]. Models of four-dimensional field theory with fuzzy extra dimensions were studied in [6,7].

[2]A GW Dirac operator without gauge field backgrounds was given earlier in [14].

down to a smaller one, bifundamental fermions are obtained from an adjoint one, but fields with the conjugate representations arise in pairs. Whether they give chiral spectrum in our spacetime in total is a problem and will be also discussed in this paper.

In Sec. II, we formulate the GW relation for matter in the adjoint representation of the gauge group. In Sec. III, we introduce TP monopole configurations and provide the index theorem for those backgrounds. We then extend it to general configurations in Sec. IV. We study configurations with $U(\sum_p k_p)/\prod_p U(k_p)$ in Sec. V. In Sec. VI, we discuss whether topological configurations in the extra dimensions really provide chiral fermions in our spacetime. Section VII is devoted to conclusions and discussions. In Appendix A, we show calculations for taking the commutative limits of the Dirac operator and the topological charge. In Appendix B, we study general configurations with $U(\sum_p k_p)/\prod_p U(k_p)$. In Appendix C, we study the charge conjugation and the Majorana condition in ten dimensions in detail.

II. GW RELATION ON FUZZY S^2 WITH ADJOINT MATTER

In this section, we provide a GW Dirac operator and an index theorem for matter in the adjoint representation of the gauge group, by following the general prescription given in [12].

Noncommutative coordinates of a fuzzy 2-sphere are given by $x_i = \alpha L_i$, where α is a noncommutative parameter, and L_i is the n-dimensional irreducible representation matrix of the $SU(2)$ algebra. One then has the relation $(x_i)^2 = \alpha^2 \frac{n^2-1}{4} \mathbb{1}_n = \rho^2 \mathbb{1}_n$, where $\rho = \alpha\sqrt{(n^2-1)/4}$ expresses the radius of the sphere. The commutative limit is taken by $\alpha \to 0$, $n \to \infty$ with ρ fixed.

In our formulation of the GW relation, we first define two chirality operators as[3]

$$\Gamma = \frac{H_r}{\sqrt{(H_r)^2}}, \qquad H_r = \sigma_i A_i^R - \frac{1}{2}, \tag{2.1}$$

$$\hat{\Gamma} = \frac{H_l}{\sqrt{(H_l)^2}}, \qquad H_l = \sigma_i A_i^L + \frac{1}{2}, \tag{2.2}$$

with covariant coordinates

$$A_i = L_i + \rho a_i. \tag{2.3}$$

The superscript R (L) in A_i^R (A_i^L) means that this operator acts from the right (left) on matrices: $A^L M \equiv AM$, $A^R M \equiv MA$. The matrices σ_i are the Pauli matrices acting on the spinor indices, and the matrices a_i in (2.3) represent the gauge fields. $U(k)$ gauge symmetry is introduced by taking $L_i = L_i \otimes \mathbb{1}_k$ and $a_i = a_i^a t^a$ in (2.3), where t^a's are the generators of $U(k)$ and a_i^a's are functions of the coordinates L_i.

The gauge transformation for the fermionic fields ψ in the adjoint representation is given by

$$\psi \to U\psi U^\dagger, \tag{2.4}$$

where U is $U(nk)$ matrices. The gauge field a_i is transformed as $a_i \to U a_i U^\dagger + \frac{1}{\rho}(U L_i U^\dagger - L_i)$, so that the covariant coordinate A_i is transformed as

$$A_i \to U A_i U^\dagger. \tag{2.5}$$

Hence, both $\Gamma\psi$ and $\hat{\Gamma}\psi$ are transformed covariantly as $\Gamma\psi \to U\Gamma\psi U^\dagger$ and $\hat{\Gamma}\psi \to U\hat{\Gamma}\psi U^\dagger$, where a relation $(AB)^R\psi = B^R A^R \psi = \psi AB$ was used.

The chirality operators (2.1) and (2.2) satisfy

$$\Gamma^\dagger = \Gamma, \qquad \hat{\Gamma}^\dagger = \hat{\Gamma}, \qquad \Gamma^2 = \hat{\Gamma}^2 = 1. \tag{2.6}$$

In the commutative limit, both Γ and $\hat{\Gamma}$ become the chirality operator on the commutative 2-sphere, $\gamma = n_i\sigma_i$, where $n_i = x_i/\rho$ is a unit vector.

We then define a GW Dirac operator as

$$D_{\rm GW} = -a^{-1}\Gamma(1 - \Gamma\hat{\Gamma}), \tag{2.7}$$

where $a = 2/n$ is a noncummutative analog of the lattice spacing. By the definition, a GW relation

$$\Gamma D_{\rm GW} + D_{\rm GW}\hat{\Gamma} = 0 \tag{2.8}$$

is satisfied. Hence, the index, i.e., the difference of the numbers of the chiral zero modes, is given by the trace of the chirality operators as

$$\text{index}\,(D_{\rm GW}) = \tfrac{1}{2}\mathcal{T}r[\Gamma + \hat{\Gamma}], \tag{2.9}$$

where $\mathcal{T}r$ is the trace over the whole configuration space, that is, over the spinor index, the gauge group space, and the matrix space representing the coordinates. Since the definition of Γ and $\hat{\Gamma}$ depends on the gauge fields a_i, the right-hand side (rhs) of (2.9) is a functional of the gauge field configurations. It also takes only integer values. It then gives a noncommutative generalization of the topological charge of the gauge field backgrounds. Thus, Eq. (2.9) gives an index theorem on the fuzzy 2-sphere.

In the commutative limit, the GW Dirac operator (2.7) becomes

$$D_{\rm GW} \to \sigma_i(\mathcal{L}_i + \rho P_{ij}\tilde{a}_j) + 1, \tag{2.10}$$

as will be shown in Appendix A. Here $\mathcal{L}_i = -i\epsilon_{ijk}x_j\partial_k$ is the derivative operator along the Killing vectors on the sphere, $\tilde{a}_i$ is the adjoint operator of a_i, i.e., $\tilde{a}_i\psi = [a_i, \psi]$, and $P_{ij} = \delta_{ij} - n_i n_j$ is the projector to the tangential directions on the sphere. The gauge fields a_i can be decomposed into the tangential components on the sphere a_i' and the normal component ϕ as

$$a_i' = \epsilon_{ijk}n_j a_k, \qquad \phi = n_i a_i, \tag{2.11}$$

[3] In the case of fundamental matter, we took $\Gamma = a(\sigma_i L_i^R - \frac{1}{2})$ instead of (2.1), where $a = 2/n$ is a noncommutative analog of the lattice spacing. $\hat{\Gamma}$ was identical with (2.2).

$$\Leftrightarrow a_i = -\epsilon_{ijk} n_j a'_k + n_i \phi. \tag{2.12}$$

The normal component ϕ is a scalar field on the sphere. The operator (2.10) is the Dirac operator of the adjoint matter on the commutative 2-sphere without a coupling to the scalar field ϕ. The absence of the Yukawa coupling is reasonable since such a coupling would violate the chiral symmetry on the sphere and contradict with the GW relation.

The commutative limit of the topological charge, the rhs of (2.9), becomes

$$\frac{1}{2}\mathcal{T}r[\Gamma + \hat{\Gamma}] \to -\rho^2 \int \frac{d\Omega}{4\pi} \,\mathrm{tr}(\epsilon_{ijk} n_k F_{ij}) + \rho^2 \int \frac{d\Omega}{4\pi} \,\mathrm{tr}(\epsilon_{ijk} n_k F_{ij}), \tag{2.13}$$

as shown in Appendix A. Here tr is the trace over the gauge group space, and the field strength F_{ij} is defined as $F_{ij} = \partial_i a'_j - \partial_j a'_i - i[a'_i, a'_j]$ with a'_i given in (2.11). The first and the second terms on the rhs of (2.13) come from $\mathcal{T}r[\Gamma]$ and $\mathcal{T}r[\hat{\Gamma}]$, respectively. Each term gives the integral of the 1st Chern character on the commutative 2-sphere. They cancel each other and vanish for any gauge field configurations, which is appropriate since we now consider the adjoint matter.

In summary, our formulation manifestly has the gauge invariance and the $SO(3)$ Poincare invariance on the fuzzy 2-sphere. Because of the GW relation, the index theorem (2.9) is satisfied, and the topological charge, the rhs of (2.9), takes only integer values. The commutative limits of the chirality operators, the Dirac operator, and the topological charge have the correct forms.

III. TP MONOPOLE CONFIGURATIONS

As topologically nontrivial configurations in the $U(2)$ gauge theory on the fuzzy 2-sphere, the following configurations were provided [15,16]:

$$A_i = \begin{pmatrix} L_i^{(n+m)} & \\ & L_i^{(n-m)} \end{pmatrix}, \tag{3.1}$$

where A_i is the covariant coordinate (2.3), and $L_i^{(n\pm m)}$ are the $(n \pm m)$-dimensional irreducible representations of the $SU(2)$ algebra. The $m = 0$ case corresponds to two coincident fuzzy 2-spheres, whose effective action is the $U(2)$ gauge theory. The cases with general m correspond to two fuzzy 2-spheres with different radii. They correspond to the TP monopole configurations with magnetic charge $-|m|$, where the $U(2)$ gauge symmetry is broken down to $U(1) \times U(1)$.

For the $m = 1$ case, (3.1) is unitarily equivalent to

$$A_i \doteq L_i^{(n)} \otimes \mathbb{1}_2 + \mathbb{1}_n \otimes \frac{\tau_i}{2}. \tag{3.2}$$

Comparing with (2.3), the gauge field is

$$a_i = \frac{1}{\rho} \mathbb{1}_n \otimes \frac{\tau_i}{2}. \tag{3.3}$$

By taking the commutative limit and making the decomposition (2.11), we obtain

$$a_i'^a = \frac{1}{\rho}\epsilon_{ija} n_j, \qquad \phi^a = \frac{1}{\rho} n_a, \tag{3.4}$$

which is precisely the TP monopole configuration [16].

We now define projection operators $P^{(\pm)}$ to pick up the $n \pm |m|$-dimensional spaces that the operator (3.1) acts. It is written as

$$P^{(\pm)} = \tfrac{1}{2}(1 \pm T), \tag{3.5}$$

with

$$T = \frac{2}{n|m|}\left(A_i^2 - \frac{n^2 + m^2 - 1}{4}\right) \tag{3.6}$$

$$= \frac{m}{|m|}\begin{pmatrix} \mathbb{1}_{n+m} & \\ & -\mathbb{1}_{n-m} \end{pmatrix}. \tag{3.7}$$

Since T commutes with the chirality operators and the Dirac operator, the index theorem (2.9) is satisfied in each space projected by $P^{(\pm)}$ as

$$\mathrm{index}\,(P^{(\pm)L}P^{(\pm)R}D_{\mathrm{GW}}) = \tfrac{1}{2}\mathcal{T}r[P^{(\pm)L}P^{(\pm)R}(\Gamma + \hat{\Gamma})], \tag{3.8}$$

where the superscript L (R) means that the operator acts from the left (right) on matrices as before. The $\pm$ signs in $P^{(\pm)L}$ and $P^{(\pm)R}$ do not necessarily coincide. Each sign combination picks up one of the following blocks in the fermionic field ψ in the adjoint representation:

$$\psi = \begin{pmatrix} \psi^{(++)} & \psi^{(+-)} \\ \psi^{(-+)} & \psi^{(--)} \end{pmatrix} \tag{3.9}$$

for $m > 0$, if we decompose ψ into the blocks in the same way as (3.1). The signs in (3.9) should be reversed for $m < 0$.

For the backgrounds (3.1), the rhs of (3.8) becomes

$$\frac{1}{2}\mathcal{T}r[P^{(\pm)L}P^{(\pm)R}(\Gamma+\hat{\Gamma})] = \begin{cases} 0 & \text{for } \psi^{(++)}, \psi^{(--)}, \\ -2|m| & \text{for } \psi^{(+-)}, \\ 2|m| & \text{for } \psi^{(-+)}, \end{cases} \tag{3.10}$$

as shown by the following calculations: For (3.1), the chirality operator $\hat{\Gamma}$ becomes

$$\hat{\Gamma} = \begin{pmatrix} \frac{2}{n+m}(\sigma \cdot L^{(n+m)} + \frac{1}{2}) & \\ & \frac{2}{n-m}(\sigma \cdot L^{(n-m)} + \frac{1}{2}) \end{pmatrix}. \tag{3.11}$$

Since the terms with $\sigma \cdot L$ vanish after taking the trace, we obtain

$$\mathcal{T}r[P^{(\pm)L}P^{(\pm)R}\hat{\Gamma}] = \mathrm{Tr}_{L,\sigma}[P^{(\pm)L}\hat{\Gamma}] \cdot \mathrm{Tr}_R[P^{(\pm)R}] = \frac{1}{n \pm |m|} 2(n \pm |m|) \cdot (n \pm |m|) = 2(n \pm |m|), \quad (3.12)$$

where $\mathrm{Tr}_{L,\sigma}$ is the trace over the space on which A_i^L and σ_i act, and Tr_R is the trace over the space on which A_i^R act. The $\pm$ sign in the last line refers to that in $P^{(\pm)R}$. Similarly, we can show

$$\mathcal{T}r[P^{(\pm)L}P^{(\pm)R}\Gamma] = -2(n \pm |m|), \quad (3.13)$$

where the $\pm$ sign in the rhs refers to that in $P^{(\pm)L}$. By adding (3.12) and (3.13), we obtain (3.10).

We now give an interpretation for (3.10). In the representation (2.3), (3.6) is written as

$$T = \frac{2}{n|m|}\left(\rho\{L_i, a_i\} + \rho^2 a_i^2 - \frac{m^2}{4}\right). \quad (3.14)$$

In the commutative limit, T becomes $\frac{2\rho}{|m|}\phi$ where ϕ is the scalar field defined in (2.11). It is also normalized as $T^2 = \mathbb{1}_{2n}$. Then, T corresponds to a normalized scalar field. Recalling that the TP monopole configuration breaks the $SU(2)$ gauge symmetry down to $U(1)$, T is the generator of this unbroken $U(1)$, the electric charge operator of the unbroken $U(1)$. [The $U(1)$ of $U(2) \simeq SU(2) \times U(1)$ is ignored since it is decoupled in the commutative limit.] By the gauge symmetry braking $SU(2)/U(1)$, fields with various electric charges of the unbroken $U(1)$ arise. Equation (3.8) gives the index theorem for each field.

For instance, $\psi^{(++)}$ in (3.9) is in the adjoint representation of the unbroken $U(1)$ with electric charge $+1/2$, and it has a vanishing index. On the other hand, $\psi^{(+-)}$ is in the bifundamental representation of the unbroken $U(1)$ with charge $+1/2$ and $-1/2$, that is, the fundamental representation with charge $+1$. It therefore has the index $-2|m|$. Although the whole fermionic field ψ has a vanishing index since it is in the adjoint representation, the field in each projected block can have nonzero index. As was shown in (2.13), topological charge is an analog of the 1st Chern character, which is proportional to the electric charge of the matter. Then, $\psi^{(+-)}$ and $\psi^{(-+)}$, having the opposite electric charge, have the opposite topological charge and the opposite index.

We finally give two comments. First, we can define a topological charge multiplied by the electric charge, such as

$$\frac{1}{16}\mathcal{T}r[(T^L - T^R)(\Gamma + \hat{\Gamma})], \quad (3.15)$$

so that contributions from the blocks in (3.9) do not cancel but are added. By using the result (3.10), (3.15) becomes $-|m|$ for the backgrounds (3.1), which agrees with the topological charge of the TP monopoles. We will develop this argument further in the next section.

Second, as seen above, fermions in the conjugate representations under the unbroken gauge group have opposite indices if one considers topological configurations in two dimensions, or more generally, in 2 (mod 4) dimensions. We can then expect that by embedding these configurations in the extra dimensions, chiral spectrum is obtained in our spacetime in low energy effective theory. We will discuss this issue in Sec. VI.

IV. GENERAL CONFIGURATIONS WITH $U(2)/U(1)^2$

We now extend the formulation in the previous section to general configurations where the $U(2)$ gauge group is broken down to $U(1) \times U(1)$ through the Higgs mechanism, i.e., a nonzero vacuum expectation value of the scalar field. This will enable us to survey the whole configuration space with all topological sectors.

Since the definition of the electric charge operator T in (3.6) was specific to the backgrounds (3.1), we first generalize it as

$$T' = \frac{(A_i)^2 - \frac{n^2-1}{4}}{\sqrt{[(A_i)^2 - \frac{n^2-1}{4}]^2}}. \quad (4.1)$$

This is valid for general configurations A_i unless the denominator has zero modes. For the configurations (3.1), T' reduces to the previous one (3.7). For general configurations

$$(T')^\dagger = T', \qquad (T')^2 = 1 \quad (4.2)$$

are satisfied. The commutative limit of T' becomes the normalized scalar field as

$$T' \to 2\phi' = 2\phi'^a \frac{\tau^a}{2}, \quad (4.3)$$

where ϕ' is normalized as $\sum_a (\phi'^a)^2 = 1$.

We next define modified chirality operators as

$$\Gamma'_r = \frac{\{T'^R, \Gamma\}}{\sqrt{\{T'^R, \Gamma\}^2}}, \quad (4.4)$$

$$\hat{\Gamma}'_r = T'^R\hat{\Gamma}, \quad (4.5)$$

$$\Gamma'_l = T'^L\Gamma, \quad (4.6)$$

$$\hat{\Gamma}'_l = \frac{\{T'^L, \hat{\Gamma}\}}{\sqrt{\{T'^L, \hat{\Gamma}\}^2}}, \quad (4.7)$$

where Γ and $\hat{\Gamma}$ are defined in (2.1) and (2.2). The superscript R (L) in T'^R (T'^L) means that this operator acts from right (left) on matrices. The chirality operators satisfy the relations

$$(\Gamma'_r)^\dagger = \Gamma'_r, \quad (\hat{\Gamma}'_r)^\dagger = \hat{\Gamma}'_r, \quad (\Gamma'_r)^2 = (\hat{\Gamma}'_r)^2 = 1, \quad (4.8)$$

$$(\Gamma_l')^\dagger = \Gamma_l', \quad (\hat{\Gamma}_l')^\dagger = \hat{\Gamma}_l', \quad (\Gamma_l')^2 = (\hat{\Gamma}_l')^2 = 1. \tag{4.9}$$

Since the chirality operators are weighted by the electric charge operator T', the commutative limits of Γ_r' and $\hat{\Gamma}_r'$ become $\gamma_r' = t^R\gamma$, and those of Γ_l' and $\hat{\Gamma}_l'$ become $\gamma_l' = t^L\gamma$. Here t is the electric charge operator of the unbroken $U(1)$ gauge group, the superscript R (L) means that the operator acts from right (left) in the gauge group space, and $\gamma = n\cdot\sigma$ is the chirality operator on the 2-sphere.

We then define modified GW Dirac operators as

$$D_r' = -a^{-1}\Gamma_r'(1 - \Gamma_r'\hat{\Gamma}_r'), \tag{4.10}$$

$$D_l' = -a^{-1}\Gamma_l'(1 - \Gamma_l'\hat{\Gamma}_l'). \tag{4.11}$$

By definition, these Dirac operators satisfy GW relations

$$\Gamma_r' D_r' + D_r'\hat{\Gamma}_r' = 0, \tag{4.12}$$

$$\Gamma_l' D_l' + D_l'\hat{\Gamma}_l' = 0. \tag{4.13}$$

Then, index theorems

$$\text{index}\,(D_r') = \tfrac{1}{2}\mathcal{T}r[\Gamma_r' + \hat{\Gamma}_r'], \tag{4.14}$$

$$\text{index}\,(D_l') = \tfrac{1}{2}\mathcal{T}r[\Gamma_l' + \hat{\Gamma}_l'], \tag{4.15}$$

are satisfied as well. By using the rhs of (4.14) and (4.15), we can also define a topological charge

$$\frac{1}{16}\mathcal{T}r[\Gamma_l' + \hat{\Gamma}_l' - \Gamma_r' - \hat{\Gamma}_r'], \tag{4.16}$$

which is a generalization of (3.15).

For the configurations (3.1), since the generalized electric charge operator (4.1) reduces to the previous one (3.7), we can calculate the rhs of (4.14) and (4.15) as we did below (3.10), giving

$$\tfrac{1}{2}\mathcal{T}r[\Gamma_r' + \hat{\Gamma}_r'] = 4|m|, \tag{4.17}$$

$$\tfrac{1}{2}\mathcal{T}r[\Gamma_l' + \hat{\Gamma}_l'] = -4|m|. \tag{4.18}$$

In (3.10), $\psi^{(+-)}$ and $\psi^{(-+)}$ have index $-2|m|$ and $2|m|$, respectively. However, since the chirality operators Γ_r' and $\hat{\Gamma}_r'$ are multiplied by -1 for $\psi^{(+-)}$, we obtain (4.17). Equation (4.18) is obtained similarly. From (4.17) and (4.18), the topological charge (4.16) becomes $-|m|$, as expected since (3.15) gave $-|m|$.

In the commutative limit, the GW Dirac operator (4.10) becomes

$$D_r' \to \tfrac{1}{2}\{2\phi'^R, (\sigma_i\mathcal{L}_i + 1)\} + \tfrac{1}{2}\{2\phi'^R, \rho\sigma_i P_{ij}a_j^L\} - \tfrac{1}{2}\{2\phi'^R, \rho\sigma_i P_{ij}a_j^R\}, \tag{4.19}$$

where the superscript R (L) means that the operator acts from right (left) in the gauge group space: $\phi'^R = \phi'^a(\Omega)\times\frac{(\tau^a)^R}{2}$, etc. In the $\phi'^a(\Omega) = (0,0,1)$ gauge, (4.19) becomes

$$(\tau^3)^R\Big(\sigma_i\mathcal{L}_i + 1 + \rho\sigma_i P_{ij}\Big(a_j^3\frac{\tilde{\tau}^3}{2} + a_j^1\frac{(\tau^1)^L}{2} + a_j^2\frac{(\tau^2)^L}{2}\Big)\Big) \equiv D_{r,\text{com}}', \tag{4.20}$$

where $\tilde{\tau}^3$ means the adjoint operator of τ^3. This Dirac operator indeed has the adjoint coupling of the unbroken $U(1)$ gauge field a_i^3. It also satisfies a chiral relation

$$\{D_{r,\text{com}}', \gamma_r'\} = 0, \tag{4.21}$$

with $\gamma_r' = (\tau^3)^R\gamma$ the chirality operator multiplied by the unbroken $U(1)$ charge, as expected from the GW relation (4.12). The same arguments hold also for D_l'.

Our remarkable result is that, by the same calculations in (2.13), the commutative limit of the rhs in (4.14) becomes

$$\frac{1}{2}\mathcal{T}r[\Gamma_r' + \hat{\Gamma}_r'] \to -4\frac{\rho^2}{8\pi}\int d\Omega\,\epsilon_{ijk}n_i(\phi'^a F_{jk}^a - \epsilon_{abc}\phi'^a(D_j\phi')^b(D_k\phi')^c), \tag{4.22}$$

where $F_{jk} = F_{jk}^a\tau^a/2$ is the field strength defined as $F_{jk} = \partial_j a_k' - \partial_k a_j' - i[a_j', a_k']$, and D_j is the covariant derivative defined as $D_j = \partial_j - i[a_j', \,]$, with a_j' given in (2.11). As $\mathcal{T}r(\hat{\Gamma})$ gave the second term in the rhs of (2.13), $\mathcal{T}r(\hat{\Gamma}_r')$ gives a similar term, but with $\text{Tr}_R(\mathbb{1}) = 2n$ replaced by $\text{Tr}_R(T'^R) \sim 2m$, giving an extra $1/n$ factor. Then, $\mathcal{T}r(\hat{\Gamma}_r')$ does not contribute to the commutative limit. On the other hand, $\mathcal{T}r(\Gamma_r')$ gives a similar term as the first term in the rhs of (2.13), but with the T'^R in the same trace. Moreover, as shown in Ref. [18], the denominator in (4.4) yields the second term on the rhs of (4.22).

Similarly, we obtain

$$\frac{1}{2}\mathcal{T}r[\Gamma_l' + \hat{\Gamma}_l'] \to 4\frac{\rho^2}{8\pi}\int d\Omega\,\epsilon_{ijk}n_i(\phi'^a F_{jk}^a - \epsilon_{abc}\phi'^a(D_j\phi')^b(D_k\phi')^c). \tag{4.23}$$

Equations (4.22) and (4.23) are precisely the topological charge given by 't Hooft [19], multiplied by ∓ 4, respectively. Since each of (4.22) and (4.23) has contributions from $\psi^{(+-)}$ and $\psi^{(-+)}$, and their electric charge is twice the usual case, the result is multiplied by ∓ 4.

V. CONFIGURATIONS WITH $U(\sum_p k_p)/\prod_p U(k_p)$

We now consider configurations as follows:

$$A_i = \begin{pmatrix} L_i^{(n_1)}\otimes \mathbb{1}_{k_1} & & & \\ & L_i^{(n_2)}\otimes \mathbb{1}_{k_2} & & \\ & & \ddots & \\ & & & L_i^{(n_h)}\otimes \mathbb{1}_{k_h} \end{pmatrix}, \tag{5.1}$$

where the gauge symmetry $U(\sum_{p=1}^h k_p)$, which the configurations $A_i = L_i \otimes \mathbb{1}_{\sum_{p=1}^h k_p}$ would have, is broken down to $\prod_{p=1}^{h} U(k_p)$. They are a generalization of the configurations (3.1) with $U(2)/U(1)^2$. They are phenomenologically attractive since they have gauge group close to that of the standard model.[4] Such configurations are also used for embedding fiber bundles in matrix models [21]. We here study whether index theorems can be formulated in these backgrounds as before.

We then define projection operators as

$$P_p = \begin{pmatrix} 0_{\sum_{q=1}^{p-1} n_q k_q} & & \\ & \mathbb{1}_{n_p k_p} & \\ & & 0_{\sum_{q=p+1}^{h} n_q k_q} \end{pmatrix} \tag{5.2}$$

for $p = 1, \ldots, h$, which pick up the pth block with dimensions $n_p k_p$. Since the projection operators (5.2) commute with the chirality operators and the Dirac operator, the index theorem (2.9) is satisfied in each projected space as

$$\mathrm{index}\,(P_p^L P_q^R D_{\mathrm{GW}}) = \tfrac{1}{2}\mathcal{T}r[P_p^L P_q^R(\Gamma + \hat{\Gamma})] \tag{5.3}$$

for $1 \le p, q \le h$. Here Γ, $\hat{\Gamma}$, and D_{GW} are defined in (2.1), (2.2), and (2.7), and the superscript L (R) means that the operator acts from the left (right).

For the backgrounds (5.1), the rhs of (5.3) becomes

$$\tfrac{1}{2}\mathcal{T}r[P_p^L P_q^R(\Gamma + \hat{\Gamma})] = -k_p k_q (n_p - n_q), \tag{5.4}$$

by following the same calculations below (3.10). For $h = 2$ and $k_1 = k_2 = 1$, this reproduces the previous result (3.10). Since the field projected by P_p^L and P_q^R is in the bifundamental representation $(k_p, \bar{k}_q)$ of the unbroken gauge group $U(k_p) \times U(k_q)$, its index is multiplied by $k_p k_q$.

We can also extend the formulation to general configurations. As in (4.1), we define electric charge operators of the unbroken $U(1)$'s as

$$T'_p = \frac{(A_i)^2 - c_p}{\sqrt{[(A_i)^2 - c_p]^2}} \tag{5.5}$$

for $p = 1, \ldots, h-1$. The numbers c_p are taken between

$$\frac{n_p^2 - 1}{4} > c_p > \frac{n_{p+1}^2 - 1}{4},$$

where we assume $n_1 > n_2 > \cdots > n_h$. For the configurations (5.1), T'_p becomes

$$\begin{pmatrix} \mathbb{1}_{\sum_{q=1}^{p} n_q k_q} & \\ & -\mathbb{1}_{\sum_{q=p+1}^{h} n_q k_q} \end{pmatrix}. \tag{5.6}$$

They are the generators of $U(1)$'s contained in the unbroken gauge group $\prod_p U(k_p)$. Note that there exist the grand unified theory monopoles when a simple gauge group is broken down to a smaller group containing $U(1)$ factors. We then define modified chirality operators as (4.4)–(4.7), for each T'_p with $p = 1, \ldots, h-1$. GW Dirac operators, GW relations, and index theorems are defined as (4.10)–(4.15). As we show in Appendix B, the commutative limits of the GW Dirac operators and the topological charges have similar forms as (4.19)–(4.23).

[4]A phenomenological study based on such configurations was given in [20].

VI. EMBEDDINGS IN IIB MATRIX MODEL

As we mentioned in the Introduction, when topologically nontrivial configurations are embedded in the extra dimensions in the matrix model formulations of superstring theory, such as the IIB matrix model, chiral fermions can be obtained in our spacetime. In this section, we discuss whether this mechanism really works or not.

A. $M^4 \times X^n \subset M^{4+n}$

Let us first consider general cases, theories in $(4+n)$-dimensional Minkowski space M^{4+n}, compactified to n-dimensional space X^n with Euclidean signature, while M^4 is our spacetime with Lorentzian signature. We then embed n-dimensional topological configurations in X^n. In particular, we assume configurations of the TP monopole type, where the gauge symmetry is broken down, which yields fields that are in the conjugate representations under the unbroken gauge group. We now denote them as $\psi^{(r)}$ and $\psi^{(\bar{r})}$, which correspond to $\psi^{(+-)}$ and $\psi^{(-+)}$ in (3.9).

For $n = 2 \pmod 4$, as we mentioned at the end of Sec. III, topological charge becomes an analog of the lth Chern character with $l = n/2$ an odd integer, which gives $\psi^{(r)}$ and $\psi^{(\bar{r})}$ opposite indices. We denote the corresponding chiral zero modes as $\psi_R^{(r)}$ and $\psi_L^{(\bar{r})}$, where the subscripts R and L stand for the chirality. [Choosing $\psi_L^{(r)}$ and $\psi_R^{(\bar{r})}$ instead would give the identical results below.] Taking spinors φ in M^4 as well, we obtain four possible Weyl spinors as follows:

$$\varphi_R \otimes \psi_R^{(r)}, \tag{6.1}$$

$$\varphi_L \otimes \psi_L^{(\bar{r})}, \tag{6.2}$$

$$\varphi_L \otimes \psi_R^{(r)}, \tag{6.3}$$

$$\varphi_R \otimes \psi_L^{(\bar{r})}. \tag{6.4}$$

The spinors (6.1) and (6.2) are in the charge conjugate representations to each other. So are (6.3) and (6.4). Here one should note that Weyl spinors in Lorentzian and Euclidean spaces are as shown in Table I.

If we consider chiral theories in M^{4+n} originally, (6.1) and (6.2) are chosen. [Choosing (6.3) and (6.4) would give the identical results.] Since φ_R in (6.1) and φ_L in (6.2) are in the different representations of the gauge group, we obtain chiral spectrum in M^4, although we have a doubling of (6.1) and (6.2). If we further impose the Majorana

TABLE I. Weyl representations of $SO(d-1,1)$ and $SO(d)$.

	$SO(d-1,1)$	$SO(d)$
$d=0$ (mod 4)	Complex	Self-conjugate
$d=2$ (mod 4)	Self-conjugate	Complex

condition in M^{4+n}, which is possible for $4+n=2$ (mod 8), (6.1) and (6.2) are identified and the doubling problem is resolved.

On the contrary, for $n=0$ (mod 4), topological configurations give $\psi^{(r)}$ and $\psi^{(\bar{r})}$ the same index. We denote the corresponding chiral zero modes as $\psi_R^{(r)}$ and $\psi_R^{(\bar{r})}$. Taking spinors φ in M^4 as well, we obtain

$$\varphi_R \otimes \psi_R^{(r)}, \tag{6.5}$$

$$\varphi_L \otimes \psi_R^{(\bar{r})}, \tag{6.6}$$

$$\varphi_L \otimes \psi_R^{(r)}, \tag{6.7}$$

$$\varphi_R \otimes \psi_R^{(\bar{r})}. \tag{6.8}$$

The spinors (6.5) and (6.6) are in the charge conjugate representations. So are (6.7) and (6.8). If we consider chiral theories in M^{4+n} originally, (6.5) and (6.8) are chosen. Since φ_R in (6.5) and φ_R in (6.8) are in the conjugate representations of the gauge group to each other, we are left with nonchiral spectrum in M^4. Even if we consider the Majorana fermions in M^{4+n} instead, we obtain a nonchiral spectrum in M^4.

B. $M^4 \times S^2 \times S^2$ in IIB matrix model

We now move to the IIB matrix model. The action of the IIB matrix model is given by

$$S_{\mathrm{IIBMM}} = -\frac{1}{g^2}\mathrm{Tr}\bigg(\frac{1}{4}[A_M, A_N][A^M, A^N] + \frac{1}{2}\bar{\psi}\Gamma^M[A_M, \psi]\bigg), \tag{6.9}$$

where A_M is a ten-dimensional vector, ψ is a ten-dimensional Majorana-Weyl spinor[5] and they are also traceless Hermitian matrices. Since the action is written in terms of the commutators, matter in the adjoint representation appears naturally.

As an application of what we studied about the fuzzy 2-sphere in this paper, let us consider a compactification to $M^4\times S^2\times S^2$ and an embedding of the following configurations:

[5]They are Wick rotated to the $SO(10)$ vector and spinor. In this paper, however, we use Lorentzian notation, such as M^{10}, since we discuss spinors.

$$\begin{aligned} A_\mu &= x_\mu \otimes \mathbb{1}_{n_1^1 n_1^2 + n_2^1 n_2^2}, \\ A_i &= \mathbb{1} \otimes \begin{pmatrix} L_i^{(n_1^1)} \otimes \mathbb{1}_{n_1^2} & \\ & L_i^{(n_2^1)} \otimes \mathbb{1}_{n_2^2} \end{pmatrix}, \\ A_j &= \mathbb{1} \otimes \begin{pmatrix} \mathbb{1}_{n_1^1} \otimes L_j^{(n_1^2)} & \\ & \mathbb{1}_{n_2^1} \otimes L_j^{(n_2^2)} \end{pmatrix}, \end{aligned} \tag{6.10}$$

where $\mu = 0, 1, 2, 3$, $i = 4, 5, 6$, and $j = 7, 8, 9$. x_μ is our spacetime background. Either commutative backgrounds as $[x_\mu, x_\nu] = 0$ or noncommutative backgrounds as $[x_\mu, x_\nu] = i\theta_{\mu\nu}$ can be considered.[6]

The second factor in (6.10)[7] represents monopole configurations wrapping around $S^2\times S^2$. The off-diagonal blocks of matter, $\psi^{(+-)}$ and $\psi^{(-+)}$ in (3.9), are in the conjugate representations of the unbroken gauge group. We now write them as $\psi^{(r)}$ and $\psi^{(\bar{r})}$. Since the topological configurations in four-dimensional $S^2\times S^2$ give $\psi^{(r)}$ and $\psi^{(\bar{r})}$ the same index, we denote the corresponding chiral zero modes as $\psi_R^{(r)}$ and $\psi_R^{(\bar{r})}$.

We now introduce the following Dirac gamma matrices in M^{10}, which are suitable for $M^4\times S^2\times S^2$:

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_3, \\ \Gamma_i &= \mathbb{1}_4 \otimes \sigma_i \otimes \mathbb{1}_2 \otimes \sigma_1, \\ \Gamma_j &= \mathbb{1}_4 \otimes \mathbb{1}_2 \otimes \sigma_j \otimes \sigma_2, \end{aligned} \tag{6.11}$$

where γ_μ is the gamma matrices in M^4. The second and the third factors act on spinors on $S^2\times S^2$, such as the chiral zero modes $\psi_R^{(r)}$ and $\psi_R^{(\bar{r})}$. Besides the spinors φ in M^4, we should also introduce spinors χ on which the final factor acts. We then obtain the following possible Weyl spinors:

$$\varphi_R \otimes \psi_R^{(r)} \otimes \chi_R, \qquad \varphi_L \otimes \psi_R^{(\bar{r})} \otimes \chi_L, \tag{6.12}$$

$$\varphi_L \otimes \psi_R^{(r)} \otimes \chi_L, \qquad \varphi_R \otimes \psi_R^{(\bar{r})} \otimes \chi_R, \tag{6.13}$$

$$\varphi_R \otimes \psi_R^{(r)} \otimes \chi_L, \qquad \varphi_L \otimes \psi_R^{(\bar{r})} \otimes \chi_R, \tag{6.14}$$

$$\varphi_L \otimes \psi_R^{(r)} \otimes \chi_R, \qquad \varphi_R \otimes \psi_R^{(\bar{r})} \otimes \chi_L. \tag{6.15}$$

The two spinors in (6.12) are in the charge conjugate representations to each other. So are those in (6.13), (6.14), and (6.15). We show it in detail in Appendix C.

[6]Fluctuations around the background (6.10) provide matter fields. Expansions of the action (6.9) give superficially renormalizable theories, but with nonlocality such as noncommutativity. The maximal supersymmetry possessed by the IIB matrix model might suppress peculiar properties caused by the nonlocality, such as the UV/IR mixing.

[7]Similar backgrounds were studied in [7,22].

Since the IIB matrix model has the ten-dimensional Majorana-Weyl spinor, we now impose these conditions. By the Weyl condition, (6.12) and (6.13) or (6.14) and (6.15) are chosen. By the Majorana condition, the two spinors in (6.12), (6.13), (6.14), and (6.15) are identified. We still have two spinors, however. We then obtain nonchiral spectrum.

There are two reasons why we could not obtain chiral spectrum. First, since we now consider four-dimensional topological configurations, the zero modes of the same chirality, $\psi_R^{(r)}$ and $\psi_R^{(\bar{r})}$, are obtained. As the case $M^4 \times X^4 \subset M^8$ gave nonchiral spectrum in M^4, now the first spinor in (6.12) and the second spinor in (6.13) necessarily arise and give nonchiral spectrum.

Second, the *remainder* two dimensions $M^{10}/(M^4 \times S^2 \times S^2)$ interrupt. In the gamma matrices (6.11), the ten-dimensional chirality operator becomes

$$\Gamma_{11} = \gamma_5 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_3. \tag{6.16}$$

Then, even if $\Gamma_{11} = +$ is imposed, both $(\gamma_5, \sigma_3) = (+, +)$ and $(\gamma_5, \sigma_3) = (-, -)$ are allowed. For instance, the first spinor in (6.12) and the first spinor in (6.13) appear.

Actually, the chirality on $S^2 \times S^2$, i.e., whether one takes $\psi_R^{(r)}$ and $\psi_R^{(\bar{r})}$ or $\psi_L^{(r)}$ and $\psi_L^{(\bar{r})}$, gives no difference. Moreover, the chirality on each S^2 is irrelevant. While the chirality operator on S^2 is $\gamma = n \cdot \sigma$, the gamma matrix in the direction normal to S^2 is also $\gamma_\perp = n \cdot \sigma$, and their product gives $\gamma\gamma_\perp = \mathbb{1}_2$ in (6.16). Then, even if one considers a chiral mode on S^2, either $\gamma\psi = +\psi$ or $\gamma\psi = -\psi$, it gives no effect on (6.16).

VII. CONCLUSIONS AND DISCUSSIONS

In this paper, we provided the GW Dirac operators and the index theorems on the fuzzy 2-sphere for matter in the adjoint representation of the gauge group. We extended our formulation to topologically nontrivial configurations, such as the TP monopoles, the general configurations with $U(2)/U(1)^2$, and the configurations with $U(\sum_p k_p)/\prod_p U(k_p)$. We can also extend it to fuzzy $S^2 \times S^2$, $S^2 \times S^2 \times S^2$, and so on. The topological charge defined on fuzzy $(S^2)^l$ in this way gives us a noncommutative generalization of the lth Chern character on $(S^2)^l$, as was shown in [22] for the fundamental matter. We will report on it in a separate paper.

We then studied the embeddings of topological configurations in higher dimensional matrix models, such as the IIB matrix model, and discussed whether chiral spectrum is really obtained in our spacetime. The formulations using the GW relation gave a firm foundation to such studies. The GW relation indeed ensures the existence of chiral zero modes against any variations since the index is a topological quantity. As a practical advantage, we can calculate exact chiral zero modes, not approximate ones. Unfortunately, however, we could not obtain chiral spectrum by the $M^4 \times S^2 \times S^2$ embeddings in the IIB matrix model. We now discuss how to resolve this problem.

One may consider decoupling dynamically one of the fields $\varphi_R \otimes \psi_R^{(r)} \otimes \chi_R$ and $\varphi_R \otimes \psi_R^{(\bar{r})} \otimes \chi_R$. (See, for instance, Ref. [23].) By introducing strong coupling interactions, such as four-Fermi interactions, to only one of them, confinement may take place, which makes all the composites massive and decoupled. The other partner remains chiral and massless. However, introducing those interactions seems artificial and unnatural from the viewpoint that we derive everything from the IIB matrix model, though it is allowed for formulating chiral gauge theories on the lattice as in [23].

A simple way to obtain chiral spectrum in our spacetime is to consider topological configurations in the entire extra six dimensions, as we studied $M^4 \times X^6 \subset M^{10}$ in Sec. VI. Coset space constructions, which cause the "remainder" dimensions, are not suitable for it. Torus is possible to construct in the same way as we did in this paper.[8] Six-dimensional curved spaces can be described within six matrices in the formulation given in [28]. One may also consider situations similar to the intersecting D-branes [29], where one has no remainder dimensions normal to all of the D-branes which are intersecting to one another. By T-duality, those situations are essentially equivalent to the above ones. We can also consider orbifolds in six dimensions [30,31]. Imposing orbifold conditions plays the same role as the topological configurations giving the index. We will report on these studies in future publications.

While we assumed the specific backgrounds in this paper, we can in principle analyze whether such configurations are realized dynamically, as we did in the analyses for the spacetime structures in the IIB matrix model and in the analyses for the fuzzy spheres. From such studies, we might be able to find that the standard model or its extension is obtained as a unique solution from the IIB matrix model or its variants. Or, more complicated structures of the vacuum, such as the landscape, might be found, but with the definite measure which enables us to discuss entropy. Anyway, the matrix models make these studies possible.

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[8]The GW relation was implemented on the noncommutative torus by using the Neuberger's overlap Dirac operator [24]. In [25], this GW Dirac operator was obtained from the general prescription of [12] and analyzed. In [26], it was extended to the gauge fields in topologically nontrivial sectors. Dynamics of topological aspects in gauge theory on the noncommutative torus were studied in [27].

APPENDIX A: COMMUTATIVE LIMITS OF DIRAC OPERATOR AND TOPOLOGICAL CHARGE

In this Appendix, we take the commutative limits of the Dirac operator and the topological charge, and provide (2.10) and (2.13). While similar calculations were given in [12,16] for $\mathcal{T}r[\hat{\Gamma}]$, a coefficient becomes slightly different in this case, and the calculation of $\mathcal{T}r[\hat{\Gamma}]$ is also instructive for that of $\mathcal{T}r[\Gamma]$. We then show both calculations in a self-contained manner.

By substituting (2.3) into (2.2), we obtain

$$H_l = \sigma \cdot L^L + \tfrac{1}{2} + \rho\sigma \cdot a^L, \tag{A1}$$

$$(H_l)^2 = \frac{n^2}{4} + \rho(\{L_i^L, a_i^L\} + i\epsilon_{ijk}\sigma_k[L_i^L, a_j^L] + \sigma \cdot a^L) + \rho^2(\sigma \cdot a^L)^2, \tag{A2}$$

and

$$\begin{aligned}\hat{\Gamma} = {} & a(\sigma \cdot L^L + \tfrac{1}{2} + \rho\sigma \cdot a^L) - \tfrac{1}{2}a^3\rho\sigma \cdot L^L\{L_i^L, a_i^L\} \\ & - \tfrac{1}{2}a^3\rho\sigma \cdot L^L(i\epsilon_{ijk}\sigma_k[L_i^L, a_j^L] + \sigma \cdot a^L \\ & + \rho(\sigma \cdot a^L)^2 - \tfrac{3}{4}a^2\rho\{L_i^L, a_i^L\}^2) \\ & - \tfrac{1}{2}a^3\rho(\tfrac{1}{2} + \rho\sigma \cdot a^L)\{L_i^L, a_i^L\} + \mathcal{O}(n^{-3}),\end{aligned} \tag{A3}$$

with $a = \frac{2}{n}$. Similarly, by substituting (2.3) into (2.1), we obtain

$$\begin{aligned}\Gamma = {} & a(\sigma \cdot L^R - \tfrac{1}{2} + \rho\sigma \cdot a^R) - \tfrac{1}{2}a^3\rho\sigma \cdot L^R\{L_i^R, a_i^R\} \\ & - \tfrac{1}{2}a^3\rho\sigma \cdot L^R(i\epsilon_{ijk}\sigma_k[L_i^R, a_j^R] - \sigma \cdot a^R \\ & + \rho(\sigma \cdot a^R)^2 - \tfrac{3}{4}a^2\rho\{L_i^R, a_i^R\}^2) \\ & - \tfrac{1}{2}a^3\rho(-\tfrac{1}{2} + \rho\sigma \cdot a^R)\{L_i^R, a_i^R\} + \mathcal{O}(n^{-3}).\end{aligned} \tag{A4}$$

For the commutative limit of the Dirac operator (2.7), it is enough to take terms up to order n^{-1} in (A3) and (A4). We then easily obtain (2.10).

For the commutative limit of the topological charge, the rhs of (2.9), however, we should take terms up to order n^{-2} in (A3) and (A4), since $\mathcal{T}r$ gives a contribution of order n^2. We first consider $\mathcal{T}r[\hat{\Gamma}]$. Taking the trace over the spinor index, we obtain

$$\mathcal{T}r[\hat{\Gamma}] = \mathcal{T}r'\Bigg[\frac{2}{n} - a^3\rho\Big(L_k^L i\epsilon_{ijk}[L_i^L, a_j^L] + L_i^L a_i^L + i\rho\epsilon_{ijk}L_i^L a_j^L a_k^L + \frac{1}{2}\{L_i^L, a_i^L\}\Big)\Bigg], \tag{A5}$$

where $\mathcal{T}r'$ is the trace over the whole configuration space without the spinor index. It is rewritten as $\mathcal{T}r' = \mathrm{tr}_L\mathrm{tr}_{t_L}\mathrm{tr}_R\mathrm{tr}_{t_R}$, where tr_L is the trace over the space on which L_i^L act, tr_{t_L} is the trace over the space on which the gauge group generators $(t^a)^L$ act, and so on. In the commutative limit, $\frac{1}{n}\,\mathrm{tr}_L(M^L)$ is replaced by $\int\frac{d\Omega_L}{4\pi}M(\Omega_L)$, and $\frac{1}{n}\,\mathrm{tr}_R(M^R)$ by $\int\frac{d\Omega_R}{4\pi}M(\Omega_R)$. Then, $\mathcal{T}r'$ becomes $n^2\int\frac{d\Omega_L}{4\pi}\times\int\frac{d\Omega_R}{4\pi}\,\mathrm{tr}_{t_L}\mathrm{tr}_{t_R}$. It then follows that

$$\begin{aligned}\mathcal{T}r[\hat{\Gamma}] &\to \int\frac{d\Omega_L}{4\pi}\int\frac{d\Omega_R}{4\pi}\,\mathrm{tr}_{t_L}\mathrm{tr}_{t_R}(2n + 2\rho^2\epsilon_{ijk}n_i^L F_{jk}^L) \\ &= 2nk^2 + 2\rho^2\int\frac{d\Omega}{4\pi}\,\mathrm{tr}(\epsilon_{ijk}n_iF_{jk}),\end{aligned} \tag{A6}$$

where $F_{ij} = \partial_i a'_j - \partial_j a'_i - i[a'_i, a'_j]$ with a'_i given in (2.11). In the last line, we used a simple expression $\mathrm{tr} = \mathrm{tr}_{t_L}\mathrm{tr}_{t_R}$.

Similarly, we obtain

$$\mathcal{T}r[\Gamma] = \mathcal{T}r'\Bigg[-\frac{2}{n} - a^3\rho\Big(L_k^R i\epsilon_{ijk}[L_i^R, a_j^R] - L_i^R a_i^R + i\rho\epsilon_{ijk}L_i^R a_j^R a_k^R - \frac{1}{2}\{L_i^R, a_i^R\}\Big)\Bigg], \tag{A7}$$

and then

$$\begin{aligned}\mathcal{T}r[\Gamma] &\to \int\frac{d\Omega_L}{4\pi}\int\frac{d\Omega_R}{4\pi}\,\mathrm{tr}_{t_L}\mathrm{tr}_{t_R}(-2n - 2\rho^2\epsilon_{ijk}n_i^R F_{jk}^R) \\ &= -2nk^2 - 2\rho^2\int\frac{d\Omega}{4\pi}\,\mathrm{tr}(\epsilon_{ijk}n_iF_{jk}).\end{aligned} \tag{A8}$$

Because of the relation $[A^R, B^R] = -[A, B]^R$, there arose the minus sign in front of the field strength F_{jk} in (A8), compared with (A6). Adding (A6) and (A8), we finally obtain (2.13).

APPENDIX B: GENERAL CONFIGURATIONS WITH $U(\sum_p k_p)/\prod_p U(k_p)$

In this Appendix, we study formulations for general configurations with $U(\sum_p k_p)/\prod_p U(k_p)$. In particular, we show that the commutative limits of the GW Dirac operators and the topological charges have similar forms as (4.19)–(4.23).

As we mentioned at the end of Sec. V, for each electric charge operator T'_p with $p = 1, \ldots, h-1$, given by (5.5), we define modified chirality operators Γ'_{pr}, $\hat{\Gamma}'_{pr}$, Γ'_{pl}, and $\hat{\Gamma}'_{pl}$ by (4.4)–(4.7). We then define modified GW Dirac operators D'_{pr} and D'_{pl} by (4.10) and (4.11). They satisfy the GW relations as (4.12) and (4.13), and the index theorems as (4.14) and (4.15).

We now study the commutative limits. Following (4.3), we write the commutative limits of the electric charge operators T'_p as

$$T'_p \to 2\phi'_p = \sum_a 2\phi'^a_p t^a, \tag{B1}$$

where t^a are the generators of the gauge group $U(\sum_{p=1}^h k_p)$. Because of $(T'_p)^2 = 1$,

$$\sum_{a,b}\phi'^a_p\phi'^b_p t^a t^b = \frac{1}{4} \tag{B2}$$

should be satisfied at the commutative level as well. The rhs is the identity operator in the gauge group space and the coordinate space of the sphere. Then, unlike the $U(2)$ case, $\phi'^a_p = (1, 0, \ldots, 0)$ gauge does not exist in general, though we have gauges where all of $\phi_p'^a$ are constant and independent of the sphere coordinate Ω.

The commutative limit of the GW Dirac operator D'_{pr} becomes

$$D'_{pr} \rightarrow \tfrac{1}{2}\{2\phi_p'^R, (\sigma_i \mathcal{L}_i + 1)\} + \tfrac{1}{2}\{2\phi_p'^R, \rho\sigma_i P_{ij} a_j^L\} - \tfrac{1}{2}\{2\phi_p'^R, \rho\sigma_i P_{ij} a_j^R\}, \tag{B3}$$

as (4.19). The superscript R (L) means that the operator acts from right (left) in the gauge group space: $\phi_p'^R = \phi_p'^a(\Omega)(t^a)^R$, etc. In the gauges $\phi_p'^a(\Omega) = \phi_p'^a$, where $\phi_p'^a$ are constant, (B3) becomes

$$2\phi_p'^R(\sigma_i\mathcal{L}_i + 1 + \rho\sigma_i P_{ij} a_j^L) - \phi_p'^a \rho\sigma_i P_{ij} a_j^b \{t^a, t^b\}^R \equiv D'_{pr,\mathrm{com}}. \tag{B4}$$

This Dirac operator has the adjoint coupling of the unbroken $U(1)$ gauge field $\sum_a \phi_p'^a a_j^a (t^a)^R (\tilde{t}^a)$. It also satisfies a chiral relation

$$\{D'_{pr,\mathrm{com}}, \gamma'_{pr}\} = 0, \tag{B5}$$

where $\gamma'_{pr} = 2\phi_p'^R\gamma$ is the chirality operator multiplied by the unbroken $U(1)$ charge. The same arguments hold also for D'_{pl}.

As (4.22) and (4.23), the commutative limits of the topological charges become

$$\frac{1}{2}\mathcal{T}r[\Gamma'_{pr} + \hat{\Gamma}'_{pr}] \rightarrow -2k\frac{\rho^2}{8\pi}\int d\Omega\epsilon_{ijk} n_i(\phi_p'^a F_{jk}^a - f_{abc}\phi_p'^a (D_j\phi'_p)^b (D_k\phi'_p)^c), \tag{B6}$$

$$\frac{1}{2}\mathcal{T}r[\Gamma'_{pl} + \hat{\Gamma}'_{pl}] \rightarrow 2k\frac{\rho^2}{8\pi}\int d\Omega\epsilon_{ijk} n_i(\phi_p'^a F_{jk}^a - f_{abc}\phi_p'^a (D_j\phi'_p)^b (D_k\phi'_p)^c), \tag{B7}$$

where $k = \sum_{p=1}^h k_p$ and f_{abc} are the structure constants of $U(\sum_{p=1}^h k_p)$. The field strength $F_{jk} = F_{jk}^a t^a$ is defined as $F_{jk} = \partial_j a'_k - \partial_k a'_j - i[a'_j, a'_k]$, and the covariant derivative D_j is defined as $D_j = \partial_j - i[a'_j,]$, with a'_j given in (2.11). In the gauges $\phi_p'^a(\Omega) = \phi_p'^a$, where $\phi_p'^a$ are constant, the integrand of (B6) and (B7) indeed gives the Abelian flux in the unbroken $U(1)$ direction $\phi_p'^a(\partial_j a_k'^a - \partial_k a_j'^a)$.

We finally give a comment. We here obtained the $h-1$ topological charges $\mathcal{T}r[\Gamma'_{pr} + \hat{\Gamma}'_{pr}]$ with $1 \le p \le h-1$, while we had $\frac{h(h-1)}{2}$ ones (5.3) for $1 \le p < q \le h$. The lack of information is covered by defining chirality operators

$$\Gamma'_{p,q} = T_p'^L \frac{\{T_q'^R, \Gamma\}}{\sqrt{\{T_q'^R, \Gamma\}^2}}, \tag{B8}$$

$$\hat{\Gamma}'_{p,q} = \frac{\{T_p'^L, \hat{\Gamma}\}}{\sqrt{\{T_p'^L, \hat{\Gamma}\}^2}} T_q'^R, \tag{B9}$$

and GW Dirac operators

$$D'_{p,q} = -a^{-1}\Gamma'_{p,q}(1 - \Gamma'_{p,q}\hat{\Gamma}'_{p,q}), \tag{B10}$$

for $1 \le p, q \le h-1$. They satisfy GW relations and then index theorems

$$\mathrm{index}\,(D'_{p,q}) = \tfrac{1}{2}\mathcal{T}r[\Gamma'_{p,q} + \hat{\Gamma}'_{p,q}], \tag{B11}$$

which indeed provide $\frac{(h-1)(h-2)}{2}$ topological charges. While $\mathcal{T}r(\Gamma'_{p,q})$ and $\mathcal{T}r(\hat{\Gamma}'_{p,q})$ vanish for the $U(2)/U(1)^2$ case of Sec. IV, they give nontrivial results in the present case of $U(\sum_p k_p)/\prod_p U(k_p)$.

APPENDIX C: CHARGE CONJUGATION

In this Appendix we show that the two spinors in (6.12)–(6.15) are in the charge conjugate representations to each other. We also show that the Majorana condition in ten dimensions can be written as the decomposition into each subspace, as in the Weyl condition.

We first introduce unitary matrices B_1 and B_2 acting on $SO(9,1)$ spinors, which satisfy

$$B_1\Gamma_M B_1^{-1} = (\Gamma_M)^*, \tag{C1}$$

$$B_2\Gamma_M B_2^{-1} = -(\Gamma_M)^*, \tag{C2}$$

for $M = 0, \ldots, 9$. (We follow the notation in Appendix B.1 in [32].) For the representation of gamma matrices (6.11), they are written as

$$B_1 = B_1^{(4)} \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \tag{C3}$$

$$B_2 = B_2^{(4)} \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1, \tag{C4}$$

where $B_1^{(4)}$ and $B_2^{(4)}$ satisfy

$$B_1^{(4)}\gamma_\mu(B_1^{(4)})^{-1} = -(\gamma_\mu)^*, \tag{C5}$$

$$B_2^{(4)}\gamma_\mu(B_2^{(4)})^{-1} = (\gamma_\mu)^*. \tag{C6}$$

The charge conjugation of $SO(9,1)$ spinors is defined as

$$\zeta^C \equiv B^{-1}\zeta^*, \tag{C7}$$

for either $B = B_1$ or $B = B_2$.

For the gamma matrices (6.11), the chirality operator in M^{10} is written as

$$\Gamma_{11} = \gamma_5 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_3, \tag{C8}$$

where the chirality operator in M^4 is

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{C9}$$

As usual,

$$B\Gamma_{11}B^{-1} = (\Gamma_{11})^* \tag{C10}$$

is satisfied for both B_1 and B_2, while

$$B^{(4)}\gamma_5(B^{(4)})^{-1} = -(\gamma_5)^* \tag{C11}$$

is satisfied for both $B_1^{(4)}$ and $B_2^{(4)}$. Then, the Weyl spinor in M^{10} is self-conjugate and that in M^4 is complex.

We may define a chirality operator in the second and the third factors in (C8) as

$$\Gamma^{(R^3\times R^3)} = \mathbb{1}_2 \otimes \mathbb{1}_2. \tag{C12}$$

We can also define chirality operators in this space as

$$\Gamma^{(S^2\times S^2)} = n\cdot\sigma \otimes n\cdot\sigma, \tag{C13}$$

$$\Gamma^{(S^2)} = n\cdot\sigma \otimes \mathbb{1}_2, \tag{C14}$$

$$\Gamma^{(S^{2\prime})} = \mathbb{1}_2 \otimes n\cdot\sigma. \tag{C15}$$

The charge conjugation matrix in this space is

$$A = \sigma_2 \otimes \sigma_2 \tag{C16}$$

for either (C3) or (C4). The Weyl spinor in terms of the chirality (C12) is self-conjugate because

$$A\Gamma^{(R^3\times R^3)}A^{-1} = (\Gamma^{(R^3\times R^3)})^* \tag{C17}$$

is satisfied. That of (C13) is self-conjugate:

$$A\Gamma^{(S^2\times S^2)}A^{-1} = (\Gamma^{(S^2\times S^2)})^*, \tag{C18}$$

and those of (C14) and (C15) are complex:

$$A\Gamma^{(S^2)}A^{-1} = -(\Gamma^{(S^2)})^*. \tag{C19}$$

We should also define a chirality operator in the fourth factor in (C8) as

$$\Gamma^{(e)} = \sigma_3. \tag{C20}$$

The charge conjugation matrix in this space is

$$A_1^{(e)} = \sigma_2, \qquad A_2^{(e)} = \sigma_1 \tag{C21}$$

for (C3) and (C4), respectively. For either $A_1^{(e)}$ or $A_2^{(e)}$, the Weyl spinor is complex because

$$A^{(e)}\Gamma^{(e)}(A^{(e)})^{-1} = -(\Gamma^{(e)})^*. \tag{C22}$$

It follows from (C11), (C18), and (C22) that the two spinors in (6.12)–(6.15) are in the charge conjugate representations to each other.

In the remainder of this Appendix, we discuss the Majorana condition. The Majorana condition in ten dimensions

$$\zeta = \zeta^C \equiv B^{-1}\zeta^* \tag{C23}$$

can be imposed since $B^*B = 1$ is satisfied for either $B = B_1$ in (C1) or $B = B_2$ in (C2).

By decomposing the spinor as

$$\zeta = \varphi \otimes \psi \otimes \chi, \tag{C24}$$

the Majorana condition (C23) with B_2 in (C4) is written as

$$\varphi^* \otimes \psi^* \otimes \chi^* = B_2^{(4)}\varphi \otimes A\psi \otimes A_2^{(e)}\chi. \tag{C25}$$

This is satisfied by imposing the conditions

$$\varphi^* = \pm B_2^{(4)}\varphi, \tag{C26}$$

$$\psi^* = \pm A\psi, \tag{C27}$$

$$\chi^* = \pm A_2^{(e)}\chi, \tag{C28}$$

where the three signs should satisfy $(\pm)(\pm)(\pm) = +$. Since $(B_2^{(4)})^*B_2^{(4)} = 1$ and $(A_2^{(e)})^*A_2^{(e)} = 1$ are satisfied, (C26) and (C28) can be imposed. While the reality condition, the Euclidean version of the Majorana condition, cannot be imposed on the $SO(3)$ spinors, which are in the pseudoreal representation, the product of two pseudoreal representations is real. This trick is used in (C27), where $A^*A = 1$ is satisfied.

Similarly, the Majorana condition (C23) with B_1 in (C3) is written as

$$\varphi^* \otimes \psi^* \otimes \chi^* = B_1^{(4)}\varphi \otimes A\psi \otimes A_1^{(e)}\chi. \tag{C29}$$

This is satisfied by imposing the conditions

$$\varphi^* \otimes \chi^* = \pm B_1^{(4)}\varphi \otimes A_1^{(e)}\chi, \tag{C30}$$

$$\psi^* = \pm A\psi, \tag{C31}$$

where the two signs should satisfy $(\pm)(\pm) = +$. The trick of doubling the pseudoreal representations is used twice, in (C30) and in (C31).

We therefore find that the Majorana condition in ten dimensions can be written as the decomposition into each subspace: (C26)–(C28), or (C30) and (C31). Although these decompositions were not used directly in the present paper, they are useful when we study the Majorana condition in each subspace.

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