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Cancellation of 4-derivative terms in Volkov-Akulov action

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Recently Kuzenko and McCarty observed the cancellation of 4-derivative terms in the $D = 4$ $\mathcal{N} = 1$ Volkov-Akulov supersymmetric action for the fermionic Nambu-Goldstone field. Here is presented a simple algebraic proof of the cancellation based on using the Majorana bispinors and Fierz identities. The cancellation shows a difference between the Volkov-Akulov action and the effective superfield action recently studied by Komargodski and Seiberg and containing one 4-derivative term. We find out that the cancellation effect takes place in coupling of the Nambu-Goldstone fermion with the Dirac field. Equivalence between the Komargodski-Seiberg (KS) and the Volkov-Akulov (VA) Lagrangians is proved up to the first order in the interaction constant of the Nambu-Goldstone (NG) fermions.

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I. INTRODUCTION

A general approach to the construction of the phenomenological Lagrangians for the Nambu-Goldstone bosons associated with arbitrary group G, spontaneously broken to its subgroup H, was studied in the known papers [1,2]. Volkov's approach [2] uses the powerful Cartan's formalism of the exterior differential ω-forms resulting in the invariant phenomenological Lagrangians of the interacting Nambu-Goldstone (NG) bosons

$$\mathcal{L} = \frac{1}{2} Sp(G^{-1}dG)_k(G^{-1}dG)_k, \qquad G = KH, \tag{1}$$

where the differential 1-forms $G^{-1}dG = H^{-1}(K^{-1}dK)H + H^{-1}dH$ represent the vielbein $(G^{-1}dG)_k$, and the connection $(G^{-1}dG)_h$ associated with the vacuum symmetry subgroup H. The generalization of the NG boson conception to the fermions with spin $1/2$ associated with the spontaneous breaking of supersymmetry was proposed by Volkov in [3] and their action was consructed in [4].

The idea of the fermionic Nambu-Goldstone particles attracts much attention and was discussed in many papers. As the NG fermion field gives a nonlinear realization of supersymmetry, its connection with the linear realization and superfields is an important issue within the spontaneous symmetry breaking theory. Light onto this question was shed in papers [5–8]. In [6], Ivanov and Kapustnikov generalized the known theorems of the nonlinear realization theory of the internal symmetries [1] to the case of supersymmetry. They proved that any superfield could be split in a set of independently transforming components with the supersymmetry parameters depending on the NG field. Also, they found that the Volkov-Akulov Lagrangian happened to be discovered in the invariant integration measure, associated with x and θ variable changes in the superfield action. In [6], these changes were expressed in the form of supersymmetry transformations, but with their parameters substituted by the NG fermionic field. On the other hand, in [7], Rocek derived the Volkov-Akulov (VA) Lagrangian starting from the scalar superfield [9] with the invariant constraints put on it. As a result, he revealed the VA Lagrangian to be the component auxiliary field of the scalar superfield expressed through NG field. In [8], Lindstrom and Rocek generalized this approach to the case of the vector superfield [10]. The connection between the linear supersymmetry and constrained superfields was further developed in the recent paper by Komargodski and Seiberg [11], where a new superfield formalism for finding the low-energy Lagrangian of the NG fermionic and other fields was proposed, and its connection with the VA Lagrangian was considered.[1] The connection stimulates some questions and further studies in this direction. Our interest, in particular, is motivated by the Kuzenko and McCarty paper [12], where they observed the complete cancellation among 4-derivative terms in the $D = 4$ $\mathcal{N} = 1$ Volkov-Akulov supersymmetric action.[2] This cancellation shows a difference between the VA [4] and Komargodski-Seiberg (KS) [11] actions and gives rise to the question about the constrained superfield action generating an effective NG Lagrangian without 4-derivative and higher derivative terms. The difference between KS and VA actions originates from the different realizations of the NG fermionic field in the VA and KS actions. In view of the invariance of the both actions under supersymmetry transformations, the problem reduces to a proper redefinition of the NG field. As experience shows, the finding of the explicit redefinition formula may turn out to be an intricate problem due to the presence of higher derivative terms of the NG field (see e.g. [13]). Another question is

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[1]Paolo Di Vecchia attracted my attention to Ref. [11]
[2]Sergei Kuzenko kindly informed me about Ref. [12].

whether such a cancellation takes place in the NG fermion couplings with other fields.

Here we present an independent proof of the cancellation effect [12], based on using the Majorana bispinor representation of the $D = 4$ $\mathcal{N} = 1$ fermionic NG field and the corresponding Fierz rearrangements. We also find out that the cancellation effect occurs in interactions of the NG fermion with other fields. As a result, the 4-derivative and higher terms, associated with the fermionic NG field, are absent in the VA Volkov-Akulov Lagrangian with couplings [4]. We show that the maximal numbers of the NG fermions and their derivatives in the VA Lagrangian of interactions with the Dirac fields equal six and three, respectively. An algorithmic procedure to verify the assumption about equivalency between the KS and VA Lagrangians, based on the redefinition of the KS fermionic field, is discussed, and their equivalence up to the first order in the constant a, describing the interaction between the NG fermions themselves, is proved.

In Sections II, III, and IV, we draw attention to supersymmetry algebra in the Weyl and Majorana representations, the Volkov-Akulov action, and its generalizations including the higher derivative terms. In Section V, we present a new proof of the cancellations of 4-derivative terms in the Volkov-Akulov action. In Section VI, we find out that the cancellation effect takes place in the NG fermion couplings with the Dirac and other fields. The explicit formula, expressing the KS fermionic field through the VA fermion up to the first order in the interaction constant a, is derived in Section VII.

II. SUPERSYMMETRY AND SUPERALGEBRA

The focus here is on the case of $D = 4$, $\mathcal{N} = 1$ supersymmetry, which transformations are given by

$$\theta'_\alpha = \theta_\alpha + \xi_\alpha, \qquad \bar\theta'_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha}, \qquad x'_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} + \frac{i}{2}(\theta_\alpha\bar\xi_{\dot\alpha} - \xi_\alpha\bar\theta_{\dot\alpha}) \tag{2}$$

in the Weyl spinor representation with $x_{\alpha\dot\alpha} = x_m\sigma^m_{\alpha\dot\alpha}$.[3] The Pauli matrices σ_i and the identity matrix σ_0 form a basic set $\sigma_m = (\sigma_0, \sigma_i)$ in the space of all $SL(2C)$ matrices. The Lorentz covariant description uses the second set of the Pauli matrices with the upper indices $\tilde\sigma_m := (\tilde\sigma_0, \tilde\sigma_i) := (\sigma_0, -\sigma_i)$

$$\{\sigma_m, \tilde\sigma_n\} = -2\eta_{mn}, \qquad Sp\,\sigma_m\tilde\sigma_n = -2\eta_{mn}, \qquad \sigma^m_{\alpha\dot\alpha}\tilde\sigma_m^{\dot\beta\beta} = -2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}, \tag{3}$$

where $\eta_{mn} = \mathrm{diag}(-1, 1, 1, 1)$. The matrices σ_m and $\tilde\sigma_m$ are Lorentz invariant similarly to the tensors η_{mn} and the spinor antisymmetric metric $\varepsilon_{\alpha\beta}$ with the components $\varepsilon_{12} = \varepsilon^{21} = -1$. The supersymmetry generators Q^α and their complex conjugate $\bar Q^{\dot\alpha} := -(Q^\alpha)^*$ have the form

$$Q^\alpha = \frac{\partial}{\partial\theta_\alpha} - \frac{i}{2}\bar\theta_{\dot\alpha}\frac{\partial}{\partial x_{\alpha\dot\alpha}}, \qquad \bar Q^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - \frac{i}{2}\theta_\alpha\frac{\partial}{\partial x_{\alpha\dot\alpha}} \tag{4}$$

and form the supersymmetry algebra

$$\{Q^\alpha, \bar Q^{\dot\alpha}\} = -i\frac{\partial}{\partial x_{\alpha\dot\alpha}} = \frac{1}{2}\tilde\sigma_m^{\dot\alpha\alpha}P^m, \qquad \{Q^\alpha, Q^\beta\} = \{\bar Q^{\dot\alpha}, \bar Q^{\dot\beta}\} = [Q^\alpha, P^m] = [\bar Q^{\dot\alpha}, P^m] = 0 \tag{5}$$

together with the translation generator $P^m = i\frac{\partial}{\partial x_m}$.

The supersymmetry transformations (2) and superalgebra (5) are presented in equivalent bispinor form after transition to the Majorana spinors

$$\delta\theta = \xi, \qquad \delta\bar\theta = \bar\xi, \qquad \delta x_m = -\frac{i}{4}(\bar\xi\gamma_m\theta), \qquad \{Q_a, Q_b\} = \frac{1}{2}(\gamma_m C^{-1})_{ab}P^m, \tag{6}$$

where $\bar\theta = \theta^T C$ with the antisymmetric matrix of the charge conjugation C

$$C^{ab} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \qquad Q_a = \frac{\partial}{\partial\bar\theta^a} - \frac{i}{4}(\gamma_m\theta)_a\frac{\partial}{\partial x_m}. \tag{7}$$

The Majorana spinors and the Dirac γ-matrices are defined as in [10]

$$\theta_a = \begin{pmatrix} \theta_\alpha \\ \bar\theta^{\dot\alpha} \end{pmatrix}, \qquad \xi_a = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}, \qquad \gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \tilde\sigma_m & 0 \end{pmatrix}, \qquad \{\gamma_m, \gamma_n\} = -2\eta_{mn}. \tag{8}$$

III. THE VOLKOV-AKULOV ACTION

To construct the phenomenological Lagrangian of the Nambu-Goldstone fermions, the elegant formalism of the invariant Cartan ω-forms [2], unified with supersymmetry by Volkov, was used in [4]. The supersymmetry invariant differential ω-forms in extended superspace with the Grassmannian coordinates θ^I_α have the form

$$\omega^I_\alpha = d\theta^I_\alpha, \qquad \bar\omega_{\dot\alpha I} = d\bar\theta_{\dot\alpha I}, \qquad \omega_{\alpha\dot\alpha} = dx_{\alpha\dot\alpha} - \frac{i}{2}(d\theta^I_\alpha\bar\theta_{\dot\alpha I} - \theta^I_\alpha d\bar\theta_{\dot\alpha I}), \tag{9}$$

where $I = 1, 2, \ldots, N$ is the index of the internal $SU(N)$ symmetry.

In the Majorana representation these fermionic and bosonic 1-forms are

$$\omega = d\theta, \quad \bar\omega = d\bar\theta, \quad \omega_m = dx_m - \frac{i}{4}(d\bar\theta\gamma_m\theta). \tag{10}$$

The ω-forms (9) were used in [4] as the building blocks for the construction of supersymmetric actions for the

[3] We use algebraic agreements accepted in [14].

interacting NG fermions. Possible actions for the fermionic NG fields are constructed in the form of the wedge products of the ω-forms (9), forming hyper-volumes imbedded in the extended superspace. The action candidates have to be invariant under the Lorentz and internal (unitary) symmetries. In the case of the 4D Minkowski space the invariant action of the NG fermions must include the factorized volume element d^4x. This requirement restricts the structure of the admissible combinations of the ω-forms. If such a combination is given by a wedge product of the ω-forms (9) and their differentials, it should have the general number of the differentials equals four. The condition is satisfied by the well known invariant [4]

$$d^4V = \frac{1}{4!}\varepsilon_{mnpq}\omega^m \wedge \omega^n \wedge \omega^p \wedge \omega^q, \tag{11}$$

where $\wedge$ is the wedge product symbol, that gives the supersymmetric extension of the volume element d^4x of the Minkowski space. The supersymmetric volume (11) is invariant under the Lorentz and unitary groups. It does not contain the spinorial 1-forms ω^I_α and $\bar{\omega}_{\dot{\alpha}I}$, but they appear, e.g. in the following invariant products [4]

$$\begin{aligned}\Omega^{(4)} &= \omega^I_\alpha \wedge \bar{\omega}_{\dot{\beta}I} \wedge \tilde{\sigma}^{\dot{\beta}\alpha}_m d \wedge \omega^m,\\ \tilde{\Omega}^{(4)} &= \varepsilon^{\alpha\beta}\omega^I_\alpha \wedge \omega^J_\beta \wedge \bar{\omega}_{\dot{\alpha}I} \wedge \bar{\omega}_{\dot{\beta}J}\varepsilon^{\dot{\alpha}\dot{\beta}},\end{aligned} \tag{12}$$

where $d \wedge \omega^m$ is the exterior differential of ω^m. The additional important symmetry of the invariants (11) and (12) is their independence on the choice of the superspace coordinate realization. It means that the four dimensional hypersurfaces, associated with (11) and (12), may be parametrized by various ways. Because Volkov's idea was to identify the Grassmannian θ coordinates with the fermionic NG fields, associated with the spontaneous breaking of supersymmetry, they must be considered as functions of x. This requirement means transition to the nonlinear realization of supersymmetry.

It explains why the pullbacks of the 4-form d^4V (11) or its generalizations (12) on the 4-dimensional Minkowski subspace were proposed in [4] to generate supersymmetric actions for the fermionic NG fields. As a result of the observations, the differential forms ω_m (10) and d^4V are presented as

$$\begin{aligned}\omega_m &= \left(\delta^n_m - \frac{i}{4}\frac{\partial\bar{\theta}}{\partial x_n}\gamma_m\theta\right)dx_n = W^n_m dx_n,\\ d^4V &= \det W d^4x.\end{aligned} \tag{13}$$

The identification of θ with the fermionic NG field is achieved by the change: $\psi(x) = a^{-1/2}\theta(x)$, where a has sense of the interaction constant $[a] = L^4$ that introduces a supersymmetry breaking scale. This constant restores the correct dimension $L^{-3/2}$ of the fermionic field $\psi(x)$ and the transition to ψ in (13) and d^4V yields the original Volkov-Akulov action [4]

$$S = \frac{1}{a}\int \det W d^4x \tag{14}$$

with the 4×4 matrix $W^n_m(\psi, \partial_m\psi)$ defined by the following relations

$$W^n_m = \delta^n_m + aT^n_m, \qquad T^n_m = -\frac{i}{4}\partial^n\bar{\psi}\gamma_m\psi. \tag{15}$$

An explicit form of the action S (14) follows from the definition of $\det W$

$$\det W = -\frac{1}{4!}\varepsilon_{n_1n_2n_3n_4}\varepsilon^{m_1m_2m_3m_4}W^{n_1}_{m_1}W^{n_2}_{m_2}W^{n_3}_{m_3}W^{n_4}_{m_4}, \tag{16}$$

where we chose $\varepsilon_{0123} = 1$. Using (15) and (16) presents S (14) in the form

$$\begin{aligned}S = \int d^4x\Big[&\frac{1}{a} + T^m_m + \frac{a}{2}(T^m_mT^n_n - T^n_mT^m_n) + a^2T^{(3)}\\ &+ a^3T^{(4)}\Big],\end{aligned} \tag{17}$$

where $T^{(3)}$ and $T^{(4)}$ code the interaction terms of the NG fermions that are cubic and quartic in the fermion derivative $\partial_m\psi$. The first term in (17) provides a nonzero vacuum expectation value for the VA Lagrangian, confirming that it describes the spontaneously broken supersymmetry. In supergravity this term associates with the cosmological term [15,16]. The second term reproduces the free action for the massless NG fermion $\psi(x)$

$$S_0 = \int d^4xT^m_m = -\frac{i}{4}\int d^4x\partial^m\bar{\psi}\gamma_m\psi. \tag{18}$$

The terms $T^{(3)}$ and $T^{(4)}$ cubic and, respectively, quartic in the NG fermion derivatives were presented in [4] in the form

$$\begin{aligned}T^{(3)} &= \frac{1}{3!}\sum_p(-)^pT^m_mT^n_nT^l_l,\\ T^{(4)} &= \frac{1}{4!}\sum_p(-)^pT^m_mT^n_nT^l_lT^k_k,\end{aligned} \tag{19}$$

where the sum $\sum_p$ corresponds to the sum in all permutations of subindices in the products of the tensors T^m_n. The terms (19) describe the vertices with six and eight NG fermions.

IV. HIGHER DERIVATIVE GENERALIZATIONS OF THE VOLKOV-AKULOV ACTION

The ω-form formalism [4] yields a clear geometric way to extend the VA action by the higher degree terms in the NG fermion derivatives. In general case, the combinations of the ω-forms (10), admissible for the higher order invariant actions, have to be the homogeneous functions of the degree four in the differentials dx and $d\psi$. The latter condition guarantees the factorization of the volume element d^4x in the action integral. To restrict the number of these invariants, the minimality condition for the degree of derivatives $\partial_m \psi$ in the general action

$$S = \int d^4 x L(\psi, \partial_m \psi) \tag{20}$$

was proposed in [4]. The minimality condition takes into account only the lowest degrees of the derivatives $\partial_m \psi$ in the Lagrangian and corresponds to the low-energy limit. To count the degree of $\partial_m \psi$ in different invariants observed was that these derivatives appear from the differentials $d\psi$ in the fundamental ω-forms. From this point of view there is an important difference among the spinor and vector 1-forms (9). The spinor 1-forms contain one derivative $\partial_m \psi$, but the vector forms (13) either do not contain the ψ fields at all or contain one derivative $\partial_m \psi$ accompanied by ψ. As a result, the whole number of the derivatives $\partial_m \psi$ with respect to the whole number of the fermionic NG fields is lower in the vector differential 1-forms than in the spinor ones. The invariants including the exterior differential of the ω-forms, like $\Omega^{(4)}$ in (12), have the higher degree in $\partial_m \psi$ in comparison with the product of ω-forms themselves. The same conclusion concerns the invariant $\tilde{\Omega}^{(4)}$ including only the spinor forms.

Thus, the demand of the minimality of the number of the derivatives $\partial_m \psi$ in S (20) will be satisfied if the admissible invariants will contain only the vector differential 1-forms ω_m. The exact realization of the minimality condition by the VA action fixes the latter, and solves the problem of the effective action construction in the low-energy limit.

V. CANCELLATION OF 4-DERIVATIVE TERMS IN THE VOLKOV-AKULOV ACTION

For the case of $\mathcal{N} = 1$ supersymmetry, the algebraic structure of the terms $T^{(3)}$ and $T^{(4)}$ (19) was analyzed in [12] using the Weyl spinor basis. It was observed that the terms having the fourth degree in $\partial_m \psi$ and collected in $T^{(4)}$ completely cancel out.

Here we consider an alternative proof of the observation using the Majorana bispinor representation. In correspondence to representation (16), the term $T^{(4)}$ (19) may be written as

$$\begin{aligned} T^{(4)} &= -\frac{1}{4!}\varepsilon_{n_1 n_2 n_3 n_4}\varepsilon^{m_1 m_2 m_3 m_4} T^{n_1}_{m_1} T^{n_2}_{m_2} T^{n_3}_{m_3} T^{n_4}_{m_4} \\ &= -\frac{1}{4!}(\varepsilon_{n_1 n_2 n_3 n_4} \bar{\psi}^{,n_1}_{a_1} \bar{\psi}^{,n_2}_{a_2} \bar{\psi}^{,n_3}_{a_3} \bar{\psi}^{,n_4}_{a_4}) \\ &\quad \times (\varepsilon^{m_1 m_2 m_3 m_4} \gamma^{a_1 b_1}_{m_1} \gamma^{a_2 b_2}_{m_2} \gamma^{a_3 b_3}_{m_3} \gamma^{a_4 b_4}_{m_4}) \\ &\quad \times (\psi_{b_1} \psi_{b_2} \psi_{b_3} \psi_{b_4}), \end{aligned} \tag{21}$$

where $\gamma^{ab}_m = (C\gamma_m)^{ab}$ is a symmetric matrix in the bispinor indices ($a, b = 1, 2, 3, 4$) and the condensed notation $\bar{\psi}^{,n}_a := \partial^n \bar{\psi}_a$ is introduced. The product $\psi_{b_1} \psi_{b_2} \psi_{b_3} \psi_{b_4}$ in (21) is a completely antisymmetric spin-tensor of the maximal rank four because of the Grassmannian nature of the spinor components ψ_b. Then we find that the product may be presented in the equivalent form as

$$\begin{aligned} \psi_{b_1} \psi_{b_2} \psi_{b_3} \psi_{b_4} = -(&C^{-1}_{b_1 b_2} C^{-1}_{b_3 b_4} + C^{-1}_{b_1 b_3} C^{-1}_{b_4 b_2} \\ &+ C^{-1}_{b_1 b_4} C^{-1}_{b_2 b_3}) \psi_1 \psi_2 \psi_3 \psi_4, \end{aligned} \tag{22}$$

where the antisymmetric matrix C^{-1} is inverse of the charge conjugation matrix C (6). The representation (22) collects all spinors ψ without derivatives in the form of a scalar multiplier. The substitution of (22) in (21) transforms it into the sum of products of the bilinear spinor covariant expressions

$$\begin{aligned} &T^{(4)} = \frac{3}{4!}\Phi\Xi, \qquad \Xi := \psi_1 \psi_2 \psi_3 \psi_4, \\ &\Phi := \varepsilon_{n_1 n_2 n_3 n_4}\varepsilon^{m_1 m_2 m_3 m_4} (\bar{\psi}^{,n_1} \Sigma_{m_1 m_2} \psi^{,n_2})(\bar{\psi}^{,n_3} \Sigma_{m_3 m_4} \psi^{,n_4}), \end{aligned} \tag{23}$$

where $\Sigma_{mn} := \frac{1}{2}[\gamma_m, \gamma_n]$ are the Lorentz transformation generators.

Taking into account the well known property of Σ_{mn}

$$\begin{aligned} &\varepsilon^{m_1 m_2 m_3 m_4} \Sigma_{m_3 m_4} = -2\gamma^5 \Sigma_{m_1 m_2}, \\ &\gamma^5 := \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \end{aligned} \tag{24}$$

one can present the Lorentz invariant Φ (23) in the compact form

$$\Phi = -2\varepsilon_{n_1 n_2 n_3 n_4} (\bar{\psi}^{,n_1} \Sigma_{m_1 m_2} \psi^{,n_2})(\bar{\psi}^{,n_3} \Sigma^{m_1 m_2} \gamma^5 \psi^{,n_4}). \tag{25}$$

Using representation (25), we shall prove the vanishing of Φ. To this end, let us recall the known Fierz relation for the Grassmannian spinors χ_i

$$(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) = -\frac{1}{4} \sum_{N=1}^{16} (\bar{\chi}_1 \Gamma^A \chi_4)(\bar{\chi}_3 \Gamma_A \chi_2), \tag{26}$$

where the 16 Dirac matrices Γ^A and their inverse $\Gamma_A = (\Gamma_A)^{-1}$, defined as

$$\begin{aligned} &\Gamma^A := (1, \gamma^m, \Sigma^{mn}, \gamma^5, \gamma^5 \gamma^m), \\ &\Gamma_A := (\Gamma^A)^{-1} = (1, -\gamma_m, -\Sigma_{mn}, -\gamma^5, -\gamma^5 \gamma_m), \end{aligned} \tag{27}$$

form the complete basis in the space of 4×4 matrices. As a result, we obtain

$$\begin{aligned} \Phi = \frac{1}{2}\varepsilon_{n_1 n_2 n_3 n_4} &\sum_{A=1}^{16} (\bar{\psi}^{,n_1} \Sigma_{m_1 m_2} \Gamma^A \Sigma^{m_1 m_2} \gamma^5 \psi^{,n_4}) \\ &\times (\bar{\psi}^{,n_3} \Gamma_A \psi^{,n_2}). \end{aligned} \tag{28}$$

The right-hand side of (28) includes the products of two bilinear covariant expressions. The second (right) of them $(\bar{\psi}^{,n_3} \Gamma_A \psi^{,n_2})$ is either symmetric or antisymmetric under the permutation $n_3 \leftrightarrow n_2$. Only the antisymmetric covariant expressions generated by $\Gamma_A = (-\gamma_r, -\Sigma_{rs})$ give nonzero contribution to (28). The first (left) covariant expression in (28), corresponding to the above choice of Γ_A, includes either the matrix L_v or L_t given by the expressions

$$L_v = \Sigma_{m_1 m_2}\gamma^r \Sigma^{m_1 m_2}\gamma^5, \quad L_t = \Sigma_{m_1 m_2}\Sigma^{rs}\Sigma^{m_1 m_2}\gamma^5. \tag{29}$$

Using the representation of $\Sigma_{m_1 m_2}$ in the form $\Sigma_{m_1 m_2} = (\eta_{m_1 m_2} + \gamma_{m_1}\gamma_{m_2})$, we obtain the following relations

$$\begin{aligned} &\Sigma_{m_1 m_2}\Gamma^A \Sigma^{m_1 m_2} = 4\Gamma^A - \gamma_{m_1}\gamma_{m_2}\Gamma^A\gamma^{m_2}\gamma^{m_1}, \\ &\gamma_m \gamma^r \gamma^m = 2\gamma^r, \qquad \gamma_m \Sigma^{rs}\gamma^m = 0 \end{aligned} \tag{30}$$

which show that

$$L_v = 0, \qquad L_t = 4\Sigma^{rs}\gamma^5. \tag{31}$$

Using the results (31) permits to present (28) in the next form

$$\Phi = -2\varepsilon_{n_1 n_2 n_3 n_4}(\bar{\psi}^{,n_1}\Sigma^{rs}\gamma^5\psi^{,n_4})(\bar{\psi}^{,n_3}\Sigma_{rs}\psi^{,n_2}). \tag{32}$$

Taking into account the symmetry property $(C\Sigma^{rs}\gamma^5)^{ab} = (C\Sigma^{rs}\gamma^5)^{ba}$ and changing the summation indices $n_3 \leftrightarrow n_1$, one can present the expression (32) in the form

$$\Phi = 2\varepsilon_{n_1 n_2 n_3 n_4}(\bar{\psi}^{,n_1}\Sigma_{rs}\psi^{,n_2})(\bar{\psi}^{,n_3}\Sigma^{rs}\gamma^5\psi^{,n_4}). \tag{33}$$

The matching (25) and (33) yields the expected result

$$\Phi = -\Phi \Rightarrow \Phi = 0, \qquad T^{(4)} = 0 \tag{34}$$

which proves that the 4-derivative term $T^{(4)}$ (21) actually vanishes in agreement with the observation [12].

Thus, the maximal number of derivatives present in the Volkov-Akulov action reduces to three and the action takes the following form

$$\begin{aligned} S = \int d^4x\bigg[&\frac{1}{a} + T^m_m + \frac{a}{2}(T^m_m T^n_n - T^n_m T^m_n) \\ &+ \frac{a^2}{3!}\sum_p (-)^p T^m_m T^n_n T^l_l\bigg] \end{aligned} \tag{35}$$

with the maximal number of NG fermions in the vertices equal to six.

Matching the Lagrangian (35) and the Komargodski and Seiberg Lagrangian [11], having the form

$$\begin{aligned} \mathcal{L}_{\rm KS} = &-f^2 + i\partial_\mu \bar{G}\tilde{\sigma}^\mu G + \frac{1}{4f^2}\bar{G}^2\partial^2 G^2 \\ &- \frac{1}{16f^6}G^2\bar{G}^2\partial^2 G^2 \partial^2 \bar{G}^2, \end{aligned} \tag{36}$$

shows their difference, because of the presence of one 4-derivative term including eight NG fermions in (36). We shall explain that the difference originates from various realizations of the NG field used in the VA and KS Lagrangians. The second question concerns a possibility of such type cancellations in the NG fermion couplings with other fields.

VI. COUPLING OF THE FERMIONIC NAMBU-GOLDSTONE FIELDS WITH THE DIRAC FIELD

Here we show that the above discussed cancellation of the 4-derivative terms also occurs in the NG fermion couplings with other fields. It is easy to see by the application of the general Volkov method [2] in the construction of the phenomenological Lagrangian, describing the NG particles interacting with other fields. The extension of this method, aimed at including the supersymmetric couplings, is based on joining of the differential $d\chi$ of a given field χ, carrying arbitrary spinor and unitary indices, to the set of the supersymmetric ω-forms [4]. Then the above described procedure for the minimal VA action construction, using only the ω-forms (9), may be applied to the enlarged set of these supersymmetric 1-forms. The only restriction on the admissible χ-terms is the demand of their invariance under the Lorentz and the internal symmetry groups. The effective actions must be the homogenious functions of the degree four in the differentials dx, $d\psi$, and $d\chi$, and, generally, it has to restrict the number of the derivatives $\partial_m\psi$ to be less than four. Then the considered cancellations are not relevant. However, if $d\chi$ is absent in the couplings, then the 4-derivative cancellation may take place and will reduce the derivative $\partial_m\psi$ number in the corresponding vertices.

An instructive example of the described possibility gives the $\mathcal{N} = 1$ minimal supersymmetric coupling of the fermionic NG particle with the massive Dirac field χ in the low-energy limit [4]

$$\begin{aligned} S = \int\bigg[&\frac{i}{2}\varepsilon_{mnpq}(\bar{\chi}\gamma^m d\chi - d\bar{\chi}\gamma^m\chi)\wedge\omega^n\wedge\omega^p\wedge\omega^q \\ &+ m\bar{\chi}\chi\varepsilon_{mnpq}\omega^m\wedge\omega^n\wedge\omega^p\wedge\omega^q\bigg]. \end{aligned} \tag{37}$$

The kinetic term of the Dirac field in (37) includes the differential $d\chi$ and the cancellation is absent here. The maximal number of the NG fermions at this term $n_{\mathrm{NG}f}$ equals six and the maximal number $n_{\mathrm{NG}d}$ of their derivatives equals three, just as in the case of the VA Lagrangian (35) after 4-derivative cancellation. The mass term in (37) does not include $d\chi$ and, respectively, it includes the supervolume form d^4V (11), because of the minimality condition. Then the cancellation effect does work and results in the same maximal numbers $n_{\mathrm{NG}f} = 6$ and $n_{\mathrm{NG}d} = 3$ as in the kinetic term. To present (37) in the standard notations [4], we substitute the ω-forms (13) in (37) and obtain

$$\begin{aligned} S = \int d^4x\bigg[&R^m_m + a(R^m_m T^n_n - R^m_n T^n_m) + \frac{a^2}{2}\sum_p (-)^p R^m_m T^n_n T^l_l \\ &+ \frac{a^3}{3!}\sum_p (-)^p R^m_m T^n_n T^l_l T^k_k + m\bar{\chi}\chi\det W\bigg], \end{aligned} \tag{38}$$

where $R^m_n := \frac{i}{2}(\bar{\chi}\gamma^m\partial_n\chi - \partial_n\bar{\chi}\gamma^m\chi)$ is the kinetic term for χ. Using the expression for $\det W$ from (35), the mass term in (38) is presented as

$$m\bar{\chi}\chi\,\det W = m\bar{\chi}\chi + am\bar{\chi}\chi\Big[T_m^m + \frac{a}{2}(T_m^m T_n^n - T_m^n T_n^m) + \frac{a^2}{3!}\sum_p(-)^p T_m^m T_n^n T_l^l\Big], \quad (39)$$

where $T_m^n = -\frac{i}{4}\partial^n\bar{\psi}\gamma_m\psi$ in accordance with the definition (15).

The mass term (39) contains the maximal number of the NG fermions $n_{\mathrm{NG}f} = 6$ and, respectively, the derivative number $n_{\mathrm{NG}d} = 3$, as a consequence of the cancellation of 4-derivative terms. These maximal numbers $n_{\mathrm{NG}f} = 6$ and $n_{\mathrm{NG}d} = 3$, characterizing the structure of the interaction action (37), are the same as for the VA action (35). The considered example shows that the cancellation effect takes place in the supersymmetric couplings containing the supervolume (11). So, we obtain that a sufficient condition for the 4-derivative cancellation in the couplings of the fermionic NG particles is the presence of d^4V (11) there. The observation sets issue on the restoration of a constrained superfield action with couplings which coincide with the effective VA action.

VII. RELATION BETWEEN THE KS AND THE VA LAGRANGIANS

Despite the difference between the VA and the KS Lagrangians, it seems that they are equivalent up to the NG field redefinition. Here we outline a straightforward way to check this assumption, and prove equivalence of these Lagrangians up to the first order in the constant a. The proof is analogous with the one considered in [7], and further developed in [13], in the context of nonlinear realization of the $\mathcal{N} = 1$ Maxwell superfield and the component structure of the supersymmetric nonlinear electrodynamics [12] (see additional refs. in these papers).

To make a comparison between the VA Lagrangian (35)

$$\mathcal{L}_{\mathrm{VA}} = \frac{1}{a} - \frac{i}{4}\bar{\psi}^{,m}\gamma_m\psi - \frac{a}{32}[(\bar{\psi}^{,m}\gamma_m\psi)^2 - (\bar{\psi}^{,n}\gamma_m\psi) \times(\bar{\psi}^{,m}\gamma_n\psi)] + a^2T^{(3)} \quad (40)$$

and the KS Lagrangian (36) clearer, we present the latter in the bispinor Majorana representation omitting the terms which have the form of total derivative

$$\mathcal{L}_{\mathrm{KS}} = \frac{1}{a} - \frac{i}{4}\bar{g}^{,m}\gamma_m g - \frac{a}{16}[(\bar{g}^{,m}g)^2 + (\bar{g}^{,m}\gamma_5 g)^2] - \left(\frac{a}{16}\right)^3[(\bar{g}g)^2 + (\bar{g}\gamma_5 g)^2][(\partial^2(\bar{g}g))^2 + ((\partial^2(\bar{g}\gamma_5 g))^2], \quad (41)$$

where $g := \sqrt{2}G$, $a := -1/f^2$, and the relations [14] connecting bilinear covariant expressions in the Weyl and the Majorana representations were used. To eliminate the 4-derivative term from $\mathcal{L}_{\mathrm{KS}}$, the expression for the Majorana spinor field g_a in terms of ψ_a has to include its higher derivatives. So, we shall seek for it in the form of a polynomial in the interaction constant a

$$g = \psi + a\chi + a^2\chi_2 + a^3\chi_3, \quad (42)$$

where the sought-for Grassmannian spinors χ, χ_2, χ_3 are the spinor nilpotent monomials of the form $\psi(\partial\bar{\psi}\psi)^n$, $n = 1, 2, 3$, respectively. The substitution of the expansion (42) in the KS Lagrangian (41) and making it equal to the VA Lagrangian (40) will produce the equations defining the spinors χ, χ_2, and χ_3. Thus, the proof of the equivalency of the Lagrangians is reduced to the solutions of these equations.

The comparison of the terms, having the same degree with respect to the constant a in the redefined KS and the original VA Lagrangians, provides an algorithmic way to generate the equations under question. In this way, we observe that the spinors χ_2 and χ_3 do not contribute in the terms linear in a in the redefined L_{KS}. Thus, it is easy to obtain an equation defining the spinor χ. Actually, the substitution of (42) into (41) and omitting the total derivative term redefines the kinetic term to the form

$$-\frac{i}{4}\bar{g}^{,m}\gamma_m g = -\frac{i}{4}\bar{\psi}^{,m}\gamma_m\psi - \frac{i}{2}a(\bar{\psi}^{,m}\gamma_m\chi) + \mathcal{O}(a^2). \quad (43)$$

The next relevant term from L_{KS} (41) is the term linear in a and quartic in the field number. Summing up of the mentioned terms results in the redefined KS Lagrangian in the linear order in a

$$\mathcal{L}_{\mathrm{KS}} = \frac{1}{a} - \frac{i}{4}\bar{\psi}^{,m}\gamma_m\psi - \frac{i}{2}a(\bar{\psi}^{,m}\gamma_m\chi) - \frac{a}{16}[(\bar{\psi}^{,m}\psi)^2 + (\bar{\psi}^{,m}\gamma_5\psi)^2] + \mathcal{O}(a^2). \quad (44)$$

Matching the Lagrangians (44) and (40) yields the sought-for equation for χ

$$i(\bar{\psi}^{,m}\gamma_m\chi) = -\frac{1}{8}[(\bar{\psi}^{,m}\psi)^2 + (\bar{\psi}^{,m}\gamma_5\psi)^2] + \frac{1}{16}\times[(\bar{\psi}^{,m}\gamma_m\psi)^2 - (\bar{\psi}^{,n}\gamma_m\psi)(\bar{\psi}^{,m}\gamma_n\psi)]. \quad (45)$$

To solve Eq. (45), we observe that its terms have the multiplier $\bar{\psi}^{,m}$ which can be canceled, resulting in

$$\gamma_m\chi = \frac{i}{8}[\psi(\bar{\psi}_{,m}\psi) + \gamma_5\psi(\bar{\psi}_{,m}\gamma_5\psi)] - \frac{i}{16}[\gamma_m\psi(\bar{\psi}^{,n}\gamma_n\psi) - \gamma_n\psi(\bar{\psi}^{,n}\gamma_m\psi)] + \triangle_m, \quad (46)$$

where $\triangle_m$ is defined by the condition $i(\bar{\psi}^{,m}\triangle_m)$ = total derivative terms. Multiplication of Eq. (46) by γ^m results in the general solution

$$\chi = -\frac{i}{32}[(\gamma_m\psi)(\bar{\psi}^{,m}\psi) + (\gamma_m\gamma_5\psi)(\bar{\psi}^{,m}\gamma_5\psi)] - \frac{i}{64}\times[3\psi(\bar{\psi}^{,m}\gamma_m\psi) + (\Sigma_{mn}\psi)(\bar{\psi}^{,n}\gamma^m\psi)] - \frac{1}{4}(\gamma^m\triangle_m). \quad (47)$$

Substitution of (47) in (42) yields the explicit expression connecting the KS and the VA realizations of the NG field up to terms linear in a

$$\sqrt{2}G = \psi\Big[1 + \frac{3ia}{64}(\bar{\psi}^{,m}\gamma_m\psi)\Big] - \frac{ia}{32}\Big[(\gamma_m\psi)(\bar{\psi}^{,m}\psi) + (\gamma_m\gamma_5\psi)(\bar{\psi}^{,m}\gamma_5\psi) - \frac{1}{2}(\Sigma_{mn}\psi)(\bar{\psi}^{,n}\gamma^m\psi)\Big] - \frac{a}{4}(\gamma^m\triangle_m) + \mathcal{O}(a^2). \qquad (48)$$

The spin-vector $\triangle_{ma}$ in (48) is composed by a linear combination of the Lorentz covariant nilpotent monomials $\psi(\partial\bar{\psi}\psi)$ of the third order in ψ, analogous with the general monomials χ_n (42), of the order $(2n+1)$ in ψ and $\partial\psi$. The quadratic terms in a are restored through the substitution of χ (47) into the expansion (42), and subsequently repetition of the above considered procedure with respect to the quadratic terms in a. Having fulfilled this, one can find χ_2, and then repeat again a similar procedure with respect to the cubic terms in the constant a. As a result, one can obtain the explicit expression for the KS field G through the VA field ψ, and to conclude about the expected equivalency between the KS and VA Lagrangians.

VIII. DISCUSSIONS

Here we presented an independent algebraic proof of the cancellation of 4-derivative terms in the $D = 4$ $\mathcal{N} = 1$ VA action using the Majorana bispinor representation and the Fierz rearrangements. The Majorana representation may simplify the investigation of such cancellations in the case of extended supersymmetries and/or of the higher dimensional spaces. We observed that the cancellation results in the difference between the Komargodski-Seiberg superfield [11] and the Volkov-Akulov [4] actions.

The difference gives rise to the question of whether the KS Lagrangian is equivalent to the VA Lagrangian. The second question arising from the cancellation concerns its presence in the NG fermion interactions with other fields. We found out that the cancellation occurs in the coupling of the fermionic NG field with massive Dirac fields. It yields the maximal number of the NG fermions n_{NGf} and their derivatives n_{NGd} in the interaction Lagrangian which equals six and three, respectively. The maximal numbers $n_{NGf} = 6$ and $n_{NGd} = 3$ are the same as in the VA action describing the NG fermion interactions between themselves. The observation poses the issue of restoration of superfield Lagrangian of interactions which uses realization of the NG fermionic field coinciding with the one in the VA Lagrangian with couplings. A way to solve these issues implies the construction of the explicit expression connecting the KS and the VA realizations of the NG field. The representation of the KS fermion field through the VA field has to contain terms with its derivatives. We discussed the problem and found the explicit formula connecting the VA and the KS realizations of NG field up to the first order in the interaction constant a. The substitution of the expression into the KS action reduced it to the VA action. It points to the expected equivalence of these actions in all orders in a. The equivalency problem posed in [17], has recently been discussed in [18] with pointing to some difficulties appearing on the way.

Taking into account the recent application of the formalism of $\mathcal{N} = 1$ constrained superfields in the minimal supersymmetric standard model (MSSM), as well as its generalizations to $\mathcal{N}$-extended supersymmetric models (see e.g. [19,20]), it is interesting to study the above considered kind of cancellations in these models. Availability of an explicit formula connecting VA and KS realizations of NG field could simplify the phenomenological analysis of the mentioned and other new models.

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