

Towards spinfoam cosmologyEugenio Bianchi,^{1,*} Carlo Rovelli,^{1,†} and Francesca Vidotto^{1,2,‡}¹*Centre de Physique Théorique de Luminy[§], Case 907, F-13288 Marseille, European Union*²*Dipartimento di Fisica Nucleare e Teorica, Università degli Studi di Pavia, and Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via A. Bassi 6, I-27100 Pavia, European Union*

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We compute the transition amplitude between coherent quantum states of geometry peaked on homogeneous-isotropic metrics. We work in the context of pure gravity without matter, we use the holomorphic representations of loop quantum gravity and the Kaminski-Kisielowski-Lewandowski generalization of the new vertex, and work at first order in the vertex expansion, second order in the graph (multipole) expansion, and first order in volume⁻¹. We show that the resulting amplitude is in the kernel of a differential operator whose classical limit is the canonical Hamiltonian of a Friedmann-Robertson-Walker cosmology. This result is an indication that the dynamics of loop quantum gravity defined by the new vertex reproduces the gravity part of the Friedmann equation in the appropriate limit.

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I. INTRODUCTION

The dynamics of loop quantum gravity (LQG) can be given in covariant form by using the *spinfoam* formalism. In this paper we apply this formalism to cosmology. In other words, we introduce a spinfoam formulation of quantum cosmology, or a “spinfoam cosmology.” We obtain two results. The first is that physical transition amplitudes can be computed in an appropriate expansion. We compute explicitly the transition amplitude between homogeneous-isotropic coherent states, at first order. The second and main result is that this amplitude is in the kernel of an operator H , and the classical limit of H turns out to be precisely the gravity part of the Hamiltonian constraint of the Friedmann dynamics of homogeneous-isotropic cosmology. In other words, we show that LQG yields the Friedmann equation in a suitable limit. Since we work in the absence of matter, thought, the dynamics of the model we consider is still rather trivial.

LQG has seen momentous developments in the last few years. We make use of several of these developments here, combining them together. The first ingredient we utilize is the “new” spinfoam vertex [1–5]. The second is the Kaminski-Kisielowski-Lewandowski extension of this to vertices of arbitrary valence [6]. The third ingredient is the coherent state technology [7–20] and, in particular, the holomorphic coherent states discussed in detail in [21]. These states define a holomorphic representation of LQG [8,22], and we work here in this representation.

Our strategy is the following. We consider the standard Hilbert space of canonical LQG and we assume the dynamics to be given by the new vertex. We consider holomorphic coherent states in this Hilbert space and we work in the holomorphic representation they define.

We truncate LQG down to a graph with a finite number of links. In particular, the calculation is based on the “dipole” graph formed by two nodes connected by four links [23]. This choice determines a Hilbert space, which describes a finite number of the degrees of freedom of the gravitational field. These degrees of freedom can be identified as the lowest modes in a multipole expansion of the metric in hyper-spherical harmonics on S_3 [24]. That is, they describe a closed cosmology, with anisotropies and a few low-mode inhomogeneities.

In particular, we consider coherent states that are peaked on homogeneous-isotropic geometries. We emphasize the fact that these states are just peaked on homogeneous and isotropic geometries, but they also include fluctuations of the inhomogeneous and anisotropic degrees of freedom. So, the dynamics of the quantum theory we consider *does* include inhomogeneous and anisotropic degrees of freedom. Homogeneous-isotropic coherent states are labeled by two parameters which capture the scale factor a of standard cosmology and its time derivative \dot{a} ; or, equivalently, the p and c canonical variables used in loop quantum cosmology (LQC). In the holomorphic representation, these two quantities appear in the complex combination $z = \alpha c + i\beta p$, and therefore the states we consider are labeled by the complex number z . The transition amplitude between two such states is then an analytic function $W(z, z')$ of two complex variables.

We write this transition amplitude at first order in a vertex expansion. We view this as the analog of a first order calculation in, say, QED perturbation theory. We compute explicitly $W(z, z')$ in the limit in which the geometry is large compared to the Planck scale.

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In other words, we compute the transition amplitude between macroscopical homogeneous-isotropic cosmological spaces in LQG, taking three approximations from the complete theory: (i) the truncation of the degrees of freedom to those defined on a finite graph; (ii) the restriction to first order in the vertex expansion; (iii) the large volume limit. The validity of these approximations can only be justified *a posteriori*, from the correctness of the result.

Our next step is to notice that the transition amplitude computed solves the equation $HW(z, z') = 0$ for a certain operator $H = H(z, t\hbar \frac{d}{dz})$. This fact implies that the amplitude defines a quantum dynamics where the operator constraint $H = 0$ holds. The corresponding classical dynamics will be governed by (the $\hbar \rightarrow 0$ limit of) the classical constraint $H(z, \bar{z}) = 0$. When written in terms of p and c , this turns out to be precisely the Hamiltonian constraint that governs (the gravitational part of) the dynamics of a classical Friedmann cosmology in the limit of large volume. Therefore LQG yields the Friedmann dynamics in this limit.

Several words of caution are necessary. First, we work in the Euclidean theory. Second, the cosmological dynamics that we obtain is the one in the large volume limit and since we do not have any matter present, this has only the solution $a = \text{constant}$, which is flat space. With these caveats, our result is that there is an approximation in LQG that leads to classical cosmology.

This result can be compared with those of LQC [25,26]. In LQC, one first reduces the classical theory to a cosmological system with a finite number of degrees of freedom, and then applies a “loop quantization” to this symmetry reduced model. Thus, one has a complete quantum theory of a truncation of the classical theory. Here, instead, we start from the full quantum theory and take an approximation. Therefore, in LQC one studies the exact solution in a truncated system, while here we study approximated solutions in the (hopefully) exact quantum theory. The possibility of introducing a spinfoam-like expansion starting from LQC has been considered in the following papers: [27–29]. These papers and the present work can be seen as two converging attempts to construct a spinfoam version of quantum cosmology.

Finally, in our opinion a main reason of interest of the result we present here is that it represents an example of a complete calculation of physical transition amplitude in background independent quantum gravity. It complements the calculation of the two-point function [30–34], that has been recently completed [35].

In Sec. II, we briefly recall the definition of the full quantum theory. In Sec. III, we discuss the approximation that selects a cosmological sector, we compute the resulting transition amplitude. In Sec. IV, we study the classical limit and recover the Friedmann dynamics.

II. THE THEORY

A. Kinematics

The theory is defined on the Hilbert space

$$\mathcal{H} = \bigoplus_{\Gamma} \mathcal{H}_{\Gamma}. \quad (1)$$

The sum runs over the abstract graphs Γ . A graph Γ is here a set of L links ℓ and N nodes n , together with two relations s (source) and t (target) assigning a source node $s(\ell)$ and a target node $t(\ell)$ to every link ℓ . The Hilbert space \mathcal{H}_{Γ} is defined to be

$$\mathcal{H}_{\Gamma} = L_2[SU(2)^L/SU(2)^N], \quad (2)$$

where the action of $SU(2)^N$ on the states $\psi(U_{\ell}) \in L_2[SU(2)^L]$ is

$$\psi(U_{\ell}) \rightarrow \psi(V_{s(\ell)}U_{\ell}V_{t(\ell)}^{-1}), \quad V_n \in SU(2)^N. \quad (3)$$

Two sets of operators are defined on each space \mathcal{H}_{Γ} . For each link we have the “holonomy” multiplicative operator $\hat{U}_{\ell}\psi(U_{\ell}) = U_{\ell}\psi(U_{\ell})$ and the “triad” operator $\hat{E}_{\ell}^i\psi(U_{\ell}) = (8\pi G\hbar\gamma)L_{\ell}^i\psi(U_{\ell})$ where G is the Newton constant and L_{ℓ}^i is the left-invariant vector field acting on the variable U_{ℓ} .

Finally, the state space of the theory is obtained by factoring \mathcal{H} by an equivalence relation, defined as follows. If Γ is a subgraph of Γ' then \mathcal{H}_{Γ} can be naturally identified with a subspace of $\mathcal{H}_{\Gamma'}$. Two states are equivalent if they can be related by this identification, or if they are mapped into each other by the discrete group of the automorphisms of Γ .

These states are to be thought as “boundary states” in the quantum theory. That is, \mathcal{H} must be identified with the space $\mathcal{H}_{\text{out}}^* \otimes \mathcal{H}_{\text{in}}$ of the initial and final states of nonrelativistic quantum mechanics.

Coherent states

An overcomplete basis of coherent states in the Hilbert space \mathcal{H}_{Γ} is provided by the holomorphic states

$$\Psi_{H_{\ell}}(U_{\ell}) = \int_{SU(2)^N} dg_n \prod_{\ell} K_t(g_{s(\ell)}^{-1}U_{\ell}g_{t(\ell)}H_{\ell}^{-1}). \quad (4)$$

Here $H_{\ell} \in SL(2, \mathbb{C})$, and K_t is [the analytic continuation to $SL(2, \mathbb{C})$ of] the *heat kernel* function on $SU(2)$. This is a function concentrated on the origin of the group, with a spread of order t . Its explicit form is¹

$$K_t(U) = \sum_j (2j+1)e^{-2t\hbar j(j+1)} \text{Tr}[D^j(U)], \quad (5)$$

¹We choose a parameter t with the dimension of an inverse action, and put \hbar explicitly in the definition of the coherent states, in order to emphasize the fact that the small t limit is the classical limit, and to keep track of the corresponding dependence on \hbar . The factor of 2 is for later convenience.

where $D^j(U)$ is the Wigner matrix of the spin- j representation of $SU(2)$.

As shown in [21], these states: (i) are the basis of the holomorphic representation [8,22], (ii) are a special case of Thiemann's complexifier's coherent states [9–18], (iii) induce Speziale-Livine coherent tetrahedra [2,36,37] on the nodes, and (iv) are equal to the Freidel-Speziale coherent states [19,20] for large spins.

The states (4) are gauge-invariant semiclassical wave packets. The integral in (4) projects (“group averages”) on the gauge-invariant states. If H_ℓ is in the $SU(2)$ subgroup of $SL(2, \mathbb{C})$, the heat kernel peaks each U_ℓ on H_ℓ . The extension of H_ℓ to $SL(2, \mathbb{C})$ has the same effect as taking a Gaussian function $\psi(x) = e^{(x-z_o)^2/2} \sim e^{(x-x_o)^2/2} e^{ip_o x}$ for a complex $z_o = x_o + ip_o$; that is, it adds a phase which peaks the states on a value of the variable conjugate to U_ℓ . Thus, the states (4) are peaked on the variables U_ℓ as well as on their conjugate momenta.

The $SL(2, \mathbb{C})$ labels H_ℓ can be given two related interpretations. First, we can decompose each $SL(2, \mathbb{C})$ label in the form

$$H_\ell = e^{i4tE_\ell/8\pi G\gamma U_\ell}, \quad (6)$$

where $U_\ell \in SU(2)$ and $2tE_\ell/(8\pi G\hbar\gamma) \in su(2)$. Then it is not hard to show that U_ℓ and E_ℓ determine the expectation values of the operators \hat{U}_ℓ and \hat{E}_ℓ on the state ψ_{H_ℓ}

$$\frac{\langle \psi_{H_\ell} | \hat{U}_\ell | \psi_{H_\ell} \rangle}{\langle \psi_{H_\ell} | \psi_{H_\ell} \rangle} = U_\ell, \quad \frac{\langle \psi_{H_\ell} | \hat{E}_\ell | \psi_{H_\ell} \rangle}{\langle \psi_{H_\ell} | \psi_{H_\ell} \rangle} = E_\ell, \quad (7)$$

and that the corresponding spread is small²

$$\Delta U_\ell \sim \sqrt{t\hbar}, \quad \Delta E_\ell \sim G\sqrt{\hbar/t}. \quad (8)$$

Alternatively, we can decompose each $SL(2, \mathbb{C})$ label in the form

$$H_\ell = n_{s,\ell} e^{-i(\xi_\ell + i\eta_\ell)(\sigma_3/2)} n_{t,\ell}^{-1}, \quad (9)$$

$\vec{\sigma} = \{\sigma_i\}$, $i = 1, 2, 3$ are the Pauli matrices and $n_{s,\ell}, n_{t,\ell} \in SU(2)$. Freidel and Speziale discuss a compelling geometrical interpretation for the $(\vec{n}_s, \vec{n}_t, \xi, \eta)$ labels of each link [19] [here and below, unit-length vectors \vec{n} and $SU(2)$ elements n are related by $\vec{n} \cdot \vec{\sigma} = n\sigma_3 n^{-1}$]. For appropriate four-valent states representing a Regge 3 geometry with intrinsic and extrinsic curvature, the vectors \vec{n}_s, \vec{n}_t are the $3d$ normals to the triangles the tetrahedra bounded by the triangle; ξ is the extrinsic curvature at the triangle and η is the area of the triangle divided by $8\pi G\hbar$. For general states, the interpretation extends to a simple generalization of Regge geometries, that Freidel and Speziale have baptized “twisted geometries.”

²If we fix a length scale $L \gg \sqrt{\hbar G}$ and choose $2\hbar t = \hbar G/L^2$ we have $\Delta U_\ell \sim \sqrt{\hbar G}/L$ and $\Delta E_\ell \sim \sqrt{\hbar GL}$.

Thus, the holomorphic coherent states provide a convenient basis of wave packets with good geometrical interpretation.

B. Dynamics

The spinfoam formalism associates an amplitude

$$\langle W | \psi \rangle = \sum_\sigma \prod_f d_f(\sigma) \prod_v W_v(\sigma) \quad (10)$$

to each boundary state $\psi \in \mathcal{H}$. The sum is over the spinfoams σ bounded by ψ . See [5] for a description of this formalism and for the notation. The vertex amplitude is $W_v(\sigma) = \langle W_v | \psi_v(\sigma) \rangle$, where $\psi_v(\sigma)$ is the spin network obtained by cutting σ with a small three-sphere surrounding v and the vertex amplitude that defines the dynamics of LQG is [2,4–6]

$$\langle W_v | \psi \rangle = (f\psi)(v). \quad (11)$$

Here $f: \mathcal{H}_\Gamma \rightarrow L_2[SO(4)^L/SO(4)^N]$ is defined as follows. Let

$$j^\pm = \frac{1 \pm \gamma}{2} j \quad (12)$$

and let Y be the map

$$Y: \mathcal{H}^{(j)} \rightarrow \mathcal{H}^{(j^+, j^-)} \quad |j, m\rangle \rightarrow |j^+, m^+; j^-, m^-\rangle \quad (13)$$

whose matrix elements are given by the Clebsch-Gordan coefficients.

$$Y^{m^+ m^-} = \langle j^+, m^+; j^-, m^- | j, m \rangle. \quad (14)$$

Consider the Peter-Weyl decomposition of $L_2[SU(2)^L]$ and, respectively, $L_2[SO(4)^L]$. Then f is defined by mapping with Y each $\mathcal{H}^{(j)}$ term of the first into the corresponding $\mathcal{H}^{(j^+, j^-)}$ of the second.

Explicitly, the generalized state W_v in (11) is given by

$$W_v(U_\ell) = \int_{SO(4)^N} dG_n \prod_\ell P_o(U_\ell, G_{s(\ell)} G_{t(\ell)}^{-1}), \quad (15)$$

where

$$P_o(U, G) = \sum_j (2j+1) \text{Tr}[D^{(j)}(U) Y^\dagger D^{(j^+, j^-)}(G) Y]. \quad (16)$$

Here $D^{(j)}$ is the Wigner matrix of the spin- j representation of $SU(2)$, while $D^{(j^+, j^-)}$ is the Wigner matrix of the spin- (j^+, j^-) representation of $SO(4)$. The first has dimension $2j+1$ while the second has dimension $(2j^++1) \times (2j^-+1)$. These matrices with different dimensions are glued by the map Y .

The vertex amplitude takes a simple form on the holomorphic basis defined by the coherent states. By combining the definition (15) of the vertex and the definition (4) of the coherent states, one obtains the holomorphic form of the vertex amplitude [22]

$$\begin{aligned}
W_\nu(H_\ell) &\equiv \langle W_\nu | \psi_{H_\ell} \rangle \\
&= \int_{SU(2)^t} dU_\ell W(U_\ell) \psi_{H_\ell}(U_\ell) \\
&= \int_{SO(4)^N} dG_n \prod_\ell P_t(H_\ell, G_{s(\ell)} G_{t(\ell)}^{-1}), \quad (17)
\end{aligned}$$

where

$$\begin{aligned}
P_t(H, G) &= \sum_j (2j+1) e^{-2t\hbar j(j+1)} \text{Tr}[D^{(j)}(H)Y^\dagger \\
&\quad \times D^{(j^+, j^-)}(G)Y]. \quad (18)
\end{aligned}$$

Here $D^{(j)}$ is the analytic continuation of the Wigner matrix from $SU(2)$ to $SL(2, \mathbb{C})$. Below, we use this last expression to compute the quantum evolution in cosmology.

III. THE COSMOLOGICAL APPROXIMATION

A. Graph expansion

There is no physics without approximations. The first approximation we take is to truncate \mathcal{H} to a single fixed graph Γ . Notice that the states with support on smaller graphs (subgraphs of Γ) are all contained in \mathcal{H}_Γ ; therefore the truncation amounts to disregard all states that have support on graphs “larger” than Γ .

We choose Γ to be the graph formed by two disconnected components Γ_i and Γ_f . We think of these as carrying an initial and a final state. In particular we choose $\Gamma_i = \Gamma_f = \Delta_2^*$, where the dipole graph Δ_2^* is defined by the set of two nodes $\{n_1, n_2\}$, by the set of four links $\{\ell_1, \ell_2, \ell_3, \ell_4\}$, and by the source and target relations $s(\ell) = n_1$ and $t(\ell) = n_2, \forall \ell$. That is

$$\Delta_2^* = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} .$$

The operators defined on the Hilbert space $\mathcal{H}_{\Delta_2^*}$ are (U_ℓ, E_ℓ) . These can be interpreted as follows. Consider a space M with the topology of a three-sphere, carrying a triad field E and a connection A . Choose an immersion of Δ_2^* into M , and a cellular decomposition Δ_2 of M , dual to Δ_2^* . Δ_2 is the triangulation of the three-sphere formed by 2 tetrahedra with all their faces identified.³ We can then identify U_ℓ with the holonomy of A on the link ℓ and E_ℓ with the flux of the triad through the triangle cut by the link ℓ , parallel transported to n_1 .

³The metric structure defined by E determines a preferred immersion, up to degeneracies. Pick two points n_1 and n_2 at maximal distance from each other (the “north” and “south” pole in M), and let the equator be the set of points equidistant from the poles. Chose four points on the equator at maximal distance from one another, and connect them to the poles with geodesic links ℓ (this gives the four meridians). The “equator” gets partitioned into four (Voronoi) triangles [38], each cut by one of the links, defined by the minimal distance from the cuts.

Homogeneous-isotropic coherent states

Consider the coherent states on $\mathcal{H}_{\Delta_2^*}$. These are labeled by four $SL(2, \mathbb{C})$ elements $H_\ell = e^{iE_\ell} U_\ell$. We are interested here in homogeneous-isotropic coherent states. To find them, let us compute (U_ℓ, E_ℓ) for the case of a homogeneous-isotropic space. Let A and E define such a space. Then we can write $A = c^o \omega$ and $E = p^o \omega$ where $o\omega = g^{-1}dg$ is the fiducial connection defined by the $SU(2)$ Maurer-Cartan connection, upon identification of M with the $SU(2)$ group manifold [39]. Notice that A is here the Ashtekar-Barbero connection, which is determined by both the spin connection and the extrinsic curvature. The spin connection of a three-sphere is $\Gamma = o\omega$, so that in terms of the scale factor a we have $c = 1 + \gamma\dot{a}$. Choose n_1 to be the identity $\mathbb{1}$ and n_2 to be $-\mathbb{1}$, and choose the links to be given by exponentiating a quadruplet \vec{n}_ℓ of vectors in $su(2)$ normal to the faces of a regular tetrahedron centered on the origin. There is an $SO(3)$ freedom in choosing the normals at each node. We choose $n_{1\ell} = n_{2\ell} := n_\ell$. This gives a realization of the immersion defined in footnote³. Then we can compute U_ℓ and E_ℓ [24]

$$\begin{aligned}
U_\ell &= n_\ell e^{-i\alpha c(\sigma_3/2)} n_\ell^{-1}, \\
E_\ell &= -in_\ell \frac{2\pi G\gamma}{t} \beta p \frac{\sigma_3}{2} n_\ell^{-1}, \quad (19)
\end{aligned}$$

where n_ℓ are $SU(2)$ group elements such that $n_\ell \sigma_3 n_\ell^{-1} = \vec{n}_\ell \cdot \vec{\sigma}$, and α and β are constants that we do not determine here. This implies that in (9) we have $n_{s,\ell} = n_{t,\ell} = n_\ell$ and

$$\xi_\ell = \xi = \alpha c, \quad \eta_\ell = \eta = \beta p, \quad (20)$$

that is

$$H_\ell(\xi, \eta) = n_\ell e^{-i(\xi+i\eta)(\sigma_3/2)} n_\ell^{-1}. \quad (21)$$

The independence of ξ_ℓ and η_ℓ from ℓ can be seen as the effect of isotropy and the equality of $n_{s,\ell}$ and $n_{t,\ell}$ can be seen as the effect of homogeneity. The two numbers $c = \xi/\alpha$ and $p = \eta/\beta$ label the homogeneous-isotropic coherent states.

Remarkably, the same states can be obtained by using the Freidel-Speziale geometrical interpretation [20]. Consider a Regge geometry formed by two equal regular tetrahedra with their faces identified and where the extrinsic curvature is the same at each triangle. This fixes the labels (n_s, n_t, ξ, η) at each triangle ℓ , and determines via (9) an $SL(2, \mathbb{C})$ group element which is precisely (21).

Finally, the quantity that we want to calculate is

$$W(\xi_i, \eta_i; \xi_f, \eta_f) = W(H_i(\xi_i, \eta_i), H_f(\xi_f, \eta_f)). \quad (22)$$

Notice that this is an holomorphic function of z_i and z_f where

$$z \equiv \xi + i\eta. \quad (23)$$

Thus, we can write it as

$$W(z_i, z_f) = W(\xi_i, \eta_i; \xi_f, \eta_f). \quad (24)$$

This is the transition amplitude between a homogeneous-isotropic universe with scale factor (square) p_i and extrinsic curvature c_i , to an homogeneous-isotropic universe with scale factor (square) p_f and extrinsic curvature c_f .

By writing the in and out states

$$\psi_z(U_\ell) := \psi_{H_\ell(z(c,p))}(U_\ell) := \langle U_\ell | z \rangle, \quad (25)$$

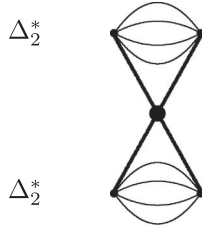
we can interpret $W(z_i, z_f)$ as the physical scalar product between (the projection on the physical state space of) the state $|z\rangle_i$ and (the projection on the physical state space of) the state $|z\rangle_f$. That is⁴

$$W(z_i, z_f) = \langle \bar{z}_f | z_i \rangle_{\text{physical}}. \quad (26)$$

We now compute this quantity to first order in the vertex expansion.

B. Vertex expansion

At first order in the vertex expansion, the amplitude (26) is given by the spinfoam formed by a single vertex bounded by four edges and eight faces;



in this approximation (22) is given by the amplitude of the single vertex v

$$W(z_i, z_f) = W_v(H_\ell(z_i), H_\ell(z_f)). \quad (27)$$

Using (17) and (18), this becomes

$$\begin{aligned} W(z_i, z_f) &= \int_{SO(4)^4} dG_1^i dG_2^i dG_1^f dG_2^f \\ &\times \prod_{\ell^i} P_t(H_\ell(z_i), G_1^i G_2^{i-1}) \prod_{\ell^f} P_t(H_\ell(z_f), G_1^f G_2^{f-1}) \\ &= \int_{SO(4)^2} dG^i dG^f \\ &\times \prod_{\ell^i} P_t(H_\ell(z_i), G^i) \prod_{\ell^f} P_t(H_\ell(z_f), G^f). \end{aligned}$$

Notice that this expression factorizes

⁴In standard quantum mechanics, the transition amplitude $Z(z_f, z_i) = \langle z_f | \exp iHt | z_i \rangle$ is antilinear in the first variable, therefore one may expect (26) to be antiholomorphic in z_f . However, since here we treat the past and future surfaces as two components of the boundary of the spacetime region between the two, the initial surface is oriented toward the future and the final surface towards the past. If we change the orientation of the final surface, z_f goes to $-\bar{z}_f$.

$$W(z_i, z_f) = W(z_i)W(z_f), \quad (28)$$

where

$$W(z) = \int_{SO(4)} dG \prod_{\ell} P_t(H_\ell(z), G). \quad (29)$$

This factorization happens only at first order. At this order, therefore, the physical projector projects on a single state, and (29) can be viewed as a Hartle-Hawking *no-boundary* state [40,41].

C. Large volume expansion

Let us now compute (29) in the limit in which the Universe is large. This limit is given by taking large p in (6). Consider the factor $D^{(j)}(H)$ in (18). Using (9), this reads

$$D^{(j)}(H_\ell) = D^{(j)}(n_\ell) D^{(j)}(e^{-iz(\sigma_3/2)}) D^{(j)}(n_\ell^{-1}). \quad (30)$$

For $p \gg 8\pi G\hbar\gamma$, namely, $\eta \gg 1$ we have

$$D^{(j)}(e^{-i(\xi+i\eta)(\sigma_3/2)}) = e^{-i(\xi+i\eta)jP}, \quad (31)$$

where P is the projector on the eigenstate of L_3 with maximum eigenvalue $m = j$. It is easy to see that this term dominates in the limit [22]. Inserting this result in the previous equation gives

$$D^{(j)}(H_\ell) = e^{-izj} D^{(j)}(n_\ell) P D^{(j)}(n_\ell^{-1}) := e^{-izj} P_\ell. \quad (32)$$

Using this in (18) gives

$$\begin{aligned} P_t(H_\ell, G) &= \sum_j (2j+1) e^{-2\hbar j(j+1) - izj} \text{Tr}[P_\ell Y^\dagger D^{(j^+, j^-)}(G) Y]. \end{aligned}$$

We show later that the trace gives a contribution polynomial in j . Therefore we can compute the sum by approximating it with a Gaussian integral. This is peaked on the value $j \sim j_o = -iz/4t\hbar$. Notice that the real part of j_o is given by the imaginary part of z , namely, p . For large p , we have $j_o \sim \beta p/4t\hbar$. The Gaussian integral gives

$$P_t(H_\ell, G) = \sqrt{\frac{\pi}{t}} e^{-(z^2/8t\hbar)} 2j_o \text{Tr}[P_\ell Y^\dagger D^{(j_o^+, j_o^-)}(G) Y]. \quad (33)$$

Using this in (29) yields

$$W(z) = \left(\sqrt{\frac{\pi}{t}} e^{-(z^2/8t\hbar)} 2j_o \right)^4 N_{j_o}, \quad (34)$$

where

$$N_{j_o} = \int_{SO(4)} dG \prod_{\ell} \text{Tr}[P_\ell Y^\dagger D^{(j_o^+, j_o^-)}(G) Y]. \quad (35)$$

This is the norm squared of the Livine-Speziale coherent regular tetrahedron of size j_o . It is given [2] by $N_{j_o} = N_o j_o^{-3}$. Using this and $j_o = -\frac{1}{4t\hbar} z$, we conclude

$$W(z) = Nz e^{-(z^2/2i\hbar)}, \quad (36)$$

where $N = -2i(\pi/i\hbar)^2 N_o$. Finally, inserting this into (28) we have

$$W(z_i, z_f) = N^2 z_i z_f e^{-(1/2i\hbar)(z_i^2 + z_f^2)}. \quad (37)$$

This is the transition amplitude between two cosmological homogeneous-isotropic coherent states.

IV. CLASSICAL LIMIT

We now observe that the transition amplitude (37) satisfies the equation

$$\frac{3}{8\pi G(4\alpha\beta\gamma)^2} \left(z^2 - t^2 \hbar^2 \frac{d^2}{dz^2} - 3t\hbar \right)^2 W(z, z') = 0 \quad (38)$$

(the square and the overall factor are for later convenience). Therefore, this amplitude describes a quantum system where the operator equation

$$\hat{H} := \frac{3}{8\pi G(4\alpha\beta\gamma)^2} \left(z^2 - t^2 \hbar^2 \frac{d^2}{dz^2} - 3t\hbar \right)^2 = 0 \quad (39)$$

holds. Let us now look for a corresponding classical system. Assume that z is the coordinate of a classical phase space with symplectic structure

$$\omega = \frac{i}{t} dz \wedge d\bar{z}; \quad (40)$$

that is

$$\{z, \bar{z}\} = it. \quad (41)$$

The corresponding operators in the quantum theory that satisfy $[z, \bar{z}] = i\hbar\{z, \bar{z}\}$ are therefore

$$\hat{z} = z, \quad \hat{\bar{z}} = t\hbar \frac{d}{dz}. \quad (42)$$

We can thus rewrite (39) as

$$\hat{H} = \frac{3}{8\pi G(4\alpha\beta\gamma)^2} (\hat{z}^2 - \hat{\bar{z}}^2 - 3t\hbar)^2 = 0. \quad (43)$$

Let us now take the classical limit of this equation. Replacing operators with classical variables, we have

$$\begin{aligned} H &= \frac{3}{8\pi G(4\alpha\beta\gamma)^2} (z^2 - \bar{z}^2 - 3t\hbar)^2 \\ &= \frac{3}{8\pi G(4\alpha\beta\gamma)^2} (4i\xi\eta - 3t\hbar)^2 = 0. \end{aligned} \quad (44)$$

Using (20), this is

$$H = \frac{3}{8\pi G(4\alpha\beta\gamma)^2} (4i\alpha\beta c p - 3t\hbar)^2 = 0. \quad (45)$$

In the $\hbar \rightarrow 0$ limit we have

$$H = -\frac{3}{8\pi G\gamma^2} c^2 p^2 = 0. \quad (46)$$

Dividing by the volume of space $\text{Vol} \sim p^{3/2} > 0$, we have

$$H = -\frac{3}{8\pi G\gamma^2} \sqrt{p} c^2 = 0. \quad (47)$$

In the large volume limit and in the absence of matter the dynamics of the universe approaches the flat ($k = 0$) case and the Friedmann hamiltonian constraint becomes

$$H_{\text{cl}} = -\frac{3}{8\pi G} \dot{a}^2 a = 0; \quad (48)$$

in this regime $c = \gamma\dot{a}$ and $p = a^2$ and this equation is precisely (47).

V. CONCLUSIONS

We have introduced a spinfoam formulation of quantum cosmology.

We have obtained two results. The first is that it is possible to compute quantum transition amplitudes explicitly in suitable approximations. In detail, we have studied three approximations: (i) cutting the theory to a finite dimensional graph (the dipole), (ii) cutting the spinfoam expansion to just one term with a single vertex, and (iii) the large volume limit. The main hypothesis on which this work is based is that the regime of validity in which these approximations are viable includes the semiclassical limit of the dynamics of large wavelengths. ‘‘Large’’ means here of the order to the size of the universe itself. This regime includes of course the standard Friedmann cosmology.

The second result is that the transition amplitude computed appears to give the correct Friedmann dynamics in the classical limit. These results must be taken with caution for a number of reasons. First, we have used the Euclidean theory, instead than the physical Lorentzian theory. Second, the dynamics we have obtained is in fact trivial due to the absence of matter. The solution of the constraint equation (47) is either $p = 0$ or $c = 0$; that is, either the Universe has no volume, or it is flat. This is physically correct, since in the absence of a matter and in the limit of an infinitely large radius one obtains precisely a flat space-time. But this is only a weak indication that the full Friedmann dynamics is recovered. Whether the result still holds with matter, or a cosmological constant, must still be checked. Also, in the derivation of the classical limit, the symplectic structure (40) has been taken as an input. It can be shown that this choice reproduces the symplectic structures of the LQC variables⁵ (c, p) or the one of the LQG variables (E_ℓ, U_ℓ). Finally, the system that the approximation defines admits obvious improvements. In particular, transitions must be computed on a larger graph, and at the next order in the vertex amplitude, in order to investigate

⁵This fixes the product $\alpha\beta = 3t/16\pi G\gamma$. Then α can be determined by noticing that for $c = 1$ the connection A is the Cartan connection and its holonomy from the identity to g is g itself. Taking $n_s = \mathbb{1}$ and $n_t = -\mathbb{1}$ gives easily $\alpha = 2\pi$.

the validity of the approximation, and it is important to understand what are the first order corrections to the Friedman equation without matter. Can these be computed and if yes, what is the expected physics from these terms?

We have noticed that at first order the transition amplitude factorizes [see Eq. (28)]. To this order, the “projector” on physical states $P = \sum_n \langle n|n\rangle$ defined by $\langle \psi_f | P | \psi_i \rangle = \langle W | \bar{\psi}_f \otimes \psi_i \rangle$ projects on a single state, say $|0\rangle$, which can be identified as the Hartle-Hawking “wave function of the Universe” defined by the so called “no-boundary proposal” [42]. We do not expect this factorization to survive higher orders, where the projector can regain its general form.

From the point of view of cosmology, the system we have described opens in principle the way to the description of inhomogeneous degrees of freedom at the bounce, circumventing the difficulties of the model given in [23]. In particular, the covariant dynamics used here can readily be extended to larger graphs. Coherent states have been largely used in loop quantum cosmology (see for instance [43–46]), in particular, in relation to the problem of finding effective equations or in numerical simulations [47–49]. Here, however, homogeneous and isotropic states appear naturally as states peaked on homogeneous and isotropic *mean values* of the quantum states, in the context of a formalism which—we stress—is not a reduction of the dynamics to homogeneous and isotropic degrees of freedom. In physical terms, these states represent a universe where inhomogeneous and anisotropic degrees of freedom are taken into account but fluctuate around zero. This provides also an elegant solution of the problem of having

to choose between coordinate or momenta in imposing a symmetry reduction in cosmology [50,51]. Ideally, this formalism could describe inhomogeneous and anisotropic quantum fluctuations of the geometry at the bounce.

The comparison of our results with those of standard loop quantum cosmology requires more work and opens several issues. In LQC the “natural” quantum dynamics based on c and p requires care, because of the consequent p dependence of the affine parameter, and the possible dependence of the formalism on the “fiducial cell” [52]. Here the same problem does not appear since we do not use such fiducial structure; however the issue deserves to be investigated in more detail. One of the purposes of the present paper is precisely to open a path for understanding the relation between these peculiar aspects of loop quantum cosmology and the full loop theory of quantum gravity.

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