# $SL(2, C)$ gravity on noncommutative space with Poisson structure

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The Einstein's gravity theory can be formulated as an $SL(2, C)$ gauge theory in terms of spinor notations. In this paper, we consider a noncommutative space with the Poisson structure and construct an $SL(2, C)$ formulation of gravity on such a space. Using the covariant coordinate technique, we build a gauge invariant action in which, according to the Seiberg-Witten map, the physical degrees of freedom are expressed in terms of their commutative counterparts up to the first order in noncommutative parameters.

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## I. INTRODUCTION

In recent years, the idea of noncommutative spacetimes has attracted much attention although it was proposed by Synder [1] as early as in 1947 in order to remove the divergence in quantum field theories. It has been argued that at a very small scale, say the Planck length, coordinates of spacetimes cannot be measured at any accuracy [2], i.e., the measurement should satisfy a set of uncertainty relations that can be well realized within the context of noncommutative sapcetimes. Starting from string theory, Seiberg and Witten [3] suggested that the $D$-brane dynamics with a $B$-field background can be described by some noncommutative field theory. Recently, the idea of noncommutative spacetimes has penetrated into various fields in physics. The research on the construction of quantum field theories on noncommutative spacetimes is fruitful (for reviews, see Ref. [4]) and it is remarkable to note that noncommutative quantum field theories can be applied to the study related to strong background fields, such as the quantum Hall effect (for a review, see Ref. [5]).

It is interesting to study gravity on noncommutative spacetimes. Actually, the noncommutative formulation of gravity has been considered [6] to be a necessity for quantization of gravity. The main obstacle for this formulation is on dealing with the general coordinate invariance. Recently, there have been some approaches proposed for solving this problem. In Ref. [7], a deformation of Einstein's gravity is constructed based on gauging the noncommutative $SO(4, 1)$ de Sitter group and then contracting it to $ISO(3, 1)$ in terms of the Seiberg-Witten map [3]. Another effort [8] also from the point of view of the Seiberg-Witten map is made to build the $SO(3, 1)$ noncommutative formulation of gravity. In Refs. [9,10], noncommutative gravity models are established by the reduction of the constrained $U(2, 2)$ to $SO(3, 1)$. Moreover, the theory of gravity can also be expressed in a $GL(2, C)$ formulation with complex vierbeins [11]. Within the framework of the gauge theory of gravity, the authors of Ref. [12] have given a noncommutative formulation of gravity based on a class of restricted diffeomorphism symmetries that preserves the noncommutative algebra. On the other hand, Wess and collaborators [13] have proposed a gravitational theory considering a twisted diffeomorphism algebra from a purely geometrical point of view.

The formulations of noncommutative gravity mentioned above are established in the canonical noncommutative spacetime in which coordinates $\hat{x}^\mu$ satisfy the following commutation relations,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is an antisymmetric constant tensor and its elements are called noncommutative parameters. The noncommutativity can be realized in the ordinary spacetime with the replacement of the ordinary product by the star-product between functions. It is reasonable to consider a more general noncommutative spacetime where noncommutative parameters are coordinate-dependent. For the most general noncommutative parameters $\theta^{\mu\nu}(\hat{x})$, the star product between functions may become nonassociative, which gives rise to quite complicated problems. Fortunately, there exists a class of noncommutative manifolds on which the star product between functions is associative, like the Poisson manifold. In the present paper, we focus our attention on such a manifold. There are already some works that deal with gravity on coordinate-dependent noncommutative spacetimes. In Ref. [14], the theory of gravity is constructed based on the work of Ref. [12] on a noncommutative spacetime with the Lie algebraic structure, which is in fact a special case of the Poisson manifold. In terms of the twisted differential geometry, another theory of gravity is proposed [15] which is invariant under diffeomorphism as well as under a $GL(2, C)$ $\star$-gauge transformation on a general noncommutative spacetime. We prefer to carry out our analysis in the framework of the $SL(2, C)$ formulation [11,16]. The advantage of this formulation lies in a tetrad formalism where the tetrad transforms covariantly under gauge transformations so that we can construct a gauge invariant action in a natural way. As a gauge theory, it enables one to use the machinery

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of noncommutative gauge theories elaborately developed in the literature [17–20]. In terms of the covariant coordinate technique [17], a rank-two tensor $\hat{R}^{\mu\nu}$ is constructed at first. Because of the coordinate-dependence of $\theta^{\mu\nu}(\hat{x})$, it is not straightforward to write the covariantly transformed curvature tensor $\hat{R}_{\mu\nu}$ through the relation $\hat{R}^{\mu\nu} = \theta^{\mu\lambda}\theta^{\nu\sigma}\hat{R}_{\lambda\sigma}$ as was done in the canonical noncommutative spacetime. To this end, a modified function $\hat{\theta}_{\mu\nu}$ (see Eq. (25) and (26)) is introduced in order for the gauge field strength to transform covariantly. As a result, the action can be constructed in terms of the curvature tensor $\hat{R}_{\mu\nu}$ and the vierbein $\hat{e}_\mu$. Furthermore, the Seiberg-Witten map [3] in our case is derived for noncommutative physical quantities up to the first order in the coordinate-dependent noncommutative parameters; we can therefore express the noncommutative theory in terms of ordinary physical quantities completely.

The paper is organized as follows. In the next section, we give a brief introduction to the $SL(2, C)$ formulation of gravity on the ordinary spacetime. In the first subsection of Sec. III, the formulation is extended to a noncommutative space with the Poisson structure, and a gauge invariant action is thus constructed. In the second subsection, the Seiberg-Witten map of the noncommutative formulation is derived up to the first order in the coordinate-dependent noncommutative parameters. The last section is devoted to the conclusion. As to notations, we use the Latin letters, $a, b, \cdots = 0, 1, 2, 3$, to denote Lorentz indices and the Greek letters, $\mu, \nu, \cdots = 0, 1, 2, 3$, spacetime indices.

## II. A BRIEF INTRODUCTION TO $SL(2, C)$ FORMULATION OF GRAVITY

Before discussing its noncommutative formulation, let us recall briefly the $SL(2, C)$ formulation of gravity [11,16,21] on the ordinary spacetime.

We have to introduce some physical quantities in order to construct the action of gravity. At first, the $SL(2, C)$ gauge field $\omega$ is introduced as

$$\omega = \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}dx^\mu = \omega_\mu dx^\mu, \tag{2}$$

where $\sigma_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$ are $SL(2, C)$ generators and $\gamma_a$ are Dirac gamma matrices satisfying the anticommutation relations $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$. Then the curvature tensor is given in terms of $\omega$ from its definition,

$$R \equiv \frac{1}{4}R_{\mu\nu}{}^{ab}\sigma_{ab}dx^\mu \wedge dx^\nu = d\omega - i\omega \wedge \omega. \tag{3}$$

In addition, the vierbein $e$ is introduced as follows,

$$e = e^a_\mu \gamma_a dx^\mu = e_\mu dx^\mu. \tag{4}$$

Under the $SL(2, C)$ transformation, $\omega$ and $e$ transform as

$$e \to \Omega e \Omega^{-1}, \tag{5}$$

$$\omega \to \Omega\omega\Omega^{-1} + i\Omega d\Omega^{-1}, \tag{6}$$

where the transformation parameter $\Omega = \exp(i\frac{1}{2}\Lambda^{ab}\sigma_{ab}) \equiv \exp(i\Lambda)$. In infinitesimal forms, Eq. (5) and (6) can be written as

$$\delta_\Lambda e = i[\Lambda, e], \tag{7}$$

$$\delta_\Lambda \omega = d\Lambda + i[\Lambda, \omega]. \tag{8}$$

Using Eq. (3) and (6), one can show that the curvature tensor $R$ transforms covariantly under the $SL(2, C)$ gauge transformation,

$$R \to \Omega R \Omega^{-1}, \tag{9}$$

where its infinitesimal form is

$$\delta_\Lambda R = i[\Lambda, R]. \tag{10}$$

Now it is straightforward to write an $SL(2, C)$ invariant action

$$\begin{aligned} S &= \int_M \mathrm{Tr}((c_0 + c_1\gamma_5)e \wedge e \wedge R + c_2\gamma_5 e \wedge e \wedge e \wedge e), \\ &= \int_M d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}((c_0 + c_1\gamma_5)e_\mu e_\nu R_{\rho\sigma} \\ &\quad + c_2\gamma_5 e_\mu e_\nu e_\rho e_\sigma), \end{aligned} \tag{11}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, $c_0$, $c_1$, and $c_2$ are arbitrary constants. Integrating out $\omega$ and endowing appropriate values for $c_0$, $c_1$, and $c_2$ in Eq. (11), one can obtain the Einstein-Hilbert action plus a cosmological constant.

In the next section, we generalize this formulation of gravity to a noncommutative space with the Poisson structure. For the sake of convenience, we do not write actions in forms but in components.

## III. $SL(2, C)$ GRAVITY ON NONCOMMUTATIVE SPACE

Consider a noncommutative spacetime whose coordinates satisfy the following commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}(\hat{x}), \tag{12}$$

where the coordinate-dependent $\theta^{\mu\nu}(\hat{x})$ is a Poisson bivector[1] and it can be used to define a Poisson bracket on the manifold,

$$\{f(x), g(x)\}_{\mathrm{Poisson}} \equiv \theta^{\mu\nu}(x)\partial_\mu f(x)\partial_\nu g(x), \tag{13}$$

where $f(x)$ and $g(x)$ are arbitrary functions on the manifold. The Jacobi identity of the Poisson bracket imposes the following conditions on the bivector $\theta^{\mu\nu}(x)$,

[1]In the sense of the star product [see Eq. (15)], the algebraic relations of the noncommutative spacetime can be written as $[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}(x)$. Thus, we can also utilize $\theta^{\mu\nu}(x)$ to denote a Poisson bivector.

$$\theta^{\mu\rho}(x)\partial_\rho\theta^{\nu\sigma}(x) + \theta^{\nu\rho}(x)\partial_\rho\theta^{\sigma\mu}(x) + \theta^{\sigma\rho}(x)\partial_\rho\theta^{\mu\nu}(x) = 0. \tag{14}$$

Suppose that the bivector $\theta^{\mu\nu}(x)$ is nondegenerate; therefore, we can define its inverse $\theta_{\mu\nu}(x)$ as: $\theta^{\mu\nu}\theta_{\nu\rho} = \delta^\mu_\rho$. With the Jacobi identity Eq. (14), we can show that the twoform $\Theta = \frac{1}{2}\theta_{\mu\nu}dx^\mu \wedge dx^\nu$ is closed ($d\Theta = 0$) and then the manifold is symplectic. In this paper, we shall consider only the case in which the manifold is symplectic.

According to Kontsevich's deformation [22], there exists an associative star product to a given Poisson bivector $\theta^{\mu\nu}(x)$, and it can be written as the following symmetric form up to the first order in $\theta^{\mu\nu}(x)$,

$$f(x)\star g(x) = f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}(x)\partial_\mu f(x)\partial_\nu g(x) + \mathcal{O}(\theta^2). \tag{15}$$

Note that it is not unique for higher order terms. In order to avoid this ambiguity, we shall restrict our discussion only to the first order in $\theta^{\mu\nu}(x)$.

In the following subsections, we construct the $SL(2,C)$ gravity on the spacetime with the structure Eq. (12) and derive the Seiberg-Witten map of the noncommutative gravity up to the first order in $\theta^{\mu\nu}(x)$.

### A. Construction of noncommutative gravity

Let us follow the covariant coordinate approach proposed in Ref. [17]. The covariant coordinates $\hat{X}^\mu = x^\mu + \hat{B}^\mu$ are defined by the gauge transformation property[2]

$$\delta_{\hat{\Lambda}}(\hat{X}^\mu \star \hat{\Psi}) = i\hat{\Lambda}\star(\hat{X}^\mu\star\hat{\Psi}), \tag{16}$$

where $\hat{\Lambda}$ is the transformation parameter and $\hat{\Psi}$ is a matter field with the gauge transformation

$$\delta_{\hat{\Lambda}}\hat{\Psi} = i\hat{\Lambda}\star\hat{\Psi}. \tag{17}$$

This requires that the field $\hat{B}^\mu$ should transform as

$$\delta_{\hat{\Lambda}}\hat{B}^\mu = i[\hat{\Lambda}, x^\mu]_\star + i[\hat{\Lambda}, \hat{B}^\mu]_\star = \theta^{\mu\nu}\partial_\nu\hat{\Lambda} + i[\hat{\Lambda}, \hat{B}^\mu]_\star, \tag{18}$$

and the covariant coordinates as

$$\delta_{\hat{\Lambda}}\hat{X}^\mu = i[\hat{\Lambda}, \hat{X}^\mu]_\star. \tag{19}$$

The noncommutative spin-connection $\hat{\omega}_\mu$ is given by [14,17]

$$\hat{\omega}_\mu = \theta_{\mu\nu}\hat{B}^\nu, \tag{20}$$

where $\theta_{\mu\nu}$ is the inverse of $\theta^{\mu\nu}$: $\theta_{\mu\nu}\theta^{\nu\rho} = \delta^\rho_\mu$. Because of the coordinate-dependence of the noncommutative structure depicted by $\theta^{\mu\nu}$, it is not possible to find the transformation of $\hat{\omega}_\mu$ in a closed form, but one can obtain it correct up to any order required in the noncommutative parameter. To the first order in $\theta$, we have

$$\delta_{\hat{\Lambda}}\hat{\omega}_\mu = \partial_\mu\hat{\Lambda} + i[\hat{\Lambda}, \hat{\omega}_\mu] - \frac{1}{2}\theta^{\lambda\sigma}\{\partial_\lambda\hat{\Lambda}, \partial_\sigma\hat{\omega}_\mu\} - \frac{1}{2}\theta_{\mu\alpha}\theta^{\lambda\sigma}\partial_\sigma\theta^{\alpha\beta}\{\partial_\lambda\hat{\Lambda}, \hat{\omega}_\beta\}, \tag{21}$$

where $\{\cdot,\cdot\}$ stands for an anticommutator.

Using the covariant coordinates, we can define a rank-two tensor

$$\hat{R}^{\mu\nu} \equiv -i([\hat{X}^\mu, \hat{X}^\nu]_\star - i\theta^{\mu\nu}(\hat{X})), \tag{22}$$

and using Eq. (19), we find it transforms as follows,

$$\delta_{\hat{\Lambda}}\hat{R}^{\mu\nu} = i[\hat{\Lambda}, \hat{R}^{\mu\nu}]_\star. \tag{23}$$

Now it is time to look for the relation between the rank-two tensor $\hat{R}^{\mu\nu}$ and the gauge field strength $\hat{R}_{\mu\nu}$. In the case of canonical noncommutative spaces where $\theta$ is constant, the relation is trivial: $\hat{R}^{\mu\nu} = \theta^{\mu\rho}\theta^{\nu\sigma}\hat{R}_{\rho\sigma}$. But in our case, $\theta$ is coordinate-dependent; we should modify the relation and make sure the gauge field strength $\hat{R}_{\mu\nu}$ transforms covariantly,

$$\delta_{\hat{\Lambda}}\hat{R}_{\mu\nu} = i[\hat{\Lambda}, \hat{R}_{\mu\nu}]_\star. \tag{24}$$

Suppose that there exists a function $\hat{\theta}_{\mu\nu}(\hat{X})$ which ensures that $\hat{R}_{\mu\nu}$ with the definition

$$\hat{R}_{\mu\nu} \equiv \hat{\theta}_{\mu\lambda}\star\hat{\theta}_{\nu\sigma}\star\hat{R}^{\lambda\sigma} \tag{25}$$

satisfies the transformation property Eq. (24). This requires that $\hat{\theta}_{\mu\nu}(\hat{X})$ should transform as

$$\delta_{\hat{\Lambda}}\hat{\theta}_{\mu\nu}(\hat{X}) = i[\hat{\Lambda}, \hat{\theta}_{\mu\nu}]_\star. \tag{26}$$

In the next subsection, we can see the function $\hat{\theta}_{\mu\nu}(\hat{X})$ indeed exists, and we shall give its expansion expression.

As in the commutative case, we introduce the noncommutative analogue of vierbeins $\hat{e}_\mu$ with the gauge transformation

$$\delta_{\hat{\Lambda}}\hat{e}_\mu = i[\hat{\Lambda}, \hat{e}_\mu]_\star. \tag{27}$$

Now it is straightforward for us to write a gauge invariant action by using Eqs. (24) and (27),

$$S = \int d^4x(\det\theta^{\mu\nu})^{-(1/2)}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}((c_0 + c_1\gamma_5)\hat{e}_\mu \star\hat{e}_\nu\star\hat{R}_{\rho\sigma} + c_2\gamma_5\hat{e}_\mu\star\hat{e}_\nu\star\hat{e}_\rho\star\hat{e}_\sigma). \tag{28}$$

[2] Coordinates are suppressed from subsection III A to the end of this paper for the sake of simplicity.

Here, the volume form on the symplectic manifold, i.e., $(\det\theta^{\mu\nu})^{-(1/2)}d^4x$, appears naturally with which the following trace property of the integral is satisfied to any order in $\theta^{\mu\nu}$,[3]

$$\int d^4x(\det\theta^{\mu\nu})^{-(1/2)}\,\mathrm{Tr}(f\star g) = \int d^4x(\det\theta^{\mu\nu})^{-(1/2)}\,\mathrm{Tr}(g\star f). \tag{29}$$

It is obvious that the action Eq. (28) is indeed gauge invariant because we can verify

$$\delta_{\hat{\Lambda}}S = \int d^4x(\det\theta^{\mu\nu})^{-(1/2)}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(i[\hat{\Lambda},(c_0+c_1\gamma_5)\hat{e}_\mu \star\hat{e}_\nu\star\hat{R}_{\rho\sigma}+c_2\gamma_5\hat{e}_\mu\star\hat{e}_\nu\star\hat{e}_\rho\star\hat{e}_\sigma]_\star)=0, \tag{30}$$

where Eq. (29) has been used.

In the next subsection, we connect the noncommutative theory with its commutative counterpart using the so-called "Seiberg-Witten map."

### B. Seiberg-Witten map to the first order

A general gauge group with an algebra $G$ does not close on noncommutative spacetimes with the exception to unitary groups. For consistency, the algebra $G$ should be enlarged to its universal enveloping algebra $\mathcal{U}(G)$ [18].

First, let us take a close look at the infinitesimal gauge transformation of the spin-connection $\hat{\omega}$ in Eq. (21) where both commutators and anticommutators appear for the case of noncommutative spacetimes. Second, note that the commutator $[\hat{\Lambda},\hat{e}]_\star$ in the infinitesimal gauge transformation of vierbeins in Eq. (27) can be written as

$$[\hat{\Lambda},\hat{e}]_\star = \frac{1}{4}\{\hat{\Lambda}^{ab},\hat{e}^c\}_\star[\sigma_{ab},\gamma_c]+\frac{1}{4}[\hat{\Lambda}^{ab},\hat{e}^c]_\star\{\sigma_{ab},\gamma_c\}. \tag{31}$$

Using the identities of the Dirac gamma matrices,

$$[\sigma_{ab},\gamma_c]=i(\eta_{ac}\gamma_b-\eta_{bc}\gamma_a), \tag{32}$$

$$\{\sigma_{ab},\gamma_c\}=-\epsilon_{abc}{}^{d}\gamma_5\gamma_d, \tag{33}$$

$$\{\sigma_{ab},\sigma_{cd}\}=\frac{1}{2}(i\epsilon_{abcd}\gamma_5+\eta_{ac}\eta_{bd}-\eta_{ad}\eta_{bc}), \tag{34}$$

we can see that $SL(2,C)$ does not close on noncommutative spacetimes and it should be enlarged to a bigger gauge group including the additional generators 1 and $\gamma_5$, i.e., $SL(2,C)$ is enlarged to $GL(2,C)$, and that the vierbeins should be extended to include the additional generator $\gamma_5\gamma_a$. Thus, the $GL(2,C)$ spin-connection $\hat{\omega}_\mu$ and gauge parameter $\hat{\Lambda}$ can be decomposed as

$$\hat{\omega}_\mu=\frac{1}{2}\hat{\omega}^{(0)ab}_\mu\sigma_{ab}+\hat{a}^{(1)}_\mu+i\hat{b}^{(1)}_{\mu5}\gamma_5,$$
$$\hat{\Lambda}=\frac{1}{2}\hat{\Lambda}^{(0)ab}\sigma_{ab}+\hat{\Lambda}^{(1)}+i\hat{\Lambda}^{(1)}_5\gamma_5, \tag{35}$$

and the vierbein can be generalized to be

$$\hat{e}_\mu=\hat{e}^{(0)a}_\mu\gamma_a+\hat{e}^{(1)a}_{\mu5}\gamma_5\gamma_a. \tag{36}$$

As a result, additional degrees of freedom appear in the noncommutative case. However, there exists a map, the Seiberg-Witten map [3], which relates noncommutative degrees of freedom $\hat{\omega}_\mu$, $\hat{e}_\mu$, and $\hat{\Lambda}$ to their commutative counterparts $\omega_\mu$, $e_\mu$, and $\Lambda$. For the transformation parameter $\hat{\Lambda}$ and the field $\hat{B}^\mu$, the map has been derived [20] up to the first order in $\theta^{\mu\nu}$,

$$\hat{\Lambda}=\Lambda+\frac{1}{4}\theta^{\mu\nu}\{\partial_\mu\Lambda,\omega_\nu\}, \tag{37}$$

$$\hat{B}^\mu=\theta^{\mu\nu}\omega_\nu-\frac{1}{4}\theta^{\rho\sigma}\{\omega_\rho,\partial_\sigma(\theta^{\mu\nu}\omega_\nu)+\theta^{\mu\nu}R_{\sigma\nu}\}, \tag{38}$$

where $R_{\mu\nu}\equiv\partial_\mu\omega_\nu-\partial_\nu\omega_\mu-i[\omega_\mu,\omega_\nu]$ is the curvature tensor for the spin connection $\omega_\mu$.

The function $\hat{\theta}_{\mu\nu}(\hat{X})$ is calculated to the zeroth order in $\theta^{\mu\nu}$ [20],

$$\hat{\theta}_{\mu\nu}=\theta_{\mu\nu}+\theta^{\rho\sigma}\partial_\rho\theta_{\mu\nu}\omega_\sigma+\mathcal{O}(\theta^{\mu\nu}), \tag{39}$$

and this is sufficient to compute the curvature tensor $\hat{R}_{\mu\nu}$ up to the first order in $\theta^{\mu\nu}$ according to Eq. (25). Using Eq. (20) and (38), we can obtain the map between $\hat{\omega}_\mu$ and $\omega_\mu$ up to the first order in $\theta^{\mu\nu}$,

$$\hat{\omega}_\mu=\omega_\mu-\frac{1}{4}\theta^{\lambda\sigma}\{\omega_\lambda,\partial_\sigma\omega_\mu+R_{\sigma\mu}\} -\frac{1}{4}\theta_{\mu\nu}\theta^{\lambda\sigma}\partial_\sigma\theta^{\nu\delta}\{\omega_\lambda,\omega_\delta\}, \tag{40}$$

where the components take the forms,

$$\hat{\omega}^{(0)ab}_\mu=\omega_\mu{}^{ab}, \tag{41}$$

$$\hat{a}^{(1)}_\mu=-\frac{1}{16}\theta^{\lambda\sigma}\omega_\lambda{}^{ab}\Big(\partial_\sigma\omega_\mu{}^{cd}+\frac{1}{2}R_{\sigma\mu}{}^{cd}\Big)\eta_{ac}\eta_{bd} -\frac{1}{16}\theta_{\mu\nu}\theta^{\lambda\sigma}\partial_\sigma\theta^{\nu\delta}\omega_\lambda{}^{ab}\omega_\delta{}^{cd}\eta_{ac}\eta_{bd}, \tag{42}$$

$$\hat{b}^{(1)}_{\mu5}=-\frac{1}{32}\theta^{\lambda\sigma}\omega_\lambda{}^{ab}\Big(\partial_\sigma\omega_\mu{}^{cd}+\frac{1}{2}R_{\sigma\mu}{}^{cd}\Big)\epsilon_{abcd} -\frac{1}{32}\theta_{\mu\nu}\theta^{\lambda\sigma}\partial_\sigma\theta^{\nu\delta}\omega_\lambda{}^{ab}\omega_\delta{}^{cd}\epsilon_{abcd}. \tag{43}$$

[3] If there exists a function $\Omega(x)$ satisfying the relation $\partial_\mu(\Omega\theta^{\mu\nu})=0$, then we have the trace property of the integral [20,23]: $\int d^4x\Omega(x)(f(x)\star g(x))=\int d^4x\Omega(x)(g(x)\star f(x))$. For a symplectic manifold, there exists [24,25] a natural choice for the function $\Omega(x)$ which satisfies the above requirement: $\Omega=(\det\theta^{\mu\nu})^{-(1/2)}$. Moreover, this is also shown in Ref. [26] from a different point of view.

The Seiberg-Witten map for the vierbein $\hat{e}_\mu$ is

$$\hat{e}_\mu + \delta_{\hat{\Lambda}}\hat{e}_\mu = \hat{e}_\mu(e + \delta_\Lambda e, \omega + \delta_\Lambda \omega). \tag{44}$$

Thus, the solution to the above equation up to the first order in $\theta^{\mu\nu}$ can be obtained,

$$\hat{e}_\mu = e_\mu - \frac{1}{2}\theta^{\lambda\sigma}\left\{\omega_\lambda, \partial_\sigma e_\mu + \frac{i}{2}[e_\mu, \omega_\sigma]\right\}, \tag{45}$$

whose components have the forms,

$$\hat{e}_\mu^{(0)a} = e_\mu^a, \tag{46}$$

$$\hat{e}_{\mu 5}^{(1)a} = \frac{1}{4}\theta^{\lambda\sigma}\omega_\lambda{}^{eb}\left(\partial_\sigma e_\mu^c - \frac{1}{2}\omega_\sigma{}^{cd}e_{\mu d}\right)\epsilon_{ebc}{}^a. \tag{47}$$

Note that Eq. (45) can be verified straightforwardly when it is substituted into Eq. (44).

Considering the definition Eq. (22) and (20), we get the following expression of $\hat{R}^{\mu\nu}$,

$$\hat{R}^{\mu\nu} = \theta^{\mu\lambda}\theta^{\nu\sigma}(\partial_\lambda\hat{\omega}_\sigma - \partial_\sigma\hat{\omega}_\lambda - i[\hat{\omega}_\lambda, \hat{\omega}_\sigma]) + \frac{1}{2}\theta^{\mu\lambda}\theta^{\nu\sigma}\theta^{\delta\eta}\{\partial_\delta\hat{\omega}_\lambda, \partial_\eta\hat{\omega}_\sigma\} + \frac{1}{2}\theta^{\mu\lambda}\partial_\eta\theta^{\nu\sigma}\theta^{\delta\eta}\{\partial_\delta\hat{\omega}_\lambda, \hat{\omega}_\sigma\}$$
$$+ \frac{1}{2}\partial_\delta\theta^{\mu\lambda}\theta^{\nu\sigma}\theta^{\delta\eta}\{\hat{\omega}_\lambda, \partial_\eta\hat{\omega}_\sigma\} + \frac{1}{2}\partial_\delta\theta^{\mu\lambda}\partial_\eta\theta^{\nu\sigma}\theta^{\delta\eta}\{\hat{\omega}_\lambda, \hat{\omega}_\sigma\} + \theta^{\sigma\lambda}\partial_\sigma\theta^{\mu\nu}\hat{\omega}_\lambda + \theta^{\mu\nu} - \theta^{\mu\nu}(\hat{X}). \tag{48}$$

In terms of the Taylor expansion of $\hat{X}(x + \hat{B})$, the last three terms of the above equation are simplified as

$$\theta^{\sigma\lambda}\partial_\sigma\theta^{\mu\nu}\hat{\omega}_\lambda + \theta^{\mu\nu} - \theta^{\mu\nu}(\hat{X}) = \frac{1}{2}\theta^{\sigma\alpha}\theta^{\lambda\beta}\partial_\sigma\partial_\lambda\theta^{\mu\nu}\hat{\omega}_\alpha\hat{\omega}_\beta, \tag{49}$$

which contains higher order derivatives of $\theta^{\mu\nu}(x)$. When $\theta^{\mu\nu}(x)$ is a linear function of $x$, i.e., the noncommutativity is of the Lie algebraic structure, Eq. (49) vanishes.

In accordance with the Seiberg-Witten map [Eq. (40)], the noncommutative curvature tensor $\hat{R}^{\mu\nu}$ in Eq. (48) can be expressed in term of the commutative spin-connection $\omega_\mu$,

$$\hat{R}^{\mu\nu} = \theta^{\mu\lambda}\theta^{\nu\sigma}R_{\lambda\sigma} + \frac{1}{2}\theta^{\mu\lambda}\theta^{\nu\sigma}\theta^{\alpha\beta}\{R_{\lambda\alpha}, R_{\sigma\beta}\} - \frac{1}{4}\theta^{\mu\lambda}\theta^{\nu\sigma}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)R_{\lambda\sigma}\} + \frac{1}{2}\theta^{\mu\lambda}\partial_\alpha\theta^{\nu\sigma}\theta^{\alpha\beta}\{R_{\lambda\sigma}, \omega_\beta\}$$
$$+ \frac{1}{2}\partial_\alpha\theta^{\mu\lambda}\theta^{\nu\sigma}\theta^{\alpha\beta}\{R_{\lambda\sigma}, \omega_\beta\} + \frac{1}{4}\theta^{\lambda\alpha}\theta^{\sigma\beta}\partial_\lambda\partial_\sigma\theta^{\mu\nu}\{\omega_\alpha, \omega_\beta\}, \tag{50}$$

where $D_\beta R_{\lambda\sigma} \equiv \partial_\beta R_{\lambda\sigma} - i[\omega_\beta, R_{\lambda\sigma}]$. With the relation between $\hat{R}^{\mu\nu}$ and $\hat{R}_{\mu\nu}$ [see Eq. (25)] and the expression of $\hat{\theta}_{\mu\nu}$ [see Eq. (39)], we obtain $\hat{R}_{\rho\sigma}$,

$$\hat{R}_{\rho\sigma} = R_{\rho\sigma} + \frac{1}{2}\theta^{\alpha\beta}\{R_{\rho\alpha}, R_{\sigma\beta}\} - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)R_{\rho\sigma}\} - \frac{1}{2}\theta_{\sigma\nu}\partial_\alpha\theta^{\mu\nu}\theta^{\alpha\beta}[R_{\rho\mu}, \omega_\beta] - \frac{1}{2}\theta_{\rho\mu}\partial_\alpha\theta^{\mu\nu}\theta^{\alpha\beta}[R_{\sigma\nu}, \omega_\beta]$$
$$+ \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\theta_{\rho\mu}\partial_\beta\theta^{\mu\nu}R_{\nu\sigma} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\theta_{\sigma\nu}\partial_\beta\theta^{\mu\nu}R_{\mu\rho} + \frac{i}{2}\theta^{\alpha\beta}\theta^{\mu\nu}\partial_\alpha\theta_{\rho\mu}\partial_\beta R_{\nu\sigma} + \frac{i}{2}\theta^{\alpha\beta}\theta^{\mu\nu}\partial_\alpha\theta_{\sigma\nu}\partial_\beta R_{\mu\rho}$$
$$+ \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\theta_{\rho\mu}\partial_\beta\theta_{\sigma\nu}\theta^{\mu\lambda}\theta^{\nu\delta}R_{\lambda\delta} + \frac{1}{4}\theta_{\rho\mu}\theta_{\sigma\nu}\theta^{\lambda\alpha}\theta^{\delta\beta}\partial_\lambda\partial_\delta\theta^{\mu\nu}\{\omega_\alpha, \omega_\beta\}, \tag{51}$$

which can also be decomposed by its components as follows:

$$\hat{R}_{\rho\sigma} = \frac{1}{4}\hat{R}_{\rho\sigma}^{(0)ab}\sigma_{ab} + \frac{1}{2}\hat{R}_{\rho\sigma}^{(1)} + \frac{i}{2}\hat{R}_{\rho\sigma 5}^{(1)}\gamma_5. \tag{52}$$

Therefore, the components take the forms,

$$\hat{R}_{\rho\sigma}^{(0)ab} = R_{\rho\sigma}{}^{ab} + i\theta_{\sigma\nu}\partial_\alpha\theta^{\mu\nu}\theta^{\alpha\beta}R_{\rho\mu}{}^{ad}\omega_\beta{}^{cb}\eta_{dc} + i\theta_{\rho\mu}\partial_\alpha\theta^{\mu\nu}\theta^{\alpha\beta}R_{\sigma\nu}{}^{ad}\omega_\beta{}^{cb}\eta_{dc} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\theta_{\rho\mu}\partial_\beta\theta^{\mu\nu}R_{\nu\sigma}{}^{ab}$$
$$+ \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\theta_{\sigma\nu}\partial_\beta\theta^{\mu\nu}R_{\mu\rho}{}^{ab} + \frac{i}{2}\theta^{\alpha\beta}\theta^{\mu\nu}\partial_\alpha\theta_{\rho\mu}\partial_\beta R_{\nu\sigma}{}^{ab} + \frac{i}{2}\theta^{\alpha\beta}\theta^{\mu\nu}\partial_\alpha\theta_{\sigma\nu}\partial_\beta R_{\mu\rho}{}^{ab}$$
$$+ \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\theta_{\rho\mu}\partial_\beta\theta_{\sigma\nu}\theta^{\mu\lambda}\theta^{\nu\delta}R_{\lambda\delta}{}^{ab}, \tag{53}$$

$$\hat{R}^{(1)}_{\rho\sigma} = \frac{1}{16}\theta^{\alpha\beta}R_{\rho\alpha}{}^{ab}R_{\sigma\beta}{}^{cd}\eta_{ac}\eta_{bd} - \frac{1}{8}\theta^{\alpha\beta}\omega_{\alpha}{}^{ab}(\partial_\beta R_{\rho\sigma}{}^{cd} + \omega_\beta{}^{ec}R_{\rho\sigma}{}^{fd}\eta_{ef})\eta_{ac}\eta_{bd} + \frac{1}{8}\theta_{\rho\mu}\theta_{\sigma\nu}\theta^{\lambda\alpha}\theta^{\delta\beta}\partial_\lambda\partial_\delta\theta^{\mu\nu}\omega_\alpha{}^{ab}\omega_\beta{}^{cd}\eta_{ac}\eta_{bd}, \quad (54)$$

$$\hat{R}^{(1)}_{\rho\sigma 5} = \frac{1}{32}\theta^{\alpha\beta}R_{\rho\alpha}{}^{ab}R_{\sigma\beta}{}^{cd}\epsilon_{abcd} - \frac{1}{16}\theta^{\alpha\beta}\omega_{\alpha}{}^{ab}(\partial_\beta R_{\rho\sigma}{}^{cd} + \omega_\beta{}^{ec}R_{\rho\sigma}{}^{fd}\eta_{ef})\epsilon_{abcd} + \frac{1}{16}\theta_{\rho\mu}\theta_{\sigma\nu}\theta^{\lambda\alpha}\theta^{\delta\beta}\partial_\lambda\partial_\delta\theta^{\mu\nu}\omega_\alpha{}^{ab}\omega_\beta{}^{cd}\epsilon_{abcd}, \quad (55)$$

where the following identity has been used in the derivation of the above equations,

$$[\sigma_{ab}, \sigma_{cd}] = i(\eta_{ac}\sigma_{bd} - \eta_{bc}\sigma_{ad} - \eta_{ad}\sigma_{bc} + \eta_{bd}\sigma_{ac}). \quad (56)$$

In terms of the above expression of $\hat{R}_{\rho\sigma}$ and the vierbein Eq. (45), the action Eq. (28) is thus expressed by the spin-connection $\omega_\mu$ and vierbein $e_\mu$ completely,

$$S = \int d^4x(\det\theta^{\mu\nu})^{-(1/2)}\epsilon^{\mu\nu\rho\sigma}\Big[(c_0\eta_{ac}\eta_{bd} + c_1\epsilon_{abcd}) \times \Big(e^a_\mu e^b_\nu \hat{R}^{(0)cd}_{\rho\sigma} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha(e^a_\mu e^b_\nu)\partial_\beta R_{\rho\sigma}{}^{cd}\Big) + c_2(e^a_\mu e^b_\nu e^c_\rho e^d_\sigma\epsilon_{abcd} - 2ie^a_\mu e^b_\nu e^c_\rho \hat{e}^{(1)d}_{\sigma 5}\eta_{ac}\eta_{bd})\Big], \quad (57)$$

where $\hat{R}^{(0)cd}_{\rho\sigma}$ can be simplified as the following form in terms of the symmetries of indices,

$$\hat{R}^{(0)ab}_{\rho\sigma} = R_{\rho\sigma}{}^{ab} + 2i\theta_{\sigma\nu}\partial_\alpha\theta^{\mu\nu}\theta^{\alpha\beta}R_{\rho\mu}{}^{ad}\omega_\beta{}^{cb}\eta_{dc} + i\theta^{\alpha\beta}\partial_\alpha\theta_{\rho\mu}\partial_\beta\theta^{\mu\nu}R_{\nu\sigma}{}^{ab} + i\theta^{\alpha\beta}\theta^{\mu\nu}\partial_\alpha\theta_{\rho\mu}\partial_\beta R_{\nu\sigma}{}^{ab}. \quad (58)$$

Incidentally, the identities below have been used in the calculation of Eq. (57),

$$\mathrm{Tr}(\gamma_a\gamma_b) = 4\eta_{ab}, \quad (59)$$

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_5) = 0, \quad (60)$$

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d) = 4(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}), \quad (61)$$

$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c\gamma_d\gamma_5) = -4i\epsilon_{abcd}. \quad (62)$$

## IV. CONCLUSION

In this paper, a model of gravity based on the $SL(2, C)$ group is constructed on a noncommutative space with the Poisson structure. In order to have a covariantly transformed curvature tensor, a modified function $\hat{\theta}_{\mu\nu}$ is introduced. Here, different from the approach utilized in [20] where a kind of covariant coordinates is defined, we use $\hat{\theta}_{\mu\nu}$ to lower the indices of the rank-two tensor $\hat{R}^{\mu\nu}$ straightforwardly. Therefore, the gauge invariant action is obtained naturally. By using the Seiberg-Witten map, we can express the noncommutative physical quantities in terms of their commutative counterparts, and then the action is completely dependent on the commutative quantities $e$ and $\omega$.

It is noted [12,14,27] that the first order correction in actions vanishes. It is interesting to point out that in our case the first order correction to the Einstein-Hilbert term in action Eq. (57) is a total derivative and thus equals o zero when $\theta^{\mu\nu}$ being constant, which agrees with the result in Refs. [12,27]. However, for a general noncommutativity parameter, it is evident in our case that the first order correction [see Eq. (57)] does not vanish. The reason lies probably in the different approaches utilized in Refs. [12,14,27] and in the present paper. In the former approach, which is based on the Poincare gauge theory, the vierbein is required to be real. This requirement leads to a gauge noninvariant action although it preserves the volume from violation of diffeomorphism in the action. In the latter approach, a $\star$-gauge invariant action is proposed, and it is thus inevitable to introduce a complex vierbein $\hat{e}_\mu$ (see Ref. [11] for the canonical noncommutative case). As a result, the vanishing first order correction claimed in Refs. [12,14,27] remains in doubt as to whether it happens to other noncommutative gravity models built in a different way from that of Refs. [12,14,27], such as to our case.

As a further consideration, we may integrate out $\omega_\mu$ in Eq. (57) and therefore write the action only in the vierbein. However, it is quite a challenge to solve the equation of motion for $\omega_\mu$. Moreover, it might be worthwhile to apply our method to other formulations of noncommutative gravity.

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