Nonperturbative tests for asymptotic freedom in the $\mathcal{PT}$-symmetric $(-\phi^4)_{3+1}$ theory

Abouzeid Shalaby[1,2,*]

[1]*Department of Mathematics and Physics, College of Arts and Sciences, Qatar University, Doha, Qatar*
[2]*Physics Department, Faculty of Science, Mansoura University, Mansoura, Egypt*

Suleiman S. Al-Thoyaib[3,4]

[3]*Physics Department, Faculty of Science, Qassim University, Buraidah, Saudi Arabia*
[4]*The National Center of Mathematics and Physics, King Abdulaziz City for Science and Technology, Riyadh, Saudi Arabia*

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In the literature, the asymptotic freedom property of the $(-\phi^4)$ theory is always concluded from real-line calculations while the theory is known to be a non-real-line one. In this article, we test the existence of the asymptotic freedom in the $(-\phi^4)_{3+1}$ theory using the mean field approach. In this approach and contrary to the original Hamiltonian, the obtained effective Hamiltonian is rather a real-line one. Accordingly, this work resembles the first reasonable analysis for the existence of the asymptotic freedom property in the $\mathcal{PT}$-symmetric $(-\phi^4)$ theory. In this respect, we calculated three different amplitudes of different positive dimensions (in mass units) and find that all of them go to very small values at high energy scales (small coupling) in agreement with the spirit of the asymptotic freedom property of the theory. To test the validity of our calculations, we obtained the asymptotic behavior of the vacuum condensate in terms of the coupling, analytically, and found that the controlling factor Λ has the value $\frac{(4\pi)^2}{6} = 26.319$ compared to the result $\Lambda = 26.3209$ from the literature, which was obtained via numerical predictions. We assert that the nonblowup of the massive quantities at high energy scales predicted in this work strongly suggests the possibility of the solution of the famous hierarchy puzzle in a standard model with the $\mathcal{PT}$-symmetric Higgs mechanism.

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One of the biggest puzzles in the theory of particle interactions is the hierarchy problem [1]. Because of this conceptual problem, all quantities of positive mass dimensions in the scalar sector of the standard model (e.g. the Higgs mass) blow up to unacceptable values at high energy scales. The worst manifestation of this problem is in the vacuum energy as it behaves like μ^4 where μ is a unit mass. This leads to the most unacceptable discrepancy between theory and experiment in the prediction of the cosmological constant (vacuum energy). Indeed, the root of the hierarchy problem stems from the fact that the scalar Higgs mechanism played by the Hermitian ϕ^4 theory has a positive beta function, which up to second order in the ϕ^4 coupling g is given by

$$\beta(g) = \frac{3g^2}{(4\pi)^2}.$$

One can easily show that the positiveness of the beta function of the ϕ^4 theory will lead to a huge Higgs mass at high energy scales [2]. By catching the main reason for the hierarchy problem, one may wonder whether the existence of a scalar theory with a rather negative beta function (i.e. asymptotically free) will help in solving the hierarchy problem in the standard model of particle interactions. In Ref. [2], we argued that such reasoning becomes legitimate since the discovery of the physical acceptability of non-Hermitian and $\mathcal{PT}$-symmetric theories [2–9].

In spite of the beauty of the idea of employing a now physically acceptable theory with bounded from above potential to play the role of the Higgs mechanism, in the literature, all the claims about the asymptotic freedom property of the $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory were built on a real-line calculation (inaccurate) while the theory is well known to be a non-real-line one [10–12]. In view of this realization, our aim in this work is to test the existence of the more than important asymptotic freedom property for $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory but this time using algorithms that proved to be reliable for the study of non-real-line problems.

To shed light on how important it is to employ an asymptotically free $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory to play the role of the Higgs mechanism in the standard model, we mention some possible problems existing in the standard model and its extensions. In the standard model, the spontaneous symmetry breaking adds a large shift to the vacuum energy of the form $\Delta\langle 0|H|0\rangle \sim -CB^4$, where H is the Hamiltonian operator, C is dimensionless, and B is the vacuum condensate [13]. This shift is finite but still large. As we will see in this work, contrary to the corresponding Hermitian theory, the vacuum energy of the $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory is tiny for all energy scales and thus gives a clue to benefits that may be drawn from the employment of this theory in the

*amshalab@mans.edu.eg

standard model. This discrepancy between the features of the two theories (the Hermitian and the non-Hermitian theories) can be understood from the fact that for a theory with negative beta function, the coupling will be dragged to the origin at high energy scales [14], which in turn drag all the dimensionful quantities to small values while the reverse is correct for a theory with positive beta function. Therefore, being of positive beta function, the Hermitian ϕ^4 theory, which plays the role of the Higgs mechanism, originates the gauge hierarchy problem. For the solution of this problem, different algorithms have been introduced. For instance, in the supersymmetry (SUSY) regime there exists natural cancellation in the dimensionful parameters that turned those parameters protected against perturbations even for very high energy scales [15]. However, SUSY introduces an upper limit to the Higgs mass by 130 GeV, and some of its mass spectra are of 1 TeV, which expose this theory directly to the fire of the LHC experiments. Another algorithm for the solution of the hierarchy problem is to consider the Higgs particle as a composite state bounded by a new set of interactions (technicolors) [16]. However, the technicolor model is strongly constrained from precision tests of electroweak theory at LEP and the Stanford Linear Collider experiments [17]. Besides, this algorithm has mass spectra of about 1 TeV and it is under the direct test of the LHC experiments. On the other hand, there exist certain models that do not incorporate the Higgs mechanism at all. For instance, a recent algorithm is suggested for which the $SU(2) \times U(1)$ symmetry is broken via the compactification of an extra dimension [18]. In fact, particles in this model attain masses through the expectation value of the fifth (for instance) component of the gauge field. To some physicists, however, the digestion of the extra dimension is not that easy and can be accepted by them at most as a mathematical modeling to the problem.

The introduction of a new scenario that may overcome the hierarchy problem but does not introduce extra problems may be possible. Indeed, the scalar field in the standard model is the source of the hierarchy problem, and thus it would be very important to have a scalar field with unproblematic features. In this work, we study the flow of different quantities of the positive mass dimension in the $\mathcal{PT}$-symmetric $(-\phi^4)$ theory and show that they do not blow up at high energy scales, which is the reverse of the behavior of the Hermitian ϕ^4 at those scales. Although our calculations are carried out for a one component field while the one used in the standard model has a higher group structure, it is not expected that the group structure will change the amazing asymptotic freedom property of the theory, which means that the results in our work strongly recommend the replacement of the conventional Higgs mechanism by a $\mathcal{PT}$-symmetric one, which then is expected to overcome the hierarchy problem.

The attractive idea of a possible safe employment of the $\mathcal{PT}$-symmetric Higgs mechanism is, in fact, confronted by two main technical problems. The first problem is the belief of the lack of a reliable calculational algorithm to follow for $\mathcal{PT}$-symmetric $(-\phi^4)$ field theory. Among the well known nonperturbative methods that work for $\mathcal{PT}$-symmetric theories is the complex contour treatment applied successfully for the $0+1$ space-time dimensions (quantum mechanics) [7]. However, this method is not willing to be applicable for higher space-time dimensions. Regarding this problem, in a previous work [19], we discovered (for the first time) that the mean field approach, which is famous in field theory calculations, works well for the $\mathcal{PT}$-symmetric $(-\phi^4)$ theory. While its applicability in higher space-time dimensions is not questionable, in Ref. [19], we exposed the effective field approach to a quantitative test and showed that it is accurate even at the level of first order in the coupling, which means that the first problem has been solved. The second problem that is confronting the progress in the study of the $\mathcal{PT}$-symmetric $(-\phi^4)$ theory in the real world of $3+1$ space-time dimensions is that the metric operator for this theory is very hard to be obtained. However, in the mean field regime in Ref. [19], one can realize that the propagator of the $\mathcal{PT}$-symmetric $(-\phi^4)$ theory has the correct sign, which may lead us to think that mean field calculations may have a self-employment of the metric. Recently, Jones *et al* argued that mean field theory may know about the metric. In fact, Jones *et al.* showed that the Green functions in the mean field approach are taking into account the employment of the metric operator [20,21]. Another way to show that the mean field approach knows about the metric is to consider the study in Ref. [22]. In this reference, the authors showed that variational calculations can be done successfully for the ground state energy (effective potential) as long as the coupling of the non-Hermitian term is small. Since the metric operator η_+ has the property $\eta_+ H \eta_+^{-1} = H^\dagger$ where $\eta_+ = \exp(-Q)$ and $Q = Q_0 + \epsilon Q_1 + \epsilon^2 Q_2 + +\epsilon^3 Q_3 + \cdots$, we can get

$$\begin{aligned} H^\dagger &= \exp(-Q) H \exp(Q) \\ &= H + [-Q, H] + \frac{[-Q,[-Q,H]]}{2!} \\ &\quad + \frac{[-Q,[-Q,[-Q,H]]]}{3!} + \cdots. \end{aligned}$$

If the coupling ϵ is small, one can have the relation $[H_0, Q_0] = 0$, where Q_0 is the zeroth order correction to the operator Q and H_0 is the free Hamiltonian. Also, the full metric operator is given by $T = V^\dagger \eta_+$ [22], and in this case the vacuum expectation value can take the form

$$\langle 0|T|0\rangle = \langle \tilde{0}|\eta_+|\tilde{0}\rangle,$$

where $|\tilde{0}\rangle$ is another variational wave function for the ground state. If the coupling of the non-Hermitian term is small, the operator η_+ may be approximated by the unity operator. Accordingly, at least at the limit of applicability of the above mentioned algorithm, the variational calculations employ the full metric of the theory. In fact, for the theory under consideration, at high energy scales the coupling of the non-Hermitian term is tiny and one can trust the variational calculations at least at this limit. In view of these explanations, we think that a nonproblematic $\mathcal{PT}$-symmetric Higgs mechanism is now possible and it is just a matter of known calculations. In fact, the version of the mean field approach used by one of us in Ref. [19] mimics the way of breaking the symmetry in the standard model, and thus we assert that it is the most known plausible method to use for the study of the $\mathcal{PT}$-symmetric Higgs mechanism. However, as a first step toward the employment of the $\mathcal{PT}$-symmetric Higgs mechanism, we need to check the existence of the asymptotic freedom property in the prototype example of the one component $\mathcal{PT}$-symmetric $(-\phi^4)$ theory using the mean field approach.

The motivation behind the application of the mean field approach for the $\mathcal{PT}$-symmetric $(-\phi^4)$ theory is that, in the literature, all the claims about the asymptotic freedom property of the $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory were built on a real-line calculation while the theory is well known to be a non-real-line one. In view of this realization, our main target in this work is to assure the existence of the asymptotic freedom property for $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory but this time using the reliable mean field approach (as used by one of us in Ref. [19]), which implements the use of the metric as well. Although our work stresses the one component $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory, its extension to charged $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory is direct.

Since the calculational algorithm we will follow in this work is the mean field approach as presented in our work in Ref. [19], as a reminder, we summarize its results. To do that, we consider the Lagrangian density of the massless $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory:

$$H(x) = \frac{1}{2}((\nabla\phi(x))^2 + \pi^2(x)) - \frac{g}{2}\phi^4(x),$$

where $\phi(x)$ is the field variable, $\pi(x)$ is the canonical conjugate momentum field, and g is the coupling constant. The effective field approach uses the application of canonical transformations of the form

$$\phi = \psi + B, \qquad \pi = \Pi = \dot{\psi},$$

where B is a vacuum condensate and ψ is a fluctuating field. Thus one obtains the form

$$H = H_0 + H_I,$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}((\nabla\psi)^2 + \Pi^2 + M^2\psi^2), \\ H_I &= -\frac{g}{2}(\psi^4 + 4B\psi^3) + \left(-\frac{1}{2}M^2 - 3gB^2\right)\psi^2 \\ &\quad - 2gB^3\psi, \end{aligned} \tag{1}$$

and M is the mass of the field ψ.

We used the known relations of the effective potential of the form

$$\frac{\partial V_{\text{eff}}}{\partial B} = 0, \qquad \frac{\partial^2 V_{\text{eff}}}{\partial B^2} = M^2, \tag{2}$$

where $V_{\text{eff}} = \langle 0|H|0\rangle$ and M is the mass of the ψ field. Up to first order in the coupling and in $0 + 1$ dimensions, we were able to obtain the equations

$$(-2g)B^3 + \left(-\frac{3}{M}g\right)B = 0, \quad (-6g)B^2 - \frac{3}{M}g = M^2. \tag{3}$$

For $B \neq 0$, one can get the parametrization

$$B = -\sqrt{\frac{M^2}{-4g}}, \qquad M = \sqrt[3]{6g}. \tag{4}$$

Since B is imaginary, the effective Hamiltonian in Eq. (1) is non-Hermitian and $\mathcal{PT}$-symmetric. Moreover, while the original theory is a non-real-line one, the effective form is a real-line theory [23,24]. In fact, this is a very important realization since the real-line effective field calculations can be extended to higher dimensional cases (field theory) for which non-real-line problems cannot be treated using the complex contour method.

To connect the method we used to other algorithms, we mention that the above results have been obtained in Ref. [20] using the Schwinger-Dyson approach. Besides, the conditions $\frac{\partial V_{\text{eff}}}{\partial B} = 0$, $\frac{\partial^2 V_{\text{eff}}}{\partial B^2} = M^2$, we used in Ref. [19] are coincident with the variational conditions $\frac{\partial V_{\text{eff}}}{\partial B} = 0$, $\frac{\partial^2 V_{\text{eff}}}{\partial M} = 0$. In fact, this method not only gives accurate results for the energy spectra and the condensate but also results in the correct sign of the propagator (no ghosts), which is assured by the positiveness of the M^2 parameter. This result is a clue to the possibility of the disappearance of the metric operator from the calculation of the Green functions in the effective field approach. In fact, the assertion that the mean field approach knows about the metric has been clearly proved by Jones *et al* in Refs. [20,21].

According to the above discussion, the effective field approach applied by one of us for the first time in Ref. [19] is a satisfactory algorithm for the calculations in the $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory, which does not need the difficult calculation of the metric operator or the maybe impossible complex contour integrations followed in the quantum mechanical case.

Unlike the quantum mechanical case, in quantum field theory one always is confronted by infinities in the amplitude calculations. Since we restrict ourselves to first order calculations, normal ordering is a valuable tool to exclude infinities at this order of calculations. With this tool, the first order vacuum energy can be obtained by normal ordering of the operators in the effective Hamiltonian with respect to two different mass scales [25–27]. The benefit of this method is that it accounts not only for first order diagrams but also for all the higher order cactus diagrams [28,29]. Moreover, in Ref. [30], it has been shown that the method is equivalent to the first order calculations with the regularization carried out via the introduction of counter terms.

To start the algorithm, consider a normal-ordered Hamiltonian density with respect to a mass parameter m of the form

$$H = N_m\left(\frac{1}{2}((\nabla\phi)^2 + \pi^2 + m^2\phi^2) - \frac{g}{4!}\phi^4\right), \tag{5}$$

where N_m denotes a normal ordering with respect to mass m. We can use the relation [25]

$$N_m \exp(i\beta\phi) = \exp(-\tfrac{1}{2}\beta^2\Delta)N_{M=t\cdot m}\exp(i\beta\phi), \tag{6}$$

with

$$\Delta = \frac{1}{(4\pi)^{(d+1)/2}}\left(\frac{\Gamma(1-\frac{d+1}{2})}{(M^2)^{1-((d+1)/2)}}\right) - \frac{1}{(4\pi)^{(d+1)/2}}\left(\frac{\Gamma(1-\frac{d+1}{2})}{(m^2)^{1-((d+1)/2)}}\right),$$

and d is the spatial dimension, to rewrite the Hamiltonian normal ordered with respect to a new mass parameter $M = \sqrt{t}\cdot m$. In Eq. (6), expanding both sides and equating the coefficients of the same power in β yields the result

$$\begin{aligned} N_m\phi &= N_M\phi, \\ N_m\phi^2 &= N_M^2\phi^2 + \Delta, \\ N_m\phi^3 &= N_M\phi^3 + 3\Delta N_M\phi, \\ N_m\phi^4 &= N_M\phi^4 + 6\Delta N_M\phi^2 + 3\Delta^2. \end{aligned} \tag{7}$$

Also, it is easy to obtain the result [25,27]

$$N_m(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\pi^2) = N_M(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\pi^2) + E_0(M) - E_0(m), \tag{8}$$

where

$$E_0(\Omega) = \frac{1}{4}\int\frac{d^dk}{(2\pi)^d}\left(\frac{2k^2+\Omega^2}{\sqrt{k^2+\Omega^2}}\right) = I_1 + I_2,$$

with

$$\begin{aligned} I_1(\Omega) &= \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{p^2}{\sqrt{p^2+\Omega^2}} \\ &= \frac{1}{2}\frac{1}{(4\pi)^{3/2}}\frac{d}{2}\left(\frac{\Gamma(\frac{1}{2}-\frac{d}{2}-1)}{\Gamma(\frac{1}{2})}\left(\frac{1}{\Omega^2}\right)^{(1/2)-(d/2)-1}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} I_2 &= \frac{1}{4}\int\frac{d^dp}{(2\pi)^d}\frac{\Omega^2}{\sqrt{p^2+\Omega^2}} \\ &= \frac{\Omega^2}{4}\frac{1}{(4\pi)^{3/2}}\left(\frac{\Gamma(\frac{1}{2}-\frac{d}{2})}{\Gamma(\frac{1}{2})}\left(\frac{1}{\Omega^2}\right)^{(1/2)-(d/2)}\right). \end{aligned} \tag{10}$$

Here, Γ is the gamma function. Accordingly,

$$\Delta E_0 = \begin{pmatrix} \frac{3}{8\sqrt{\pi}}\frac{M^4}{\epsilon(4\pi)^{3/2}} - \frac{1}{2\sqrt{\pi}}M^4\left(\left(\frac{\frac{3}{4}\gamma-\frac{5}{8}}{(4\pi)^{3/2}} - \frac{3}{4}\frac{\ln 4\pi}{(4\pi)^{3/2}}\right) - \frac{3}{4}m^4\frac{\ln t}{(4\pi)^{3/2}}\right) + O(\epsilon) \\ -\frac{3}{8\sqrt{\pi}}\frac{m^4}{\epsilon(4\pi)^{3/2}} + \frac{1}{2\sqrt{\pi}}\left(m^4\left(\frac{\frac{3}{4}\gamma-\frac{5}{8}}{(4\pi)^{3/2}} - \frac{3}{4}\frac{\ln 4\pi}{(4\pi)^{3/2}}\right) - \frac{3}{4}m^4\frac{\ln t}{(4\pi)^{3/2}}\right) + O(\epsilon) \\ -\frac{1}{4\sqrt{\pi}}\frac{M^4}{\epsilon(4\pi)^{3/2}} - \frac{1}{4\sqrt{\pi}}M^4\left(\left(\frac{\ln 4\pi}{(4\pi)^{3/2}} - \frac{\gamma-1}{(4\pi)^{3/2}}\right) + \frac{\ln t}{(4\pi)^{3/2}}\right) + O(\epsilon) \\ -\left(-\frac{1}{4\sqrt{\pi}}\frac{m^4}{\epsilon(4\pi)^{3/2}} - \frac{1}{4\sqrt{\pi}}m^4\left(\left(\frac{\ln 4\pi}{(4\pi)^{3/2}} - \frac{\gamma-1}{(4\pi)^{3/2}}\right) + \frac{\ln t}{(4\pi)^{3/2}}\right)\right) + O(\epsilon) \end{pmatrix}, \tag{11}$$

where $\epsilon = \frac{3-d}{2}$, $\Delta E_0 = E_0(M) - E_0(m)$, γ is the Euler number given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln n\right),$$

and

$$t = \frac{\mu^2}{m^2} = \frac{\nu^2}{M^2}.$$

Here μ and ν are unit masses chosen to make the argument of the logarithm dimensionless. Also, the relation $\frac{\mu^2}{m^2} = \frac{\nu^2}{m^2}$ has been employed to fix the renormalization scheme [31]. As $\epsilon \to 0$, ΔE_0 can be simplified as

$$\Delta E_0 = \frac{1}{64\pi^2}(M^4 - m^4)(1 - \gamma + \ln 4\pi + \ln t).$$

The mass shift $m \to M$ should be accompanied by the canonical transformation [27]

$$(\phi, \pi) \to (\psi + B, \Pi). \tag{12}$$

The field ψ has mass $M = \sqrt{t}\cdot m$, B is a constant, the field condensate, and Π is the conjugate momentum ($\dot{\psi}$). Therefore, the Hamiltonian in Eq. (5) can be written in the form

$$H = \bar{H}_0 + \bar{H}_I + \bar{H}_1 + E, \tag{13}$$

where

$$\bar{H}_0 = N_M\left(\frac{1}{2}(\Pi^2 + (\nabla\psi)^2)\right) + \frac{1}{2}N_M\left(m^2 - \frac{g}{2}(B^2+\Delta)\psi^2\right),$$
$$\bar{H}_I = \frac{-g}{4!}N_M(\psi^4 + 4B\psi^3),$$

and $\bar{H}_1$ can be found as

$$\bar{H}_1 = N_M\left(m^2 - \frac{g}{4!}(4B^2+3\Delta)\right)B\psi. \tag{14}$$

Also, the field independent terms can be regrouped as

$$E = \frac{1}{2}\left(m^2 - \frac{12g\Delta}{4!}\right)B^2 - \frac{g}{4!}B^4 + \Delta E_0 - \frac{3g\Delta^2}{4!} + \frac{1}{2}m^2\Delta. \tag{15}$$

In taking $d = 3 - 2\epsilon$, we get

$$\Delta = \frac{1}{(4\pi)^{(d+1)/2}}\left(\frac{\Gamma(1-\frac{d+1}{2})}{(M^2)^{1-((d+1)/2)}}\right) - \frac{1}{(4\pi)^{(d+1)/2}} \times\left(\frac{\Gamma(1-\frac{d+1}{2})}{(m^2)^{1-((d+1)/2)}}\right) = \frac{1}{16\pi^2}(M^2-m^2)(\gamma - 1 - \ln 4\pi + \ln t).$$

Substituting for ΔE_0 and Δ in Eq. (15), we get the effective potential $E = \langle 0|H|0\rangle$ as

$$E = \frac{1}{2}\left(m^2 - \frac{12g}{4!}\frac{1}{16\pi^2}(M^2-m^2) \times(\gamma - 1 - \ln 4\pi + \ln t)\right)B^2 - \frac{g}{4!}B^4 + \left(\frac{1}{64\pi^2}(M^4-m^4)(1-\gamma+\ln 4\pi + \ln t)\right) - \frac{3g(\frac{1}{16\pi^2}(M^2-m^2)(\gamma-1-\ln 4\pi+\ln t))^2}{4!} + \frac{1}{2}m^2\left(\frac{1}{16\pi^2}(M^2-m^2)(\gamma-1-\ln 4\pi+\ln t)\right). \tag{16}$$

Or,

$$\frac{E}{m^4} = \frac{1}{2}\left(1 - \frac{12g}{4!}\frac{1}{16\pi^2}(t-1)(\gamma-1-\ln 4\pi+\ln t)\right) \times\frac{b^2}{(16\pi^2)} - \frac{g}{4!}\left(\frac{b}{4\pi}\right)^4 + \left(-\frac{1}{64\pi^2}(t^2-1)(\gamma+\ln t-\ln 4\pi-1)\right) - \frac{3g(\frac{1}{16\pi^2}(t-1)(\gamma-1-\ln 4\pi+\ln t))^2}{4!} + \frac{1}{2}\left(\frac{1}{16\pi^2}(t-1)(\gamma-1-\ln 4\pi+\ln t)\right), \tag{17}$$

where we used the dimensionless parameter b such that $b = \frac{4\pi B}{m}$. Also, in using dimensionless quantities of the form $e = \frac{32\pi^2}{m^4}E$ and $g = (4\pi)^2 G$, we have the dimensionless vacuum energy of the form

$$e = -\left(\frac{1}{2}G(t-1)(\gamma+\ln t-\ln 4\pi-1)-1\right)b^2 - \frac{G}{12}b^4 + \left(-\frac{1}{2}(t^2-1)(\gamma+\ln t-\ln 4\pi-1)\right) - \frac{G((t-1)(\gamma-1-\ln 4\pi+\ln t))^2}{4} + ((t-1)(\gamma-1-\ln 4\pi+\ln t)). \tag{18}$$

To understand well the features of this result, let us note that the effective potential is the generating functional of the one-particle irreducible amplitudes [13]. This fact results in the relation

$$\frac{\partial^n}{\partial b^n}E(b,t,G) = g_n, \tag{19}$$

where g_n is related to the n-point Green function. For instance, the two-point function can be generated from the effective potential via the second derivative of the effective potential with respect to the condensate, i.e.,

$$\frac{\partial^2 E}{\partial B^2} = -iD^{-1} = M^2, \tag{20}$$

where D is the propagator. Since B does not depend on the position (zero momentum), we have $D = i/(p^2 - M^2) = -i/M^2$. Thus, $g_2 = M^2 = -\frac{1}{2}gB^2 + m^2 - \frac{1}{2}g\Delta$. Since in our work M^2 is constrained to be positive, this shows that the propagator has the correct sign (no ghosts). This unexpected result has been explained by Jones *et al.* as the mean field approach in $\mathcal{PT}$-symmetric theories knows about the metric.

To analyze our results, we note that the stability condition $\frac{\partial E}{\partial B} = 0$ enforces $\bar{H}_1$ in Eq. (14) to be zero. Accordingly, we get the results

$$\frac{1}{3}b(-Gb^2 + 3G(t-1) \times(-\gamma + 2\ln 2 + \ln\pi + 1 - \ln t) + 6) = 0,$$
$$G(t-1)(-\gamma+2\ln 2+\ln\pi+1-\ln t) + 2 - Gb^2 = 2t. \tag{21}$$

From these equations one can obtain the reparametrizations

$$b = \sqrt{-\frac{3t}{G}}, \tag{22}$$

$$2G(t-1)(\gamma - 2\ln 2 - \ln\pi + \ln t - 1) - 4 = 2t. \tag{23}$$

Note that these results show that the vacuum condensate predicted from Eq. (22) is imaginary, and thus the Hamiltonian in Eq. (13) is non-Hermitian but $\mathcal{PT}$-symmetric.

One can solve Eq. (23) for t as a function of G and thus can obtain the dependence of the vacuum energy e and the vacuum condensate b on the coupling G. In Figs. 1–3 we show the calculations of the vacuum energy, Higgs mass squared, and the vacuum condensate, respectively, as a function of the coupling G. Before we go on, let us first check the accuracy of our calculations. In Ref. [9], Bender *et al.* obtained the behavior of the one point function b as $G \to 0^+$ (numerically). They showed that b goes to zero as $\exp(-\frac{\Lambda}{\epsilon})$, where Λ is called the controlling factor and ϵ there is related to the coupling. In $3+1$ space-time dimensions, they obtained the numerical value $\Lambda = 26.3209$. Now, Eq. (23) can be rewritten as

$$G = \frac{t+2}{(t-1)\ln\frac{t}{c}},$$

where c is given by

$$c = \exp(-(\gamma - 2\ln 2 - \ln\pi - 1)).$$

Accordingly, as the parameter $t \to 0^+$, the coupling $G \to 0^+$. Now writing Eq. (23) in the form

$$G(\gamma - 2\ln 2 - \ln\pi + \ln t - 1) = \frac{t+2}{(t-1)}.$$

Since $t \to 0^+$ as $G \to 0^+$, then at this limit one can approximate this equation by

$$\begin{aligned} G(\gamma - 2\ln 2 - \ln\pi + \ln t - 1) &\approx -(t+2)(1+t), \\ &\approx -2 - 3t, \end{aligned} \tag{24}$$

which can be solved to give

$$t = \frac{1}{3} G\omega\left(\frac{3}{G} e^{-((Gc+2)/G)}\right), \tag{25}$$

where $\omega(x)$ is the Lambert ω function defined by $\omega(x)e^{\omega(x)} = x$. Note that, for small arguments $\omega(x) \approx x$ and thus as $G \to 0^+$, the parameter t can be approximated by

$$t = \exp(-c)\exp\left(\frac{-2}{G}\right), \tag{26}$$

and in using Eq. (22), we obtain the asymptotic behavior of the one point function as $G \to 0^+$ in the form

$$b = \sqrt{-\frac{3\exp(-c)\exp(\frac{-2}{G})}{G}} = \pm i\sqrt{\frac{3}{G}} e^{-(1/2)c} e^{-1/G}. \tag{27}$$

This shows that we were able to obtain the exponential behavior for the condensate (analytically and for the first time) predicted numerically in Ref. [9]. Moreover, in accounting for the different coupling used in our work from that in Ref. [9] (they used an interaction term of the form $\frac{g}{4}\phi^4$ while we used $\frac{g}{4!}\phi^4$), we find that the controlling factor is given by $\Lambda = \frac{(4\pi)^2}{6} = 26.319$ compared to the numerical prediction of Λ in Ref. [9] as 26.3209. This result assures the reliability of our analytic calculations.

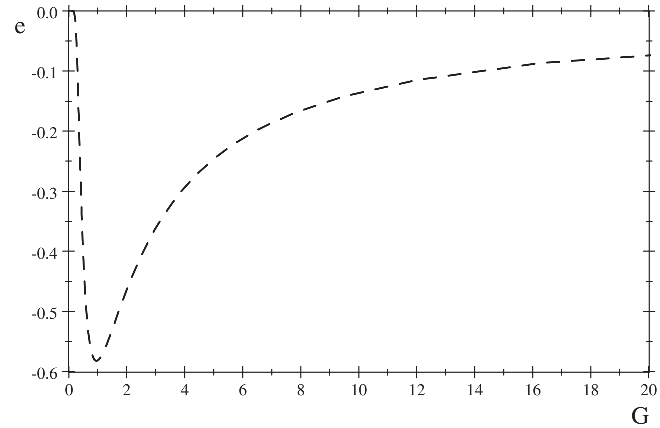

FIG. 1. The vacuum energy (has mass dimensions of 4) of the $(-\phi^4)$ scalar field theory. For either G small (high energy scales) or large (IR energy scales) the vacuum energy is finite.

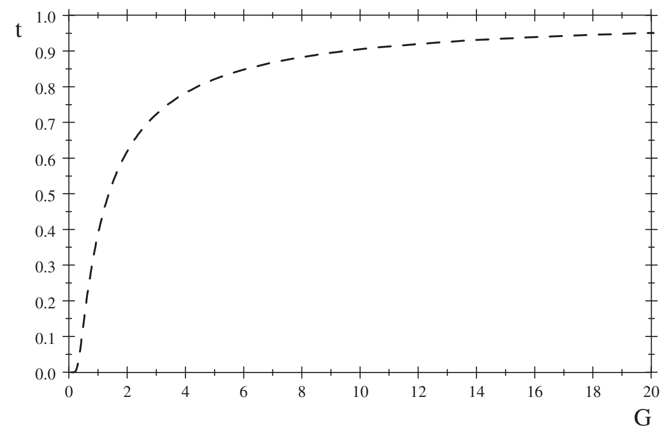

FIG. 2. The mass squared t ($t = \frac{M^2}{m^2}$) of the $(-\phi^4)_{3+1}$ scalar field theory, which has mass dimensions of 2. In this case also for either G small (high energy scales) or large (IR energy scales) t is finite.

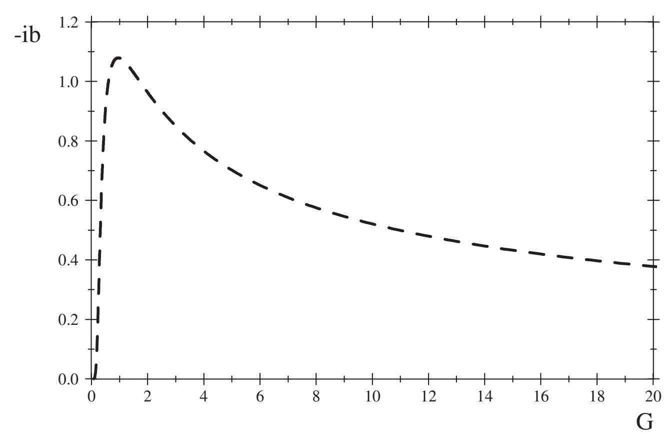

FIG. 3. The absolute value of the vacuum condensate (has mass dimensions of 1) $|b|$ of the $(-\phi^4)$ scalar field theory. For either G small (high energy scales) or large (IR energy scales) the vacuum condensate is finite too.

The result $t \to 0^+$ as $G \to 0^+$ obtained above seems to be strange as one expects that $t \to 1$ (no quantum corrections) at this limit. To explain this result, we mention that, up to the first order correction, G here is the normalized coupling, and if the theory is asymptotically free, it means that at the UV scales $G \to 0$. Accordingly, at this limit all quantities of positive mass dimensions go to zero as well. Therefore, the result $t \to 0$ as $G \to 0^+$ coincides with the spirit of the asymptotic freedom property concluded from the real-line perturbative calculations. In other words, Eq. (23) agrees with the renormalization group flow of the coupling, small values of the coupling correspond to high energy scales and vice versa.

While quantities that have positive dimension in terms of mass unit for the Hermitian ϕ^4 theory blow up at high energy scales (large coupling in this case), the corresponding quantities in the $\mathcal{PT}$-symmetric $(-\phi^4)$ theory tend to tiny values at high energy scales (small coupling in this case). This realization pushes us to believe that the employment of the $\mathcal{PT}$-symmetric Higgs mechanism in the standard model may solve the famous hierarchy problem. In fact, the interesting asymptotic freedom feature is appearing in our calculations presented in Figs. 1–3. For instance, in Fig. 1, we plotted the vacuum energy, which has a mass dimension of 4, and it is clear that the vacuum energy goes to zero as $G \to 0^+$. This is a very important result because in the corresponding Hermitian theory, vacuum energy represents the worst case of the hierarchy problem, which introduces the cosmological constant problem. In view of our analysis, we strongly believe that the employment of the $\mathcal{PT}$-symmetric Higgs mechanism will solve the cosmological constant problem too.

In Fig. 2, a quantity of mass dimension 2 is represented, which assures the existence of the asymptotic freedom in the $\mathcal{PT}$-symmetric $(-\phi^4)$ field theory. Another quantity of mass dimension of 1, the vacuum condensate, has been plotted in Fig. 3; again it confirms the existence of the asymptotic freedom property in the respective theory.

Let us now speculate about the actual case in the standard model where the Higgs mass dominant contribution has the form [1]

$$M_H^2 = M_0^2 - \frac{\Lambda^2}{8\pi^2 v^2}[M_H^2 + 2M_w^2 + M_Z^2 - 4M_t^2], \quad (28)$$

where the different parameters are defined in Ref. [1] (Λ here represents a momentum cutoff). Since all the species in the standard model attain their masses from the vacuum condensate, the masses then will have even sharper than an exponential decrease in Λ [32]. Hence, one expects that all the particles in the standard model (including the Higgs) have finite masses at high energy scales. In fact, if the view introduced in this work persists even in the actual case, we may think about the existence of a composite structure of the Higgs particle (Higgs balls) and to guess the search of the Higgs (it will be then a strongly interacting particle) to be twisted to mimic the same way we search for quarks and gluons [33].

To conclude, we have calculated the vacuum energy for the non-Hermitian and $\mathcal{PT}$-symmetric $(-\phi^4)$ scalar field theory in $3 + 1$ dimensions by using the reliable effective field approach. We find that the vacuum energy is small for the whole range of energy scales, which enhances the belief that this theory is a very good candidate to play the role of the Higgs mechanism in the standard model of particle interactions. We assert that the conventional Hermitian (ϕ^4) has a vacuum energy that blows up at high energy scales as a manifestation of the famous hierarchy puzzle. In fact, this adds a large value to the cosmological constant and thus enhances the unacceptable discrepancy between theoretical and experimental predications of the cosmological constant. Accordingly, the discussions here support the fruitfulness of replacing the Hermitian Higgs mechanism by the $\mathcal{PT}$-symmetric $(-\phi^4)$ one.

Since in the literature the asymptotic freedom is always assumed for the theory under consideration by the aid of real-line calculations, we generated the diagrams in Figs. 1–3 using our effective field calculations. Rather than the original Hamiltonian model, the effective Hamiltonian is a real-line theory, and thus conclusions from our calculations are then reliable. In fact, the quantities generated in the graphs have positive mass dimensions, and according to asymptotic freedom they have to vanish at very high energy scales (small coupling), which is very clear from the figures. While the naive perturbation analysis in Ref. [2] shows the same result, the Higgs mass blows up at small energy scales (large coupling). However, at large couplings perturbations are meaningless and conclusions have to be drawn from nonperturbative treatments of the theory. In fact, the nonperturbative effective field approach we used has cured this problem, as we can realize from Fig. 2 that the mass parameter is finite for the whole energy range (or the whole range of the coupling G).

The effective field approach used in this work also knows about the metric as one can realize that the propagator has its correct sign. This prediction was recently proved by Jones *et al.* in Refs. [20,21]. Accordingly, there exists no need to take care about the so far unattained metric operator. In view of this realization and the fact that the effective Hamiltonian is real-line, one can claim that the algorithm we followed here is sufficient to tackle the theory and can be employed easily to the realistic case of the standard model. Moreover, the accuracy of the algorithm has been tested in a quantitative manner in Ref. [19], and it was found that the effective field approach is reliable for the study of $\mathcal{PT}$-symmetric theories.

To check the validity of our calculations, we obtained the asymptotic behavior of the vacuum condensate as a function of the coupling when $G \longrightarrow 0^+$. At this limit, we found that the condensate behaves like $\exp(\frac{-1}{G})$, a result that was predicted by Bender *et al.* in Ref. [9] using seminumerical calculations. Moreover, we obtained the controlling factor $\Lambda = \frac{(4\pi)^2}{6}$, which is very close to the numerical prediction obtained by Bender *et al.* in Ref. [9].

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