BPS Skyrme model and baryons at large N_c

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Within the class of field theories with the field content of the Skyrme model, one submodel can be found which consists of the square of the baryon current and a potential term only. For this submodel, a Bogomol'nyi-Prasad-Sommerfield bound exists, and the static soliton solutions saturate this bound. Further, already on the classical level, this Bogomol'nyi-Prasad-Sommerfield Skyrme model reproduces some features of the liquid drop model of nuclei. Here, we investigate the model in more detail and, besides, we perform the rigid rotor quantization of the simplest Skyrmion (the nucleon). In addition, we discuss indications that the viability of the model as a low-energy effective field theory for QCD is further improved in the limit of a large number of colors N_c.

I. INTRODUCTION

The derivation of the correct description of low-energy hadron dynamics is undoubtedly one of the most prominent challenges of QCD. The difficulty of the problem originates in the nonperturbative nature of the quark and gluon interactions in the low-energy limit. In the large N_c limit, however, it is known that QCD becomes equivalent to an effective theory of mesons [1]. Baryons (hadrons as well as atomic nuclei) appear as solitonic excitations with an identification between the baryon number and the topological charge [2]. One of the most popular realizations of this idea is the Skyrme model [3], i.e., a version of a phenomenological chiral Lagrangian, where the primary ingredients are meson fields. Static properties of baryons as well as nuclei are derived with the help of the semiclassical quantization of the solitonic zero modes [4–6]. While many phenomenological properties of baryonic (and nuclear) matter seem to fit perfectly in the framework of the Skyrme theory, there are still some results which are in disagreement with the experimental or lattice data.

First of all, there is a conceptual problem with the derivation of the Skyrme model from the underlying quantum theory. Assuming the large N_c expansion, one may derive the pure pseudoscalar low-energy dynamics from QCD in the leading order in derivatives usually truncated at fourth [7,8] or sixth [9–11] order terms. However, baryons being solitons, i.e., extended collective excitations of the chiral field, they contain regions with rather large values of the gradients of the field. Thus, any truncation does not seem to be justified by a small derivative expansion, and one should instead consider more terms [12–17] (also an

infinite series of terms [18]) or find another acceptable principle which could provide a selection criterium for a simple effective action. The situation is further complicated by the fact that, already at the fourth order in the derivative expansion, one gets not only the standard Skyrme term but also two additional terms which contribute at the same level to the effective action. In other words, there is no reason to omit them in this kind of expansion. Moreover, one of these terms contains the second time derivative squared and, therefore, leads to serious problems in the dynamics (the standard Cauchy data are not sufficient to determine the time evolution uniquely) as well as in the quantization procedure (no obvious Hamiltonian). Additionally, this term enters with a wrong sign, destabilizing soliton configurations. All this seems to indicate that a simultaneous large N_c and small derivative expansion might be problematic.

Another source of criticism of the Skyrme model is related to certain phenomenological features of solutions of the model [19–21].

(i) *Large binding energies.*—As the Skyrme model is not an exact BPS theory, its soliton solutions do not saturate the corresponding linear energy-topological charge relation, which results in the appearance of binding energies. Unfortunately, their value is significantly bigger than experimental energies which do not exceed 1% of the nuclei masses [22–27]. From the point of view of the large N_c expansion, this seems to indicate that the binding energies scale like $N_c \Lambda_{\mathrm{QCD}}$, instead of $\Lambda_{\mathrm{QCD}}/N_c$ as expected for the weakly bound nuclear matter.

(ii) *Crystal state of matter.*—The matter described by the Skyrme model in the limit of large baryon charge behaves like a crystal, not as a liquid, for $N_c \to \infty$ [28] as well as for finite N_c [22–26]. Moreover, shell-like structures are preferred rather than core-type or ball-type configurations. This can

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be improved by including the potential term (that is, massive pions), but still for a fixed value of the mass parameter, the first few Skyrmions possess shell-like structures.

(iii) *Strong forces at intermediate distances.*—Because of the enhancement of the pion coupling constant in the Skyrme model $g_{\pi NN} \sim N_c^{3/2}$, the axial coupling constant grows linearly with N_c. This leads to strong spin-isospin forces at distances larger than the size of the nucleus, which is in contradiction to experimental as well as lattice data [14].

Let us remark that in Ref. [20] a critical evaluation of large N_c properties has been conducted not only for the standard Skyrme model but also for the standard nonrelativistic quark model. In the latter case, the solution proposed in [20] consists in the so-called dichotomous nucleon model, where one assumes that for odd N_c all quarks are paired in diquarks except for one which carries the quantum numbers of the nucleon and has much larger spatial extension than the diquarks. The resulting very small overlap in wave functions ("dichotomy") tames the strong forces at large N_c.

The problems mentioned above may suggest that the Skyrme model (at least in the versions usually considered), under certain circumstances, is probably not the right starting point for the effective low-energy description of baryons for large N_c.

On the other hand, model-independent results, which are related to topological and geometrical aspects of the solutions rather than to a particular form of the action, may indicate the correctness of the original topological concept of the Skyrme model. One can still expect that mesonic matrix fields (or possibly their generalizations) are the right low-energy degrees of freedom (in agreement with [1]), even when the proper effective action is not accessible via the small derivative expansion. This motivates the philosophy of our approach. We use the $SU(2)$ matrix field, i.e., keep the topological content of the Skyrme model but change the Lagrangian. In fact, the unique principles we are left with, if we do not want to rely on the derivative expansion, are, again, topology and the need for a BPS theory with chiral Skyrme fields.

The BPS Skyrme model we propose [29] is, by construction, even more topological in nature than any of the standard versions of the Skyrme model. As in the standard Skyrme model, soliton solutions are stabilized by a higher order term in derivatives (a sextic term in our case). It is an example of a BPS theory with topological solitons saturating the pertinent Bogomol'nyi bound and, therefore, with zero binding energies. Further, this BPS model possesses a huge number of symmetries which lead to its integrability. What is more important, among its symmetries are the volume-preserving diffeomorphisms on the base space. This allows us to interpret the BPS Skyrme matter as an incompressible liquid. Moreover, the solitons of this theory

are of compact type, which results in a finite range of interactions (contactlike interactions). Therefore, the BPS Skyrme model cures the above-mentioned problems of the standard Skyrme model, at least on a qualitative level, although it does so in a rather radical way. In addition, these properties are N_c independent. The zero binding energy, the liquid state of matter, and the contact interaction occur for all N_c, no matter how N_c enters into the parameters of the BPS model. As a consequence, the BPS Skyrme model apparently is a good guess for certain aspects of a low-energy effective action for $N_c = \infty$, although there does not seem to exist an obvious large N_c limit yet based directly on QCD, which would produce just the BPS Skyrme model as its leading order.

We want to remark that there exists another BPS generalization of the Skyrme model recently proposed by Sutcliffe [27] based on dimensional reduction of the $(4 + 1)$-dimensional Yang-Mills (YM) theory, where the $SU(2)$ Skyrme field is accompanied by an infinite tower of vector and tensor mesons. It also gives the linear energy-charge relation, as an property inherited from the pertinent self-dual sector of the YM theory, and potentially zero binding energy. Whether the large N_c problems of the standard Skyrme model can be cured in this model, and whether the resulting nuclear matter is in a crystal or liquid state, still has to be investigated. Also, the addition of a potential seems to be difficult if one wants to maintain the relation with $4 + 1$ Yang-Mills theory.

II. THE BPS SKYRME MODELS

Let us, then, consider the proposed family of models, which we will call *the BPS Skyrme models*:

$$L_{06} = \frac{\lambda^2}{24^2} [\mathrm{Tr}(\epsilon^{\mu\nu\rho\sigma} U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\rho U)]^2 - \mu^2 V(U, U^\dagger). \tag{1}$$

The subindex 06 refers to the fact that in the above Lagrangian only a potential term without derivatives and a term sextic in derivatives are present.[1] The sextic part of the action is nothing but the topological current density squared. Here, we use the standard parametrization by means of a real scalar ξ and a three-component unit vector $\vec{n}$ field ($\vec{\tau}$ are the Pauli matrices),

$$U = e^{i\xi\vec{n}\cdot\vec{\tau}}.$$

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[1]We remark that models which are similar in some aspects, although with a different target space geometry, have been studied in [30,31]. Further, the model studied here and in Ref. [29] as well as its "baby Skyrme" version in $2 + 1$ dimensions have already been introduced in [32], where the main aim was to study more general properties of Skyrme models in any dimension. For further discussion of the $(2 + 1)$-dimensional version, see [33–35] and, from a more geometric point of view, [36].

The vector field may be related to a complex scalar u by the stereographic projection

$$\vec{n} = \frac{1}{1 + |u|^2}(u + \bar{u}, -i(u - \bar{u}), |u|^2 - 1).$$

Assuming for the potential

$$V = V(\text{tr}(U + U^\dagger)) = V(\xi)$$

(which we assume for the rest of the paper), we get

$$L_{06} = -\frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4}(\epsilon^{\mu\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2 - \mu^2 V(\xi). \quad (2)$$

The pertinent equations of motion read

$$\frac{\lambda^2 \sin^2 \xi}{(1 + |u|^2)^4}\partial_\mu(\sin^2 \xi H^\mu) + \mu^2 V_\xi' = 0,$$

$$\partial_\mu\left(\frac{K^\mu}{(1 + |u|^2)^2}\right) = 0,$$

where

$$H_\mu = \frac{\partial(\epsilon^{\alpha\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \xi^\mu}, \qquad K_\mu = \frac{\partial(\epsilon^{\alpha\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \bar{u}^\mu}.$$

These objects obey the useful formulas

$$H_\mu u^\mu = H_\mu \bar{u}^\mu = 0, \qquad K_\mu \xi^\mu = K_\mu u^\mu = 0,$$

$$H_\mu \xi^\mu = K_\mu \bar{u}^\mu = 2(\epsilon^{\alpha\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2.$$

A. Symmetries

Apart from the standard Poincare symmetries, the model has an infinite number of target space symmetries. The sextic term alone is the square of the pullback of the volume form on the target space S^3, where this target space volume form reads explicitly

$$dV = -i\frac{\sin^2 \xi}{(1 + |u|^2)^2}d\xi du d\bar{u} \quad (3)$$

and the exterior (wedge) product of the differentials is understood. Therefore, the sextic term alone is invariant under all target space diffeomorphisms which leave this volume form invariant (the volume-preserving diffeomorphisms on the target S^3). The potential term, in general, does not respect all these symmetries, but depending on the specific choice, it may respect a certain subgroup of these diffeomorphisms. Concretely, for $V = V(\xi)$, the potential is invariant under those volume-preserving target space diffeomorphisms which do not change ξ, that is, which act nontrivially only on $u, \bar{u}$. Since u spans a two-sphere in target space, these transformations form a one-parameter family of the groups of the area-preserving diffeomorphisms on the corresponding target space S^2 (one-parameter family because the transformations may still depend on ξ, although they act nontrivially only on $u, \bar{u}$). Both the Poincare transformations and this family of area-preserving target space diffeomorphisms are symmetries of the full action, so they are Noether symmetries with the

corresponding conserved currents. The latter symmetries may, in fact, be expressed in terms of the generalized integrability, as we briefly discuss in the next subsection.

The energy functional for static fields has an additional group of infinitely many symmetry transformations, as we discuss now. These symmetries are not symmetries of the full action, so they are not of the Noether type, but nevertheless they are very interesting from a physical point of view, as we will see below. The energy functional for static fields reads

$$E = \int d^3x \left(\frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4}(\epsilon^{mnl}i\xi_m u_n \bar{u}_l)^2 + \mu^2 V\right) \quad (4)$$

and we observe that both d^3x and $\epsilon^{mnl}i\xi_m u_n \bar{u}_l$ are invariant under coordinate transformations of the base space coordinates x_j which leave the volume form d^3x invariant. So this energy functional has the volume-preserving diffeomorphisms on the base space as symmetries. These symmetries are precisely the symmetries of an incompressible ideal fluid, which makes them especially interesting in the context of applications to nuclear matter. Indeed, the resulting field theory is able to reproduce some basic features of the liquid droplet model of nuclei; see e.g. [29,37] and the discussion below.

B. Integrability

The BPS Skyrme model is integrable in the sense that there are infinitely many conserved charges. Indeed, it belongs to a family of models integrable in the sense of the generalized integrability [38,39]. To show this, we introduce

$$\mathcal{K}^\mu = \frac{K^\mu}{(1 + |u|^2)^2}.$$

The currents are

$$J_\mu = \frac{\delta G}{\delta \bar{u}}\bar{\mathcal{K}}^\mu - \frac{\delta G}{\delta u}\mathcal{K}^\mu, \qquad G = G(u, \bar{u}, \xi),$$

where $G(u, \bar{u}, \xi)$ is an arbitrary real function of its arguments. Then,

$$\partial^\mu J_\mu = G_{\bar{u}\bar{u}}\bar{u}_\mu\bar{\mathcal{K}}^\mu + G_{\bar{u}u}u_\mu\bar{\mathcal{K}}^\mu + G_{\bar{u}}\partial_\mu\bar{\mathcal{K}}^\mu$$
$$- G_{u\bar{u}}\bar{u}_\mu\mathcal{K}^\mu - G_{uu}u_\mu\mathcal{K}^\mu - G_u\partial_\mu\mathcal{K}^\mu$$
$$+ G_{\bar{u}\xi}\xi_\mu\bar{\mathcal{K}}^\mu - G_{u\xi}\xi_\mu\mathcal{K}^\mu = 0,$$

where we used that $u_\mu\mathcal{K}^\mu = \xi_\mu\mathcal{K}^\mu = 0$, $\bar{u}_\mu\mathcal{K}^\mu = u_\mu\bar{\mathcal{K}}^\mu$, which follows from the previous identities. Finally, using the field equations for the complex field, one, indeed, finds an infinite number of conserved currents. These currents are a higher-dimensional generalization of those constructed for the pure baby Skyrme model [40] and are generated by the relevant subgroup of the volume-preserving diffeomorphisms on the target space, as discussed in the previous subsection; see also [41].

We remark that the existence of infinitely many conservation laws (integrability), together with the possibility to reduce the static field equations to a system of ordinary differential equations via the right ansatz (the hedgehog ansatz in our case), leads to the existence of infinitely many exact solutions, as we shall see in the next subsections. This observation lends further credibility to the conjecture that the relation between integrability (in the sense of infinitely many conservation laws) and solvability is always true in higher-dimensional field theories. The conjecture holds true in all concrete cases we are aware of. Nevertheless, a deeper mathematical understanding or even a proof, which would advance our understanding of integrability in higher dimensions, is still under investigation.

C. Bogomol'nyi bound

In the BPS Skyrme model, the following Bogomol'nyi bound exists for the energies E of static solutions:

$$E \geq 2\lambda\mu\pi^2 C[V]|B|, \tag{5}$$

where B is the baryon number (topological charge). Further, $C[V]$ is a calculable number which depends on the potential but *not* on the specific solution. Therefore, for a given theory (i.e., a given potential), the bound is, indeed, topological. For a proof, we write the energy functional as

$$
\begin{aligned}
E &= \int d^3x \left(\frac{\lambda^2\sin^4\xi}{(1+|u|^2)^4}(\epsilon^{mnl}i\xi_m u_n \bar{u}_l)^2 + \mu^2 V \right) \\
&= \int d^3x \left(\frac{\lambda\sin^2\xi}{(1+|u|^2)^2}\epsilon^{mnl}i\xi_m u_n \bar{u}_l \pm \mu\sqrt{V} \right)^2 \mp \int d^3x \frac{2\mu\lambda\sin^2\xi\sqrt{V}}{(1+|u|^2)^2}\epsilon^{mnl}i\xi_m u_n \bar{u}_l \\
&\geq \mp \int d^3x \frac{2\mu\lambda\sin^2\xi\sqrt{V}}{(1+|u|^2)^2}\epsilon^{mnl}i\xi_m u_n \bar{u}_l = \pm(2\lambda\mu\pi^2)\left[\frac{-i}{\pi^2}\int d^3x \frac{\sin^2\xi\sqrt{V}}{(1+|u|^2)^2}\epsilon^{mnl}\xi_m u_n \bar{u}_l \right] \equiv 2\lambda\mu\pi^2 C[V]|B|,
\end{aligned}
$$

where the sign has to be chosen appropriately (upper sign for $B > 0$). If we replace V by 1 in the last expression in brackets, and $C[V]$ by 1, then this expression is just the topological charge, so its topological nature is obvious. Equivalently, this expression is just the base space integral of the pullback of the volume form on the target space S^3, normalized to 1, and this second interpretation may be easily generalized to nontrivial V. Indeed, we just have to introduce a new target space coordinate $\bar{\xi}$ such that

$$\sin^2\xi\sqrt{V(\xi)}d\xi = C[V]\sin^2\bar{\xi}d\bar{\xi}. \tag{7}$$

The constant $C[V]$ and a second constant C', which is provided by the integration of Eq. (7), are needed to impose the two conditions $\bar{\xi}(\xi = 0) = 0$ and $\bar{\xi}(\xi = \pi) = \pi$, which have to hold if $\bar{\xi}$ is a good coordinate on the target space S^3. Obviously, $C[V]$ depends on the potential $V(\xi)$. E.g., for the standard Skyrme potential $V = 1 - \cos\xi$, $C[V]$ is

$$C[V] = \frac{32\sqrt{2}}{15\pi}.$$

The corresponding Bogomol'nyi (first order) equation is

$$\frac{\lambda\sin^2\xi}{(1+|u|^2)^2}\epsilon^{mnl}i\xi_m u_n \bar{u}_l = \mp\mu\sqrt{V} \tag{8}$$

and is satisfied by all soliton solutions which we shall encounter in this article. This proof for the Bogomol'nyi bound has been given already in [29] and is repeated here for the convenience of the reader (the proof for the analogous $2 + 1$-dimensional theory has been given in [34,36,42]).

D. Exact solutions

We are interested in static topologically nontrivial solutions. Thus u must cover the whole complex plane ($\vec{n}$ must cover the target space S^2 at least once) and $\xi \in [0, \pi]$. The natural (hedgehog) ansatz is

$$\xi = \xi(r), \qquad u(\theta, \phi) = g(\theta)e^{in\phi}. \tag{9}$$

Then, the field equation for u reads

$$\frac{1}{\sin\theta}\partial_\theta\left(\frac{g^2 g_\theta}{(1+g^2)^2\sin\theta}\right) - \frac{gg_\theta^2}{(1+g^2)^2\sin^2\theta} = 0,$$

and the solution with the right boundary condition is

$$g(\theta) = \tan\frac{\theta}{2}. \tag{10}$$

Observe that this solution holds for all values of n. The equation for the real scalar field is

$$\frac{n^2\lambda^2\sin^2\xi}{2r^2}\partial_r\left(\frac{\sin^2\xi\xi_r}{r^2}\right) - \mu^2 V_\xi = 0.$$

This equation can be simplified by introducing the new variable

$$z = \frac{\sqrt{2}\mu r^3}{3|n|\lambda}. \tag{11}$$

It reads

$$\sin^2\xi\partial_z(\sin^2\xi\xi_z) - V_\xi = 0, \tag{12}$$

and may be integrated to

$$\tfrac{1}{2}\sin^4\xi\xi_z^2 = V, \tag{13}$$

where we chose a vanishing integration constant to get finite energy solutions. We remark that this first integration of the field equation is equivalent to a Bogomol'nyi equation and, thus, to a Bogomol'nyi bound for the dimensionally reduced, effectively one-dimensional problem. It can be proved that also for the full theory, without any symmetry reduction, there exists a Bogomol'nyi bound and a Bogomol'nyi equation which is obeyed by all the solutions we find below. In terms of the variable r the integrated (Bogomol'nyi) equation reads

$$\frac{n^2\lambda^2}{4\mu^2 r^4}\sin^4\xi\,\xi_r^2 = V, \tag{14}$$

and it is this form which will be useful for the discussion of the energy. Indeed, the energy is

$$E = \int d^3x\left(-\frac{\lambda^2\sin^4\xi}{(1+|u|^2)^4}(\nabla_r\xi)^2\right.$$

$$\left. \times (\nabla_\theta u\nabla_\phi\bar{u} - \nabla_\phi u\nabla_\theta\bar{u})^2 + \mu^2 V\right), \tag{15}$$

or, after inserting the hedgehog ansatz with the solution (10) for u,

$$E = 4\pi\int r^2 dr\left(\frac{\lambda^2 n^2\sin^4\xi}{4r^4}\xi_r^2 + \mu^2 V\right). \tag{16}$$

It follows from the Bogomol'nyi equation for r, (14), that the sextic term and the potential contribute the same amount to the energy density for arbitrary values of r. Therefore, we may further simplify the expression for the energy as

$$E = 4\pi \cdot 2\mu^2 \int r^2 dr V(\xi(r))$$

$$= 4\sqrt{2}\pi\mu\lambda|n| \int dz V(\xi(z)). \tag{17}$$

Further, we can already draw some qualitative conclusions about the behavior of the energy density profiles for different types of potentials. Finiteness of the energy requires that the fields take values in the vacuum manifold of the potential V in the limit $r \to \infty$. For the class of potentials $V = V(\xi)$ we consider, this just means that $\lim_{r\to\infty}V(\xi(r)) = 0$. Further, the topology of Skyrmion fields requires that the matrix field U takes a constant, direction-independent value in the limit $r \to \infty$. Within the hedgehog ansatz this implies that the field ξ must take one of its two boundary values $\xi = 0$, π in this limit. For Skyrmions with finite energy, therefore, at least one of these two boundary values must belong to the vacuum manifold of the potential. Without loss of generality, let us assume that ξ takes the value $\xi = 0$ in the limit $r \to \infty$. For a wide class of potentials this implies that ξ must take the opposite boundary value $\xi = \pi$ at $r = 0$, because it follows easily from (14) that ξ is a monotonous function of r in the region

where $V \neq 0$. These observations lead to the following conclusions. For one-vacuum potentials with the only vacuum at $\xi = 0$, the energy density cannot be zero inside the Skyrmion. If, in addition, the potential is a monotonous function of ξ in the range of ξ, then the energy density is a monotonous function of r and takes its maximum value at $r = 0$; i.e., the soliton is of the core type. If the potential has the two vacua $\xi = 0$, π, then the energy density is zero also at $r = 0$, and the soliton is of the shell type. For more complicated vacuum manifolds of V, more complicated soliton structures emerge, but they may still be found by a variant of the simple qualitative reasoning applied in this paragraph. We remark that a qualitatively similar relation between the vacuum manifold of the potential and the Skyrmion structure also is observed in the original Skyrme model with a potential. The difference is that in the latter case this relation is the result of complicated, three-dimensional numerical integrations, whereas in our case it follows from some simple analytical arguments.

E. The Skyrme potential

The first obvious possibility is to consider the standard Skyrme potential

$$V = \tfrac{1}{2}\text{Tr}(1 - U) \to V(\xi) = 1 - \cos\xi. \tag{18}$$

Imposing the boundary conditions for topologically nontrivial solutions, we get

$$\xi = \begin{cases} 2\arccos\sqrt[3]{\frac{3z}{4}} & z \in \left[0, \frac{4}{3}\right] \\ 0 & z \geq \frac{4}{3}. \end{cases} \tag{19}$$

The corresponding energy is

$$E = 8\sqrt{2}\pi\mu\lambda|n| \int_0^{4/3}\left(1 - \left(\frac{3z}{4}\right)^{2/3}\right)dz = \frac{64\sqrt{2}\pi}{15}\mu\lambda|n|. \tag{20}$$

The solution is of the compacton type; i.e., it has a finite support [43] (compact solutions of a similar type in different versions of the baby Skyrme model have been found in [33,44]). The function ξ is continuous but its first derivative is not. The jump of the derivative is, in fact, infinite at the compacton boundary $z = 4/3$, as the left derivative at this point tends to minus infinity. Nevertheless, the energy density and the topological charge density (baryon number density) are continuous functions at the compacton boundary, and the field equation (12) is well defined there. The reason is that ξ_z always appears in the combination $\sin^2\xi\xi_z$, and this expression is finite (in fact, zero) at the compacton boundary. We could make the discontinuity disappear altogether by introducing a new variable $\tilde{\xi}$ instead of ξ, which satisfies

$$\tilde{\xi}_z = \sin^2\xi\xi_z.$$

We prefer to work with ξ just because this is the standard variable in the Skyrme model. In order to extract the energy density, it is useful to rewrite the energy with the help of the rescaled radial coordinate

$$\tilde{r} = \left(\frac{\sqrt{2}\mu}{4\lambda}\right)^{1/3} r \equiv \frac{r}{R_0} = \left(\frac{3|n|z}{4}\right)^{1/3} \qquad (21)$$

(here R_0 is the compacton radius) as

$$E = 8\sqrt{2}\mu\lambda\left(4\pi\int_0^{|n|^{1/3}} d\tilde{r}\tilde{r}^2(1 - |n|^{-(2/3)}\tilde{r}^2)\right)$$

such that the energy density per unit volume (with the unit of length set by $\tilde{r}$) is

$$\mathcal{E} = 8\sqrt{2}\mu\lambda(1 - |n|^{-(2/3)}\tilde{r}^2) \quad \text{for} \quad 0 \leq \tilde{r} \leq |n|^{1/3} = 0$$
$$\text{for} \quad \tilde{r} > |n|^{1/3}. \qquad (22)$$

$\tilde{r}$ does not depend on the topological charge $B = n$, so the dependence of $\mathcal{E}$ on n is explicit.

In the same fashion we get for the topological charge (baryon number) (see e.g. Chap 1.4 of [45])

$$B = -\frac{1}{\pi^2}\int d^3x \frac{\sin^2\xi}{(1+|u|^2)^2} i\epsilon^{mnl}\xi_m u_n \bar{u}_l$$

$$= -\frac{2n}{\pi}\int dr\sin^2\xi\xi_r = -\frac{2n}{\pi}\int dz\sin^2\xi\xi_z$$

$$= \frac{4n}{\pi}\int_0^{4/3} dz\left(1 - \left(\frac{3}{4}\right)^{2/3}z^{2/3}\right)^{1/2}$$

$$= \text{sign}(n)\frac{4}{\pi^2}\left(4\pi\int_0^{|n|^{1/3}} d\tilde{r}\tilde{r}^2(1-|n|^{-(2/3)}\tilde{r}^2)^{1/2}\right) = n,$$
$$\tag{23}$$

and for the topological charge density per unit volume

$$\mathcal{B} = \text{sign}(n)\frac{4}{\pi^2}(1 - |n|^{-(2/3)}\tilde{r}^2)^{1/2} \quad \text{for}$$
$$0 \leq \tilde{r} \leq |n|^{1/3} = 0 \quad \text{for} \quad \tilde{r} > |n|^{1/3}. \tag{24}$$

Both densities are zero outside the compacton radius $\tilde{r} = |n|^{1/3}$. We remark that the values of the densities at the center $\tilde{r} = 0$ are independent of the topological charge

$B = n$, whereas the radii grow like $n^{1/3}$. For $n = 1$, we plot the two densities in Fig. 1, where we normalize both densities (i.e., multiply them by a constant) such that their value at the center is 1.

III. SOME PHENOMENOLOGY OF NUCLEI

After having discussed the properties of the Lagrangian and its classical solutions in the preceding sections, let us now try to apply it to the description of some properties of nuclei. After all, this possibility is one of the rationales for the original Skyrme model and its generalizations. We certainly do not consider this model by itself as the complete effective field theory of QCD, but we want to study whether solitons of this model can be related to some properties of real baryons. Here we shall first focus on the classical theory and solutions, and we will find that at this level the model already reproduces quite well some properties of the nuclear drop model. In the next step, we perform the semiclassical quantization of the (iso)rotational degrees of freedom of the $B = 1$ soliton, i.e., the nucleon. Further, we choose the standard Skyrme potential $V = 1 - \cos\xi$, for simplicity, throughout this section.

A. Classical aspects

We find immediately that the classical solutions of the BPS Skyrme model seem to describe surprisingly well some static properties of nuclei. As was discussed already in [29], they provide an alternative starting point for an effective soliton model of baryons, which by construction is much more topological in nature. Let us present and further elaborate on these results.

Mass spectrum and linear energy-charge relation.—As a consequence of the BPS nature of the classical solutions, the energy of the solitons is proportional to the topological (baryon) charge

$$E = E_0|B|,$$

where $E_0 = 64\sqrt{2}\pi\mu\lambda/15$. Such a linear dependence is a well-established fact in nuclear physics. For the moment (i.e., in the context of the purely classical reasoning), let us fix the energy scale by assuming that $E_0 = 931.75$ MeV.
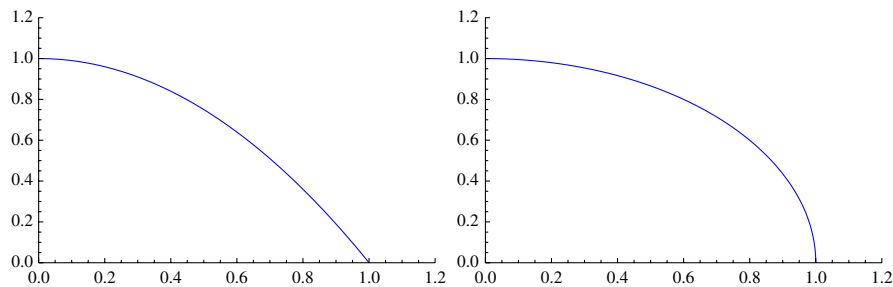

FIG. 1 (color online). Normalized energy density (left figure) and topological charge density (right figure) as a function of the rescaled radius $\tilde{r}$, for topological charge $n = 1$. For $|n| > 1$, the height of the densities remains the same, whereas their radius grows like $|n|^{1/3}$.

This is equivalent to the assumption that the mass of the solution with $B = 4$ is equal to the mass of He^4. One usually assumes this value because the ground state of He^4 has zero spin and isospin [24]. Therefore, possible corrections to the mass from spin-isospin interactions are absent. In Table I we compare energies of the solitons in the BPS model with experimental values and energies obtained in the vector-Skyrme model [17] and the standard massive Skyrme model [23]. (We use the numerical data, if accessible, or calculate them from fitted functions [23]. The energy scale is set by the same prescription.) It is interesting to note that instead of the approximate 7% accuracy typical for the soliton energies of standard Skyrme theories, we get maximally only a 0.7% discrepancy. Besides, the masses of the BPS Skyrme model solitons are slightly smaller than the experimental masses in *all* cases (except for the He^4 used for the fit, of course). This has to be expected, because the (iso)rotational excitation energies should be added to the classical soliton masses (except for the He^4, of course) for a more reliable comparison with physical masses of nuclei.

No binding energy.—It follows from the BPS nature of the model that the binding energy is zero. This is different from the standard Skyrme model, where binding energies are rather big. For example, the energy of the baryon number-2 Skyrmion exceeds the topological energy bound by 23%. Of course, such binding energies are significantly larger than the experimentally observed ones, which usually do not reach 1%. Therefore, as pointed out by P. Sutcliffe [6], a BPS Skyrme theory seems to be a better starting point to get realistic binding energies. Small (nonzero) binding energies could be produced by small perturbations around a BPS theory.

Size of nuclei and compactons.—Because of the compact nature of the solitons, their radius is well defined and can be easily computed,

$$R_B = R_0\sqrt[3]{B}, \qquad R_0 = \left(\frac{2\sqrt{2}\lambda}{\mu}\right)^{1/3},$$

TABLE I. Energies of the solutions in the pure Skyrme model, compared with masses for the vector-Skyrme and Skyrme model, as well as with the experimental data. All numbers are in MeV.

B	$E_{\mathrm{experiment}}$	E_{BPS}	$E_{\mathrm{vec\,Skyrme}}$	E_{Skyrme}
1	939	931.75	996	1024
2	1876	1863.5	1999	1937
3	2809	2795.25	2913	2836
4	3727	3727	3727	3727
6	5601	5590.5	$\cdots$	5520
8	7455	7454	$\cdots$	7327
10	9327	9317.5	$\cdots$	9113

which again reproduces the well-known experimental relation. The numerical value which best reproduces the known radii of nuclei is approximately $R_0 = 1.25$ fm. Further, the compact nature of our solutions probably can be viewed as an advantage of the model rather than a defect. In fact, the absence of interactions (or, more precisely, of *finite range interactions*) corresponds quite well with the very short range of forces between nuclei.

Core-type energy density.—For the solution of Sec. II E, which provides spherically symmetric energy densities for all baryon numbers, the resulting energy density takes its maximum value at the origin. It is of some interest to compare this result with the densities in the standard massless or massive Skyrme models. For the massless Skyrme model, solitons are geometrically complicated shell-like structures with empty space regions inside [22]. In addition, the size of the shell Skyrmions grows like $\sqrt{B}$, which is in contradiction to the experimental data. In the case of the massive Skyrme model, the situation is slightly more subtle [6,23,24]. The proper size-charge relation has been reported in [23]. Moreover, depending on the mass of the pion field and baryon number, squeezed clustered solutions, instead of shell ones, begin to be preferred. Precisely speaking, for a fixed value of the mass parameter, the first few Skyrmions possess a shell-like structure, whereas for higher baryon charge a clustered solution is the true minimum. The critical charge, for which a shell Skyrmion occurs, seems to be smaller if the mass is increased [24]. However, even for the physically acceptable value $m = 1$ (which is more or less twice the bare pion mass), Skyrmions with $B \leq 9$ are shells. In the modern interpretation this problem can be cured by treating the massive parameter as a renormalized pion mass, which should be adjusted to best reproduce observed data [24,46]. Then, increasing m, one gets rid of unwanted shell solutions, leaving only clustering ones.

Let us also notice that there is a reminiscence of this clustering phenomenon in the BPS Skyrme model, even though it is quite trivial. Namely, due to the compact and BPS nature of the solutions of the BPS Skyrme model, it is possible to construct a collection of separated components, provided they are sufficiently separated (they do not touch each other). Such a clustered configuration has a total baryon number equal to the sum of the components. In the BPS Skyrme model, none of these clustered (multicenter) solutions is energetically preferred, which again is a simple outcome of the BPS origin of the solitons.

Finally, the values of the energy and charge densities of the solutions of Sec. II E at the center do not depend on the baryon number, which, again, is a property which holds reasonably well for physical nuclei.

The liquid drop property.—The energy functional for static field configurations has the volume-preserving diffeomorphisms on the three-dimensional base space as symmetries. In physical terms, all deformations of solitons

which correspond to these volume-preserving diffeomorphisms may be performed without any cost in energy. But these deformations are exactly the allowed deformations for an ideal, incompressible droplet of liquid when surface contributions to the energy are neglected. These symmetries are not symmetries of a physical nucleus. A physical nucleus has a definite shape, and deformations which change this shape cost energy. Nevertheless, deformations which respect the local volume conservation (i.e., deformations of an ideal incompressible liquid) cost much less energy than volume-changing deformations, as an immediate consequence of the liquid drop model of nuclear matter. This last observation also further explains the nature of the approximation our model provides for physical nuclei. It reproduces some of the classical features of the nuclear liquid drop model, at least on a qualitative level, and the huge amount of symmetries of the model is crucial for this fact. Its soliton energies, e.g., correspond to the bulk (volume) contribution of the liquid drop model, with the additional feature that the energies are quantized in terms of a topological charge.

This should be contrasted with the expected behavior for large baryon number for the standard Skyrme model. In the standard Skyrme model, there remain some long-range forces between different Skyrmions, whose attractive or repulsive character depends on the relative orientation of the Skyrmions. As a consequence, it is expected that for large baryon number the energy-minimizing configurations are Skyrmion crystals, where all the Skyrmions are brought into the right positions and orientations to minimize the total energy. For physical nuclear matter, there is no sign of this crystal-type behavior. Instead, nuclear matter seems to be in a liquid state, which is well described by our BPS Skyrme model.

Let us remark that in QCD at $N_c = \infty$ the instanton liquid becomes incompressible as well [47]. Whether this appearance of an incompressible liquid at large N_c, both in the BPS Skyrme model and in the instanton liquid, is more than a mere coincidence remains to be seen.

Let us also remark that a liquid-drop-like behavior for nuclear matter has been established recently in a rather different approach. Indeed, in [48] the authors obtained both the bulk (volume) term and the surface contribution of the liquid drop model to the nuclear masses from an *ab initio* very idealized and simplified QCD lattice computation, with massless quarks and infinite coupling. In addition, they also found, as in our case, the absence of pseudoscalar meson exchange forces (see next paragraph).

Absence of pion fluctuations.—In the model, both the quadratic and the quartic kinetic terms are absent. As a consequence, neither propagating pions nor the two-body interaction between pions can be described in the model. Nevertheless, already at the classical level the model seems to describe some nuclear properties reasonably well, which seems to indicate that in certain circumstances the sextic

term could be more important than the terms L_2 and L_4. The quadratic term is kinetic in nature, whereas the quartic term provides, as a leading behavior, two-body interactions. On the other hand, the sextic term is essentially topological in nature, being the square of the topological current (baryon current). So in circumstances where our model is successful, this seems to indicate that a *collective* (topological) contribution to the nucleus is more important than kinetic or two-body interaction contributions. This behavior is, in fact, not so surprising for a system at strong coupling (or for a strongly nonlinear system). A first consequence of the absence of pions is the compact nature of the solutions, i.e., the absence of the exponentially decaying pion cloud. A second consequence is the absence of linear pion radiation, and one may wonder whether there exists classical radiation at all in this model. The answer is probably yes, although the study of radiation is inherently nonlinear in compacton-supporting models of this type (the field equations remain nonlinear in the weak-field limit). The simplest way to find some indications of radiation is the study of rotating solitons. In the standard Skyrme model (with a nonzero pion mass), it is found that rotating solutions exist for angular velocities that are not too large, but cease to exist if the angular velocity exceeds a certain limit. The reason for this behavior may be understood easily from the linearized weak-field analysis. If the corresponding angular frequency is too large (essentially larger than the pion mass), then the formal solution is oscillatory instead of exponentially decaying, and so it has infinite energy. Physically, this is interpreted as the onset of pion radiation at that frequency. So one may wonder what happens for rotating solitons in the BPS Skyrme model. Unfortunately, the field equations in this case can no longer be reduced to an ordinary differential equation. There exists, however, a baby Skyrme version of the BPS Skyrme model in one dimension lower, where the dimensional reduction of a rotating baby Skyrmion ansatz to an ordinary differential equation is possible and has been performed in [33]. The result is as follows: the rotating baby Skyrmion solution exists and can be found exactly if the angular velocity remains below a certain critical value. It remains compact, and its radius even shrinks with the angular velocity (although the moment of inertia increases, as one would expect). For frequencies above the critical value, on the other hand, a solution does not exist. This may be viewed as an indication that radiation will also set in for a sufficiently fast rotating BPS Skyrmion, although we repeat that radiation for compactons is an inherently nonlinear and, therefore, complicated problem.

Finally, let us repeat that in the large N_c expansion the meson-meson couplings are of order $1/N_c$. Hence, mesons become free and noninteracting at $N_c = \infty$ [1]. From this perspective, the BPS Skyrme model (at $N_c = \infty$) provides an acceptable, although rather radical, result. That is to say, it not only holds that mesons do not interact; they disappear

completely from the particle spectrum, their only remnants being some collective (solitonic) excitations, and the chiral symmetry breaking aspects of pion dynamics taken into account by the potential. This fact is crucial to cure the unwanted strong forces at intermediate range in the Skyrme model at large N_c.

Summary.—As announced previously, we found that the classical model already describes rather well some features of the liquid drop model of nuclei. These classical results are probably more trustworthy for nuclei that are not too small, for the following reasons:

   (i) The contribution of the pion cloud (which is absent in our model) to the size of the nucleus is of lesser significance for larger nuclei. We remind the reader that, in addition to the core of a nucleus (with a size which grows essentially with the third root of the baryon number), a surface term is known to exist for physical nuclei whose thickness is essentially independent of the baryon number.

  (ii) The description of a nucleus as a liquid drop of nuclear matter is more appropriate for larger baryon number.

 (iii) The contribution of (iso)rotational quantum excitations to the total mass of a nucleus is smaller for larger nuclei, essentially because of the larger moments of inertia of larger nuclei.

We will find further indications for this behavior in the next subsection, where a rigid rotor quantization of the (iso) rotational degrees of freedom is performed for the $B = 1$ nucleon. Indeed, as we shall see, both the corresponding (iso)rotational excitations and the (missing) pion cloud will be of some importance in this case.

B. Quantization

Let us now discuss the issue of quantization of the BPS Skyrme model. As the model is rather unusual, not containing the quadratic, sigma model kinetic part, one might doubt whether the quantization procedure can be performed. However, the sextic derivative term used in the construction (the square of the pullback of the volume form on the target space) is a very special term. It is the unique term with sextic derivatives which leads to a Lagrangian of second order in time derivatives. Therefore, we deal with a Hamiltonian of second order in time derivatives, and the system can be quantized in the standard manner.

We want to perform the semiclassical quantization about a soliton solution in the same way it is performed for the standard Skyrme model. Let us recall that for the nonzero or vibrational modes, the semiclassical quantization consists in a quantization of the quadratic oscillations about the classical solution. These oscillations presumably just amount to renormalizations of the couplings of the theory and therefore may be taken into account implicitly by fitting the model parameters to their physical values. The

zero mode fluctuations related to the symmetries, on the other hand, cannot be approximated by quadratic fluctuations and have to be treated by the method of collective coordinates. In principle, one collective coordinate has to be introduced for each symmetry transformation of the model which acts nontrivially on the soliton about which the quantization is performed. Here, nevertheless, we shall only consider the collective coordinate quantization of the rotational and isorotational degrees of freedom. The physical reason for this restriction is, of course, the fact that the excitational spectra of nuclei are classified exactly by the corresponding quantum numbers of spin and isospin. A more formal justification of this restriction could be, for instance, that the additional collective coordinates do not provide discrete spectra of excitations but, instead, just renormalize the coupling constants, like the vibrational modes do. A definite answer to this question would require a more detailed investigation of the full moduli space of the theory, where all the infinitely many symmetries are taken into account. This is probably a very difficult problem which is beyond the scope of the present paper. A second justification consists in the assumption that, in any case, the given model is just an approximation, whereas a more detailed application to the properties of nuclei requires the inclusion of additional terms in the Lagrangian which, although small in some sense, have the effect of breaking the symmetries down to the ones of the standard Skyrme model.

We start from the classical, static field configuration U_0 found in Sec. II E. For simplicity, we only consider the hedgehog configuration with baryon number $B = 1$. This configuration is invariant under a combined rotation in base and isotopic space; therefore, it is enough to introduce the collective coordinates of one of the two. The allowed excitational states will always have the corresponding quantum numbers of spin and isospin equal, as a consequence of the symmetries of the hedgehog configuration. Following the standard treatment, we introduce the collective coordinates of the isospin by including a time-dependent isorotation of the classical soliton configuration,

$$U(x) = A(t)U_0(x)A^\dagger(t), \qquad (25)$$

where $A(t) = a_0 + ia_i\tau_i \in SU(2)$ and $a_0^2 + \vec{a}^2 = 1$. Inserting this expression into the Lagrangian, we get

$$L = -E_0 + I\,\mathrm{Tr}[\partial_0 A^\dagger(t)\partial_0 A(t)], \qquad (26)$$

where the energy (mass) of the classical solution is

$$E_0 = \frac{64\sqrt{2}\pi}{15}\mu\lambda \qquad (27)$$

and the moment of inertia is

$$I = \frac{4\pi}{3}\lambda^2 \int_0^\infty dr(\sin^4\xi\,\xi_r'^2) = \frac{4\sqrt{2}\pi}{3}\sqrt{\lambda\mu}\left(\frac{3\lambda}{\sqrt{2}\mu}\right)^{2/3}\gamma, \qquad (28)$$

where

$$\gamma = \int_0^\infty dz(z^{2/3}\sin^4\xi\xi_z'^2) = 4\int_0^{4/3} z^{2/3}\left(1 - \left(\frac{3z}{4}\right)^{2/3}\right)dz$$
$$= \frac{32}{35}\left(\frac{4}{3}\right)^{2/3}.$$

Then, finally,

$$I = \frac{2^8\sqrt{2}\pi}{15\cdot 7}\lambda\mu\left(\frac{\lambda}{\mu}\right)^{2/3}. \quad (29)$$

Introducing the conjugate momenta π_n to the coordinates a_n on $SU(2) \simeq S^3$, we get the Hamiltonian

$$H = E_0 - \frac{1}{8I}\sum_{n=0}^3 \pi_n^2 = E_0 - \frac{\hbar^2}{8I}\sum_{n=0}^3 \frac{\partial^2}{\partial a_n^2},$$

where the usual canonical quantization prescription $\pi_n \rightarrow -i\hbar\partial/\partial a_n$ has been performed. Finally, we get

$$H = E_0 + \frac{\hbar^2 I^2}{2I} = E_0 + \frac{\hbar^2 S^2}{2I},$$

where I^2 is the isospin operator (the spherical Laplacian on S^3). We introduced $\hbar$ explicitly because later on we want to use units where $\hbar$ is different from 1. Further, S^2 is the spin operator, and we took into account the equality of the spin and isospin for the hedgehog configuration. It is interesting to note that the isospin operator automatically allows for wave functions on S^3, both for integer isospin (homogeneous polynomials of even degree) and half-odd integer isospin (homogeneous polynomials of odd degree).

The soliton with baryon number 1 is quantized as a fermion. Concretely, the nucleon has spin and isospin 1/2, whereas the Δ resonance has spin and isospin 3/2, so we find for their masses

$$M_N = E_0 + \frac{3\hbar^2}{8I},$$
$$M_\Delta = E_0 + \frac{15\hbar^2}{8I} \Rightarrow M_\Delta - M_N = \frac{3\hbar^2}{2I}, \quad (30)$$

which is exactly like the nucleon and delta mass splitting formula of the standard Skyrme model. The difference comes only from particular expressions for E_0 and I. These expressions may now be fitted to the physical masses of the nucleon ($M_N = 938.9$ MeV) and the Δ resonance ($M_\Delta = 1232$ MeV), which determines the fitted values for the coupling constants. Concretely, we get

$$\lambda\mu = 45.66 \text{ MeV}, \qquad \frac{\lambda}{\mu} = 0.2556 \text{ fm}^3,$$

where we used $\hbar = 197.3$ MeV fm. These may now be used to "predict" further physical quantities like, e.g., the charge radii of the nucleons. For this purpose, we need the linear (i.e., per unit radius) isoscalar and isovector charge densities. These expressions have already been determined for a generalized Skyrme model, including the sextic term, in [16], so we just use these results in the appropriate limit. For the isoscalar (baryon) charge density per unit r, we find

$$\rho_0 = 4\pi r^2 B^0 = -\frac{2}{\pi}\sin^2\xi\xi_r', \quad (31)$$

and for the isovector charge density per unit r,

$$\rho_1 = \frac{1}{I}\frac{4\pi}{3}\lambda^2\sin^4\xi\xi_r'^2. \quad (32)$$

Then the electric charge densities for the proton and the neutron, $\rho_{E,p(n)} = \frac{1}{2}(\rho_0 \pm \rho_1)$, read

$$\rho_{e,p(n)} = \frac{2\sqrt{2}}{\pi}\frac{\mu}{\lambda}r^2\sqrt{1 - \left(\frac{\mu}{2\sqrt{2}\lambda}\right)^{2/3}r^2}$$
$$\times \left(1 \pm \frac{4\sqrt{2}\pi^2\lambda\mu r^2}{3I}\sqrt{1 - \left(\frac{\mu}{2\sqrt{2}\lambda}\right)^{2/3}r^2}\right). \quad (33)$$

The corresponding isoscalar and isovector mean square electric radii are

$$\langle r^2 \rangle_{e,0} = \int dr r^2 \rho_0 = \left(\frac{\lambda}{\mu}\right)^{2/3}, \quad (34)$$

$$\langle r^2 \rangle_{e,1} = \int dr r^2 \rho_1 = \frac{10}{9}\left(\frac{\lambda}{\mu}\right)^{2/3}. \quad (35)$$

Further, the isoscalar magnetic radius is defined as the ratio

$$\langle r^2 \rangle_{m,0} = \frac{\int dr r^4 \rho_0}{\int dr r^2 \rho_0} = \frac{5}{4}\left(\frac{\lambda}{\mu}\right)^{2/3}. \quad (36)$$

With the numerically determined values of the coupling constants, we find the values for the radii displayed in Table II.

In relation to Table II, some comments are appropriate. First, observe that in the BPS Skyrme model all radii are bound by the compacton radius $R_0 = \sqrt{2}(\lambda/\mu)^{(1/3)}$. This bound holds because all radii can be expressed as moments of densities $(\int dr r^n \rho_i)^{(1/n)}$, where ρ_i is a density normalized to 1. Second, all radii in the BPS Skyrme model are

TABLE II. Compacton radius and some charge radii and their ratios for the nucleon. The numbers for the massive Skyrme model are from Ref. [5]. All radii are in fm.

Radius	Experiment	BPS Skyrme	Massive Skyrme
Compacton	$\cdots$	0.90	$\cdots$
Electric isoscalar $r_{e,0}$	0.72	0.64	0.68
Electric isovector $r_{e,1}$	0.88	0.67	1.04
Magnetic isoscalar $r_{m,0}$	0.81	0.71	0.95
$r_{e,1}/r_{e,0}$	1.22	1.05	1.53
$r_{m,0}/r_{e,0}$	1.13	1.12	1.40
$r_{e,1}/r_{m,0}$	1.09	0.94	1.10

TABLE III. Proton and neutron magnetic moments. The numbers for the massive Skyrme model are from Ref. [5].

	Experiment	BPS Skyrme	Massive Skyrme
μ_p	2.79	1.92	1.97
μ_n	-1.91	-1.29	-1.24
$\lvert \mu_p/\mu_n \rvert$	1.46	1.49	1.59

significantly smaller than their physical values, as well as significantly smaller than the values predicted in the standard massive Skyrme model. This, however, has to be expected, because we already know that the pion cloud is absent in the BPS Skyrme model, and the densities strictly go to zero at the compacton radius. We also display the ratios of some radii for the following reason. If the deviations of the BPS Skyrme model radii from their physical values are mainly due to the same "systematic error" (the absence of the pion cloud in the model), then we expect that this "systematic error" will partly cancel in the ratios. This is precisely what happens. The errors in the radii themselves are of the order of 30%, whereas the errors in the ratios do not exceed 15%, providing us with a nice consistency check for our interpretation of the model.

Finally, let us display the numerical results for the magnetic moments of the proton and the neutron. The corresponding expressions are

$$\mu_{p(n)} = 2M_N\left(\frac{1}{12\,\mathcal{I}}\langle r^2\rangle_{e,0} + (-)\frac{\mathcal{I}}{6\hbar^2}\right), \qquad (37)$$

and the resulting numerical values are given in Table III.

The quality of the values is comparable to the case of the standard massive Skyrme model, so the absence of the pion cloud apparently does not have such a strong effect on the magnetic moments.

IV. CONCLUSIONS

In this work we proposed an integrable limit within the space of generalized Skyrme models, which we called *the BPS Skyrme model*. The model consists of two terms: the square of the pullback of the target space volume form (or, equivalently, of the topological density) and a nonderivative part, i.e., a potential. Both terms are required to guarantee the stability of static solutions. This theory possesses rather striking properties. It is integrable; that is, there are infinitely many conserved charges. It is also solvable for any form of the potential with solutions given by quadratures. Further, all solutions are of the BPS type. They obey a first order differential equation and saturate a Bogomol'nyi bound. These properties provide the model with some independent mathematical interest of its own, although in this paper our main concern is its possible relevance as a low-energy effective field theory for strong interaction physics and for the phenomenology of nuclei.

First, let us emphasize again the possible relevance of the BPS Skyrme model in the limit of a large number of colors N_c of the underlying QCD-type theory. Indeed, as was pointed out, e.g., in [20], some problems of the standard Skyrme model when applied to QCD-like theories become more severe in the large N_c limit. For instance, in the Skyrme model rather strong forces of order N_c are generated between nuclei, and the ground state of sufficiently high baryon number tends to be a Skyrmion crystal with binding energies again of order N_c. Both of these findings are in conflict with lattice simulations and with known properties of physical nuclei, respectively. On the other hand, both of these issues are absent in the BPS Skyrme model (there are no long-range forces and no binding energies). So one might speculate that the BPS Skyrme model provides more accurate results as an effective field theory in the large N_c limit. Unfortunately, however, there is no obvious large N_c limit which would produce just the BPS Skyrme model as its leading order, so the rather good large N_c properties of the model must be due to some more subtle mechanism. A better theoretical understanding of the conditions under which the BPS Skyrme theory provides a reasonable limit as an effective field theory for large N_c QCD-like theories would be highly desirable. For instance, the two terms might be enhanced by two different physical mechanisms, where the sextic term is related to some collective or topological excitations, whereas the potential is related, e.g., to the chiral quark condensate of QCD.

Second, in our attempts for direct physical applications of the BPS Skyrme theory, analogously to what is usually done for the standard Skyrme theory, we found that the classical solitons in the BPS Skyrme model already have properties which make this idea worthy of discussion. The mass (energy) of the solitons is proportional to the baryon charge $E \sim |B|$, which, as we know from experimental data, is, to a good degree, a feature of physical nuclei. Moreover, the radii of the solitons also follow the standard experimental law $R_0 \sim |B|^{1/3}$. Additionally, the energy density is of nucleus type. The linear energy-charge relation is valid for all potentials. Thus, it is, in principle, possible to specify a particular potential by fitting the resulting energy density to the experimental data. We find it quite intriguing that this simple BPS Skyrme model gives significantly better approximations—at least on a purely classical level—to the experimental energies (and densities) of baryons than the original Skyrme model (and its generalizations). However, as we neglected the standard kinetic term for the chiral field, there are no obvious pseudoscalar degrees of freedom (η, $\vec{\pi}$). Fortunately, the model contains terms which are of maximally second order in time derivatives, and therefore, it has the standard time dynamics and Hamiltonian interpretation. Thus, it is possible to investigate interactions of such solitons and

identify effective forces between them (and effective degrees of freedom carrying the interactions).

Third, for an application to the physics of nuclei the issue of quantization of the BPS Skyrme solitons certainly has to be investigated. The result is that the standard semiclassical rigid rotor quantization of the rotational and isorotational degrees of freedom of a Skyrmion can be performed in a completely equivalent fashion for the BPS Skyrme model. We explicitly did this rigid rotor quantization for the $B = 1$ Skyrmion (nucleon), and the numerical results conform well with the physical interpretation of the BPS Skyrme model when applied to nuclei. For instance, the absence of the pion cloud is clearly reflected in the values that are too small for the resulting charge radii, whereas ratios of these radii, where the pion cloud effect partially cancels, agree quite well with their experimental counterparts. We did not explicitly calculate the (iso)rotational excitation spectra of Skyrmions with higher baryon charge, but as we still deal with exact solutions having continuous symmetries, one may expect that this task should not be too difficult.

On the other hand, for the standard Skyrme theory and its generalizations, higher Skyrmions have rather complicated discrete symmetries and are known only in numerical form, so their quantization is a rather complicated procedure. Nevertheless, recently the rotational and isorotational excitations of the rigid rotor quantization of the solitons of the standard massive Skyrme model have been applied quite successfully to the corresponding spectra of excitations of light nuclei [6]. As the solutions in the standard Skyrme model are sometimes quite different from ours, one might think that this fact casts some serious doubts on the applicability of our model to the phenomenology of nuclei. Here we just want to point out that this does not have to be the case. In fact, the information which is most important for the spectra of excitations consists in the *symmetries* of the solitons, and not in the full dynamical content of the soliton solutions. These symmetries determine the Finkelstein-Rubinstein constraints on the allowed excited states and, therefore, the spectra of excitations for each baryon number. Further, the solutions in our model typically have higher symmetry due to the special properties of this model.

As a consequence, the following picture is quite plausible. Our model as it stands already describes quite well some bulk properties of nuclei-like masses and charge and energy densities. A more detailed description does require the addition of further terms, but these will be small in some sense (e.g. their contribution to the total mass is small). On the other hand, these additional terms will break the symmetries of the resulting soliton solutions, and these solutions probably have the symmetries of the standard Skyrme model and, consequently, their spectra of excitations. If this symmetry breaking is small, then the spectral lines should still show some approximate degeneracy; that

is, some spectral lines should be spaced more narrowly than others. A detailed investigation of this issue is beyond the scope of the present paper and will be presented in future publications. Of course, in the simplest baryon number-1 case (the hedgehog), the symmetries and the excitational spectra coincide.

There are certainly some further applications of the BPS Skyrme model beyond the realm of nuclear physics. One may, for instance, consider it as a laboratory for Skyrmions of the standard Skyrme model. The BPS model allows for the analytical investigation of problems which can be studied only by advanced 3D numerical simulations in the original Skyrme theory. Or one may consider the BPS model as a lowest order approximation for more complicated generalized Skyrme models and calculate the properties of the latter by a kind of perturbative expansion about the BPS model. These issues are, however, beyond the scope of the present work and shall be investigated elsewhere.

Summarizing, we believe that we have identified and solved an important submodel in the space of Skyrme-type effective field theories, which is singled out both by its capacity to reproduce qualitative properties of the liquid drop approximation of nuclei and by its unique mathematical structure. The model directly relates the nuclear mass to the topological charge, and it naturally provides a finite size of the nuclei, as well as the liquid drop behavior, which probably is not easy to get from an effective field theory. (One may wonder whether it is also possible to get the surface contribution to the energy of the nuclear drop model from an effective field theory, as the BPS Skyrme model does for the volume contribution.) So our model solves a conceptual problem by explicitly deriving said properties from a (simple and solvable) effective field theory. Last but not least, our exact solutions might provide a calibration for the demanding numerical computations in physical applications of more general Skyrme models.

Note added.—After finishing this article we became aware of a simultaneous paper [49], where the authors use a version of the BPS Skyrme model with a different potential to describe the binding energies of higher nuclei. Concretely, they first calculate the exact static soliton solutions plus the (iso)rotational energies in the rigid rotor quantization for general baryon charge $B = n$ for the

spherically symmetric ansatz (Sec. II D in our paper). Then they allow for small contributions to the total energies from the quadratic and quartic Skyrme terms and fit the resulting binding energies to the experimental binding energies of the most abundant isotopes of higher nuclei, assuming, as is usually done, that these correspond to the states with the lowest possible value of the isospin. The resulting agreement between calculated and experimentally determined masses and binding energies is impressive, lending further support to the viability of the BPS Skyrme model as the leading contribution to an effective theory for the properties of nuclear matter.

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