# $\mathcal{N} = 4$, 3D supersymmetric quantum mechanics in a non-Abelian monopole background

Evgeny Ivanov*
*Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia*

Maxim Konyushikhin†
*SUBATECH, Université de Nantes, 4 rue Alfred Kastler, BP 20722, Nantes 44307, France and Institute for Theoretical and Experimental Physics, B. Cheremushkinskaya 25, Moscow 117259, Russia*
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Using the harmonic superspace approach, we construct the 3D $\mathcal{N} = 4$ supersymmetric quantum mechanics of the supermultiplet (**3**, **4**, **1**) coupled to an external SU(2) gauge field. The off-shell $\mathcal{N} = 4$ supersymmetry requires the gauge field to be a static form of the 't Hooft ansatz for the 4D self-dual SU(2) gauge fields, that is a particular solution of Bogomolny equations for Bogomolny-Prasad-Sommerfeld monopoles. We present the explicit form of the corresponding superfield and component actions, as well as of the quantum Hamiltonian and $\mathcal{N} = 4$ supercharges. The latter can be used to describe a more general $\mathcal{N} = 4$ mechanics system, with an arbitrary Bogomolny-Prasad-Sommerfeld monopole background and on-shell $\mathcal{N} = 4$ supersymmetry. The essential feature of our construction is the use of semidynamical spin (**4**, **4**, **0**) multiplet with the Wess-Zumino type action.

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## I. INTRODUCTION

The models of supersymmetric quantum mechanics (SQM) with background gauge fields are of obvious interest for a few reasons. One reason is the close relation of these systems to the renowned Landau problem and its generalizations (see e.g., [1]). The Landau-type models constitute a basis of the theoretical description of quantum Hall effect (QHE), and it is natural to expect that their supersymmetric extensions, with extra fermionic variables added, may be relevant to spin versions of QHE. Also, these systems can provide quantum-mechanical realizations of various Hopf maps closely related to higher-dimensional QHE (see e.g., [2] and references therein). At last, they exhibit $d = 1$ prototypes of couplings to higher-$p$ forms in superbranes and so offer a simplified framework to study these couplings.

$\mathcal{N} = 4$ SQM models with the background Abelian gauge fields were treated in the pioneer papers [3,4] and, more recently, e.g., in [5–8]. In particular, in [6] an off-shell Lagrangian superfield formulation of the general models associated with the multiplets (**4**, **4**, **0**) and (**3**, **4**, **1**) was given in the $\mathcal{N} = 4$, $d = 1$ harmonic superspace.[1] It was found that $\mathcal{N} = 4$, $d = 1$ supersymmetry requires the gauge field to be self-dual in the 4D (**4**, **4**, **0**) case, or to obey a "static" version of the self-duality condition in the three-dimensional (**3**, **4**, **1**) case. In the papers [7,8] it was observed (in a Hamiltonian approach) that the Abelian (**4**, **4**, **0**) $\mathcal{N} = 4$ SQM admits a simple generalization to arbitrary self-dual non-Abelian background.[2] In [11] an off-shell Lagrangian formulation was shown to exist for a particular class of such non-Abelian $\mathcal{N} = 4$ SQM models, with SU(2) gauge group and 't Hooft ansatz [12] for the self-dual SU(2) gauge field (see also [13]). As in the Abelian case, it was the use of $\mathcal{N} = 4$, $d = 1$ harmonic superspace that allowed us to construct such an off-shell formulation. A new nontrivial feature of the construction of [11] is the involvement of an auxiliary "semidynamical" (**4**, **4**, **0**) multiplet with the Wess-Zumino type action possessing an extra gauged U(1) symmetry. After quantization, the corresponding bosonic $d = 1$ fields become a sort of spin SU(2) variables to which the background gauge field naturally couples.[3]

In the present paper, we exploit a similar method to construct $\mathcal{N} = 4$ supersymmetric coupling of the multiplet (**3**, **4**, **1**) to an external non-Abelian gauge field. Like in the (**4**, **4**, **0**) case, it is the $d = 1$ harmonic superspace which makes it possible to perform such a construction in a general form. Off-shell $\mathcal{N} = 4$ supersymmetry is shown to restrict the external gauge field to a static version of the 't Hooft ansatz for 4D self-dual SU(2) gauge field, that is to a particular solution of the general monopole Bogomolny equations [18].[4] A new feature of the (**3**, **4**, **1**)

*eivanov@theor.jinr.ru
†konush@itep.ru

[1]The first superfield formulation of general (**3**, **4**, **1**) SQM (without background gauge field couplings) was given in [9].

[2]The presence of $\mathcal{N} = 4$ supersymmetry in the Dirac operator with a self-dual gauge field was established first in [10], though in an implicit way.

[3]The use of such auxiliary bosonic variables for setting up coupling of a particle to Yang-Mills fields can be traced back to [14]. In the context of $\mathcal{N} = 4$ SQM, they were employed in [2,15–17].

[4]Some BPS monopole backgrounds in the framework of $\mathcal{N} = 2$ SQM were considered, e.g., in [19].

 

case is the appearance of "induced" potential term in the on-shell action as a result of eliminating the auxiliary field of the (**3**, **4**, **1**) multiplet. This term is bilinear in the SU(2) gauge group generators. As a particular "spherically symmetric" case of our construction (with the exact SU(2) $R$ symmetry) we recover, up to an essentially different treatment of the spin variables, the $\mathcal{N}=4$ mechanics with Wu-Yang monopole [20] recently considered in [17].

## II. SUPERFIELD FORMULATION

In the $\mathcal{N}=4$, $d=1$ harmonic superspace (HSS) approach [6], the superfields depend on bosonic variables $t$, $u^{\pm\alpha}$, where the harmonics $u^{+\alpha}$, $u^-_\alpha = (u^{+\alpha})^*$, $u^{+\alpha}u^-_\alpha = 1$ parametrize the $R$ symmetry group SU(2) of the $\mathcal{N}=4$ superalgebra, and on fermionic variables $\theta^\pm = \theta^\alpha u^\pm_\alpha$, $\bar\theta^\pm = \bar\theta^\alpha u^\pm_\alpha$. The most important feature of HSS is the presence of an *analytic subspace* $\{t_{\rm A}, \theta^+, \bar\theta^+, u^\pm_\alpha\}$ in it involving the "analytic time" $t_{\rm A} = t + i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)$ and containing twice as less fermionic coordinates. Spinor derivatives $D^+$ and $\bar D^+$ in the *analytic basis* $\{t_{\rm A}, \theta^\pm, \bar\theta^\pm, u^\pm_\alpha\}$ are [21]

$$D^+ = \frac{\partial}{\partial\theta^-}, \qquad \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}. \tag{1}$$

Other important objects used in what follows are the harmonic derivatives $D^{++}$, $D^{--}$ preserving the $\mathcal{N}=4$ analyticity:

$$D^{++} = u^+_\alpha\frac{\partial}{\partial u^-_\alpha} + \theta^+\frac{\partial}{\partial\theta^-} + \bar\theta^+\frac{\partial}{\partial\bar\theta^-} + 2i\theta^+\bar\theta^+\frac{\partial}{\partial t_{\rm A}}, \tag{2}$$

$$D^{--} = u^-_\alpha\frac{\partial}{\partial u^+_\alpha} + \theta^-\frac{\partial}{\partial\theta^+} + \bar\theta^-\frac{\partial}{\partial\bar\theta^+} + 2i\theta^-\bar\theta^-\frac{\partial}{\partial t_{\rm A}}. \tag{3}$$

Also, for further use, we give how the coordinates of the analytic subspace transform under $\mathcal{N}=4$ supersymmetry:

$$\begin{aligned}
\delta\theta^+ &= \epsilon^\alpha u^+_\alpha,\\
\delta\bar\theta^+ &= \bar\epsilon^\alpha u^+_\alpha,\\
\delta t_A &= 2i(\epsilon^\alpha u^-_\alpha\bar\theta^+ - \bar\epsilon^\alpha u^-_\alpha\theta^+),\\
\delta u^\pm_\alpha &= 0,\\
\bar\epsilon^\alpha &= (\epsilon_\alpha)^*.
\end{aligned}\tag{4}$$

In this paper, we shall deal with the analytic superfields $L^{++}$ and $v^+$, $\tilde v^+$ which encompass, respectively, the multiplets (**3**, **4**, **1**) and (**4**, **4**, **0**) and are subjected to the constraints

$$D^+L^{++} = \bar D^+L^{++} = 0, \tag{5a}$$

$$D^{++}L^{++} = 0, \qquad \widetilde{L} = -L^{++}, \tag{5b}$$

$$D^+(v^+, \tilde v^+) = \bar D^+(v^+, \tilde v^+) = 0, \tag{6a}$$

$$(D^{++} + iV^{++})v^+ = (D^{++} - iV^{++})\tilde v^+ = 0. \tag{6b}$$

The U(1) gauge superfield $V^{++}$ appearing in Eq. (6b) is analytic,

$$D^+V^{++} = \bar D^+V^{++} = 0, \tag{7}$$

and pseudoreal, $V^{++} = \widetilde{V}^{++}$. It ensures the covariance of (6b) under the gauge U(1) transformations with the analytic parameter $\Lambda$ [22]

$$\begin{aligned}
V^{++} &\to V^{++} + D^{++}\Lambda, \qquad v^+ \to e^{-i\Lambda}v^+,\\
\tilde v^+ &\to e^{i\Lambda}\tilde v^+, \qquad D^+\Lambda = \bar D^+\Lambda = 0.
\end{aligned}\tag{8}$$

In what follows, we shall use the Wess-Zumino (WZ) gauge for $V^{++}$,

$$V^{++} = 2i\theta^+\bar\theta^+B. \tag{9}$$

Here $B(t)$ is a real $d=1$ "gauge field", it transforms as $B \to B + \dot\lambda$, with $\lambda(t)$ being the parameter of the residual gauge U(1) symmetry.

The constraints (5a), (6a), and (7) are the $\mathcal{N}=4$ Grassmann analyticity conditions just implying that the superfields $L^{++}$, $v^+$, $\tilde v^+$, $V^{++}$ live on the analytic superspace $\{t_{\rm A}, \theta^+, \bar\theta^+, u^\pm_\alpha\}$. The basic conditions are those with the harmonic derivatives, i.e., (5b) and (6b). They constrain the analytic superfields $L^{++}$ and $v^+$, $\tilde v^+$ to have the appropriate off-shell component field contents, namely, (**3**, **4**, **1**) and (**4**, **4**, **0**):

$$\begin{aligned}
L^{++} = {}&\ell^{\alpha\beta}u^+_\alpha u^+_\beta + i\theta^+\chi^\alpha u^+_\alpha + i\bar\theta^+\bar\chi^\alpha u^+_\alpha\\
&+ \theta^+\bar\theta^+[F - 2i\dot\ell^{\alpha\beta}u^+_\alpha u^-_\beta],
\end{aligned}\tag{10}$$

with $(\ell_{\alpha\beta})^* = -\ell^{\alpha\beta}$, $(\chi^\alpha)^* = \bar\chi_\alpha$, and

$$\begin{aligned}
v^+ = {}&\phi^\alpha u^+_\alpha + \theta^+\omega_1 + \bar\theta^+\bar\omega_2\\
&- 2i\theta^+\bar\theta^+(\dot\phi^\alpha + iB\phi^\alpha)u^-_\alpha,
\end{aligned}\tag{11}$$

$$\begin{aligned}
\tilde v^+ = {}&\bar\phi^\alpha u^+_\alpha + \theta^+\omega_2 - \bar\theta^+\bar\omega_1\\
&- 2i\theta^+\bar\theta^+(\dot{\bar\phi}^\alpha - iB\bar\phi^\alpha)u^-_\alpha,
\end{aligned}\tag{12}$$

with $\bar\phi^\alpha = (\phi_\alpha)^*$, $\bar\omega_{1,2} = (\omega_{1,2})^*$. The multiplet $L^{++}$ involves the 3D target space coordinates $\ell^{\alpha\beta} = \ell^{\beta\alpha}$, their fermionic partners, and a real auxiliary field $F$, while $v^+$ accommodates the auxiliary degrees of freedom needed to arrange a coupling to the external non-Abelian SU(2) Yang-Mills field [11].

The full Lagrangian $\mathcal{L}$ entering the $\mathcal{N}=4$ invariant off-shell action $S = \int dt\mathcal{L}$ consists of the three pieces:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm FI}\\
&= \int dud^4\theta\, R_{\rm kin}(L^{++}, L^{+-}, L^{--}, u)\\
&\quad - \frac{1}{2}\int dud\bar\theta^+d\theta^+\, K(L^{++}, u)v^+\tilde v^+\\
&\quad - \frac{ik}{2}\int dud\bar\theta^+d\theta^+\, V^{++},
\end{aligned}\tag{13}$$

where $L^{+-} = \frac{1}{2}D^{--}L^{++}$ and $L^{--} = D^{--}L^{+-}$. The superfield functions $R_{\rm kin}$ and $K$ bear an arbitrary dependence on their arguments. The meaning of three terms in (13) will be explained in the next section.

## III. FROM HARMONIC SUPERSPACE TO COMPONENTS

The first, sigma-model-type term in Eq. (13), after integrating over Grassmann and harmonic variables, yields the generalized kinetic terms for $\ell^{\alpha\beta}$, $\chi^\alpha$, $\bar{\chi}_\alpha$:

$$\begin{aligned}\mathcal{L}_{\text{kin}} = &\frac{1}{8}f^{-2}(-2\dot{\ell}_{\alpha\beta}\dot{\ell}^{\alpha\beta} + F^2) \\ &+ \frac{i}{8}f^{-2}(\bar{\chi}_\alpha\dot{\chi}^\alpha - \dot{\bar{\chi}}_\alpha\chi^\alpha) + \frac{1}{64}(\partial_{\alpha\beta}\partial^{\alpha\beta}f^{-2})\chi^4 \\ &+ \frac{i}{4f^3}\dot{\ell}^{\alpha\beta}\{\partial_{\alpha\gamma}f\chi_\beta\bar{\chi}^\gamma + \partial_{\beta\gamma}f\chi^\gamma\bar{\chi}_\alpha\} \\ &- \frac{1}{4f^3}F\chi^\alpha\bar{\chi}^\beta\partial_{\alpha\beta}f, \end{aligned} \tag{14}$$

where $\chi^4 = \chi^\alpha\chi_\alpha\bar{\chi}^\beta\bar{\chi}_\beta$, $\partial_{\alpha\beta} \equiv \frac{\partial}{\partial\ell^{\alpha\beta}}$ and $f(\ell)$ is a conformal factor.[5] The fermionic kinetic term can be brought to the canonical form by the change of variables

$$\chi^\alpha = 2f\psi^\alpha, \qquad \bar{\chi}_\alpha = 2f\bar{\psi}_\alpha. \tag{15}$$

It is worth pointing out that the $R$ symmetry SU(2) group amounts to the rotational SO(3) group in the $\mathbb{R}^3$ target space parametrized by $\ell^{\alpha\beta}$. The conformal factor $f(\ell)$ can bear an arbitrary dependence on $\ell^{\alpha\beta}$, so this SO(3) can be totally broken in the Lagrangian (14).

The second piece in Eq. (13) describes the coupling to an external non-Abelian gauge field. Performing the integration over $\theta^+$, $\bar{\theta}^+$ and $u^\pm_\alpha$, eliminating the auxiliary fermionic fields $\omega_{1,2}$ and, finally, rescaling the bosonic doublet variables as $\varphi_\alpha = \phi_\alpha\sqrt{h(\ell)}$, where

$$h(\ell) = \int du K(\ell^{\alpha\beta}u^+_\alpha u^+_\beta, u^\pm_\gamma), \tag{16}$$

after some algebra we obtain

$$\begin{aligned}\mathcal{L}_{\text{int}} = &i\bar{\varphi}^\alpha(\dot{\varphi}_\alpha + iB\varphi_\alpha) + \bar{\varphi}^\gamma\varphi^\delta\tfrac{1}{2}(\mathcal{A}_{\alpha\beta})_{\gamma\delta}\dot{\ell}^{\alpha\beta} \\ &- \tfrac{1}{2}F\bar{\varphi}^\gamma\varphi^\delta U_{\gamma\delta} + \tfrac{1}{4}\chi^\alpha\bar{\chi}^\beta\bar{\varphi}^\gamma\varphi^\delta\nabla_{\alpha\beta}U_{\gamma\delta}. \end{aligned} \tag{17}$$

Here the non-Abelian background gauge field and the scalar (matrix) potential are fully specified by the function $h$ defined in (16):

$$\begin{aligned}(\mathcal{A}_{\alpha\beta})_{\gamma\delta} &= \frac{i}{2h}\{\varepsilon_{\gamma\beta}\partial_{\alpha\delta}h + \varepsilon_{\gamma\alpha}\partial_{\beta\delta}h + \varepsilon_{\delta\beta}\partial_{\alpha\gamma}h + \varepsilon_{\delta\alpha}\partial_{\beta\gamma}h\}, \\ U_{\gamma\delta} &= \frac{1}{h}\partial_{\gamma\delta}h. \end{aligned} \tag{18}$$

[5]The calculations are most easy in the central basis, where $L^{++} = u^+_\alpha u^+_\beta L^{\alpha\beta}(t, \theta_\gamma, \bar{\theta}^\delta)$. Then

$$f^{-2}(\ell) = -\partial_{\alpha\beta}\partial^{\alpha\beta}\int R_{\text{kin}}(\ell^{\alpha\beta}u^+_\alpha u^+_\beta, \ell^{\alpha\beta}u^+_\alpha u^-_\beta, \ell^{\alpha\beta}u^-_\alpha u^-_\beta)du.$$

By definition, the function $h$ obeys the 3D Laplace equation,

$$\partial^{\alpha\beta}\partial_{\alpha\beta}h = 0. \tag{19}$$

Using the explicit expressions (18), it is straightforward to check the relation

$$(\mathcal{F}_{\alpha\beta})_{\gamma\delta} = 2i\nabla_{\alpha\beta}U_{\gamma\delta}, \tag{20}$$

where

$$\begin{aligned}(\mathcal{F}_{\alpha\beta})_{\gamma\delta} = &-2\partial_\alpha{}^\lambda(\mathcal{A}_{\lambda\beta})_{\gamma\delta} + i(\mathcal{A}_\alpha{}^\lambda)_{\gamma\sigma}(\mathcal{A}_{\lambda\beta})_\delta{}^\sigma \\ &+ (\alpha \leftrightarrow \beta), \end{aligned} \tag{21}$$

$$\begin{aligned}\nabla_{\alpha\beta}U_{\gamma\delta} = &-2\partial_{\alpha\beta}U_{\gamma\delta} + i(\mathcal{A}_{\alpha\beta})_{\gamma\lambda}U_\delta{}^\lambda \\ &+ i(\mathcal{A}_{\alpha\beta})_{\delta\lambda}U_\gamma{}^\lambda, \end{aligned} \tag{22}$$

and $(\mathcal{F}_{\alpha\beta})_{\gamma\delta}$ is related to the standard gauge field strength in the vector notation, see below. As we shall see soon, the condition (20) is none other than the static form of the general self-duality condition for the SU(2) Yang-Mills field on $\mathbb{R}^4$, i.e., the Bogomolny equations for BPS monopoles [18], while (18) provides a particular solution to these equations, being a static form of the renowned 't Hooft ansatz [12].

Note that the relation (20) is covariant and the Lagrangian (17) is form-invariant under the following "target space" SU(2) gauge transformations:

$$\begin{gathered}\varphi_\alpha \to (U^\dagger\varphi)_\alpha, \qquad \bar{\varphi}^\alpha \to (\bar{\varphi}U)^\alpha \\ \mathcal{A}_{\alpha\beta} \to \Lambda^\dagger\mathcal{A}_{\alpha\beta}\Lambda + i\Lambda^\dagger\partial_{\alpha\beta}\Lambda, \qquad U \to \Lambda^\dagger U\Lambda, \end{gathered} \tag{23}$$

with $\Lambda(\ell) \in$ SU(2). This is not a genuine symmetry; rather, it is a reparametrization of the Lagrangian which allows one to cast the background potentials (18) in some different equivalent forms. It is worth noting that the gauge group indices coincide with those of the $R$ symmetry group, like in the 4D case [11]. Nevertheless, the "gauge" reparametrizations (23) do not affect the doublet indices of the target space coordinates $\ell^{\alpha\beta}$ and their superpartners accommodated by the superfield $L^{++}$. They act only on the semidynamical spin variables $\varphi_\alpha$, $\bar{\varphi}^\alpha$ and gauge and scalar potentials (18).

Finally, the last piece in Eq. (13) yields the Fayet-Iliopoulos term,

$$\mathcal{L}_{\text{FI}} = kB. \tag{24}$$

In the quantum case, the coefficient $k$ is quantized, $k \in \mathbb{Z}$, on the same ground as in the 4D case [11]. As is obvious from Eqs. (17) and (24), the auxiliary gauge field $B$ serves as a Lagrange multiplier for the constraint

$$\bar{\varphi}^{\alpha}\varphi_{\alpha} = k. \tag{25}$$

In the classical case it implies (together with the residual U(1) gauge freedom) that $\bar{\varphi}^{\alpha}$, $\varphi_{\alpha}$ describe coordinates on a sphere $S^2$ in the target space, while in the quantum case the constraint (25) is imposed on the wave function requiring it to span an irreducible SU(2) multiplet with spin $|k|/2$ [11].

It is instructive to rewrite the above relations and expressions, including the full Lagrangian (13) in a vector notation. To this end, we associate a vector $v_i$ to any traceless bispinor $v_{\alpha}^{\beta}$ by the general rule

$$v_{\alpha}^{\beta} = v_i(\sigma_i)_{\alpha}{}^{\beta}, \qquad v_i = \tfrac{1}{2}v_{\beta}^{\alpha}(\sigma_i)_{\alpha}{}^{\beta}, \qquad i = 1, 2, 3, \tag{26}$$

where $\sigma_i$ are Pauli matrices. In particular, the 3D spinor coordinates $\ell^{\alpha\beta}$ (restricted by the condition $(\ell^{\alpha\beta})^* = -\ell_{\alpha\beta}$) correspond to real vector coordinates $\ell_i$. The only exception from the rule (26) is the relation between the partial derivatives $\partial_{\alpha\beta} = \partial/\partial\ell^{\alpha\beta}$ and $\partial_i = \partial/\partial\ell_i$,

$$\partial_{\alpha\beta} = -\tfrac{1}{2}(\sigma_i)_{\alpha\beta}\partial_i, \qquad \partial_i = -(\sigma_i)_{\alpha}{}^{\beta}\partial_{\beta}^{\alpha}. \tag{27}$$

We also make a similar conversion of the gauge group indices,

$$\begin{aligned} M_{\gamma}{}^{\delta} &= \tfrac{1}{2}M^a(\sigma_a)_{\gamma}^{\delta}, \\ M^a &= M_{\delta}{}^{\gamma}(\sigma_a)_{\gamma}{}^{\delta}, \\ a &= 1, 2, 3, \end{aligned} \tag{28}$$

for any Hermitian traceless $2 \times 2$ matrix $M$, and define

$$T^a = \tfrac{1}{2}\bar{\varphi}^{\alpha}(\sigma_a)_{\alpha}{}^{\beta}\varphi_{\beta}. \tag{29}$$

In the new notations, the total Lagrangian (13) takes the following form:

$$\begin{aligned} \mathcal{L} = {}& \tfrac{1}{2}f^{-2}\dot{\ell}_i^2 + \mathcal{A}_i^a T^a\dot{\ell}_i + i\bar{\varphi}^{\alpha}(\dot{\varphi}_{\alpha} + iB\varphi_{\alpha}) \\ &+ kB + i\bar{\psi}_{\alpha}\dot{\psi}^{\alpha} + f^2\nabla_i U^a T^a\psi\sigma_i\bar{\psi} \\ &+ \tfrac{1}{4}\{f\partial_i^2 f - 3(\partial_i f)^2\}\psi^4 + 2f^{-1}\varepsilon_{ijk}\partial_i f\dot{\ell}_j\psi\sigma_k\bar{\psi} \\ &+ \tfrac{1}{8}f^{-2}F^2 + \tfrac{1}{2}F(U^aT^a - f^{-1}\partial_i f\psi\sigma_i\bar{\psi}). \end{aligned} \tag{30}$$

Here

$$\nabla_i U^a = \partial_i U^a + \varepsilon^{abc}\mathcal{A}_i^b U^c \tag{31}$$

and the Bogomolny Eqs. (20) relating $\mathcal{A}_i^a$ and $U^a$ are equivalently rewritten in the more familiar form,

$$\mathcal{F}_{ij}^{\,a} = \varepsilon_{ijk}\nabla_k U^a, \tag{32}$$

where $\mathcal{F}_{ij}^a = \partial_i\mathcal{A}_j^a - \partial_j\mathcal{A}_i^a + \varepsilon^{abc}\mathcal{A}_i^b\mathcal{A}_j^c$. Finally, the gauge field and the matrix potential defined in (18) are rewritten as

$$\mathcal{A}_i^{\,a} = -\varepsilon_{ija}\partial_j \ln h, \qquad U^a = -\partial_a \ln h, \qquad \Delta h = 0. \tag{33}$$

The component action corresponding to the Lagrangian (30) is partly on shell since we have already eliminated the fermionic fields of the auxiliary $v^+$ multiplet by their algebraic equations of motion. The fields of the coordinate multiplet $L^{++}$ are still off shell. The $\mathcal{N} = 4$ transformations leaving invariant the action $S = \int dt\mathcal{L}$ look most transparent in terms of the component fields $\ell_i$, $F$, $\chi^{\alpha}$, $\bar{\chi}^{\alpha}$, $\phi^{\beta}$, $\bar{\phi}^{\beta}$:

$$\begin{aligned} \delta\ell_i &= -\frac{i}{2}(\epsilon\sigma_i\chi + \bar{\epsilon}\sigma_i\bar{\chi}), \qquad \delta F = \epsilon^{\alpha}\dot{\chi}_{\alpha} + \bar{\epsilon}^{\alpha}\dot{\bar{\chi}}_{\alpha}, \\ \delta\chi^{\alpha} &= iF\bar{\epsilon}^{\alpha} + 2(\bar{\epsilon}\sigma_i)^{\alpha}\dot{\ell}_i, \\ \delta\bar{\chi}^{\alpha} &= -iF\epsilon^{\alpha} - 2(\epsilon\sigma_i)^{\alpha}\dot{\ell}_i, \\ \delta\phi^{\alpha} &= \frac{i}{2}(\epsilon^{\alpha}\chi\sigma_i\phi + \bar{\epsilon}^{\alpha}\bar{\chi}\sigma_i\phi)\partial_i \ln h, \\ \delta\bar{\phi}^{\alpha} &= \frac{i}{2}(\epsilon^{\alpha}\chi\sigma_i\bar{\phi} + \bar{\epsilon}^{\alpha}\bar{\chi}\sigma_i\bar{\phi})\partial_i \ln h. \end{aligned} \tag{34}$$

These transformations can be deduced from the analytic subspace realization of $\mathcal{N} = 4$ supersymmetry (4), with taking into account the compensating U(1) gauge transformations of the superfields $v^+$, $\tilde{v}^+$ and $V^{++}$ needed to preserve the WZ gauge (9). Note that $\delta B = 0$ under $\mathcal{N} = 4$ supersymmetry.[6]

After eliminating the auxiliary field $F$ by its equation of motion,

$$F = 2f^2(f^{-1}\partial_i f\psi\sigma_i\bar{\psi} - U^aT^a), \tag{35}$$

the Lagrangian (30) takes the form

$$\begin{aligned} \mathcal{L} = {}& \frac{1}{2}f^{-2}\dot{\ell}_i^2 + \mathcal{A}_i^a T^a\dot{\ell}_i + i\bar{\varphi}^{\alpha}(\dot{\varphi}_{\alpha} + iB\varphi_{\alpha}) \\ &+ kB + i\bar{\psi}_{\alpha}\dot{\psi}^{\alpha} + f^2\psi\sigma_i\bar{\psi}(\nabla_i + f^{-1}\partial_i f)U^aT^a \\ &+ \frac{1}{4}\{f\partial_i^2 f - 4(\partial_i f)^2\}\psi^4 + 2f^{-1}\varepsilon_{ijk}\partial_i f\dot{\ell}_j\psi\sigma_k\bar{\psi} \\ &- \frac{1}{2}f^2(U^aT^a)^2. \end{aligned} \tag{36}$$

It is invariant, modulo a total time derivative, under the transformations (34) in which $F$ is expressed from (35). We see that this Lagrangian involves three physical bosonic fields $\ell_i$ and four physical fermionic fields $\psi_{\alpha}$. It is fully specified by two independent functions: the metric conformal factor $f(\ell)$ which can bear an arbitrary dependence on $\ell_i$ and the function $h(\ell)$ which satisfies the 3D Laplace equation and determines the background non-Abelian gauge and scalar potentials. The representation (16) for $h$ in terms of the analytic function $K(\ell^{++}, u)$ yields in fact a

[6]This transformation law matches with the $\mathcal{N} = 4$, $d = 1$ superalgebra in WZ gauge, taking into account that the $d = 1$ translation of $B$ looks as a particular U(1) gauge transformation of the latter.

general solution of the 3D Laplace equation [21]. The Lagrangian (36) also contains the "semidynamical" spin variables $\varphi_\alpha$, $\bar{\varphi}^\alpha$, the role of which is the same as in the 4D case [11]: after quantization they ensure that $T^a$ defined in (29) become matrix SU(2) generators corresponding to the spin $|k|/2$ representation.

## IV. HAMILTONIAN AND SUPERCHARGES

The Lagrangian (36) is the point of departure for setting up the Hamiltonian formulation of the model under consideration and quantizing the latter. The main peculiarity of the quantization procedure in the present case is related to the spin variables $\varphi_\alpha$, $\bar{\varphi}^\alpha$. The corresponding commutation relations are

$$[\varphi_\alpha, \bar{\varphi}^\beta] = \delta^\alpha_\beta, \qquad [\varphi_\alpha, \varphi_\beta] = [\bar{\varphi}^\alpha, \bar{\varphi}^\beta] = 0, \quad (37)$$

whence, e.g., $\varphi_\alpha \to \hat{\varphi}_\alpha \equiv \partial/\partial\bar{\varphi}^\alpha$ and the constraint (25) becomes the condition on the wave functions

$$\bar{\varphi}^\alpha \frac{\partial}{\partial\bar{\varphi}^\alpha}\Psi = k\Psi \quad (38)$$

(hereafter, without loss of generality, we assume that $k > 0$). It implies that $\Psi$ is a collection of homogeneous monomials of $\bar{\varphi}^\alpha$ of an integer degree $k$ and, thus, carries an irreducible SU(2) multiplet with spin $k/2$ (the number of such independent monomials is equal just to $k + 1$). The SU(2) vector $T^a$ defined in (27) satisfies the SU(2) commutation relations

$$[T^a, T^b] = i\varepsilon^{abc}T^c, \quad (39)$$

and, as a consequence of the constraint (38), is subject to the condition

$$T^aT^a = \frac{k}{2}\left(\frac{k}{2} + 1\right). \quad (40)$$

In this way, $T^a$ can be treated as generators of the irreducible unitary representation of SU(2) with spin $k/2$.[7]

The system (36) is a generalization, to the non-Abelian case, of the Abelian $\mathcal{N} = 4$ 3D system found in [4], which, in turn, is a generalization, to the conformal metric, of the system in a flat space invented by de Crombrugghe and Rittenberg [3]. After substitution of SU(2) spin-$k/2$ generators instead of $T^a$ [11], the (quantum) Hamiltonian of this system takes the form

[7]The crucial role of the constraint (38) is to restrict the space of quantum states of the considered model to the *finite* set of irreducible SU(2) multiplets of fixed spins (e.g., of the spin $k/2$ in the bosonic sector). This is an essential difference of our approach from that employed, e.g., in [14] (and, lately, in [13,17]) where no any analog of the constraints (25) and (38) is imposed, thus allowing the space of states to involve an *infinite* number of SU(2) multiplets of all spins. The quantization scheme which we follow here can be traced back to the work [23]. In the SQM context, it was already used in [11,16].

$$\begin{aligned} H = {} & \tfrac{1}{2}f(\hat{p}_i - \mathcal{A}_i)^2 f + \tfrac{1}{2}f^2U^2 - f^2\nabla_i U\psi\sigma_i\bar{\psi} \\ & + \Big\{\varepsilon_{ijk}f\partial_i f(\hat{p}_j - \mathcal{A}_j) - f\partial_k fU\Big\}\psi\sigma_k\bar{\psi} \\ & + f\partial^2 f\Big\{\psi^\gamma\bar{\psi}_\gamma - \tfrac{1}{2}(\psi^\gamma\bar{\psi}_\gamma)^2\Big\}, \end{aligned} \quad (41)$$

which is just a static 3D reduction of the 4D Hamiltonian given in [8]. In this expression, the gauge field $\mathcal{A}_i = \mathcal{A}^a_i T^a$ and the scalar potential $U = U^aT^a$ are SU(2) matrices subjected to the constraint (32). It is also easy to find the supercharges $Q_\alpha$, $\bar{Q}^\beta$,

$$\begin{aligned} Q_\alpha &= f(\sigma_i\bar{\psi})_\alpha(\hat{p}_i - \mathcal{A}_i) - \psi^\gamma\bar{\psi}_\gamma(\sigma_i\bar{\psi})_\alpha i\partial_i f \\ &\quad - ifU\bar{\psi}_\alpha, \\ \bar{Q}^\alpha &= (\psi\sigma_i)^\alpha(\hat{p}_i - \mathcal{A}_i)f + i\partial_i f(\psi\sigma_i)^\alpha\psi^\gamma\bar{\psi}_\gamma \\ &\quad + ifU\psi^\alpha, \end{aligned} \quad (42)$$

$$\{Q_\alpha, \bar{Q}^\beta\} = 2\delta^\beta_\alpha H, \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}^\alpha, \bar{Q}^\beta\} = 0. \quad (43)$$

The ordering ambiguity arising in the case of the general conformal factor $f(\ell)$ can be fixed, as in [8], by the arguments of Ref. [24].

We would like to emphasize that the only condition required from the background matrix fields $\mathcal{A}_i$ and $U$ for the generators $Q_\alpha$ and $\bar{Q}^\beta$ to form $\mathcal{N} = 4$ superalgebra (43) is that these fields satisfy the Bogomolny Eqs. (32). Thus, the expressions (41) and (42) define the $\mathcal{N} = 4$ SQM model in the field of *arbitrary* BPS monopole, not necessarily restricted to the ansatz (33). Also, one can extend the gauge group SU(2) to SU($N$) in (41) and (42). The 't Hooft type ansatz (33) and the choice of SU(2) as the gauge group are required for the existence of *off-shell* Lagrangian formulation of this SQM system. We do not know whether the most general system can be derived from some off-shell superfield formalism, though the corresponding component Lagrangian with the on-shell realization of $\mathcal{N} = 4$ supersymmetry can certainly be constructed. It is a straightforward extension of the Lagrangian (30) or (36), with the properly enlarged set of semidynamical spin variables, and the external potentials $\mathcal{A}_i$, $U$ taking values in the $su(N)$ algebra and obeying Eq. (32). This situation is quite similar to what was observed in [8,11] in the case of 4D SQM with self-dual gauge fields.

Finally, as a simple example of the monopole background consistent with the off- and on-shell $\mathcal{N} = 4$ supersymmetry, let us consider a particular 3D spherically symmetric case. It corresponds to the most general SO(3) invariant solution of the Laplace equation for the function $h$

$$h_{\mathrm{so}(3)}(\ell) = c_0 + c_1\frac{1}{\sqrt{\ell^2}}. \quad (44)$$

The corresponding potentials calculated according to Eqs. (33) read

$$\mathcal{A}_i^a = \varepsilon_{ija}\frac{\ell_j}{\ell^2}\frac{c_1}{c_1 + c_0\sqrt{\ell^2}},$$
$$U^a = \frac{\ell_a}{\ell^2}\frac{c_1}{c_1 + c_0\sqrt{\ell^2}}. \tag{45}$$

This configuration becomes the Wu-Yang monopole [20] for the choice $c_0 = 0$. It is easy to find the analytic function $K(\ell^{++}, u)$ which generates the solution (44) (see [6]):

$$h_{\mathrm{so}(3)}(\ell) = \int du K_{\mathrm{so}(3)}(\ell^{++}, u),$$
$$K_{so(3)}(\ell^{++}, u) = c_0 + c_1(1 + a^{--}\hat{\ell}^{++})^{-(3/2)},$$
$$\ell^{++} \equiv \hat{\ell}^{++} + a^{++}, \quad a^{\pm\pm} = a^{\alpha\beta}u^{\pm}_{\alpha}u^{\pm}_{\beta}, \quad a^{\alpha}_{\beta}a^{\beta}_{\alpha} = 2. \tag{46}$$

One could equally choose as $h(\ell)$, e.g., the well-known multicenter solution to the Laplace equation, with the broken SO(3). Note that the $\mathcal{N} = 4$ mechanics with coupling to Wu-Yang monopole was recently constructed in [17], proceeding from a different approach, with the built-in SO(3) invariance and the treatment of spin variables in the spirit of Ref. [14]. Our general consideration shows, in particular, that the demand of SO(3) symmetry is not necessary for the existence of $\mathcal{N} = 4$ SQM models with non-Abelian monopole backgrounds.

## V. RELATION TO THE 4D $\mathcal{N} = 4$ SQM MODEL

It is instructive to show that (33) can indeed be viewed as a 3D reduction of 't Hooft ansatz for the solution of general self-duality equation in $\mathbb{R}^4$ for the gauge group SU(2), with the identification $U^a = \mathcal{A}^a_0$, while the condition (32) as 3D reduction of this equation.

To establish this relation, we use the following dictionary between the SO(4) ~ SU(2) × SU(2) spinor formalism of Refs. [8,11] and its SU(2) reduction:

$$(\sigma_\mu)_{\alpha\dot\beta} \to \{i\delta^\beta_\alpha, (\sigma_i)^\beta_\alpha\}, \qquad \varepsilon^{\dot\alpha\dot\beta} \to -\varepsilon_{\alpha\beta},$$
$$\varepsilon_{\dot\alpha\dot\beta} \to -\varepsilon^{\alpha\beta} \qquad x_{\alpha\dot\beta} \to \ell^\beta_\alpha,$$
$$x^{\alpha\dot\beta} \to -\ell^\alpha_\beta \qquad \psi_{\dot\alpha} \to \psi^\alpha. \tag{47}$$

This reflects the fact that the $R$ symmetry SU(2) in the (**3**, **4**, **1**) models can be treated as a diagonal subgroup in the symmetry group SO(4) ~ SU(2) × SU(2) of the (**4**, **4**, **0**) models, with the SU(2) factors acting, respectively, on the undotted and dotted indices.

The self-dual $\mathbb{R}^4$ SU(2) gauge field in the 't Hooft ansatz used in [11] can be written in the spinor notation as

$$(\mathcal{A}_{\alpha\dot\rho})_\beta{}^\gamma = -\frac{2i}{h}\left(\varepsilon_{\alpha\beta}\partial^\gamma_{\dot\rho}h - \frac{1}{2}\delta^\gamma_\beta\partial_{\alpha\dot\rho}h\right), \qquad \partial_{\alpha\dot\rho} \equiv \frac{\partial}{\partial x^{\alpha\dot\rho}},$$
$$h = h(x^{\alpha\dot\beta}), \qquad \partial^{\alpha\dot\beta}\partial_{\alpha\dot\beta}h = 0. \tag{48}$$

Then, using the rules (47), one performs the reduction $\mathbb{R}^4 \to \mathbb{R}^3$ as

$$(\mathcal{A}_{\alpha\dot\beta})_\gamma{}^\delta \to iU_\gamma{}^\delta\delta^\beta_\alpha + (\mathcal{A}^\beta_\alpha)_\gamma{}^\delta, \qquad (\mathcal{A}^\alpha_\alpha)_\gamma{}^\delta = 0,$$
$$h(x) \to h(\ell), \qquad \partial^\alpha_\beta\partial^\beta_\alpha h = 0. \tag{49}$$

Upon this reduction, the 4D ansatz (48) yields precisely (18), while the general self-duality condition

$$2\partial_{\alpha\dot\rho}(\mathcal{A}^{\dot\rho}_\beta)_\gamma{}^\delta + i(\mathcal{A}_{\alpha\dot\rho})_\gamma{}^\lambda(\mathcal{A}^{\dot\rho}_\beta)_\lambda{}^\delta + (\alpha \leftrightarrow \beta) = 0 \tag{50}$$

goes over into the Bogomolny Eqs. (20). Of course, the same reduction can be performed in the vector notation, with $\mathcal{F}_{\mu\nu} \to \{\mathcal{F}_{ij}, \mathcal{F}_{0k} = \nabla_k U\}$, and Eqs. (32) and (33) as an output.

Thus, the general gauge field background prescribed by the off-shell $\mathcal{N} = 4$ supersymmetry in our (**3**, **4**, **1**) model is a static form of the 't Hooft ansatz for the self-dual SU(2) gauge field in $\mathbb{R}^4$. As was shown in [11], this particular form of the self-dual field is prescribed by the same off-shell $\mathcal{N} = 4$ supersymmetry in the 4D SQM model based on the supermultiplet (**4**, **4**, **0**). This suggests that the above bosonic target space reduction has its superfield counterpart relating the model of [11] to the one considered in the present paper.

Indeed, the superfield (**3**, **4**, **1**) action (13) can be obtained from the (**4**, **4**, **0**) multiplet action of Ref. [11] via the "automorphic duality" [25] by considering a restricted class of the (**4**, **4**, **0**) actions with U(1) isometry and performing a superfield gauging of this isometry by an extra gauge superfield $V^{++\prime}$ along the general line of Ref. [22]. Actually, the bosonic target space reduction we have just described corresponds to the shift isometry of the analytic superfield $q^{+\dot\alpha}$ accommodating the (**4**, **4**, **0**) multiplet, namely, to $q^{+\dot\alpha} \to q^{+\dot\alpha} + \omega u^{+\dot\alpha}$. It is the invariant projection $q^{+\dot\alpha}u^+_{\dot\alpha}$ which is going to become the (**3**, **4**, **1**) superfield $L^{++}$ upon gauging this isometry and choosing the appropriate manifestly $\mathcal{N} = 4$ supersymmetric gauge. Another type of possible isometry of the $q^{+\dot\alpha}$ actions of Ref. [11] is the phase one, with $q^{+\dot 1}q^{+\dot 2}$ as the appropriate invariant. It can also be gauged, with the same $L^{++}$ action as a result.

An important impact of this superfield reduction on the structure of the component action is the appearance of the new induced potential bilinear in the gauge group generators $\sim U^2 = U^aU^bT^aT^b$. It comes out as a result of eliminating the auxiliary field $F$ in the off-shell (**3**, **4**, **1**) multiplet, and so is necessarily prescribed by $\mathcal{N} = 4$ supersymmetry. It is interesting that analogous potential terms were introduced in [26] at the bosonic level for ensuring the existence of some hidden symmetries in the models of the 3D particle in a non-Abelian monopole background.

The same reduction $\mathbb{R}^4 \to \mathbb{R}^3$ can be performed at the level of Hamiltonian and supercharges. In particular, the reduction of the Hamiltonian of the 4D system of [8] yields the 3D Hamiltonian (41).

## VI. CONCLUSIONS

In this paper, we constructed some rather general off-shell $\mathcal{N} = 4$ supersymmetric coupling of the $d = 1$ multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ to an external SU(2) gauge field. The off-shell $\mathcal{N} = 4$ supersymmetry restricts the latter to be a 3D reduction of the 't Hooft ansatz for self-dual SU(2) gauge field in $\mathbb{R}^4$, that is a particular solution of the Bogomolny monopole equations. At the component level, the coupling to a gauge field is necessarily accompanied by an induced potential which is bilinear in the SU(2) generators and arises as a result of eliminating an auxiliary field. Our main devices, as in [11], were the HSS approach and the use of an analytic "semidynamical" multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ with the WZ type action. This multiplet incorporates SU(2) doublet bosonic spin variables which are crucial for arranging couplings to non-Abelian gauge fields. We also presented the explicit form of the corresponding Hamiltonian and $\mathcal{N} = 4$ supercharges which can be equally used for an arbitrary monopole BPS background, though with the on-shell realization of $\mathcal{N} = 4$ supersymmetry.

Like in the case of 4D, $\mathcal{N} = 4$ mechanics coupled to a self-dual non-Abelian gauge field [11], in the 3D case considered here there remains a problem of extending the model to a generic SU($N$) gauge group, as well as to general monopole backgrounds obtained as a 3D reduction of Atiyah-Hitchin-Drinfeld-Manin construction [27]. It would be also interesting to study SQM models with nonlinear counterparts of the target space multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ [6,28] and/or of the semidynamical multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ [22]. Such models exhibit more general target geometries as compared to the conformally flat ones associated with the linear $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet and are capable to yield also more general background gauge fields.

Finally, it is worthwhile to note that similar constraints (Bogomolny equations) on the external non-Abelian 3D gauge field were found in [29], while considering an $\mathcal{N} = 4$ extension of Berry phase in quantum mechanics. However, no invariant actions and/or the explicit expressions for the Hamiltonian and $\mathcal{N} = 4$ supercharges were presented there.

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