State-space correlations and stabilities

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The state-space pair correlation functions and the notion of stability of extremal and nonextremal black holes in string theory and M theory are considered from the viewpoint of thermodynamic Ruppeiner geometry. We have explicitly analyzed the state-space configurations for (i) the two- and three-charge extremal black holes, (ii) the four- and six-charge nonextremal black branes, which both arise from the string theory solutions. An extension is considered for the D_6-D_4-D_2-D_0 multicentered black branes, fractional small black branes, and two-charge rotating fuzzy rings in the setup of Mathur's fuzzball configurations. The state-space pair correlations and the nature of stabilities have been investigated for three-charged bubbling black-brane foams, and thereby the M-theory solutions are brought into the present consideration. Novel aspects of the state-space interactions have been envisaged from the coarse graining counting entropy of underlying conformal field theory microstates.

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I. INTRODUCTION

State-space configurations involving thermodynamics of extremal and nonextremal black branes in string theory [1–8] and M theory [9–12] possess rich intrinsic geometric structures [13–16]. The present article thus focuses attention on thermodynamic perspectives of black branes and thereby explicates the nature of the concerned state-space pair correlations and associated stability of the solutions containing large number of branes and antibranes. Besides several general notions which have earlier been analyzed in condensed matter physics [17–21], we shall here consider specific string theory and M-theory configurations thus mentioned with few thermodynamic parameters and analyze possible state-space pair correlation functions and their scaling relations. Basically, the investigation which we shall follow here entails certain intriguing features of underlying statistical fluctuations which can be defined in terms of thermodynamic parameters. Given the definite covariant state-space description of a consistent macroscopic black-brane solution, we shall expose (i) for what conditions the considered configuration is stable, (ii) how its state-space correlation functions scale in terms of the chosen thermodynamic parameters. In this process, we shall also enlist the complete set of nontrivial relative state-space correlation functions of the configurations considered in [15,16]. It may further be envisaged in this direction that similar considerations indeed remain valid for the other black holes in general relativity [22–25], attractor black holes [26–31] and Legendre transformed finite parameter chemical configurations [32,33], quantum field theory, and QCD backgrounds [34].

On other hand, the state-space configurations of four-dimensional $\mathcal{N}=2$ black holes can be characterized by electric and magnetic charges q_J and p^I which arise from usual flux integrals of the field strength tensors and their Poincaré duals. In such cases, the near horizon geometry of an extremal black hole turns out to be an $\mathrm{AdS}_2 \times S^2$ manifold which describes the Bertotti-Robinson vacuum associated with the black hole. The area of the black hole horizon A and thus the macroscopic entropy [26–29] is given as $S_{\mathrm{macro}} = \pi |Z_\infty|^2$. Such attractor solutions and their critical properties have further been explored from an effective potential defined in $\mathcal{N}=2$, $D=4$ supergravities coupled to n_V abelian vector multiplets for an asymptotically flat extremal black holes describing $(2n_V+2)$-dyonic charges and n_V number of complex scalar fields which parameterize a n_V dimensional special Kähler manifold [30,35–37]. The statistical entropy of the supersymmetric charge black holes coming from counting the degeneracy of bound states has been examined against the macroscopic Wald entropy [38–40] which further agrees term by term with the higher derivative supergravity corrections, as well [8]. In order to study the respective cases of nonextremal black branes of such black holes, one may add some total mass or corresponding antibranes to the chosen extremal black-brane configuration, and thereby a possible specific computation of the black-brane entropy can either be performed in the macroscopic setup, or in the associated microscopic considerations. Furthermore, the investigation of [41,42] shows a match between the S_{micro} and S_{macro} for the nonextremal configurations carrying definite brane and antibrane charges.

There have been various extremal black holes [2,3,8,43,44] and nonextremal black-brane space-times [41], multicentered black-brane configurations [45,46], small black holes with fractional branes [5–8], fuzzy rings

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in Mathur's fuzzballs, as well as the subensemble theoretic setup [47–50] for which the notion of state-space geometry has been introduced in [15,16]. Similar properties have further been explored for the three-charge bubbling black-brane solutions in M theory as well [51]. We shall thus systematically present the status of lower dimensional black hole thermodynamic configurations from the viewpoint of an intrinsic Riemannian geometry. In this connection, the microscopic perspectives of black branes have been analyzed in [4,52], and their thermodynamic geometry whose basics have been motivated in [19,53] has recently been investigated [15,16]. We shall show that similar phenomena may further be explored to exhibit how state-space local correlations scale and under what conditions a chosen black-brane solution corresponds to a stable state-space configuration. In this case, it has been possible to outline that there is an explicit correspondence between the parameters of the microscopic spectrum and macroscopic properties of a class of extremal [43,44] and associated nonextremal black-brane systems [54]. It has there after been pointed out that there should exist definite microscopic origins of underlying statistical and thermodynamic interactions among the microstates of black-brane configurations which give rise to an intrinsic Riemannian geometry. In turn, we apperceive from [1] that such notions may in turn be investigated from the perspective of statistical fluctuations which arise from the coarse graining entropy of chosen configuration.

The state-space geometry thus defined introduces, in particular, that the thermodynamic interactions considered as the function of charges, angular momenta, and mass of a given black-brane configuration may be characterized by an ensemble of equilibrium microstates of underlying microscopic configurations. Furthermore, it has been observed in all such cases that there exists a clear mechanism on the black-brane side that describes the notion of interactions on the state-space which turns out to be a regular intrinsic Riemannian manifold or vice versa. In fact, one may therefore exhibit well-defined intrinsic thermodynamic geometries associated to each other via conformal transformations. The physical observations thus found are consistent with the existing picture of the microscopic conformal field theory (CFT) [54–57] that the microscopic entropy S_{micro} counts the states of the black-brane configuration in a field theory description dual to the gravitational description. In fact, such a conventional understanding of the entropy is based on coarse graining over a large number of microstates [48,58], and thus it turns out to be a crucial ingredient in realizing equilibrium macroscopic thermodynamic geometry. It has been shown [15–17,53] that the components of the state-space metric tensor defined as Hessian matrix of the entropy signify state-space pair correlation functions, and the associated state-space curvature scalar implies the nature of the global correlation volume of the underlying statistical system. Such intrinsic geometric local and global correlations have initially been studied for the thermodynamic configurations of general relativity black holes [22–25], and thereafter they have been brought into focus to the string theory or M-theory black holes [14,59]. Furthermore, the state-space interactions generically remain finite and nonzero when small thermal fluctuations in the canonical ensemble are taken into account [13].

As mentioned above, we shall analyze the intrinsic geometric relative correlations and the notion of stability for a large class of extremal as well as nonextremal black hole configurations at an attractor fixed point solution. The study of chosen thermodynamic systems has rather been already intimated in [15,16], and in this paper our specific goal shall thereby be to explicate their further state-space properties for large charge rotating and nonrotating black-brane configurations. A similar application has indeed been performed for the spherical and nonspherical horizon M-theory configurations, see for details [60,61]. Moreover, it has been demonstrated that various state-space geometric notions turn out to be well-defined, even at zero temperature [59]. Interestingly, such geometric studies in connection with attractor fixed point(s) are expected to give further motivations for analyzing a large class of extremal and nonextremal black-brane configurations under Planck length corrections [62] or that of the higher derivative stringy α' corrections [38–40,59] being incorporated via definite Sen entropy functions [63–71] which are obtained over an underlying supergravity effective action for a given moduli space configuration. These notions shall however be left for future investigation.

The present article is organized as follows. The first Sec. I motivates the study of state-space configurations of the string theory and M-theory black-brane solutions. In either of the subsequent state-space configurations, we shall analyze scaling properties of the state-space pair correlation functions, possible positivity of heat capacities, and nontrivial state-space stabilities. In Sec. II, we have introduced very briefly what is the black-brane thermodynamic geometry, based on the consideration of large number of equilibrium microstates. In Sec. III, we provide the state-space properties for the two- and three-charge extremal black holes and four- and six-charge nonextremal black holes. We consider the two-charge D_1-D_5 solution as an excited string carrying n_1 number of winding modes and n_p number of momentum modes. In fact, we demonstrate that the nature of state-space correlations has a similar pattern for the three-charge extremal black holes. Subsequently, we analyze the state-space geometry for the nonextremal black branes corresponding to three-charge D_1-D_5-P extremal solutions with the addition of an antibrane charge. Furthermore, we show explicitly that similar conclusions hold for the six charge nonextremal black branes as well. In Sec. IV, we focus on the state-space correlations of multicentered D_6-D_4-D_2-D_0 configurations and thereby expose the respective cases for the single center and double center four-charge solutions. In Sec. V, we demonstrate state-space correlation properties for the two-cluster, three-cluster, and then arbitrary finite-cluster D_0-brane fractionations in the D_0-D_4 black branes. In Sec. VI, we explicate that similar state-space geometric notions hold for three-parameter fuzzy rings introduced in the setup of Mathur's fuzzball solutions. In Sec. VII, we extend our state-space analysis to the three-charge bubbling black-brane solutions in M theory. Finally, Sec. VIII contains some concluding issues and the other implications of the state-space geometry of string theory and M-theory black-brane solutions for a future study.

II. STATE-SPACE GEOMETRY

The present section provides a brief introduction to the Ruppeiner's thermodynamic state-space geometry [13,17,25]. In this framework, the state-space metric tensor is defined as the negative Hessian matrix of the black hole entropy $S(\vec{x})$. In general, the components of the metric tensor are thus expressed as

$$g_{ij} := -\frac{\partial^2 S(\vec{x})}{\partial x^j \partial x^i}. \tag{1}$$

In the above setup, a state-space covariant vector $\vec{x} \in M_n$ is considered as the collection of the electric charges, magnetic and angular momenta (q_i, p^i, J_i) of the black hole. We shall show that the state-space geometry thus defined takes an account of the thermodynamic interactions and possible phase transitions. It is worth mentioning that such an interaction for a non-Bogomol'nyi-Prasad-Sommerfeld (BPS) black hole can in principle be considered as the function of the charges, angular momenta, and mass of the black hole. On the other hand, the microscopic viewpoint of the state-space geometry may be considered by introducing fluctuations over the ensemble of equilibrium microstates which characterize the underlying statistical configurations. This is left as an open problem for the future investigation. To illustrate the consideration of state-space geometry, let us explicate the case of two-parameter black-brane configurations. To be concrete, let the parameters be electric charge q and the magnetic charge p, then the components of the Ruppeiner metric tensor are given by

$$g_{qq} = -\frac{\partial^2 S}{\partial q^2}, \quad g_{qp} = -\frac{\partial^2 S}{\partial q \partial p}, \quad g_{pp} = -\frac{\partial^2 S}{\partial p^2}. \tag{2}$$

In this case, the components of the state-space metric tensor are associated to the respective statistical pair correlation functions. It is worth mentioning that the coordinates on the state-space manifold are the parameters of the microscopic boundary conformal field theory which is dual the black hole space-time solution. This is because the underlying state-space metric tensor comprises of the Gaussian fluctuations of the entropy which is the function of the parameters of the black hole configuration. For a given black hole solution, the local stability of the underlying statistical system requires both the principle minors to be positive. In this concern, the diagonal components of the state-space metric tensor $\{g_{x_a x_a} \mid x_a = (q, p)\}$ signify the heat capacities of the system, and thus we require them to be remain positive definite

$$g_{x_i x_i} > 0, i = q, p. \tag{3}$$

In the case of extremal black-brane configurations, we have pointed out that the ratio of diagonal space-state correlations varies as the inverse square of the chosen parameters, while the off diagonal components vary as the inverse of the chosen parameters. We further discuss the significance of this observation for the nonextremal black-brane configurations and find the similar conclusion that the state-space correlations extenuate as the chosen parameters are increased. In both the extremal and nonextremal configurations, we subsequently demonstrate that the notion of scaling property suggests that the brane-brane pair correlations, which find an asymmetric nature in comparison with the other state-space pair correlations, weaken relatively faster and relatively swiftly come into an equilibrium statistical configuration.

From the perspective of intrinsic Riemannian geometry, the stability properties of these black branes are thus divulged from the positivity of principle minors of the space-state metric tensor. For the Gaussian fluctuations over the two-charge equilibrium statistical configurations, the existence of positive definite volume form on the state-space manifold $(M_2(R), g)$ imposes such a stability condition. In particular, the above configuration is said to be stable if the determinant of the state-space metric tensor

$$\|g\| = S_{qq} S_{pp} - S_{qp}^2 \tag{4}$$

remains positive. For the two-charge configurations, the geometric quantities corresponding to the chosen state-space elucidates typical features of the Gaussian fluctuations about an ensemble of equilibrium brane microstates. Subsequently, we can further calculate the Christoffel connection Γ_{ijk}, Riemann curvature tensor R_{ijkl}, Ricci tensor R_{ij}, and Ricci scalar R for the chosen state-space manifold. From the viewpoint of state-space geometry, the intrinsic scalar curvature, as a global invariant, accompanies information of the correlation volume of the underlying statistical systems. In this case, the scalar curvature R is explicitly given by

$$\begin{aligned} R = \frac{1}{2}(S_{qq}S_{pp} - S_{qp}^2)^{-2}(S_{pp}S_{qqq}S_{qpp} + S_{qp}S_{qqp}S_{qpp} \\ + S_{qq}S_{qqp}S_{ppp} - S_{qp}S_{qqq}S_{ppp} - S_{qq}S_{qpp}^2 - S_{pp}S_{qqp}^2). \end{aligned} \tag{5}$$

Under these premises, the zero scalar curvature indicates that the information on the event horizon of the black hole fluctuates independent of each other, while a divergent scalar curvature signifies a sort of phase transition indicating an ensemble of highly correlated pixels of information on the horizon. Ruppeiner has further interpreted that the assumption "that all the statistical degrees of freedom of a black hole live on the black hole event horizon" indicates that the state-space scalar curvature signifies the average number of correlated Planck areas on the event horizon of the black hole [17]. This picture takes an account of the proposal that the area of the event horizon is an integral multiple of the Planck area [57] and Mathur's fuzzballs [58]. For the case of two-parameter state-space configuration, the state-space scalar curvature and curvature tensor are related as

$$R(q, p) = \frac{2}{\|g\|} R_{qpqp}. \tag{6}$$

The state-space scalar curvature thus defined explicates the nature of long range global correlation and phase transitions. In this sense, Ruppenier has introduced an ensemble of micostates corresponding to the black hole, and these states are statistically (i) interacting, if the underlying state-space configuration has a nonzero scalar curvature and (ii) noninteracting, if the scalar curvature vanishes identically. Incrementally, one may note that the configurations under present analysis are effectively attractive or repulsive and weakly interacting in general, while they are stable only if at least one of the parameter, viz., the electric charges, magnetic charges, and angular momenta remains fixed. It is worth mentioning that the finding of a statistical mechanical models with the like behavior could yield a further insight into the microscopic properties of the string theory black hole.

The state-space analysis could provide a set of physical indications encoded in the state-space quantities, e.g., scalar curvature and higher state-space invariants. For given black-brane solutions, the state-space analysis involves an ensemble or subensemble equilibrium microstates forming the statistical basis for the Gaussian fluctuations. With this introduction to the thermodynamic state-space geometry of black-brane solutions, we shall proceed to systematically analyze the underlying geometric structures. This paper considers a case by case study of the intrinsic geometric properties of various possible black holes and black branes. The subsequent analysis is devoted to the state-space geometric implications arising from the theory of extremal and nonextremal black branes in string theory, multicentered black-brane configurations, fractionation of the electric branes, fuzzy rings in Mathur's subensemble theory, and the bubbling black-brane solutions in M theory.

III. BLACK HOLES IN STRING THEORY

In this section, we analyze the nontrivial state-space interactions among various parameters of the black-brane configurations. To illustrate the basic idea of state-space geometry of string theory black holes, we shall explore the consideration for the two- and three-charge extremal configurations and subsequently for the four- and six-charge nonextremal configurations. The notions of the relative correlations and associated stabilities are determined from the perspective state-space investigation.

A. Two-charge extremal configurations

In this subsection, we consider the case of the two-charge extremal D_1-D_5 configurations. It turns out that the state-space geometry of such configurations may be analyzed in terms of the winding modes and the momentum modes of an excited string carrying n_1 number of winding modes and n_p number of momentum modes. To be concrete, we consider the state-space geometry arising from an extremal black hole whose microstates are characterized by the momentum and winding numbers. The microscopic entropy formula [2,3,43,44] obtained from large charge degeneracy of states reduces to

$$S_{\text{micro}} = 2\sqrt{2n_1 n_p} \tag{7}$$

Macroscopically, the entropy of such two-charged black holes may be computed by considering the electric-magnetic charges on the D_4- and D_0-branes, with ascertained compactifications to obtain $M_{3,1}$ black hole space-time. There certainly exist higher derivative corrections in string theory, like for instance the R^2 corrections or R^4 corrections to the standard Einstein action, and thus these corrections make the horizon area nonzero, as the horizon of vanishing Bekenstein-Hawking entropy black holes is being stretched by such higher derivative α' corrections. The computation of the corresponding macroscopic entropy is usually accomplished by assuming spherically symmetric Ansatz for the noncompact spatial directions [4]. On the other hand, the microscopic entropy may be evaluated by considering an ensemble of weakly interacting D-branes [72]. One indeed finds for n_4 number of D_4-branes and n_0 number of D_0-branes that both entropies do match with

$$S_{\text{micro}} = 2\pi\sqrt{n_0 n_4} = S_{\text{macro}} \tag{8}$$

First of all, an immediate goal would be to understand state-space geometric notions associated with the leading order two-charge black-brane solutions, which we shall thus consider via an analysis of the state-space configurations of either an excited string or that of the D_0, D_4 black holes. The analysis follows directly by computing the Hessian matrix of the entropy with respect to concerned extensive thermodynamic variables of the either configurations. It is worth mentioning that the respective entropy can simply be defined as a function of the winding and momentum charges of the string or that of the two-charge

D_0, D_4 black branes. Such configurations are uniquely related to each other and have the same expressions for their entropy. Thus, we focus our attention on the two-charge D_0-D_4 configurations.

From the given expression of the entropy of the two-charge D_0 -D_4 configuration, we observe that the statistical pair correlations may easily be accounted for by simple geometric descriptions, being expressed in terms of the brane numbers connoting an ensemble of microstates of the D_0 -D_4 black hole solutions. Furthermore, it is not difficult to see that the components of the state-space metric tensor describing equilibrium statistical pair correlations may be computed from the negative Hessian matrix of the entropy. As an easy result, we deduce for all allowed values of the parameters of the two-charge D_0 -D_4 black holes that the components of the underlying state-space metric tensor[1] are given as

$$g_{n_0n_0} = \frac{\pi}{2n_0}\sqrt{\frac{n_4}{n_0}}, \qquad g_{n_0n_4} = -\frac{\pi}{2}\frac{1}{\sqrt{n_0n_4}}, \\ g_{n_4n_4} = \frac{\pi}{2n_4}\sqrt{\frac{n_0}{n_4}} \tag{9}$$

It is thus evident that the principle components of the state-space metric tensor $\{g_{n_in_i}|i = 0, 4\}$ essentially signify a set of definite heat capacities (or the related compressibilities) whose positivity in turn apprises that the D_0-D_4 black-brane solutions comply with an underlying equilibrium statistical configuration. In particular, it is further clear for an arbitrary number of D_0- and D_4-branes that the associated state-space metric constraints as the diagonal pair correlation functions remain positive definite, viz., we have

$$g_{n_in_i} > 0 \quad \forall\ i \in \{0, 4\} \mid n_i > 0 \tag{10}$$

The case of finitely many D_0-D_4-branes indeed agrees with an expectation that the nondiagonal component $g_{n_0n_4}$ of the state-space metric tensor, respectively, finds some nonzero negative value. Furthermore, we visualize from the definition of state-space metric tensor that the ratios of principle components of Gaussian statistical pair correlations vary as the inverse square of the concerned brane charges, while that of the off diagonal correlations modulate only inversely. Interestingly from just designated state-space pair correlations of these two-charge black hole configurations, it follows for distinct i, $j \in \{0, 4\}$ that the following expressions define a possible set of admissible scaling relations:

[1]In the present and subsequent sections, we shall invariably use for a given set of brane and antibrane charges and angular momenta $X_a = (X_1, X_2, \cdots, X_k) \in M_k$ that the tensor notations $g_{X_iX_j}$ and g_{ij} signify the same intrinsic state-space object.

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ij}}{g_{ii}} = -\frac{n_i}{n_j} \tag{11}$$

In order to determine the global properties of fluctuating two-charge D_0-D_4 extremal configurations, we need to determine stabilities along each intrinsic direction, each intrinsic plane, and intrinsic hyper plane, if any, as well as on the full intrinsic state-space manifold. Nevertheless, we notice that the underlying state-space manifold in the present case is just an ordinary intrinsic surface, and thus the set of stability criteria on various possible state-space configurations could simply be determined by the two possible principle minors, viz., p_1 and p_2. For all n_0 and n_4, we find that the first minor constraint $p_1 > 0$ directly follows from the positivity of the first component of the metric tensor

$$p_1 = \frac{\pi}{2n_0}\sqrt{\frac{n_4}{n_0}} \tag{12}$$

Moreover, the minor constraint $p_2 > 0$ becomes the positivity of the determinant of the metric tensor which nevertheless vanishes identically for all allowed values of the n_0 and n_4. In this case, we explicitly see that the minor constraint is not fulfilled, viz., the minor $p_2 := g(n_0, n_4)$ takes the null value, and thus the leading order consideration of degeneracy of the states of large charge D_0-D_4 extremal black branes or excited strings with n_1 number of winding and n_p number of momenta finds a degenerate intrinsic state-space configuration.

B. Three charge extremal configurations

To have a test of a higher charge black hole state-space configuration, we may add n_5 number of D_5-branes to the above excited string configuration. Then, it turns out that the leading order entropy of three-charge extremal black hole may be obtained from the two derivative level Einstein-Hilbert action. It is well-known [1] that the entropy of extremal D_1-D_5 solutions arising from Einstein-Hilbert action is proportional to the area of the horizon and the corresponding microscopic entropy may as well be counted by considering an ensemble of weakly interacting D-branes. It turns out that the two entropies match and they take the following form:

$$S_{\mathrm{micro}} = 2\pi\sqrt{n_1n_5n_p} = S_{\mathrm{macro}} \tag{13}$$

The state-space geometry describing the correlations between the equilibrium microstates of the three charged rotating extremal D_1-D_5 black holes resulting from the degeneracy of the microstates may easily be computed as before from the Hessian matrix of the entropy with respect to the number of D_1-, D_5-branes and the Kaluza-Klein momentum, viz, n_1, n_5, and n_p. We then see that the components of state-space metric tensor are given by

$$g_{n_1n_1} = \frac{\pi}{2n_1}\sqrt{\frac{n_5n_p}{n_1}}, \qquad g_{n_1n_5} = -\frac{\pi}{2}\sqrt{\frac{n_p}{n_1n_5}}$$
$$g_{n_1n_p} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1n_p}}, \qquad g_{n_5n_5} = \frac{\pi}{2n_5}\sqrt{\frac{n_1n_p}{n_5}} \tag{14}$$
$$g_{n_5n_p} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5n_p}}, \qquad g_{n_pn_p} = \frac{\pi}{2n_p}\sqrt{\frac{n_1n_5}{n_p}}.$$

The statistical pair correlations thus ascertained could in turn be accounted for by simple microscopic descriptions being expressed in terms of the number of D_1-D_5-branes and Kaluza-Klein momentum connoting an ensemble of microstates of the extremal black hole configuration. Furthermore, it is evident that the principle components of the pair correlation functions remain positive definite for all the allowed values of concerned three parameters of the black holes. As a result, we thus easily observe that the concerned state-space metric constraints are satisfied with

$$g_{n_in_i} > 0 \quad \forall\ i \in \{1, 5, p\} \mid n_i > 0. \tag{15}$$

We thus see in this case that the principle components of state-space metric tensor $\{g_{n_in_i}, g_{n_pn_p} | i = 1, 5\}$ essentially signify a set of definite heat capacities (or related compressibilities) whose positivity demonstrates that the three-charge D_1-D_5-P black holes comply with the underlying locally stable equilibrium statistical configuration. Furthermore, we suspect that an addition of Kaluza-Klein momentum charge does not alter the conclusion of excited string system that the D_1-D_5-P configuration with finitely many D_1-D_5-branes and momentum excitations agrees with the naive expectation that respective nondiagonal components, viz., g_{ij} and g_{ip} of the state-space metric tensor can find some nonpositive values.

Interestingly, the ratios of the principle components of metric tensor describing Gaussian statistical pair correlations vary as inverse square of the brane numbers and momentum charge, while that of the off diagonal rations of the state-space correlations modulate only inversely. It further follows from the above expressions that we may explicitly visualize for distinct $i, j \in \{1, 5\}$, and p that the list of relative correlation functions thus described is consisting of the following scaling properties

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ii}}{g_{pp}} = \left(\frac{n_p}{n_i}\right)^2, \qquad \frac{g_{ii}}{g_{ij}} = -\left(\frac{n_j}{n_i}\right)$$
$$\frac{g_{ii}}{g_{ip}} = -\left(\frac{n_p}{n_i}\right), \qquad \frac{g_{ip}}{g_{jp}} = \left(\frac{n_j}{n_i}\right), \qquad \frac{g_{ii}}{g_{jp}} = -\left(\frac{n_jn_p}{n_i^2}\right)$$
$$\frac{g_{ip}}{g_{pp}} = -\left(\frac{n_p}{n_i}\right), \qquad \frac{g_{ij}}{g_{ip}} = \left(\frac{n_p}{n_j}\right), \qquad \frac{g_{ij}}{g_{pp}} = -\left(\frac{n_p^2}{n_in_j}\right). \tag{16}$$

Along with the positivity of principle components of state-space metric tensor, we need to demand, in order to accomplish the local stability of associated system, that all the principle minors should be positive definite. It is nevertheless not difficult to compute the principle minors of the Hessian matrix of the entropy of three-charge D_1-D_5-P extremal black holes. In fact, after some manipulations one encounters that the local stability conditions along the principle line and that of the respective two-dimensional surface of the concerned state-space manifold be simply measured by the following equations

$$p_1 = \frac{\pi}{2n_1}\sqrt{\frac{n_5n_p}{n_1}}, \qquad p_2 = -\frac{\pi^2}{4n_1n_5^2n_p}(n_p^2n_1 + n_5^3) \tag{17}$$

For the physically allowed values of brane numbers and momentum charge of the D_1-D_5-P extremal black holes, we thus notice that the minor constraint $p_2(n_1, n_5, n_p) > 0$ never gets satisfied for any real positive physical parameters. In particular, we may easily suspect that the nature of state-space geometry for the three-charge D_1-D_5-P extremal black holes is that these solutions are stable along the line on state-space but have planar instabilities. It is easy to stipulate that our conclusion holds for an arbitrary number of D_1-D_5-branes and Kaluza-Klein momentum.

In the viewpoint of the simplest two-charge extremal solutions, it tuns out that the local stability on the entire equilibrium phase-space configurations of the D_1-D_5-P extremal black holes may clearly be determined by computing the determinant of the underlying state-space metric tensor. As in the previous example, it is easy to observe that the state-space metric tensor is a nondegenerate and everywhere regular function of the brane charges n_1 and n_5 and Kaluza-Klein momentum charge n_p. In particular, we find under the present consideration that the determinant of the metric tensor as the highest principle minor $p_3 := g$ of the Hessian matrix of the entropy takes a simple form

$$\|g\| = -\frac{1}{2}\pi^3(n_1n_5n_p)^{-1/2} \tag{18}$$

Moreover, we observe that the determinant of the metric tensor does not take a positive definite, well-defined form, and thus there is no positive definite globally well-defined volume form on the state-space manifold (M_3, g) of the concerned three-charge D_1-D_5-P extremal system. In turn, the nonzero value of the determinant of state-space metric tensor $g(n_1, n_5, n_p)$ indicates that the extremal D_1-D_5-P solution may decay into some other degenerate vacuum state configurations procuring the same corresponding entropy or microscopic degeneracy of states. Here, we further notice, independent of the microscopic type-II string description or heterotic string description, that the three-charge D_1-D_5-P black holes, when considered as a bound state of the D_1-D_5-brane microstates and Kaluza-Klein excitations, do not correspond to an intrinsically stable statistical configuration. It is worth mentioning, as introduced in [15,59] in order to divulge phase transitions and related global state-space properties, that this statistical system remains everywhere regular as long as the number of brane and the Kaluza-Klein momentum charge take finite values.

In the next two sections, we shall deal with the state-space geometry of nonextremal black branes in string theory with two/three charges and two/three anticharges of leading order entropy configurations. After defining state-space metric tensor, we shall analyze scaling properties of possible state-space pair correlation functions and stability requirements for the chosen nonextremal black-brane solution.

C. Four-charge nonextremal configurations

The present subsection examines the state-space configuration of nonextremal black holes and extends our intrinsic geometric assessments for the D_1-D_5 black holes having nonzero momenta along the clockwise and anticlockwise directions of the Kaluza-Klein compactification circle S^1. For the purpose of critical ratifications, we shall focus our attention on the state-space geometry arising from the entropy of nonextremal black holes, which one can simply achieve just by adding corresponding antibranes to the chosen extremal black-brane solution. It follows precisely that we shall first consider the simplest example of such systems, viz., a string having a large amount of winding and D_5-brane charges: n_1, n_5 with extra energy, which in the microscopic description creates an equal amount of momenta running in opposite directions of the S^1. In this case, the entropy has been calculated from both the microscopic and macroscopic perspective [41] and matches for given total mass and brane charges. In particular, it has been shown in [41] that the either of the above entropies satisfy

$$S_{\text{micro}} = 2\pi\sqrt{n_1 n_5}(\sqrt{n_p} + \sqrt{\bar{n}_p}) = S_{\text{macro}} \qquad (19)$$

We may then analyze the state-space covariant metric tensor defined as a negative Hessian matrix of entropy with respect to the number of D_1-, D_5-branes $\{n_i \mid i = 1, 5\}$ and opposite Kaluza-Klein momentum charges $\{n_p, \bar{n}_p\}$.[2] The associate components of the state-space metric tensor and stability parameters are thus easy to compute for the nonextremal D_1-D_5 black holes. In fact, a direct computation finds that the components of the metric tensor take the following expression:

[2]In this section, the notations $\bar{n}_p$ and $n_{\bar{p}}$ shall imply the same Kaluza-Klein momentum charges which are in opposite direction of the n_p momentum charge and flow along the S^1.

$$\begin{aligned}
g_{n_1 n_1} &= \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}(\sqrt{n_p} + \sqrt{\bar{n}_p}), \quad g_{n_1 n_5} = -\frac{\pi}{2\sqrt{n_1 n_5}}(\sqrt{n_p} + \sqrt{\bar{n}_p}) \\
g_{n_1 n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 n_p}}, \quad g_{n_1 \bar{n}_p} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 \bar{n}_p}} \\
g_{n_5 n_5} &= \frac{\pi}{2}\sqrt{\frac{n_1}{n_5^3}}(\sqrt{n_p} + \sqrt{\bar{n}_p}), \quad g_{n_5 n_p} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 n_p}} \\
g_{n_5 \bar{n}_p} &= -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 \bar{n}_p}}, \quad g_{n_p n_p} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{n_p^3}} \\
g_{n_p \bar{n}_p} &= 0, \quad g_{\overline{n_p}\,\overline{n_p}} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{\bar{n}_p^3}}.
\end{aligned} \qquad (20)$$

It is clear that there exists an intriguing intrinsic geometric enumeration which describes the possible nature of statistical pair correlations. The present framework affirms in turn that the concerned state-space pair fluctuations determined in terms of the brane and antibrane numbers (or brane charges) of the D_1-D_5-P nonextremal black holes demonstrate definite expected behavior of the underlying heat capacities. Hitherto, we see apparently that the principle components of statistical pair correlations remain positive definite quantities for all admissible values of underlying configuration parameters of the black branes. It may easily be observed that the following state-space metric constraints are satisfied:

$$g_{n_i n_i} > 0 \quad \forall\ i = 1, 5; \qquad g_{n_a n_a} > 0 \quad \forall\ a = p, \bar{p}. \qquad (21)$$

We thus physically note that the principle components of the state-space metric tensor $\{g_{n_i n_i}, g_{n_a n_a} | i = 1, 5; a = p, \bar{p}\}$ signify a set of heat capacities (or the associated compressibilities) whose positivity exhibits that the underlying black hole system is in local equilibrium statistical configuration of the branes and antibranes. The present analysis thus complies with the fact that the positivity of $g_{n_a n_a}$ obliges that the dual conformal field theory living on the boundary must be associated with a nonvanishing value of the momentum charges associated with large integers n_p, $\bar{n}_p$ defining the degeneracy of microscopic conformal field theory.

It follows from the above expressions of the components of state-space metric tensor that the ratios of principle components of statistical pair correlations vary as the inverse square of the brane numbers, while one finds in specific limit of leading order entropy that the off diagonal correlations vary only inversely. Interestingly, we may as a sequel visualize for the distinct $i, j \in \{1, 5\}$, and $k, l \in \{p, \bar{p}\}$ describing four-charge nonextremal D_1-D_5-P-$\bar{P}$ black holes that the statistical pair correlations thus proclaimed consists of the following set of scaling relations

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ii}}{g_{kk}} = \frac{n_k}{n_i^2}\sqrt{n_k}(\sqrt{n_p} + \sqrt{\bar{n}_p}),$$
$$\frac{g_{ii}}{g_{ij}} = -\frac{n_j}{n_i} \qquad \frac{g_{ii}}{g_{ik}} = -\frac{\sqrt{n_k}}{n_i}(\sqrt{n_p} + \sqrt{\bar{n}_p}),$$
$$\frac{g_{ik}}{g_{jk}} = \frac{n_j}{n_i}, \qquad \frac{g_{ii}}{g_{jk}} = -\frac{n_j}{n_i^2}\sqrt{n_k}(\sqrt{n_p} + \sqrt{\bar{n}_p}) \tag{22}$$
$$\frac{g_{ik}}{g_{kk}} = -\frac{n_k}{n_i}, \qquad \frac{g_{ij}}{g_{ik}} = \frac{\sqrt{n_k}}{n_j}(\sqrt{n_p} + \sqrt{\bar{n}_p}),$$
$$\frac{g_{ij}}{g_{kk}} = -\frac{n_k}{n_i n_j}\sqrt{n_k}(\sqrt{n_p} + \sqrt{\bar{n}_p}).$$

We further see that the list of other relative correlation functions concerning the nonextremal D_1-D_5-P-$\bar{P}$ black holes is

$$\frac{g_{ik}}{g_{il}} = \sqrt{\frac{n_l}{n_k}}, \qquad \frac{g_{ik}}{g_{jl}} = \frac{n_j}{n_i}\sqrt{\frac{n_l}{n_k}}, \qquad \frac{g_{kl}}{g_{ij}} = 0$$
$$\frac{g_{kl}}{g_{ii}} = 0, \qquad \frac{g_{kk}}{g_{ll}} = \left(\frac{n_l}{n_k}\right)^{3/2}, \qquad \frac{g_{kl}}{g_{kk}} = 0. \tag{23}$$

To investigate the entire set of geometric properties of fluctuating nonextremal D_1-D_5 configurations, we need to determine stability along each intrinsic direction, each intrinsic plane, as well as on the full intrinsic state-space manifold. Here, we may adroitly compute the principle minors from the Hessian matrix of the associated entropy concerning the four-charge string theory nonextremal black hole solution carrying D_1, D_5 charges and left and right Kaluza-Klein momenta. In fact, a simple manipulation discovers that the set of local stability criteria on various possible surfaces and hyper-surfaces of the underlying state-space configuration is, respectively, determined by the following set of equations:

$$p_0 = 1, \qquad p_1 = \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}(\sqrt{n_p} + \sqrt{\bar{n}_p}) \qquad p_2 = 0,$$
$$p_3 = -\frac{1}{2n_p}\frac{\pi^3}{\sqrt{n_1 n_5}}(\sqrt{n_p} + \sqrt{\bar{n}_p}). \tag{24}$$

For all physically admissible values of the brane and antibrane charges (or concerned brane numbers) of the nonextremal D_1-D_5 black holes, we can thus easily ascertain that the minor constraint, viz., $p_2(n_i, n_p, \bar{n}_p) = 0$ exhibits that the two-dimensional state-space configurations are not stable for any value of the brane numbers and assigned Kaluza-Klein momenta. Similarly, the positivity of $p_1(n_i, n_p, \bar{n}_p)$ for arbitrary number of branes shows that the underlying fluctuating configurations are locally stable because of the linewise positive definiteness.

The constraint $p_3(n_i, n_p, \bar{n}_p) > 0$ respectively imposes the condition that the system may never attain stability on three-dimensional subconfigurations for all given positive Kaluza-Klein momenta and given positive n_i's. In particular, these constraints enable us to investigate the potential nature of the state-space geometric stability for leading order nonextremal D_1-D_5 black branes. We thus observe that the presence of planar and hyper-planar instabilities exist for the spherical horizon nonextremal D_1-D_5 solutions. We expect altogether in the viewpoint of subleading higher derivative contributions in the entropy that the involved systems demand some restriction on the allowed value of the Kaluza-Klein momenta and number of branes and antibranes.

Moreover, it is not difficult to enquire the complete local stability of the full state-space configuration of nonextremal D_1-D_5 black branes, and in fact it may simply be acclaimed by computing the determinant of the state-space metric tensor. Nevertheless, it is possible to enumerate a compact formula for the determinant of the metric tensor. For the different allowed values of brane numbers, viz., $\{n_1, n_5\}$ and Kaluza-Klein momenta $\{n_p, \bar{n}_p\}$, one apparently discovers from concerned intrinsic geometric analysis that the nonextremal D_1-D_5 system admits the following expression for the determinant of the state-space metric tensor

$$g(n_1, n_5, n_p, \bar{n}_p) = -\frac{1}{4}\frac{\pi^4}{(n_p\bar{n}_p)^{3/2}}(\sqrt{n_p} + \sqrt{\bar{n}_p})^2. \tag{25}$$

Furthermore, we may exhibit that the nature of the statistical interactions and the other global properties of the D_1-D_5 nonextremal configurations are indeed not really perplexing to anatomize. In this regard, one computes certain global invariants of the state-space manifold (M_4, g) which in the present case can easily be determined in terms of the parameters of underlying brane configurations. Here, we may work in the large charge limit in which the asymptotic expansion of the entropy of nonextremal D_1-D_5 system is valid. In particular, we notice that the state-space scalar curvature as indicated in [15,16] generically remains nonvanishing for all finite values of the brane charges and Kaluza-Klein momenta. Thus for physically acceptable parameters, the large charge nonextremal D_1-D_5 black branes having nonvanishing scalar curvature function on their state-space manifold (M_4, g) imply an almost everywhere weakly interacting statistical basis.

D. Six-charge nonextremal configurations

In this subsection, we shall consider the state-space configuration for the six-parameter nonextremal string theory black holes and focus our attention to analyze concerned state-space pair correlation functions and present stability analysis in detail. In order to do so, we extrapolate the expression of the entropy of the four-charge nonextremal D_1-D_5 solution to a nonlarge charge domain, where we are no longer close to an ensemble of supersymmetric states. It is known that the leading order entropy [73], which includes all such special extremal and near-extremal cases, can be written as a function of charges $\{n_i\}$ and anticharges $\{m_i\}$ to be

$$S(n_1, m_1, n_2, m_2, n_3, m_3) := 2\pi(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}) \quad (26)$$

Incidentally, we notice from the simple brane and antibrane description that there exists an interesting state-space interpretation which covariantly describes various statistical pair correlation formulae arising from corresponding microscopic entropy of the aforementioned (non) supersymmetric (non) extremal black-brane configurations. Furthermore, we see for given charges $i,\ j \in A_1 := \{n_1, m_1\}$; $k,\ l \in A_2 := \{n_2, m_2\}$; and $m,\ n \in A_3 := \{n_3, m_3\}$ that the intrinsic state-space pair correlations turn out to be in precise accordance with the underlying macroscopic attractor configurations being disclosed in the special leading order limit of the nonextremal solutions.

It is again not difficult to explore the state-space geometry of equilibrium microstates of the six-charge anticharge nonextremal black holes in $D = 4$ arising from the entropy expression emerging from the consideration of Einstein-Hilbert action. As stated earlier, the state-space Ruppeiner metric is defined by negative Hessian matrix of the nonextremal Bekenstein-Hawking entropy with respect to the extensive variables. These variables in this case are in turn the conserved charges-anticharges carried by the nonextremal black hole. Explicitly, we find that the components of covariant state-space metric tensor over generic nonlarge charge domains are

$$\begin{aligned}
&g_{n_1n_1} = \frac{\pi}{2n_1^{3/2}}(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}), \qquad g_{n_1m_1} = 0 \qquad g_{n_1n_2} = -\frac{\pi}{2\sqrt{n_1n_2}}(\sqrt{n_3} + \sqrt{m_3}),\\
&g_{n_1m_2} = -\frac{\pi}{2\sqrt{n_1m_2}}(\sqrt{n_3} + \sqrt{m_3}) \qquad g_{n_1n_3} = -\frac{\pi}{2\sqrt{n_1n_3}}(\sqrt{n_2} + \sqrt{m_2}), \qquad g_{n_1m_3} = -\frac{\pi}{2\sqrt{n_1m_3}}(\sqrt{n_2} + \sqrt{m_2})\\
&g_{m_1m_1} = \frac{\pi}{2m_1^{3/2}}(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}), \qquad g_{m_1n_2} = -\frac{\pi}{2\sqrt{m_1n_2}}(\sqrt{n_3} + \sqrt{m_3})\\
&g_{m_1m_2} = -\frac{\pi}{2\sqrt{m_1m_2}}(\sqrt{n_3} + \sqrt{m_3}), \qquad g_{m_1n_3} = -\frac{\pi}{2\sqrt{m_1n_3}}(\sqrt{n_2} + \sqrt{m_2}) \qquad g_{m_1m_3} = -\frac{\pi}{2\sqrt{m_1m_3}}(\sqrt{n_2} + \sqrt{m_2}),\\
&g_{n_2n_2} = \frac{\pi}{2n_2^{3/2}}(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_3} + \sqrt{m_3}) \qquad g_{n_2m_2} = 0, \qquad g_{n_2n_3} = -\frac{\pi}{2\sqrt{n_2n_3}}(\sqrt{n_1} + \sqrt{m_1})\\
&g_{n_2m_3} = -\frac{\pi}{2\sqrt{n_2m_3}}(\sqrt{n_1} + \sqrt{m_1}), \qquad g_{m_2m_2} = \frac{\pi}{2m_2^{3/2}}(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_3} + \sqrt{m_3})\\
&g_{m_2n_3} = -\frac{\pi}{2\sqrt{m_2n_3}}(\sqrt{n_1} + \sqrt{m_1}), \qquad g_{m_2m_3} = -\frac{\pi}{2\sqrt{m_2m_3}}(\sqrt{n_1} + \sqrt{m_1})\\
&g_{n_3n_3} = \frac{\pi}{2n_3^{3/2}}(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2}), \qquad g_{n_3m_3} = 0 \qquad g_{m_3m_3} = \frac{\pi}{2m_3^{3/2}}(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2}).
\end{aligned} \quad (27)$$

In the entropy representation, we thus see for the nonvanishing entropy that the Hessian matrix of entropy illustrates the nature of possible Gaussian state-space correlations between the set of space-time parameters, which in this case are nothing but the charges on the brane and antibranes, if nonextremality is violated in general. Substantially, we articulate for given nonzero values of large charges and anti charges $\{n_i, m_1 \mid i = 1, 2, 3\}$ that the nonvanishing principle component of underlying intrinsic state-space metric tensor are positive definite quantities. It is in fact not difficult to see for distinct $i,\ j,\ k \in \{1, 2, 3\}$ that the components involving brane-brane state-space correlations $g_{n_in_i}$ and antibrane-antibrane state-space correlations $g_{m_im_i}$ satisfy

$$\begin{aligned}
&g_{n_in_i} > 0 \quad \forall \text{ finite } n_i,\ i = 1, 2, 3\\
&g_{m_im_i} > 0 \quad \forall \text{ finite } m_i,\ i = 1, 2, 3.
\end{aligned} \quad (28)$$

Furthermore, it has been observed that the ratios of diagonal components vary inversely with a multiple of a well-defined factor in the underlying parameters which change under the Gaussian fluctuations, whereas the ratios involving off diagonal components in effect uniquely inversely vary in the parameters of the chosen set A_i of equilibrium black-brane configurations. This suggests that the diagonal components weaken in relatively controlled fashion into an equilibrium, in contrast with the off diagonal components which vary over the domain of associated parameters defining the D_1-D_5-P- nonextremal nonlarge charge configurations. Importantly, we can easily substantiate for the distinct $n_i, m_i \mid i \in \{1, 2, 3\}$ describing six-charge string theory black holes that the relative pair correlation functions have three types of relative correlation functions. In particular, we firstly see for $i,\ j \in \{n_1, m_1\}$, and $k,\ l \in \{n_2, m_2\}$ that the relative correlation functions satisfy the following list of scaling relations

$$\frac{g_{ii}}{g_{jj}}=\left(\frac{j}{i}\right)^{3/2}, \qquad \frac{g_{ii}}{g_{kk}}=\left(\frac{k}{i}\right)^{3/2}\left(\frac{\sqrt{n_2}+\sqrt{m_2}}{\sqrt{n_3}+\sqrt{m_3}}\right),$$
$$\frac{g_{ij}}{g_{ii}}=0\,\frac{g_{ii}}{g_{ik}} \quad =-\tfrac{\sqrt{k}}{i}(\sqrt{n_2}+\sqrt{m_2}), \qquad \tfrac{g_{ik}}{g_{jk}}=\sqrt{\tfrac{j}{i}},$$
$$\frac{g_{ii}}{g_{jk}}=-\frac{\sqrt{jk}}{i^{3/2}}(\sqrt{n_2}+\sqrt{m_2}) \qquad \frac{g_{kk}}{g_{ik}}=-\frac{\sqrt{i}}{k}(\sqrt{n_2}+\sqrt{m_2}),$$
$$\frac{g_{ij}}{g_{ik}}=0, \qquad \frac{g_{ij}}{g_{kk}}=0. \tag{29}$$

The other concerned relative correlation functions are

$$\frac{g_{ik}}{g_{il}}=\sqrt{\frac{l}{k}}, \qquad \frac{g_{ik}}{g_{jl}}=\sqrt{\frac{jl}{ik}}, \qquad \frac{g_{ij}}{g_{kl}}=n.d.$$
$$\frac{g_{kl}}{g_{ii}}=0, \qquad \frac{g_{kk}}{g_{ll}}=\left(\frac{l}{k}\right)^{3/2}, \qquad \frac{g_{kl}}{g_{kk}}=0. \tag{30}$$

For $k,\ l\in\{n_2, m_2\}$, and $m,\ n\in\{n_3, m_3\}$, we have

$$\frac{g_{kk}}{g_{mm}}=\left(\frac{m}{k}\right)^{3/2}\left(\frac{\sqrt{n_3}+\sqrt{m_3}}{\sqrt{n_2}+\sqrt{m_2}}\right), \quad \frac{g_{kl}}{g_{kk}}=0,$$
$$\frac{g_{kk}}{g_{km}}=-\frac{\sqrt{m}}{k}(\sqrt{n_3}+\sqrt{m_3}) \quad \frac{g_{km}}{g_{lm}}=\sqrt{\frac{l}{k}},$$
$$\frac{g_{kk}}{g_{lm}}=-\frac{\sqrt{lm}}{k^{3/2}}(\sqrt{n_3}+\sqrt{m_3}), \quad \frac{g_{mm}}{g_{km}}=-\frac{\sqrt{k}}{m}(\sqrt{n_2}+\sqrt{m_2})$$
$$\frac{g_{kl}}{g_{km}}=0, \quad \frac{g_{kl}}{g_{mm}}=0. \tag{31}$$

The other concerned relative correlation functions are

$$\frac{g_{km}}{g_{kn}}=\sqrt{\frac{n}{m}}, \qquad \frac{g_{km}}{g_{ln}}=\sqrt{\frac{ln}{km}}, \qquad \frac{g_{kl}}{g_{mn}}=n.d.$$
$$\frac{g_{mn}}{g_{kk}}=0, \qquad \frac{g_{mm}}{g_{nn}}=\left(\frac{n}{m}\right)^{3/2}, \qquad \frac{g_{mn}}{g_{mm}}=0. \tag{32}$$

While for $i,\ j\in\{n_1, m_1\}$, and $m,\ n\in\{n_3, m_3\}$, we have

$$\frac{g_{ii}}{g_{mm}}=\left(\frac{m}{i}\right)^{3/2}\left(\frac{\sqrt{n_3}+\sqrt{m_3}}{\sqrt{n_1}+\sqrt{m_1}}\right), \quad \frac{g_{ij}}{g_{ii}}=0,$$
$$\frac{g_{ii}}{g_{im}}=-\frac{\sqrt{m}}{i}(\sqrt{n_3}+\sqrt{m_3}) \quad \frac{g_{im}}{g_{jm}}=\sqrt{\frac{j}{i}},$$
$$\frac{g_{ii}}{g_{jm}}=-\frac{\sqrt{jm}}{i^{3/2}}(\sqrt{n_3}+\sqrt{m_3}), \quad \frac{g_{mm}}{g_{im}}=-\frac{\sqrt{i}}{m}(\sqrt{n_1}+\sqrt{m_1})$$
$$\frac{g_{ij}}{g_{im}}=0, \quad \frac{g_{ij}}{g_{mm}}=0, \quad \frac{g_{im}}{g_{in}}=\sqrt{\frac{n}{m}}$$
$$\frac{g_{im}}{g_{jn}}=\sqrt{\frac{jn}{im}}, \quad \frac{g_{ij}}{g_{mn}}=n.d., \quad \frac{g_{mn}}{g_{ii}}=0, \quad \frac{g_{mn}}{g_{mm}}=0. \tag{33}$$

For given $i,\ j\in A_1:=\{n_1, m_1\}$; $k,\ l\in A_2:=\{n_2, m_2\}\}$; and $m,\ n\in A_3:=\{n_3, m_3\}$; we thus see by utilizing $g_{n_1m_1}=0$, $g_{n_2m_2}=0$, and $g_{n_3m_3}=0$ that there are seven nontrivial relative correlation functions for each set A_i, where $i=1, 2, 3$, and one nontrivial ratio in each chosen family A_i. It is worth mentioning that the scaling relations remain similar to those obtained in the previous case, except when (i) the number of relative correlation functions has been increased, and (ii) the set of cross ratios, viz., $\{\frac{g_{ij}}{g_{kl}}, \frac{g_{kl}}{g_{mn}}, \frac{g_{ij}}{g_{mn}}\}$ being zero in the previous case becomes ill-defined for the six-charge state-space configuration. Inspecting a specific pair of distinct charge sets A_i and A_j, one finds in this case that there are thus 24 types of nontrivial relative correlation functions.

Specifically, we see for three-brane and three-antibrane solutions that the ratios involving diagonal components in the numerator with nondiagonal components in the denominator vanish identically $\forall i, j, k\in\{n_1, m_1, n_2, m_2, n_3, m_3\}$. Alternatively, we thereby appraise in this case that the set of principle components denominator ratios computed from above state-space metric tensor reduce to

$$\frac{g_{ij}}{g_{kk}}=0 \quad \forall\ i, j, k\in\{n_1, m_1, n_2, m_2, n_3, m_3\}. \tag{34}$$

In particular, for given $i,\ j\in A_1:=\{n_1, m_1\}$; $k,\ l\in A_2:=\{n_2, m_2\}\}$; and $m,\ n\in A_3:=\{n_3, m_3\}$, we confirm the above fact by utilizing $g_{n_1m_1}=0$, $g_{n_2m_2}=0$, and $g_{n_3m_3}=0$ that there are a total of 15 types of trivial relative correlation functions. It is not difficult to see there are five such trivial ratios in each chosen family $\{A_i \mid i=1, 2, 3\}$. It is worth mentioning for each set A_i that the trivial ratios reduce to the scaling relations which are nevertheless similar to those realized in the previous case, except for the fact that the number of relative correlation functions has been ill-defined. Inspecting a pair of distinct charge sets A_i and A_j, one finds in this case that there is a unique kind of ill-defined relative correlation, and thus there are in total three types of divergent relative correlation functions.

As noticed in the previous configuration, it is not difficult to analyze the local stability for the higher charged string theory nonextremal black holes as well. In particular, one can easily determine the principle minors associated with the state-space metric tensor and thus we argue that all the principle minors must be positive definite. However, it may not be the case for all the black holes that they are stabile in all the dimensions of the state-space manifold. In this case, we have computed the principle minors from the Hessian matrix of associated entropy concerning the three-charge and three-anticharged black holes and observe that some of them are indeed nonpositive. In fact, we discover that the local stability criteria on the lower dimensional hyper-surfaces and two-dimensional surface of underlying state-space manifold are, respectively, given by the following relations:

$$p_1 = \frac{\pi}{2n_1^{3/2}}(\sqrt{n_2}+\sqrt{m_2})(\sqrt{n_3}+\sqrt{m_3})$$
$$p_2 = \frac{1}{4}\frac{\pi^2}{(n_1m_1)^{3/2}}(\sqrt{n_2}+\sqrt{m_2})^2(\sqrt{n_3}+\sqrt{m_3})^2$$
$$p_3 = \frac{1}{8}\frac{\pi^3}{(n_1m_1n_2)^{3/2}}\sqrt{m_2}(\sqrt{n_3}+\sqrt{m_3})^3(\sqrt{n_2}+\sqrt{m_2}) \times(\sqrt{n_1}+\sqrt{m_1})$$
$$p_4 = 0. \quad (35)$$

For the physically admitted values of associated charges and anticharges of the nonextremal string theory black holes, we thus ascertain that the minor constraint, viz., $p_2 > 0$ inhibits the domain of assigned brane antibrane charges, so that it must be a positive definite real number, while the constraint $p_3 > 0$ imposes that the charges must, respectively, satisfy desired state-space minor conditions. In particular, these constraints enables us to investigate the nature of the state-space geometry of string theory black holes. We have further observed that the presence of planar and hyper-planar instabilities exist for the nonextremal black holes. It is worth mentioning that the $p_4(n_i, m_i) = 0$ exhibits that the four-dimensional state-space configurations are not stable for any value of the brane and antibrane numbers. This altogether demands definite restriction on the allowed value of the parameters.

Similarly we find that the principle minor p_5 remains nonvanishing for all values of charges on the constituent brane and antibranes. The generic expression of the minor p_5 may further be easily computed from the general minor formula [60]. An explicit calculation specifically finds that the hyper-surface minor p_5 takes a fairly nontrivial value in general. However, the simplest values of the brane and antibrane charges that they be identical implies that the minor p_5 reduces to the specific value of

$$p_5(k) = -64\frac{\pi^5}{k^{5/2}}. \quad (36)$$

Thus for the identical values of the brane antibrane charges, the minor $p_5 < 0$, respectively, implies that the nonextremal black hole solutions under consideration are not stable over the possible choice of the state-space configurations. In order to obtain the highest minor p_6, we in general need to compute the determinant of the metric tensor, which finally reduces to a function of the brane and antibrane charges. Moreover, it is not difficult to demonstrate the global stability on the full state-space configuration, which may in fact be carried forward by computing determinant of the state-space metric tensor. In this case, one observes that the exact expression of the determinant of the intrinsic state-space metric tensor is

$$\|g\| = -\frac{\pi^6}{16}(n_1m_1n_2m_2n_3m_3)^{-3/2}(\sqrt{n_2}+\sqrt{m_2})^2(\sqrt{n_3}+\sqrt{m_3})^3(\sqrt{n_1}+\sqrt{m_1})^3(n_2\sqrt{m_1n_3}+n_2\sqrt{m_1m_3} + 2\sqrt{n_2m_1m_2n_3}+2\sqrt{n_2m_1m_2m_3}+m_2\sqrt{m_1n_3}+m_2\sqrt{m_1m_3}+n_2\sqrt{n_1n_3}+n_2\sqrt{n_1m_3} + 2\sqrt{n_1n_2m_2n_3}+2\sqrt{n_1n_2m_2m_3}+m_2\sqrt{n_1n_3}+m_2\sqrt{n_1m_3}), \quad (37)$$

which in turn never vanishes for a domain of given nonzero brane antibrane charges, except for the following state-space extreme values of the charges, when the brane and antibrane charges n_i, m_i belong to

$$B := \{(n_1, n_2, n_3, m_1, m_2, m_3) \mid n_2\sqrt{m_1n_3}+n_2\sqrt{m_1m_3}+2\sqrt{n_2m_1m_2n_3}+2\sqrt{n_2m_1m_2m_3}+m_2\sqrt{m_1n_3} + m_2\sqrt{m_1m_3}+n_2\sqrt{n_1n_3}+n_2\sqrt{n_1m_3}+2\sqrt{n_1n_2m_2n_3}+2\sqrt{n_1n_2m_2m_3}+m_2\sqrt{n_1n_3}+m_2\sqrt{n_1m_3}=0\}. \quad (38)$$

We may further note that the entire state-space configuration remains positive definite for potential value of the n_i, m_i. We thus observe that the underlying state-space geometry of six-charge nonextremal string theory configurations are well in compliance, and in turn they generically correspond to a nondegenerate fluctuating statistical basis as an intrinsic Riemannian manifold $N := M_6 \setminus B$. Furthermore, we see that the components of the covariant Riemann tensors may become zero for definite values of the charges on branes and antibranes. In addition, the Ricci scalar curvature diverges at the same set of points on state-space manifold (M_6, g) as that of the roots of the determinant of metric tensor, viz., the points defined by the set B.

There exists an akin single higher degree polynomial equation on which we precisely find that the Ricci scalar curvature becomes null, and exactly to these points defining the state-space configuration of the underlying (extremal or near-extremal or general) nonlarge charge black hole system, it corresponds to a noninteracting statistical system. Here, the state-space manifold (M_6, g) is curvature free. A systematic calculation further shows that the general expression for the Ricci scalar is quite involved, and even for equal brane charges $n_1 := n$; $n_2 := n$; $n_3 := n$ and equal antibrane charges $m_1 := m$; $m_2 := m$; $m_3 := m$ the result does not sufficiently simplify. Nevertheless, we find for the identical large values of brane and antibrane charges $n := k$ and $m := k$ [15,16] that there exists an attractive state-space configuration for which the expression of corresponding curvature scalar reduces to a particularly small negative value of

$$R(k) = -\frac{15}{16}\frac{1}{\pi k^{3/2}}. \quad (39)$$

IV. MULTI-CENTERED $D_6D_4D_2D_0$ BLACK BRANES

The present section explores the state-space manifold containing both single-centered black-brane solutions and multicentered black-brane configurations, viz., we shall study the state-space geometry whose co-ordinates are defined in terms of four charges of the $D_6D_4D_2D_0$ black-brane configurations. Here, we shall explicitly present the analysis of the state-space correlations arising from the entropy of stationary single-centered systems as well as that of the double centered black hole molecule configurations. Such multicentered black hole configurations have recently been examined by the so called pin-sized D-brane systems [45,46] and thus we intend to realize underlying state-space geometry arising from the counting entropy of the number of microstates of zoo of entropically dominant multicentered black-brane configurations along with usual single-centered black holes.

It has been shown [45,46] in suitable parameter regimes that the multicentered entropy dominates the single-centered entropy in the uniform large charge limit. Following [15,16], we shall here investigate the state-space geometric implication for the single center and two centers of the multicentered $D_6D_4D_2D_0$ systems. In this connection, we may consider a charge $\Gamma = \sum_i \Gamma_i$ obtained by wrapping the D_4-, D_2-, and D_0-branes around various cycles of a compact space X, and the concerned charges are scaled as $\Gamma \to \Lambda\Gamma$, and then there exists a two-centered brane solution with horizon entropy scaling as Λ^3, while that of the single-centered entropy simply scales as Λ^2. More properly as analyzed in [45,46], let us consider the type-IIA string theory compactified on a product of three two-tori $X := T_1^2 \times T_2^2 \times T_3^2$. Then, the entropy as a function of the charge Γ corresponding to $p0$ D_6-branes on X, p D_4-branes on $(T_1^2 \times T_2^2) + (T_2^2 \times T_3^2) + (T_3^2 \times T_1^2)$, q D_2-branes on $(T_1^2 + T_2^2 + T_3^2)$ and $q0$ D_0-branes is given by

$$S(\Gamma) := \pi\sqrt{-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2}. \quad (40)$$

The state-apace geometry constructed out of the equilibrium state of the four-charged $D_6D_4D_2D_0$ black branes resulting from the entropy may thus be easily computed as earlier from the negative Hessian matrix of the entropy with respect to the D_6-, D_4-, D_2-, D_0-brane charges $\Gamma_i := (p_i^\Lambda, q_{\Lambda,i})$, which in effect form the co-ordinates of the intrinsic state-space manifold. Explicitly, we find that the components of the covariant metric tensor are given as

$$\begin{aligned}
g_{p0p0} &= -4\pi\frac{-3p^2q^2q0^2 + 3pq^4q0 - q^6 + p^3q0^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{p0p} &= 6\pi\frac{-p^3q0^2q + 2p^2q0q^3 + p^2q0^3p0 - pq^5 - 2pq^2p0q0^2 + p0q^4q0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{p0q} &= -12\pi\frac{2p^3q^2q0 + p^2qq0^2p0 - 2pq^3q0p0 - q^4p^2 + q^5p0 - p^4q0^2}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{p0q0} &= -\pi\frac{-6p^4qq0 + 3p^2q^2q0p0 - 9pqq0^2p0^2 + 5q^3p^3 - 6q^4p0p + 6q^3p0^2q0 + 6p0q0^2p^3 + p0^3q0^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{pp} &= -12\pi\frac{p^4q0^2 - p^3q^2q0 - 3p^2qq0^2p0 + 4pq^3q0p0 - p0^2q0^2q^2 + p0^2q0^3p - q^5p0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \cdot g_{pq}\\
&= 3\pi\frac{2p^4qq0 - 2p0q0^2p^3 + 3p^2q^2q0p0 - 3q^3p^3 + 2q^4p0p - pqq0^2p0^2 - 2q^3p0^2q0 + (p0q0)^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{pq0} &= -12\pi\frac{p^5q0 - 2p^3q0p0q - p^4q^2 + 2p^2q^3p0 + pq^2p0^2q0 - p0^2q^4}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{qq} &= -12\pi\frac{4p^3q0p0q - p^2q^3p0 - p^2q0^2p0^2 - 3pq^2p0^2q0 + p0^2q^4 - p^5q0 + p0^3qq0^2}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{qq0} &= 6\pi\frac{-p^5q + 2p^3q^2p0 - 2p^2qp0^2q0 + p0p^4q0 - p0^2q^3p + p0^3q^2q0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}\\
g_{q0q0} &= -4\pi\frac{-p^6 + 3p^4p0q - 3p0^2p^2q^2 + p0^3q^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}
\end{aligned} \quad (41)$$

In order to simplify the presentation, we shall define $X_a = (p0, p, q, q0)$ and subsequently use the following set of notations $1 \leftrightarrow p_0$, $2 \leftrightarrow p$, $3 \leftrightarrow q$, $4 \leftrightarrow q_0$. Employing either of the above notations, we observe from the definition that the ascertained statistical pair correlations may in turn be accounted by simple microscopic descriptions, which can be expressed in terms of the brane charges connoting an ensemble of microstates of the multicentered black hole configuration. Furthermore, it is in fact evident that the principle components of statistical pair correlations are positive definite for all allowed values of the concerned parameters of the D_6-D_4-D_2-D_0 black holes. As a result, we can easily see that the concerned state-space metric constraints are defined by

$$g_{ii}(X_a) > 0 \quad \forall\ i \in \{1, 2, 3, 4\} \mid m_{ii} < 0. \tag{42}$$

The principle components of state-space metric tensor $\{g_{ii}(X_a) | i = 1, 2, 3, 4\}$ essentially signify a set of definite heat capacities (or the related compressibility) whose positivity apprises that the black-brane solution complies an underlying local equilibrium statistical configuration. It is intriguing to note that the positivity of the components g_{ii} requires that the brane charges of associated multicentered D_6-D_4-D_2-D_0 black holes should satisfy the above constraints. This is indeed admissible because of the fact that the brane configuration divulges physically stable system for all values of the brane charges satisfying Eq. (42) with

$$\begin{aligned}
m_{11} &:= -3p^2q^2q0^2 + 3pq^4q0 - q^6 + p^3q0^3\\
m_{22} &:= p^4q0^2 - p^3q^2q0 - 3p^2qq0^2p0 + 4pq^3q0p0 - p0^2q0^2q^2 + p0^2q0^3p - q^5p0\\
m_{33} &:= 4p^3q0p0q - p^2q^3p0 - p^2q0^2p0^2 - 3pq^2p0^2q0 + p0^2q^4 - p^5q0 + p0^3qq0^2\\
m_{44} &:= -p^6 + 3p^4p0q - 3p0^2p^2q^2 + p0^3q^3.
\end{aligned} \tag{43}$$

From the above expressions of metric tensor, we visualize that the ratios of the principle components of statistical pair correlations vary as definite functions of the asymptotic charges; while those of the off diagonal correlations modulate slightly differently. Interestingly, it follows for the distinct $i, j, k, l \in \{1, 2, 3, 4\}$ that the admissible statistical pair correlations thus connoted are consisting of diverse scaling properties. The set of nontrivial relative correlations signifying possible scaling relations of the state-space correlations may nicely be depicted by

$$\begin{aligned}
C_r = \Big\{&\frac{g_{11}}{g_{12}}, \frac{g_{11}}{g_{13}}, \frac{g_{11}}{g_{14}}, \frac{g_{11}}{g_{22}}, \frac{g_{11}}{g_{23}}, \frac{g_{11}}{g_{24}}, \frac{g_{11}}{g_{33}}, \frac{g_{11}}{g_{34}}, \frac{g_{11}}{g_{44}}, \frac{g_{12}}{g_{13}}, \frac{g_{12}}{g_{14}}, \frac{g_{12}}{g_{22}}, \frac{g_{12}}{g_{23}}, \frac{g_{12}}{g_{24}}, \frac{g_{12}}{g_{33}}, \frac{g_{12}}{g_{34}}, \frac{g_{12}}{g_{44}}, \frac{g_{13}}{g_{14}}, \frac{g_{13}}{g_{22}}, \frac{g_{13}}{g_{23}}, \frac{g_{13}}{g_{24}}, \frac{g_{13}}{g_{33}},\\
&\frac{g_{13}}{g_{34}}, \frac{g_{13}}{g_{44}}, \frac{g_{14}}{g_{22}}, \frac{g_{14}}{g_{23}}, \frac{g_{14}}{g_{24}}, \frac{g_{14}}{g_{33}}, \frac{g_{14}}{g_{34}}, \frac{g_{14}}{g_{44}}, \frac{g_{22}}{g_{23}}, \frac{g_{22}}{g_{24}}, \frac{g_{22}}{g_{33}}, \frac{g_{22}}{g_{34}}, \frac{g_{22}}{g_{44}}, \frac{g_{23}}{g_{24}}, \frac{g_{23}}{g_{33}}, \frac{g_{23}}{g_{34}}, \frac{g_{23}}{g_{44}}, \frac{g_{24}}{g_{33}}, \frac{g_{24}}{g_{34}}, \frac{g_{24}}{g_{44}}, \frac{g_{33}}{g_{34}}, \frac{g_{33}}{g_{44}}, \frac{g_{34}}{g_{44}}\Big\}
\end{aligned} \tag{44}$$

The local stability condition of the underlying statistical configuration under the Gaussian fluctuations requires that all the principle components of the fluctuations should be positive definite, i.e. for a given set of state-space variables $\Gamma_i := (p_i^\Lambda, q_{\Lambda,i})$ one must demand that $\{g_{ii}(\Gamma_i) > 0; \forall i = 1, 2\}$. In particular, it is important to note that this condition is not sufficient to insure the global stability of the chosen multicentered configuration and thus one may only accomplish certain local equilibrium statistical configurations. It is however worth mentioning that the complete stability condition requires that all the principle components of the Gaussian fluctuations should be positive definite and the other components of the fluctuations should vanish. In order to ensure this condition, we can observe that all the principle components and all the principle minors of the metric tensor must be strictly positive definite. This implies that the global stability condition constrains the allowed domain of the parameters of black hole configurations, which are interestingly expressed by the following set of simultaneous equations:

$$\begin{aligned}
p_1 &= -4\pi \frac{(-3p^2q^2q0^2 + 3pq^4q0 - q^6 + q0^3p^3)}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - p0^2q0^2)^{3/2}}\\
p_2 &= -12\pi^2 \frac{(q0^4p^4 - 4q^2q0^3p^3 + 6q^4q0^2p^2 - 4q^6q0p + q^8)}{(4p^3q0 - 3p^2q^2 - 6p0pqq0 + 4p0q^3 + p0^2q0^2)^2}\\
p_3 &= -36\pi^3 \frac{(-3p^2q^2q0^2 + 3pq^4q0 - q^6 + q0^3p^3)}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - p0^2q0^2)^{3/2}}.
\end{aligned} \tag{45}$$

In this case, we thus observe that the parameters p, q, $q0$, and $p0$ of the solution remain (i) real in the domain in which the entropy $S(\Gamma)$ remains a real valued function and (ii) positive when the following constraints are simultaneously satisfied:

$$3p^2q^2q0^2 - 3pq^4q0 + q^6 - q0^3p^3 > 0$$
$$q0^4p^4 - 4q^2q0^3p^3 + 6q^4q0^2p^2 - 4q^6q0p + q^8 < 0. \tag{46}$$

It is worth mentioning that the hyper-surface minor constraints, up to the scaling of $9\pi^2$, turn out to be the same as the minor constraint $p_1 > 0$. Thus, the condition $p_3 > 0$ does not introduce extra constraint towards the ensemble stability of the D_6-D_4-D_2-D_0 black holes.

In addition, it is evident that the local stability of the full state-space configuration can likewise be determined by computing the determinant of the metric tensor of concerned state-space geometry. Here, we may easily compute a compact formula for the determinant of the metric tensor as the function of various possible values of brane charges. In particular, our intrinsic geometric analysis assigns the following constant expression to the determinant of the metric tensor

$$\|g\| = 9\pi^4. \tag{47}$$

As the determinant of basic state-space metric tensor is a constant and positive quantity in the viewpoints of large charge consideration, one acquires a nonvanishing central charge of corresponding D_6-D_4-D_2-D_0 CFT configurations [45,46]. Our analysis herewith discovers that there exists a nondegenerate state-space geometry for the leading multicentered configurations. Furthermore, it is worth noting that the determinant of the metric tensor takes a positive definite form, which in turn shows that there is a positive definite volume form on the concerned state-space manifold (M_4, g) of the multicentered D_6-D_4-D_2-D_0 black-brane configurations at the leading order contributions.

Intelligibly, the positivity further follows from the fact that the responsible equilibrium entropy tends to its maximum value, while the same culmination may not remain valid on the chosen planes or hyper planes of the entire state-space manifolds of the single-centered and doubly-centered configurations. It is thus envisaged for either the single or double center descriptions or the dual CFT descriptions that the multicentered black branes do correspond to intrinsically stable statistical configurations. Thus, it is indeed plausible that the underlying ensemble of CFT microstates upon subleading higher derivative corrections lives in the same basis of D_6-D_4-D_2-D_0-brane charges.

A. State-space correlations of the single center configurations

For the charges, $p0 := 0$; $p := 6\Lambda$; $q := 0$; $q0 := -12\Lambda$, describing the single center configurations considered in [45,46], we see that the above state-space correlation functions reduce to the following values:

$$g_{11} = \pi\sqrt{2}, \qquad g_{13} = \frac{3}{2}\pi\sqrt{2} = -g_{22}$$
$$g_{24} = \frac{3}{4}\pi\sqrt{2} = -g_{33}, \qquad g_{44} = \frac{1}{8}\pi\sqrt{2} \tag{48}$$
$$g_{12} = 0 = g_{14} = g_{23} = g_{34}.$$

Following the previously acclaimed notations, we observe that the statistical pair correlations being accounted by simple state-space characterization can be expressed in terms of the brane charges. Furthermore, an easy analysis finds that all the principle components of the statistical pair correlations are positive definite for chosen value of parameters of the single center black holes. In particular, we see for $p0 := 0$; $p := 6\Lambda$; $q := 0$; $q0 := -12\Lambda$ that the concerned state-space metric constraints can for all Λ be depicted as

$$g_{ii}(X_a) > 0 \ \ \forall\, i = 1, \quad 3g_{jj}(X_a) < 0 \ \ \forall\, j = 2, 4. \tag{49}$$

The principle components of state-space metric tensor $\{g_{ii} | i = 1, 2, 3, 4\}$ signifying a set of heat capacities (or the related compressibility) do not all find positive values. Here, a violation of the positivity of heat capacity apprises that the corresponding single center black-brane solution corresponds to a locally unstable statistical configuration over the Gaussian fluctuations. It is thus important to mention that the positivity of principle components does not hold for the above set of brane charges associated with the single-centered $D_6D_4D_2D_0$ black branes.

In analyzing the other state-space constraints, we see that the relative correlations defined as $c_{ijkl} := g_{ij}/g_{kl}$ reduce to the three sets of constant values, viz., finite, zero, and infinite. First, following the proclaimed procedure, we find that there are only 15 nonvanishing finite ratios defining the relative state-space correlation functions for the single center configuration

$$c_{1113} = \frac{2}{3} = -c_{1122}, \qquad c_{1124} = \frac{4}{3} = -c_{1133}c_{1144} = 8,$$
$$c_{1322} = -1 = c_{2433} \qquad c_{1324} = 2 = -c_{1333},$$
$$c_{2233} = 2 = -c_{2224} \qquad c_{1344} = 12 = -c_{2244},$$
$$c_{2444} = 6 = -c_{3344}. \tag{50}$$

Furthermore, an observation finds that the set of vanishing ratios of relative correlation functions is

$$C_R^0 := \{c_{1213}, c_{1224}, c_{1222}, c_{1224}, c_{1233}, c_{1244}, c_{1422}, c_{1424}, c_{1433}, c_{1444}, c_{2324}, c_{2333}, c_{2344}, c_{3444}\} = \{0\}. \tag{51}$$

In the Eq. (51), the symbol C_R^0 indicates the set of vanishing state-space relative correlations, i.e., all the elements of the set C_R^0 are identically zero. On the other hand, we notice for $p0 := 0$; $p := 6\Lambda$; $q := 0$; $q0 := -12\Lambda$ that there exist limiting ill-defined relative correlations. In particular, the concerned ratios get numeric exception, and they receive a division by zero when approaching the single-centered configuration. Thus, the characterization of the relative state-space pair correlation may be accomplished by the set

$$C_R^\infty := \{c_{1112}, c_{1114}, c_{1123}, c_{1134}, c_{1223}, c_{1234}, c_{1314}, c_{1323}, c_{1334}, c_{1423}, c_{1434}, c_{2223}, c_{2234}, c_{2334}, c_{2434}, c_{3334}\} = \{\infty\}. \tag{52}$$

In the Eq. (52), the set C_R^∞ indicates a set of infinite state-space relative correlations, i.e., all the elements of the set C_R^∞ are infinite.

State-space stability of single center D_6-D_4-D_2-D_0 configurations

Furthermore, we see that the entropy corresponding to single center specification takes a constant value $S(\Gamma = \Lambda(0, 6, 0, -12)) = \pi\sqrt{10368}\Lambda^2$. Whilst it is interesting to note that the possible stability of internal configurations under the Gaussian fluctuations is reduced to the positivity of

$$p_1 = \sqrt{2}\pi, \qquad p_2 = -3\pi^2, \qquad p_3 = 9\sqrt{2}\pi^3. \tag{53}$$

We thus find for the chosen values of brane charges which physically describe single center systems that the statistical configurations have definite stability and instability characteristics. In particular, positivity of $p_1(0, 6\Lambda, 0, -12\Lambda)$ shows that the underlying brane configurations are locally stable on an intrinsic state-space line. Nevertheless, we observe for the chosen value of brane charges that the two-dimensional surfaces of the single center state-space configurations are not stable. This has in turn been easily ascertained via the fact that the associated surface minor constraint is not satisfied. Specifically, it turns out for chosen set of charge $(0, 6\Lambda, 0, -12\Lambda)$ that the system exhibits a negative surface minor, viz., we have $p_2(0, 6\Lambda, 0, -12\Lambda) < 0$. Similarly, one may however notice that the system turns out to be stable on three-dimensional hyper-surfaces against the single center configurations. The argument follows directly from the fact that the hyper-surface minor $p_3(0, 6\Lambda, 0, -12\Lambda)$ picks up a positive definite value.

More generally, it is interesting to note that the general expression of the determinant of the metric tensor defined as $g = 9\pi^4$ remains constant for the entire domain of brane parameters. In addition, we find for all nonzero entropy solutions that the state-space scalar curvature signifying global correlation volume of an underlying statistical system has no divergence. As expected further from [15,16] the leading order entropy solutions defining the single solutions confirm for all admissible values of brane charges that their state-space correlation volume varies as an inverse function of the single center brane entropy arising from the degeneracy of equilibrium statistical configurations.

B. State-space correlations of the two-center configurations

In this subsection, we shall explicitly present the state-space geometry of D_6-D_4-D_2-D_0 black holes in string theory carrying a set of respective D-brane charges. We notice that these solutions carry different state-space pair correlations. For given Λ, this follows from the fact that the two-center black holes in general have particular correlations which are neither the same for both of the centers, nor the same as that of the single center counterparts. We nevertheless find that the global state-space correlations which characterize stability of the vacuum string theory configurations of either center do not change for the choice of brane charges considered in [45,46].

1. State-space correlations at the first center

For the value of brane charges $p0 := 1$; $p := 3\Lambda$; $q := 6\Lambda^2$; and $q0 := -6\Lambda$ defining first center of the two-center D_6-D_4-D_2-D_0 configurations, we have the following components of the state-space metric tensor

$$\begin{aligned}
&g_{11} = 108\pi\Lambda^3\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{12} = -54\pi\Lambda^2\frac{7\Lambda^2 + 16\Lambda^4 + 12\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}} \\
&g_{13} = 54\pi\Lambda^3\frac{4\Lambda^2 + 4\Lambda^4 + 1}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{14} = -\pi\frac{18\Lambda^4 + 27\Lambda^6 - 1}{(3\Lambda^4 - 1)^{3/2}} \qquad g_{22} = 18\pi\Lambda\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{(3\Lambda^4 - 1)^{3/2}}, \\
&g_{23} = -3\pi\frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}} \qquad g_{24} = 9\pi\Lambda^3\frac{1 + 2\Lambda^2}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{33} = 3\pi\Lambda\frac{2 + 9\Lambda^2 + 12\Lambda^4}{(3\Lambda^4 - 1)^{3/2}} \\
&g_{34} = -\frac{3}{2}\pi\Lambda^2\frac{1 + 3\Lambda^2}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{44} = \frac{1}{2}\pi\Lambda^3\frac{1}{(3\Lambda^4 - 1)^{3/2}}.
\end{aligned} \tag{54}$$

Simplifying subsequent notations by defining $c_{ijkl} := g_{ij}/g_{kl}$, we then see at the first of the two-centers D_6-D_4-D_2-D_0 that the relative state-space correlations describing concerned statistical system are physically sound in nature. We in this case see that it is not difficult to compute the c_{ijkl}. Nevertheless, the exact expression for the set of c_{ijkl} is quite involved and thus we relegate it to Appendix A 1.

2. State-space correlations at the second center

The value of charges $p0 := -1$; $p := 3\Lambda$; $q := -6\Lambda^2$; and $q0 := -6\Lambda$ defines the second center of the two-center configurations for which we have the following limiting values of the state-space pair correlation functions

$$g_{11} = 108\pi\Lambda^3 \frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{12} = 54\pi\Lambda^2 \frac{7\Lambda^2 + 16\Lambda^4 + 12\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}}$$
$$g_{13} = 54\pi\Lambda^3 \frac{4\Lambda^2 + 4\Lambda^4 + 1}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{14} = \pi \frac{18\Lambda^4 + 27\Lambda^6 - 1}{(3\Lambda^4 - 1)^{3/2}} \qquad g_{22} = 18\pi\Lambda \frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{(3\Lambda^4 - 1)^{3/2}},$$
$$g_{23} = 3\pi \frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}} \qquad g_{24} = 9\pi\Lambda^3 \frac{1 + 2\Lambda^2}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{33} = 3\pi\Lambda \frac{2 + 9\Lambda^2 + 12\Lambda^4}{(3\Lambda^4 - 1)^{3/2}} \tag{55}$$
$$g_{34} = \frac{3}{2}\pi\Lambda^2 \frac{1 + 3\Lambda^2}{(3\Lambda^4 - 1)^{3/2}}, \qquad g_{44} = \frac{1}{2}\pi\Lambda^3 \frac{1}{(3\Lambda^4 - 1)^{3/2}}.$$

Employing the previously defined notations, it is similarly seen that the relative correlations $c_{ijkl} := g_{ij}/g_{kl}$ of the state-space configuration concerning second center of the D_6-D_4-D_2-D_0 system simplify to the one which has been presented in Appendix A 2.

3. State-space stability of double center D_6-D_4-D_2-D_0 configurations

We shall now consider state-space stability for the two-center black-brane configurations and analyze the related positivity properties of their underlying statistical pair correlation functions and correlation volumes for the basins of D_6-D_4-D_2-D_0 configurations. At the first and second centers of the two-center D_6-D_4-D_2-D_0 configurations, we find in turn that the mentioned statistical pair correlations can be simply accounted by a common factor of the charges Γ_i. These notions further receive support from microscopic descriptions, viz., an ensemble of microstates of the multicentered black hole configurations could effectively be expressed in terms of Λ as such a basis, which is simply connoted via the invariant brane charges $\{\Gamma_i\}$.

As indicated by Denef and Moore in [45,46], the two-centered bound state configurations arise with charge centers $\Gamma_1 = (1, 3\Lambda, 6\Lambda^2, -6\Lambda)$ and $\Gamma_2 = (-1, 3\Lambda, -6\Lambda^2, -6\Lambda)$. Thus, we focus our attention for these charge centers and analyze their state-space quantities as the function of Λ. It is apparent for some given Λ that the entropies at either of the two centers Γ_1, Γ_2 match, and, in particular, we find that the double center entropy varies as

$$S(\Gamma_1) = S(\Gamma_2) = \pi\sqrt{108\Lambda^6 - 36\Lambda^2} \sim \Lambda^3. \tag{56}$$

For given charge centers Γ_1 and Γ_2, we may, apart from definite scaling in Λ, appreciate over an equilibrium statistical basis that either of the state-space pair correlation functions as defined in the Eqs. (54) and (55) can be realized as an even function of the parameter Λ. Similarly, one can contemplate the possible nature of the pair correlation functions over the jump of one center to the other. In turn for chosen Γ_1 and Γ_2, we find that the above two-center D_6-D_4-D_2-D_0 configurations form two types of state-space pair correlation functions. In particular, we see from the Eqs. (54) and (55) that the two proclaimed set of pair correlations are

$$C^{(1)}_{ij}(\Gamma) := \{g_{ij}(\Gamma_1) = g_{ij}(\Gamma_2); (i, j) \in \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 3), (3, 4), (4, 4)\}\}$$
$$C^{(2)}_{ij}(\Gamma) := \{g_{ij}(\Gamma_1) = -g_{ij}(\Gamma_2); (i, j) \in \{(1, 2), (1, 4), (2, 3)\}\}. \tag{57}$$

It has explicitly been seen for nonvanishing Λ that the state-space pair correlations belonging to $C^{(1)}$ remain the same for both the centers, while the pair correlations belonging to the set $C^{(2)}$ change their signature. The present analysis implies that the principle components of the metric tensor defining equilibrium statistical pair correlations are positive definite for all allowed values of the parameter Λ. In fact, the Λ being the single parameter for both the first and second centers of the two-center D_6-D_4-D_2-D_0 black branes describes potential stability and state-space correlation properties of the D_6-D_4-D_2-D_0 multicenter configurations. As a result, we see for all Λ and for either of the two centers, viz., Γ_1 and Γ_2 that the respective state-space metric constraints satisfy

$$g_{ii}(X_a) > 0 \quad \forall\ i \in \{1, 2, 3, 4\}. \tag{58}$$

Furthermore, it is intriguing to note that the double center black hole configurations arising with two different charge vectors have the same set of principle minors. In particular, we find that both the first center carrying charges $p0 := 1$; $p := 3\Lambda$; $q := 6\Lambda^2$; $q0 := -6\Lambda$ and the second center carrying charges $p0 := -1$; $p := 3\Lambda$; $q := -6\Lambda^2$; $q0 := -6\Lambda$ have the same principle minors

$$p_1 = 108\pi|\Lambda|^3\left(\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}}\right)$$
$$p_2 = -972\pi^2|\Lambda|^4\left(\frac{1 + 8\Lambda^2 + 24\Lambda^4 + 32\Lambda^6 + 16\Lambda^8}{(3\Lambda^4 - 1)^{-2}}\right)$$
$$p_3 = 972\pi^3|\Lambda|^3\left(\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{-3/2}}\right). \tag{59}$$

Although a state-space singularity can exist in general. However, in the above case, this happens precisely for the set of charges for which entropy of the black brane vanishes. Specifically, the singularity occurs when the scaling parameter Λ satisfies $3\Lambda^4 = 1$. Finally, it interesting to note in general that the determinant of the metric tensor remains positive, implying a well-defined state-space manifold (M_4, g). In addition, we find that the corresponding state-space scalar curvature signifying global correlation properties of the underlying statistical system has no divergence for all nonzero entropy solutions. As expected, this confirms for all admissible values of brane charges that the correlation volume of both the single center solution and double center solutions modulates as inverse function of the entropy associate with chosen basin of the D_6-D_4-D_2-D_0 multicentered black brane configurations.

V. FRACTIONATION OF BRANES: D_0-D_4 BLACK HOLES

The present configurations in this section, as an elucidation of the state-space geometry of small black holes in string theory, carry a set of electric charges and a magnetic charge. We notice that the state-space geometry of these solutions is natural to analyze in the type-II string theory description. In particular, it is known that these black holes in general may carry a finite number of clusters parameters, viz., electric charges and magnetic charge which characterize the vacuum string theory configurations made out of D_0-branes and D_4-branes [5–8]. Furthermore, the general details of [4,74–77] have been noteworthy towards some of our subsequent considerations.

In order to make contact of state-space geometry with definite microscopic perspective, let us consider the chiral primaries of $SU(1, 1 | 2)_Z$. Then, the associated supersymmetric ground states of $\mathcal{N} = 4$ supersymmetric quantum mechanics [52] furnish an understanding of the microscopics of small black holes and concerned electric brane fractionations. In this consideration, we may easily see that there are $24p$ bosonic chiral primaries with total D_0-brane charge N in the background with fixed magnetic D_4 charge p. Then, the degeneracy involved with the counting of microstates arises from the combinatorics of total N number of the D_0-brane charge splitting into k-small clusters with n_i number of D_0-branes on each cluster such that the sum $\sum_{i=1}^{k} n_i = N$ corresponds to the wrapped D_2-branes residing on any of the $24p$ bosonic chiral primary states. Here, the counting is done with the degeneracy d_N of states having level function N in the $(1 + 1)$ CFT with $24p$ bosons, and thus one renders with the celebrated leading order microscopic entropy formula

$$S = \ln d_N = 4\pi\sqrt{\sum_{i=1}^{k} n_i p}. \tag{60}$$

Below, we shall sequentially compute the components of state-space covariant metric tensor, which are defined as the negative Hessian matrix with respect to given k-electric charges $\{n_i\}_{i=1}^{k}$ on D_0-branes and the magnetic charge p on D_4-branes, and thereby divulge the state-space notion of metric positivity, relative correlations as well as planar and hyper-planar stability for the finite cluster small black-brane configurations. Note that the properties of single cluster configurations are already considered in the very beginning of the present investigation.

An illustration of the basic idea of state-space geometry of these particular black holes remains the same as that of the excited string carrying n_1 number of winding modes and n_p number of momentum modes. As we have first considered the case of simplest two-charge extremal configurations, it turns out once again that the state-space geometry of single cluster configurations can be analyzed in terms of the net electric charges replacing the winding modes and net magnetic charge replacing the momentum modes of an excited string. In the next subsection, we shall first consider the two-cluster configuration and analyze respective state-space scaling relations and stability properties.

A. Two electric charge fractionation

In order to find the general pattern of state-space geometric objects of brane fractionated small black holes, we shall in this subsection explain the state-space geometry for some potential values by restricting the number of electric clusters in which the total N number of D_0-brane charge splits into specific finite partitions. In particular, we shall first explore the case for $k = 2$ for which the entropy with two clusters takes the form

$$S = 4\pi\sqrt{p(n_1 + n_2)}. \tag{61}$$

In this case, there are two sets of charges carried by the small black holes which can form co-ordinate charts on the underlying state-space configuration. We shall take the first set of state-space variables to be the fractionated D_0-brane numbers $\{n_1, n_2\}$ which are simply proportional to the available fraction of the electric charges present in respective clusters, while the other state-space variable is the number of D_4-branes which is represented by the magnetic charge p. We then find that the components of state-space metric tensor arising from the Hessian of entropy of D_0-D_4 black holes are

$$g_{pp} = \frac{\pi}{p}\sqrt{\frac{n_1+n_2}{p}} \qquad g_{pn_1} = g_{pn_2} = -\frac{\pi}{\sqrt{p(n_1+n_2)}}$$
$$g_{n_1n_1} = g_{n_1n_2} = g_{n_2n_2} = \frac{\pi}{(n_1+n_2)}\sqrt{\frac{p}{n_1+n_2}}. \qquad (62)$$

For $i, j \in \{n_1, n_2\}$, and p, we observe that the statistical pair correlations just accounted may in turn be simply ascertained by microscopic descriptions which are being expressed in terms of large integers (or associated brane charges) of the D_0-D_4 small black-brane solutions connoting an ensemble of microstates. Furthermore, it is evident for the small black-brane configurations that the principle components of the statistical pair correlations are positive definite for allowed values of the concerned parameters of small black-brane solution. As a result, we can easily see for all admissible sets of n_1, n_2, and p that the components of state-space metric tensor as given above comply

$$g_{pp}(n_1, n_2, p) > 0, \quad g_{n_in_i}(n_1, n_2, p) > 0 \ \ \forall\, i = 1, 2. \qquad (63)$$

The principle components of state-space metric tensor $\{g_{n_in_i}, g_{pp} | i = 1, 2\}$ in effect signify a set of positive definite heat capacities (or the related compressibilities) of the two-cluster configurations. In fact, the positivity constraint apprises that the D_0-D_4 black branes comply an underlying local equilibrium statistical configuration. Furthermore, it is intriguing to note that the nondiagonal component $g_{n_in_j}$ also takes a positive value, viz., we have

$$g_{n_1n_2} > 0 \quad \forall\ (n_1, n_2, p). \qquad (64)$$

This shows that the correlations between the associated number of D_0-branes in (1 + 1) CFT (or the charges in dual description) remain positive in the limit of large electric charges. This is clearly perceptible because of the fact that the leading order fractionated small black-brane configuration becomes unphysical for these values of the brane parameters.

Interestingly, it follows that the ratios of the principle components of statistical pair correlations involving electric charges or one electric charge in either correlations are identical, whereas the ratios involving both the electric and magnetic charges vary as inverse square of the connoted charges, while the ratios involving the off diagonal pair correlations modulate only inversely. From the above expressions, it is not difficult to visualize, for the distinct $i,\ j \in \{n_1, n_2\}$, and the magnetic charge p, that the admissible statistical pair correlations as described above obey the following scaling properties:

$$\frac{g_{ii}}{g_{jj}} = \frac{g_{ii}}{g_{ij}} = \frac{g_{ip}}{g_{jp}} = 1\,\frac{g_{ii}}{g_{ip}} = \frac{g_{ii}}{g_{jp}} = \frac{g_{ip}}{g_{pp}} = \frac{g_{ij}}{g_{ip}}$$
$$= -\left(\frac{p}{n_1+n_2}\right)\frac{g_{ii}}{g_{pp}} = \frac{g_{ij}}{g_{pp}} = \left(\frac{p}{n_1+n_2}\right)^2. \qquad (65)$$

Apart from the positivity of principle components of state-space metric tensor, one demands, in order to accomplish the locally stable statistical configuration, that all associated principle minors of the configuration should be positive definite. It is furthermore not difficult to compute the list of the principle minors, viz., $\{p_1, p_2\}$ from the Hessian matrix of the associated entropy of fractional D_0-D_4 black branes. In fact, after some simple manipulations, one encounters that the concerned stability conditions at a point along one dimensional lines and the two-dimensional surfaces of state-space manifold are, respectively, measured by

$$p_0 = 1, \qquad p_1 = \frac{\pi}{p}\sqrt{\frac{n_1+n_2}{p}}, \qquad p_2 = 0. \qquad (66)$$

For all physically allowed values of invariant electric-magnetic charges of the D_0-D_4 black holes, one thus stipulates that the minor constraint $p_1 > 0$ obliges that the domain of ascribed magnetic charge must, respectively, take positive values, while the surface constraint $p_2 = 0$ implies that there are no electric-magnetic charges such that the p_2 remains a positive real number, and thus the two-dimensional subconfigurations of leading order D_0-D_4 black holes are not stable. In effect, we can further inspect the complete nature of state-space configuration for the D_0-D_4 black branes with electric fractionations that the entire stabilities of the system do not hold for any value of electric and magnetic charges. This follows from the nonexistence of a positive definite value of the determinant of the metric tensor. In particular, we see easily for all $i, j \in \{n_1, n_2\}$, and p that the determinant of the state-space metric tensor finds vanishing value. In two-cluster fractionations, the leading order D0-D4 system is unstable over the Gaussian statistical fluctuations.

B. Three electric charge fractionation

We focus our attention on an extension of state-space analysis for larger number of electric cluster for the D_0-D_4 black-brane configurations. The exploration begins by considering three clusters of D_0-branes, and single cluster of D_4 magnetic brane for the spherical horizon four-dimensional small black hole solutions. What follows here is that the magnetic charge is quantized in terms of the number D_4-branes, while that of the electric charges render as the number of brane present in the chosen cluster of configurations. More precisely, the underlying electric and magnetic charges take large integer values in terms of the net number of constituent D_0- and D_4-branes. In turn, one arrives at the simple quantization condition that the existing charges may be inscribed as fractionated brane configuration. Such space-time solutions appear quite naturally in the string theory, see for example [4,74–77]. In this case, one finds from the general entropy expression that the three cluster entropy of D_0-D_4 black branes is given to be

$$S = 4\pi\sqrt{p(n_1 + n_2 + n_3)}. \tag{67}$$

Thus, the intrinsic Riemannian geometry as the equilibrium state-space configuration may immediately be introduced as earlier from the negative Hessian matrix of the entropy of three electric charges and one magnetic charge extremal small black holes with D_0-brane fractionations. We find that the components of the state-space metric tensor are easily obtained with respect to the underlying electric charges $\{n_1, n_2, n_3\}$ and the magnetic charge p as

$$\begin{aligned} g_{pp} &= \frac{\pi}{p}\sqrt{\frac{n_1+n_2+n_3}{p}} \\ g_{pn_1} &= g_{pn_2} = g_{pn_3} = -\frac{\pi}{\sqrt{p(n_1+n_2+n_3)}} \\ g_{n_1n_1} &= g_{n_1n_2} = g_{n_1n_3} = g_{n_2n_2} = g_{n_2n_3} = g_{n_3n_3} \\ &= \frac{\pi}{(n_1+n_2+n_3)}\sqrt{\frac{p}{n_1+n_2+n_3}}. \end{aligned} \tag{68}$$

For all $i,\ j \in \{n_1, n_2, n_3\}$, and p, we notice that the similar set of positivity conditions and state-space scaling relations are followed as that of the two electric charge fractionation. Hitherto, we see apparently that the principle components of state-space pair correlations remain positive definite quantities for all admissible values of underlying electric-magnetic charges of the black-brane configuration. It is easy to observe for given n_1, n_2, n_3, and p that the following state-space metric constraints are satisfied

$$\begin{aligned} &g_{pp}(n_1, n_2, n_3, p) > 0, \\ &g_{n_in_i}(n_1, n_2, n_3, p) > 0 \quad \forall\ i = 1, 2, 3. \end{aligned} \tag{69}$$

Physically, one may thus note that the principle components of state-space metric tensor $\{g_{ii}, g_{pp} | i = n_1, n_2, n_3\}$ signify a set of heat capacities (or the associated compressibilities) whose positivity exhibits that the underlying D_0-D_4 small black-brane system is in the locally equilibrium statistical configurations. Our analysis further complies that the positivity of g_{pp} obliges that the associated dual conformal field theory living on the boundary must prevail a nonvanishing value of the magnetic charge defining an associated degeneracy of a large number of conformal field theory microstates. It is worth mentioning for given $i,\ j \in \{n_1, n_2, n_3\}$, and p that the inter cluster state-space correlation functions are again nontrivial in nature. In particular, we see in this case that the nondiagonal components $g_{n_in_j}$ of the metric tensor take definite positive values

$$g_{n_in_j}(n_1, n_2, n_3, p) > 0 \quad \forall\ i \neq j \in \{1, 2, 3\}. \tag{70}$$

We may notice further that the ratio of principle components of state-space pair correlations form three different sets of relations, and specifically we find in a chosen cluster that they remain the same, vary as the inverse of the involved electric-magnetic charges, and the others vary as inverse square of the involved parameters. It is in fact not difficult to inspect for nonidentical $i,\ j,\ k \in \{1, 2, 3\}$, and p that the state-space pair correlations are consisting of the following type of scaling relations

$$\begin{aligned} &\frac{g_{ij}}{g_{jj}} = \frac{g_{ii}}{g_{jj}} = \frac{g_{ik}}{g_{ii}} = \frac{g_{ik}}{g_{jj}} = \frac{g_{ik}}{g_{ij}} = \frac{g_{ip}}{g_{jp}} = 1 \\ &\frac{g_{ii}}{g_{ip}} = \frac{g_{ij}}{g_{jp}} = \frac{g_{ij}}{g_{kp}} = \frac{g_{ii}}{g_{jp}} = \frac{g_{ip}}{g_{pp}} = -\left(\frac{p}{n_1+n_2+n_3}\right) \\ &\frac{g_{ii}}{g_{pp}} = \frac{g_{ij}}{g_{pp}} = \left(\frac{p}{n_1+n_2+n_3}\right)^2. \end{aligned} \tag{71}$$

An investigation of definite global properties of three electric clustered D_0-D_4 black-brane configurations determines certain stability considerations along each direction, each plane, and each hyper plane, as well as on the entire intrinsic state-space manifold. Specifically, we can determine whether the underlying D_0-D_4 configuration is locally stable on state-space planes and hyper planes, and thus one needs to compute corresponding principle minors of negative Hessian matrix of the D_0-D_4 black hole entropy. In this case, we may easily appraise, for all physically likely values of magnetic charge and electric charges, that the possible principle minors computed from the above state-space metric tensor are

$$\begin{aligned} &p_0 = 1, \qquad p_1 = \frac{\pi}{p}\sqrt{\frac{n_1+n_2+n_3}{p}}, \\ &p_i = 0, \qquad i = 2, 3. \end{aligned} \tag{72}$$

In the entropy representation, it could thus be seen that the principle minors defined by

$$\begin{aligned} p_2(n_1, n_2, n_3, p) &:= g_{11}g_{22} - g_{12}^2 \\ p_3(n_1, n_2, n_3, p) &:= g_{n1n1}(g_{n2n2}g_{n3n3} - g_{n2n3}^2) \\ &\quad - g_{n1n2}(g_{n1n2}g_{n3n3} - g_{n1n3}g_{n2n3}) \\ &\quad + g_{n1n3}(g_{n1n2}g_{n2n3} - g_{n1n3}g_{n2n2}) \end{aligned} \tag{73}$$

vanish identically for all admissible values of the electric charges n_1, n_2, n_3, and magnetic charge p. In turn, one can easily observe that the vanishing condition $p_{i>1}(n_1, n_2, n_3, p) = 0$ signifying the state-space configurations corresponding to the three clusters of electric D_0-branes indicates that the statistical system remains unstable over possible surfaces and hyper-surfaces. Furthermore, we indeed find for the entire system that the positivity of final minor is just the positivity condition of the determinant of the metric tensor. Then, an easy inspection observes further that the determinant of the metric tensor vanishes as well for all three clusters of electric charges and magnetic charge $\{n_1, n_2, n_3, p\}$, which form co-ordinates on its state-space configuration.

C. Multi electric charge fractionation

Now, we shall consider the state-space configuration for the most general case of brane fractionation in the finite cluster of D_0-D_4 small black branes and present our analysis from the viewpoints of associated microscopic entropy obtained for k clusters. It turns out that the involved entropy can be defined via an appropriate degeneracy formula and the concerned expression reduces to the entropy as ascribed in Eq. (60).

The state-space geometry describing the local pair correlations between the equilibrium microstates of multiclustered charged extremal D_0-D_4 black holes resulting from degeneracy of microstates may then be computed as earlier from the Hessian matrix of Eq. (60) with respect to the parameters, viz., the D_0 electric charges $\{n_1, n_2, \ldots n_k\}$ and the D_4 magnetic charge p. At this juncture, we obtain that the components of underlying state-space covariant metric tensor are generically given by

$$g_{pp} = \frac{\pi}{p}\sqrt{\frac{\sum_{i=1}^{k} n_i}{p}} \qquad g_{pn_i} = -\frac{\pi}{\sqrt{p(\sum_{i=1}^{k} n_i)}}$$
$$g_{n_i n_j} = \frac{\pi}{(\sum_{i=1}^{k} n_i)}\sqrt{\frac{p}{\sum_{i=1}^{k} n_i}}, \quad \forall\ i, j = 1, 2, \ldots, k. \tag{74}$$

For finite number of parameters of the D_0-D_4 black-brane configurations, viz., the electric charges $i, j, k, l \in \{n_1, n_2, \ldots n_k\}$, and the magnetic charge p, we observe that the specific inspections observed in previous subsections for the two- and three-cluster systems hold as well. In effect, it is evident in general that the principle components of equilibrium statistical pair correlations are positive definite for all allowed values of concerned parameters of the D_0-D_4 small black branes in each of the electric clusters. As an immediate result, one finds from the present analysis that the concerned state-space metric constraints are satisfied with

$$g_{n_i n_i} > 0 \quad \forall\ i = 1, 2, \ldots, k$$
$$g_{pp} > 0 \quad \forall\ (n_1, n_2, \ldots, n_k, p). \tag{75}$$

Interestingly, it is worth mentioning that our geometric expressions arising from the entropy of small black holes indicate that some of the brane charges can be safely turned off, say $n_i = 0$, while having a well-defined state-space geometry. However, it is unfeasible to have an intrinsic state-space configuration of small black holes with no electric charge or no magnetic charge, say $n_i = 0\ \forall\ i$ or $p = 0$, since the objects inside the square root of the statistical entropy vanish, and thus the argued small black hole configurations with vanishing number of either D_0-branes or D_4-branes are no more well-defined state-state configurations. The case of finitely many electric branes indeed agrees with our expectation that the nondiagonal components $g_{n_i n_j}$ find their respective positive values

$$g_{n_i n_j} > 0 \quad \forall\ (n_1, n_2, \ldots, n_k, p). \tag{76}$$

Under the present considerations, we thus observe for given fraction of the electric charges $i \neq j \neq k \neq l \in \{n_1, n_2, \ldots, n_k\}$ and the magnetic charge p that the relative state-space pair correlation functions form the same scaling qualifications as in the case of three clusters of D_0 electric branes dealt with in Eq. (71), except for the fact that now the sum in the denominator runs over $\{1, 2, \cdots, k\}$. Furthermore, one may now easily see for the D_0-D_4 configurations involving four or higher clusters of D_0-branes that there exists an extra identical scaling relation

$$\frac{g_{ij}}{g_{kl}} = 1. \tag{77}$$

We thus see for the most general leading order brane fractionation in D_0-D_4 system that there are in total 14 types of relative correlation functions at the chosen state-space basis. It is worth mentioning that an appraisal of exhaustive state-space stability constraints demands that all the associated principle minors must be positive definite, as the positivity of principle components of metric tensor defines the local linear stability in the neighborhood of the chosen local co-ordinate chart on an underlying (M_{k+1}, g) describing concerned state-space manifold of finite clustered D_0-D_4 solutions.

Specifically for $i, j \in \{n_1, n_2, \ldots n_k\}$, and p, we find that there are no extra types of planar and hyper-planar stabilities as that of the relative state-space correlation functions other than the linearly stable multiclustered D_0-D_4 small black-brane system. It is rather easy to divulge the physical picture of the solution set, and in fact after some simplifications one discovers that the planar stability criteria on the two-dimensional surfaces and hyper-planar stability criteria on the three or higher dimensional surfaces of the state-space manifold may simply be rendered from the definition of the state-space geometry.

Intriguingly, it is not difficult to compute from the consideration of the Hessian matrix of k-clustered D_0-D_4 black-brane leading order entropy solutions that the list of nonzero principle minors remains the same as that of the two- or three-clustered configurations. In addition, as in the case of two and three clusters of electric charges, we observe for general k electric charge configurations that the set of all possible principle minors $\{p_i(n_1, n_2, \ldots n_k, p) | \forall\, i > 1\}$ remains zero on the state-space manifold

(M_{k+1}, g) as well as on respective lower dimensional associated systems of multiclustered D_0-D_4 black branes.

It is worth mentioning, in particular, that the local stability of full small black-brane state-space configuration is determined by computing the determinant of concerned state-space metric tensor. Herewith, we may in principle as well compute a compact formula for the determinant of the metric tensor, and, indispensably, our intrinsic state-space geometric analysis arising from the leading order entropy consideration demonstrates that the determinant of the metric tensor does not find a nonvanishing value for any admissible finite electric clusters of the D_0-D_4 black-brane configurations.

In the next section, we shall consider implications of state-space geometry arising from the fuzzball solutions and explicate the nature of scaling properties of possible state-space pair correlation functions and stability requirements of the fuzzy ring solutions in the setup of Mathur's fuzzball consideration [49].

VI. THE FUZZBALL SOLUTIONS: FUZZY RINGS

The viewpoints of the Mathur's fuzzball solutions [58] are considered in this section. To be specific, we shall analyze concerned aspects of state-space geometry for the most exhaustively studied two-charge extremal black branes having electric-magnetic charges (Q, P) and an angular momentum J. We shall focus, in particular, on the analysis of the state-space observations in terms of concerned parameters of the fuzzball solution and thereby shed light on the state-space quantities from Mathur's recent proposal to find an ensemble of microstates, which form an equilibrium statistical basis, over which we shall define the associated thermodynamic intrinsic state-space geometry.

It is worth mentioning in the fuzzball picture that one can construct the classical space-time geometry with definite horizon topology when many of the quanta of the underlying three-parameter D_1-D_5-P CFT lie in the same mode. Nevertheless, it turns out in general that the generic states will not have all the quanta placed in a few modes, so the throat of concerned black hole space-times ends in a very quantum fuzzball, see for the introduction of the fuzzball solutions [47–49].

It is however interesting to note in the fuzzball picture that the actual microstates of such black branes do not have an event horizon. Rather, it is the area of the boundary of the fuzzy region where microstates start differing from each other that satisfies a Bekenstein-Hawking type relation and thereby defines an entropy inside the chosen boundary. Moreover, it turns out, according to the string theory picture, that the different microstates are ''capped off'' before reaching the end of an infinite throat, and thus they give rise to different near horizon space-time geometries. In particular, the average throat behaves as the inverse of the average radius of the fuzzballs. Thus the Bekenstein-Hawking entropy [47–49] has been obtained from the area of such a stretched horizon whose state-space interpretation may be obtained from the coarse graining statistical entropy

$$S(Q, P, J) = C\sqrt{QP - J}. \tag{78}$$

The associated state-space geometry of the rotating two-charge fuzzy ring system can then be constructed out of the parameters which characterize the microstates of the black brane. In particular, we can perform an investigation either in terms of the D_1-brane electric charge Q and D_5-brane magnetic charge P or correspondingly the n_1 number of D_1-branes and n_5 number of D_5-branes. Then, the dimension of the state-space manifold is equal to the number of actual parameters which define the fuzzy black ring solution. We shall then study the state-space configurations whose co-ordinates deal with the charges or number of constituent branes. In particular, we shall consider the electric-magnetic charges (Q, P) and angular momentum J that define co-ordinates on the concerned state-space manifold of the two-charge fuzzy black ring solution.

The state-apace geometry constructed out of the equilibrium state of the rotating two-charged extremal black ring resulting from the entropy can now easily be computed as before from the negative Hessian matrix of the entropy with respect to the charges and angular momentum. Note that an understanding of the state-apace geometry based on the stretched horizon requires the classical time scale limit of the fuzzball. This is because the size of the fuzzball is made by the generic states, such that its surface area to the leading order satisfies a Bekenstein-Hawking type relation with the entropy of the fuzzball, whose boundary surface becomes like a horizon only over classical time scales. We may therefore see that the components of the metric tensor are explicitly given as

$$\begin{aligned}
g_{PP} &= \frac{1}{4}CQ^2(PQ-J)^{-3/2},\\
g_{PQ} &= -\frac{1}{4}C(PQ-2J)(PQ-J)^{-3/2}\\
g_{PJ} &= -\frac{1}{4}CQ(PQ-J)^{-3/2},\\
g_{QQ} &= \frac{1}{4}CP^2(PQ-J)^{-3/2}\\
g_{QJ} &= -\frac{1}{4}CP(PQ-J)^{-3/2},\\
g_{JJ} &= \frac{1}{4}C(PQ-J)^{-3/2}.
\end{aligned} \tag{79}$$

From the simple D-brane description, we observe that there exists an interesting brane interpretation which describes the state-space correlation formulae arising from the corresponding microscopic entropy of the aforementioned two-charge rotating D_1-D_5 solutions. Furthermore, the state-space correlations turn out to be in precise accordance with an associated attractor configuration being disclosed in the limiting special Bekenstein-Hawking solution. In the entropy representation, it has then been noticed that the Hessian matrix of the entropy illustrates the basic nature of possible state-space correlations between the set of extensive variables, which in this case are nothing more than the D_1- and D_5-brane charges and angular momentum. As mentioned before, we can articulate in this case as well that, for all nonzero admissible values of P, Q, J, the principle components of the intrinsic state-space metric tensor satisfy

$$g_{PP} > 0, \qquad g_{QQ} > 0, \qquad g_{JJ} > 0. \tag{80}$$

Substantially, the principle components of the state-space metric tensor signify heat capacities or the associated compressibility, whose positivity indicates that the underlying statistical system is in a local equilibrium, consisting of the D_1- and D_5-brane configurations. Furthermore, we perceive that the ratio of possible diagonal components varies as the inverse square which weakens faster, and thus relatively quickly comes into an equilibrium configuration, than those involving the off diagonal components varying inversely in the involved parameters. Incidentally, the ratios of nondiagonal components varying inversely remain comparable for a longer domain with respect to the parameters varying under the Gaussian fluctuations. We have, in particular, inspected $\forall i \neq j \in \{P, Q\}$ and J that the relative pair correlation functions satisfy the following scaling relations:

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{j}{i}\right)^2, \qquad \frac{g_{ii}}{g_{JJ}} = j^2, \qquad \frac{g_{ij}}{g_{ii}} = -\frac{1}{j^2}(PQ - 2J)$$
$$\frac{g_{ii}}{g_{iJ}} = -j, \qquad \frac{g_{iJ}}{g_{jJ}} = \frac{j}{i}, \qquad \frac{g_{ii}}{g_{jJ}} = -\frac{j^2}{i} = \frac{g_{iJ}}{g_{JJ}} j,$$
$$\frac{g_{ij}}{g_{iJ}} = \frac{1}{j}(PQ - 2J), \qquad \frac{g_{ij}}{g_{JJ}} = -(PQ - 2J). \tag{81}$$

An investigation of definite global properties of two-charged fuzzball configurations determines certain stability approximations along each direction, each plane, each hyper plane, and on the entire intrinsic state-space manifold. In this case, as we intend to determine whether the underlying fuzzball configuration is locally stable on state-space planes and hyper planes, we are thus required to compute corresponding principle minors of the negative Hessian matrix of the entropy. Specifically, we may easily appraise, for all physically likely values of the brane charge and angular momentum, that the possible principle minors computed from the above state-space metric tensor are nonzero definite functions of the electric-magnetic charges $\{P, Q\}$ and the angular momentum J. We in effect see, for all admissible parameters describing the three-parameter fuzzball solutions, that the list of concerned state-space stability functions is

$$p_1 = \frac{1}{4} C Q^2 (PQ - J)^{-3/2}, \qquad p_2 = \frac{1}{4} C^2 J (PQ - J)^{-2}. \tag{82}$$

In the Eq. (82), we thus notice that the cases $J > 0$ and $J < 0$ are different solutions. In the present case, these regimes describe the two different ergo branches and occur as per definition of the fuzzy ring entropy Eq. (78). The minor constraints on p_1, p_2 imply that the two-charge fuzzball solutions under consideration are stable over the lines, planes of the state-space configuration, for all values of the D_1-, D_5-brane charges, and any positive value of the angular momentum. As we have shown in the previous examples, the determinant of the metric tensor thus defined is nonzero for nonzero brane charges and angular momentum. In fact, it is easy to observe that the determinant of the metric tensor reduces to

$$\|g\| = -\frac{1}{16} C^3 (PQ - J)^{-5/2}. \tag{83}$$

Similarly, the constraint $p_3 := g(Q, P, J) < 0$ results in an interpretation that this configuration is globally unstable over the full intrinsic state-space configurations. This is also intelligible from the fact that the responsible equilibrium entropy tends to its maximum value, while the same culmination does not remain valid over the entire state-space manifold. It may in turn be envisaged in the D_1-D_5-P description that the fuzzball black rings do not correspond to intrinsically stable statistical basis, when all the configuration parameters fluctuate. Thus, it is very probable that the underlying ensemble of chosen CFT microstates upon subleading higher derivative corrections may smoothly move into more stable brane configurations.

Finally, in order to elucidate the universal nature of the statistical interactions and the other properties concerning fuzzball rotating black rings, one needs to determine definite global state-space geometric invariant quantities on their intrinsic state-space manifold. Indeed, we notice that the indicated simplest invariant is achieved just by computing the state-space scalar curvature, which, as explained in [15,16], can be obtained in a straightforward fashion by applying the standard method of our intrinsic geometry. In the large charge limit, in which the asymptotic expansion of the entropy of the two-charge rotating ring solution is valid, we notice, in particular, that the state-space scalar curvature can rather be expressed as an inverse function of the entropy.

An exact analysis in turn finds that the constant of proportionality between the state-space scalar curvature and the entropy is negative, and thus we find the fuzzy ring to be an attractive statistical configuration, see for related interpretations [15,16]. Most importantly, it turns out, in the limit when the fuzzy ring is viewed in the perspective of many fuzzballs, that the present analysis relies on the corrected averaged horizon configuration. Finally, it is worth mentioning that the statistical systems of the D_1-D_5-P fuzzy rings find an intriguing conclusion in the Gaussian approximations, and, consequently, the present description vindicates physically sound containments that the state-space configuration of fuzzy rings is nondegenerate, curved, and an everywhere regular intrinsic Riemannian manifold (M_3, g).

VII. BUBBLING BLACK-BRANE SOLUTIONS: BLACK-BRANE FOAMS

In this section, we finally analyze the state-space geometry of an ensemble of equilibrium microstates characterizing three charge foamed black-brane configurations in M theory [51]. These supergravity bubbling solutions naturally appear in the string theory and M theory, see for concerned details [51,78–80]. The study of bubbled space-time geometries and axis-symmetric merger solutions then turns out to be interesting for further investigation from the viewpoints of our state-space geometry. We shall here show that the possible characterization of the state-space geometry has herewith been accomplished in terms of the parameters describing an ensemble of microstates for the three-charge black-brane foam solutions [81].

A. A Toy model: Single Gibbons-Hawking center

The state-space geometry arising from entropy of the foam configurations having single Gibbons-Hawking (GH) center can be divulged by considering the M-theory background [51] compactified on T^6. In the large N limit, one then finds a set of flux parameters which may be written in terms of brane charges. It is worth remarking that the associated topological entropy is independent of the number and charges on the Gibbons-Hawking base points. The origin of such an entropy lies solely in the possible number of choices of positive quantized fluxes on topologically nontrivial cycles.

Indeed, it turns out that these cycles satisfy definite constraints, viz one finds, in particular, that the supergravity and worldvolume descriptions have the same relation between the brane parameters, which determined the entropy of the bubbled black-brane foam, see for instance [51]. Note that an understanding of the state-apace geometry based on the bubbling black branes requires the knowledge of the Bekenstein-Hawking entropy, which could be obtained from the area of the horizon of the chosen solution. A microscopic interpretation may then be offered as the coarse graining of the concerned combinatorial entropy of the foam.

In order to describe the state-space geometry of the single center Gibbons-Hawking configuration, we shall, in particular, consider a set of flux parameters $\{k_i^1, k_i^2, k_i^3\}$ to be positive half integers [51]. Then, the topological entropy coming from the leading order contributions of the fluxes $\{k_i^1\}$, where the index i defines the positions of the Gibbons-Hawking base points, has been written as

$$S(Q_1, Q_2, Q_3) := \frac{\pi}{3}\sqrt{6}\left(\frac{Q_2Q_3}{Q_1}\right)^{1/4}. \tag{84}$$

As proclaimed in the previous subsections, we notice in this case as well, that the state-apace geometry describing the nature of equilibrium brane microstates can be constructed out of the three charges of the bubbled black-brane foams. The covariant metric tensor, as invoked earlier, can immediately be computed from the negative Hessian matrix of the foam entropy resulting from the underlying statistical configuration. Thus, the brane charges, viz., $\{Q_1, Q_2, Q_3\}$ form the co-ordinate charts for the state-space manifold of our interest. Thus, with respect to the brane parameters, we may describe the typical intrinsic geometric features of the bubbled black-brane foams having single GH-center. In fact, we notice that the components of the covariant metric tensor can easily be presented to be

$$\begin{aligned}
g_{Q_1Q_1} &= -\frac{5\pi\sqrt{6}}{48Q_1^2}\left(\frac{Q_2Q_3}{Q_1}\right)^{1/4},\\
g_{Q_1Q_2} &= \frac{\pi\sqrt{6}Q_3}{48Q_1^2}\left(\frac{Q_1}{Q_2Q_3}\right)^{3/4}\\
g_{Q_1Q_3} &= \frac{\pi\sqrt{6}Q_2}{48Q_1^2}\left(\frac{Q_1}{Q_2Q_3}\right)^{3/4},\\
g_{Q_2Q_2} &= \frac{\pi\sqrt{6}}{16}\left(\frac{Q_3}{Q_1}\right)^{2}\left(\frac{Q_1}{Q_2Q_3}\right)^{7/4}\\
g_{Q_2Q_3} &= -\frac{\pi\sqrt{6}}{48Q_1}\left(\frac{Q_1}{Q_2Q_3}\right)^{3/4},\\
g_{Q_3Q_3} &= \frac{\pi\sqrt{6}}{16}\left(\frac{Q_2}{Q_1}\right)^{2}\left(\frac{Q_1}{Q_2Q_3}\right)^{7/4}.
\end{aligned} \tag{85}$$

One thus appreciates, for all $i, j, k \in \{1, 2, 3\}$ describing the single GH-center bubbling brane configuration, that the state-space geometry materializing from the leading order Bekenstein-Hawking entropy of the toroidally compactified M-theory configuration admits remarkably simple expressions in terms of physical charges. It may again be expected that the microscopic preliminaries would plausibly be suggested via the Cardy formula or the associated general Hardy-Ramanujan formula. As enumerated in earlier sections, we nevertheless stress, for all nonzero values of the brane charges Q_1, Q_2, Q_3, that the principle components of the concerned state-space metric tensor satisfy

$$g_{Q_1Q_1} < 0, \qquad g_{Q_2Q_2} > 0, \qquad g_{Q_3Q_3} > 0. \tag{86}$$

The present analysis physically proclaims that the principle components of the state-space metric tensor signify heat capacities or the relevant compressibilities, whose positivity connotes that the underlying statistical system is in locally stable equilibrium configurations of an ensemble of dual CFT microstates. Moreover, it is rather instructive to note that the behavior of the brane-brane statistical pair correlation defined as $g_{Q_1Q_1}$ is asymmetric, in contrast to the other existing diagonal correlations. In fact, one can understand it by arguing that an increment of the Q_1-brane charge reduces the entropy, and thus it corresponds to locally unstable state-space interactions, in contrast with the brane-brane self-interactions involving either Q_2 or Q_3 charges.

Furthermore, it has been substantiated that the ratio of diagonal components varies as the inverse square of the invariant parameters, which vary under the Gaussian fluctuations, whereas the ratios involving off diagonal components vary only inversely with the chosen charges. In particular, we see for $i, j, k \in \{Q_1, Q_2, Q_3\}$ describing the single GH-center configuration that the possible relative state-space correlation functions are defined as

$$C_{BB} := \left\{\frac{g_{ij}}{g_{jj}}, \frac{g_{ii}}{g_{jj}}, \frac{g_{ik}}{g_{ii}}, \frac{g_{ik}}{g_{jj}}, \frac{g_{ik}}{g_{ij}}\right\}. \tag{87}$$

This suggests that the diagonal components weaken faster, and hence relatively quickly come into an equilibrium, than the off diagonal components, which remain comparable for the longer domain of the parameters defining the single GH-center bubbling configurations. An explicit observation shows that the relative pair correlation functions satisfy a simple set of scaling relations. In particular, we can easily observe, for given distinct $i, j, k \in \{Q_1, Q_2, Q_3\}$, that the possible relative state-space correlation functions for the single GH-center find the following values:

$$\begin{aligned} C^S_{BB} &= \left\{\frac{g_{13}}{g_{22}}, \frac{g_{12}}{g_{13}}, \frac{g_{12}}{g_{22}}, \frac{g_{22}}{g_{33}}, \frac{g_{23}}{g_{22}}\right\} \\ &= \left\{\frac{1}{3}\frac{Q_2^2}{Q_1Q_3}, \frac{Q_3}{Q_2}, \frac{1}{3}\frac{Q_2}{Q_1}, \left(\frac{Q_3}{Q_2}\right)^2, -\frac{1}{3}\frac{Q_2}{Q_3}\right\}. \end{aligned} \tag{88}$$

As noticed in the previous configurations, it is not difficult to analyze the local stability for the bubbling black holes as well. In particular, one can determine the principle minors associated with the state-space metric tensor and thereby demand that all the principle minors must be positive definite. In this case, we may adroitly compute the principle minors from the Hessian matrix of the associated entropy concerning the three-charge bubbling black holes. From the Eq. (73), we find that the local stability criteria on the two-dimensional surfaces and the three-dimensional hyper-surfaces of the underlying state-space manifold are, respectively, given by the following relations:

$$\begin{aligned} p_1 &= -\frac{5\sqrt{6}\pi}{48} Q1^{-9/4} Q2^{1/4} Q3^{1/4}, \\ p_2 &= -\frac{\pi^2}{24} Q1^{-5/2} Q2^{-3/2} Q3^{1/2}. \end{aligned} \tag{89}$$

For all physically admitted values of brane charges of the bubbling black holes, we may thus easily ascertain that the minor constraint, viz., $p_1(Q_i) > 0$ inhibits the domain of assigned charges, that two of them must be positive, and the third one has to be a be negative real number, while the constraint $p_2(Q_i) > 0$ imposes that the brane charges must, respectively, satisfy the above definite brane charge conditions. In particular, these constraints enable us to investigate the nature of the state-space geometry of M-theory bubbling black holes. We thus observe that the presence of planar and hyper-planar instabilities exist for the bubbling black holes, which together impose a restriction on the allowed values of the brane charges.

As stated earlier, we find in this case that the determinant of the state-space geometry describing correlations between two chosen microstates of the bubbled black-brane foams may be characterized in terms of the extensive brane charges of the single GH-center solution. Employing the state-space consideration of the negative Hessian matrix of the foam entropy, with respect to the brane charges $\{Q_1, Q_2, Q_3\}$, we find that the determinant of the metric tensor is given by

$$\|g\| = -\frac{\pi^3\sqrt{6}}{384 Q_1^4}\left(\frac{Q_1}{Q_2Q_3}\right)^{-5/4}. \tag{90}$$

Furthermore, for equal values of the charges $Q_1 := Q$, $Q_2 := Q$, and $Q_3 := Q$, it is easy to see that the principle minor $p_1 := g_{11}$ reduces to $p_1 = -\frac{5\sqrt{6}\pi}{48} Q^{-7/4}$, while the surface minor $p_2 := g_{11}g_{22} - g_{12}^2$ shows further that the two-dimensional state-space configurations of underlying single GH-center solutions are unstable. In particular, we find an explicit expression for equal values of charges, i.e. that the surface minor is given by $p_2(Q) = -\frac{\pi^2}{24} Q^{-7/2}$.

As expected, we see for equal value of brane charges, viz., $Q_i = Q$ that the toy model single GH bubble black-brane solution remains unstable over an entire fluctuating statistical configuration. This follows from the fact that the determinant of the metric tensor, as being the highest principle minor p_3, reduces to $g(Q) = -\frac{\pi^3\sqrt{6}}{384} Q^{-21/4}$.

Interestingly, it is noteworthy from the general expression of the determinant of the metric tensor, and in addition that of the state-space scalar curvature signifying a global correlation volume of the underlying statistical system, that the single GH-center bubbled systems are unstable and find an attractive statistical nature for a given nonzero entropy solution. Finally, for all admissible values of brane charges, we come up with the fact that the state-space scalar curvature, signifying global correlation length of an underlying statistical system, confirms no divergence, and, in turn, it varies as an inverse function of the entropy of the chosen single center GH-center bubbled configurations.

B. Black-brane foams

In this subsection, we shall consider the state-space geometry of the most general three-center GH solutions, which may exhaustively be contemplated by three brane charges of the bubbled black-brane configurations. The co-ordinate chart of the underlying intrinsic state-space manifold naturally emerge from the parameters of equilibrium microstates of chosen bubbling supergravity solutions [82–85]. It turns out from the details of brane parameters that one may easily ascribe the state-space definitions to the central charge contributions associated with the rotating black branes in Minkowski space as well. In effect, our attention shall therefore be focused on possible U-dual configurations and describe a promising analysis in the viewpoints of [86–89].

Here, our very purpose will thus be to exploit the state-space meanings of symmetric factors of brane charges arising from an elementary conformal field theory living on the boundary. As we have encountered the state-space geometry of the single GH-center bubbled black branes in the previous subsection, in this subsection we shall analyze the state-space fluctuations for unrestricted 3-charge bubbled black-brane foam solutions [51]. Thereby, we shall examine the general nature of concerned state-space configurations over leading order symmetric charge contributions into the topological entropy of the three-charged bubbled black-brane foams, characterized by the charges Q_1, Q_2, and Q_3. It turns out by considering appropriate factors coming from the partitioning of concerned flux parameters, viz., $\{k_i^1, k_i^2, k_i^3\}$ that the involved topological entropy may be defined by the following formula:

$$S(Q_1, Q_2, Q_3) := \frac{2\pi}{\sqrt{6}}\left\{\left(\frac{Q_2Q_3}{Q_1}\right)^{1/4} + \left(\frac{Q_1Q_2}{Q_3}\right)^{1/4} + \left(\frac{Q_1Q_3}{Q_2}\right)^{1/4}\right\}. \tag{91}$$

It is again not difficult to explore the state-space geometry of the equilibrium microstates of the three-charge bubbled black-brane foams arising from the entropy expression, which concerns just the Einstein-Hilbert action. As stated earlier, the Ruppeiner metric on the state-space manifold is given by the negative Hessian matrix of the ring entropy with respect to the thermodynamic variables. The state-space variables in this case are the conserved brane charges, which in turn are proportional to the fluxes carried by the constituent branes. Explicitly, we find in this case that the components of covariant state-space metric tensor are

$$\begin{aligned}
g_{Q_1Q_1} &= -\pi\left\{\frac{5\sqrt{6}}{48Q_1^2}\left(\frac{Q_2Q_3}{Q_1}\right)^{1/4} - \frac{\sqrt{6}}{16Q_1}\left(\frac{Q_2}{Q_3Q_1}\right)^{1/4} - \frac{\sqrt{6}}{16Q_1}\left(\frac{Q_3}{Q_2Q_1}\right)^{1/4}\right\}\\
g_{Q_1Q_2} &= -\pi\left\{-\frac{\sqrt{6}Q_3}{48Q_1^2}\left(\frac{Q_1}{Q_2Q_3}\right)^{3/4} + \frac{\sqrt{6}}{48Q_3}\left(\frac{Q_3}{Q_1Q_2}\right)^{3/4} - \frac{\sqrt{6}Q_3}{48Q_2^2}\left(\frac{Q_2}{Q_1Q_3}\right)^{3/4}\right\}\\
g_{Q_1Q_3} &= -\pi\left\{-\frac{\sqrt{6}Q_2}{48Q_1^2}\left(\frac{Q_1}{Q_2Q_3}\right)^{3/4} - \frac{\sqrt{6}Q_2}{48Q_3^2}\left(\frac{Q_3}{Q_1Q_2}\right)^{3/4} + \frac{\sqrt{6}}{48Q_2}\left(\frac{Q_2}{Q_1Q_3}\right)^{3/4}\right\}\\
g_{Q_2Q_2} &= -\pi\left\{\frac{5\sqrt{6}}{48Q_2^2}\left(\frac{Q_1Q_3}{Q_2}\right)^{1/4} - \frac{\sqrt{6}}{16Q_2}\left(\frac{Q_3}{Q_1Q_2}\right)^{1/4} - \frac{\sqrt{6}}{16Q_2}\left(\frac{Q_1}{Q_3Q_2}\right)^{1/4}\right\}\\
g_{Q_2Q_3} &= -\pi\left\{\frac{\sqrt{6}}{48Q_1}\left(\frac{Q_1}{Q_2Q_3}\right)^{3/4} - \frac{\sqrt{6}Q_1}{48Q_3^2}\left(\frac{Q_3}{Q_1Q_2}\right)^{3/4} - \frac{\sqrt{6}Q_1}{48Q_2^2}\left(\frac{Q_2}{Q_1Q_3}\right)^{3/4}\right\}\\
g_{Q_3Q_3} &= -\pi\left\{\frac{5\sqrt{6}}{48Q_3^2}\left(\frac{Q_1Q_2}{Q_3}\right)^{1/4} - \frac{\sqrt{6}}{16Q_3}\left(\frac{Q_2}{Q_1Q_3}\right)^{1/4} - \frac{\sqrt{6}}{16Q_3}\left(\frac{Q_1}{Q_2Q_3}\right)^{1/4}\right\}.
\end{aligned} \tag{92}$$

It follows from the above expressions that the statistical pair correlations thus described can in turn be accounted for by a simple geometric description, expressed in terms of the brane charges connoting an ensemble of fluxes, for the general three GH centered bubbling black-brane configurations. Furthermore, we observe that the principle components of the underlying state-space configuration are positive definite for all allowed values of the bubbling parameters of the multicenter GH solution. In particular, it is evident for functions $f_{ii}(Q_1, Q_2, Q_3)$, as defined in Eq. (94), that the state-space metric constraints defining the positivity of concerned diagonal pair correlation functions are

$$g_{Q_iQ_i}(Q_1, Q_2, Q_3) > 0 \quad \forall\ i \in \{1, 2, 3\} \mid f_{ii} < 0. \tag{93}$$

Essentially, the principle components of the state-space metric tensor $\{g_{Q_iQ_i} | i = 1, 2, 3\}$ signify a set of definite heat capacities (or the related compressibilities), whose positivity for a range of involved charges, as presented below, apprises that the bubbled black holes comply with an underlying locally stable equilibrium statistical configuration along each direction. It is intriguing to note that the positivity of $g_{Q_iQ_i}$ holds, even if some of the brane charges of the associated brane charges become zero. This is clearly perceptible because of the fact that the brane configuration remains physical and locally stable for all brane charges (Q_1, Q_2, Q_3), such that the following relations defining Eq. (93) are satisfied:

$$\begin{aligned} f_{11}(Q_1, Q_2, Q_3) &:= \frac{5\sqrt{6}}{48Q_1^2}\left(\frac{Q_2Q_3}{Q_1}\right)^{1/4} - \frac{\sqrt{6}}{16Q_1}\left(\frac{Q_2}{Q_3Q_1}\right)^{1/4} - \frac{\sqrt{6}}{16Q_1}\left(\frac{Q_3}{Q_2Q_1}\right)^{1/4} \\ f_{22}(Q_1, Q_2, Q_3) &:= \frac{5\sqrt{6}}{48Q_2^2}\left(\frac{Q_1Q_3}{Q_2}\right)^{1/4} - \frac{\sqrt{6}}{16Q_2}\left(\frac{Q_3}{Q_1Q_2}\right)^{1/4} - \frac{\sqrt{6}}{16Q_2}\left(\frac{Q_1}{Q_3Q_2}\right)^{1/4} \\ f_{33}(Q_1, Q_2, Q_3) &:= \frac{5\sqrt{6}}{48Q_3^2}\left(\frac{Q_1Q_2}{Q_3}\right)^{1/4} - \frac{\sqrt{6}}{16Q_3}\left(\frac{Q_2}{Q_1Q_3}\right)^{1/4} - \frac{\sqrt{6}}{16Q_3}\left(\frac{Q_1}{Q_2Q_3}\right)^{1/4}. \end{aligned} \tag{94}$$

Interestingly, it is immediately observed that the ratio of the associated components of statistical pair correlations vary as a definite sum and are symmetric factors of concerned brane charges, whereas we see that there is no very direct scaling relations, as in the case of the single GH-center bubbling brane configurations. Nevertheless, we notice, for the distinct $i, j, k \in \{1, 2, 3\}$, that the number of statistical pair correlations, thus described, remains the same. Moreover, we find for the multiple GH-center black-brane foam configuration that the same type of relative correlation set is followed, except that the relative correlation functions now take realistic values over the parameters of given flux partitions. It is worth noting that the precise scaling properties are easily visualized, just by considering the set C_{BB} of the possible ratios, consisting of the components of the state-space metric tensor of the three-charge bubbling black-brane configurations.

Although there exists the positivity of the principle components of state-space metric tensor, nevertheless, in order to accomplish local state-space stability, one needs to further demand that all associated principle minors should be positive definite. It is rather easy to obtain the principle minors of the Hessian matrix of the entropy associated with multiple GH-center black-brane foams. In fact, one finds, after standard algebraic manipulations, that the local stability conditions on the one dimensional line, two-dimensional surfaces, and three-dimensional hypersurfaces on the state-space manifold are, respectively, measured by the following expressions:

$$\begin{aligned} p_1(Q_1, Q_2, Q_3) &= -\frac{\sqrt{6}\pi}{48}Q_1^{-15/4}Q_2^{-7/4}Q_3^{-7/4}(5Q_1^{3/2}Q_2^2Q_3^2 - 3Q_1^2Q_2^{3/2}Q_3^2 - 3Q_1^2Q_2^2Q_3^{3/2}) \\ p_2(Q_1, Q_2, Q_3) &= -\frac{\pi}{96}Q_1^{-7/2}Q_2^{-7/2}Q_3^{-3/2}(4Q_1^{3/2}Q_2^2Q_3^2 + Q_1^{3/2}Q_2^2Q_3^{3/2} - 8Q_1^{3/2}Q_2^{3/2}Q_3^2 \\ &\quad - 2Q_1^2Q_2^2Q_3 + Q_1^2Q_2^{3/2}Q_3^{3/2} + 4Q_1^2Q_2Q_3^2). \end{aligned} \tag{95}$$

An investigation of definite global properties of the general bubbled black-brane foam configurations determines certain stability approximations along each direction, each plane, and each hyper plane, as well as on the entire intrinsic state-space manifold. Specifically, we need to determine whether the underlying three GH-center foam configuration can be locally stable on state-space planes and hyper planes, and thus one is required to just compute the corresponding principle minors of the negative Hessian matrix of the foam entropy. Moreover, one finds that the principle minor p_1 remains positive for all (Q_1, Q_2, Q_3), such that the function $\tilde{p}_1(Q_1, Q_2, Q_3)$ satisfies

$$\begin{aligned} \tilde{p}_1(Q_1, Q_2, Q_3) &:= 5Q_1^{3/2}Q_2^2Q_3^2 - 3Q_1^2Q_2^{3/2}Q_3^2 \\ &\quad - 3Q_1^2Q_2^2Q_3^{3/2} < 0. \end{aligned} \tag{96}$$

It is further intriguing to mention, from the viewpoint of our present consideration, that the principle minor $p_2 := g_{11}g_{22} - g_{12}^2$ reduces to positive values for a domain of brane charges. In particular, we see, for given values of admissible brane charges, that the state-space stability on two-dimensional surfaces is ensured, provided the function

$$\begin{aligned} \tilde{p}_2(Q_1, Q_2, Q_3) &:= 4Q_1^{3/2}Q_2^2Q_3^2 + Q_1^{3/2}Q_2^2Q_3^{3/2} \\ &\quad - 8Q_1^{3/2}Q_2^{3/2}Q_3^2 - 2Q_1^2Q_2^2Q_3 \\ &\quad + Q_1^2Q_2^{3/2}Q_3^{3/2} + 4Q_1^2Q_2Q_3^2 \end{aligned} \tag{97}$$

finds a definite negative value for a set of given brane charges (Q_1, Q_2, Q_3). Alternatively, the linear and planar stabilities require that the given foam configurations be scarcely populated, and thus the net brane charges are effectively bounded by some maximum brane charges. Moreover, it is not difficult to investigate the global stability on the full state-space configuration, which may in fact be easily carried out by computing the determinant of the state-space metric tensor. In this case, we observe that the determinant of the intrinsic state-space metric tensor is a well behaved function of brane charges. From the definition of the highest principle minor, viz., $p_3(Q_1, Q_2, Q_3) := \|g\|$, it is in fact not difficult to compute that the determinant of the metric tensor reads

$$\|g\| = -\frac{\pi^3\sqrt{6}}{384}(Q_1Q_2Q_3)^{-13/4}\tilde{g}(Q1, Q2, Q3), \tag{98}$$

where the factor $\tilde{g}(Q_1, Q_2, Q_3)$ is defined by

$$\begin{aligned}\tilde{g}(Q_1, Q_2, Q_3) := & -Q_1^{3/2}Q_2Q_3^2 - Q_1Q_2^{3/2}Q_3^2 \\ & + 3Q_1^{3/2}Q_2^{3/2}Q_3^{3/2} - Q_1^{3/2}Q_2^2Q_3 \\ & - Q_1Q_2^2Q_3^{3/2} - Q_1^2Q_2^{3/2}Q_3 \\ & + Q_1^2Q_2^{1/2}Q_3^2 + Q_1^2Q_2^2Q_3^{1/2} \\ & - Q_1^2Q_2Q_3^{3/2} + Q_1^{1/2}Q_2^2Q_3^2. \end{aligned} \tag{99}$$

More explicitly, we see for equal values of brane charges $Q_i := Q$ that the principle minors and the determinant of the metric tensor being defined as the highest principle minor, viz., $p_3 := g(Q)$ reduce to the following set of values:

$$p_1(Q) = \frac{\sqrt{6}\pi}{48}Q^{-7/4}, \quad p_2(Q) = 0, \quad g(Q) = 0. \tag{100}$$

Thus, the minor constraint $p_1 > 0$ implies that the three GH-center foam configurations under consideration are stable over the lines of the state-space manifold. However, the vanishing of higher minor constraints, viz., $\{p_i = 0 \mid i = 2, 3\}$ implies that the system is not stable over the planes and the hyper planes of underlying state-space configurations for any positive values of the brane charges. In particular, the constraint $g = 0$ results in an interpretation that the equal charge foam is unstable over the entire three-dimensional manifold describing the full intrinsic state-space configuration. In the limit of the equal charges $Q_i = Q$, we have shown that the state-space Ricci scalar curvature of the black-brane foam diverges. This is because the determinant of the state-space metric tensor vanishes identically for $Q_i = Q$.

VIII. CONCLUSION AND DISCUSSION

We have analyzed state-space pair correlation functions and the notion of stability for the extremal and nonextremal black holes in string theory and M theory. Our consideration is from the viewpoints of thermodynamic state-space geometry. We find, from the intrinsic Riemannian geometry, that the stability of these black branes have been divulged from the positivity of principle minors of the space-state metric tensor. We have explicitly analyzed the state-space configurations for (i) the two- and three-charge extremal black holes, (ii) the four- and six-charge nonextremal black branes.

The former arises from the string theory solutions containing large number of branes, while the latter accounts for both the branes and antibranes. The numbers of branes and antibranes offer a set of parameters to define an intrinsic state-space geometry. An extension of the state-space geometry is analyzed for the D_6-D_4-D_2-D_0 multicentered black branes, small black holes with fractional electric branes, and two-charge rotating fuzzy rings in the setup of Mathur's fuzzball configurations. The state-space pair correlations and the potential nature of stabilities are thereby investigated for the three-charged bubbling black-brane foams. The state-space configuration finds further support from the consideration of [14–16,59], and thus the nature of state-space geometry of rotating and nonrotating charged black branes in string theory and M theory have, respectively, been examined.

In either of the black-brane configurations, it has been shown that there exists an intriguing property of relative space-state correlations, namely, that the ratio of diagonal components varies as the inverse square of the chosen parameters, while the off diagonal components vary as the inverse of the chosen parameters. Similarly, for the corresponding nonextremal configurations, we find that the ratio of diagonal components weakens faster then for the other, off diagonal components. Our analysis thus suggests that the brane-brane statistical pair correlation functions, which find an asymmetric nature in comparison with the other relative pair correlations, weaken relatively faster, and thus they swiftly come into an equilibrium statistical configuration. In both configurations, the underlying microscopic notion of the state-space interactions arises from coarse graining of the counting entropy over large numbers of CFT microstates of the considered black branes.

We have analyzed the state-space configurations arising from fluctuating spherical horizon string theory and M-theory black-brane solutions. In effect, the present paper has exemplified our theory held at the outset for diverse string theory extremal and nonextremal black-brane solutions, multicentered black-brane configurations, fuzzy rings, and single and multi Gibbons-Hawking center bubbling black-brane foams. It is instructive to note in this perspective that state-space investigations of string theory and M-theory black-brane configurations are based on an understanding of the microscopic entropy of various black branes, in which the present consideration requires the coarse graining phenomenon of a large number of degenerate CFT microstates, defining an equilibrium statistical basis for the chosen black-brane system. The present analysis, thus, offers a direct method to uncover statistical properties of fluctuating black-brane configurations.

An exploration finds that the crucial ingredient in analyzing the state-space manifold of black-brane configurations depends on the parameters carried by the space-time solution or that of an underlying microscopic conformal field theory. An illustration of the state-space geometry of these black branes includes the case of extremal and nonextremal configurations, which in both of the proclaimed configurations demonstrates that the stability constraints, arising from the state-space pair correlation functions, in effect determine the potential nature of the local and global correlations. It is worth mentioning that the components of a state-space metric tensor are the two-point statistical correlation functions. In general, these correlations are intertwined in the fluctuation of the parameters of the associated boundary CFT.

This is because the required parameters of black-brane configurations, which describe the microstates of the dual conformal field theory living on the boundary, may in principle be determined via an application of the AdS/CFT correspondence. Consequently, our intrinsic geometric formalism thus described deals with an ensemble of degenerate CFT ground states, which at small constant positive temperature form an equilibrium vacuum configuration, over which we have defined the Gaussian statistical fluctuations. It is interesting to note that the quadratic nature of Gaussian statistical fluctuations, about an equilibrium statistical configuration, determines the metric tensor of associated state-space manifolds. In both cases, our explicit computation shows, over a definite domain of black-brane parameters, that the principle components of state-space metric tensors are positive, while the nonidentical off diagonal ones may be or may not be positive. It has nonetheless been explicitly observed, for the case of finite electric clusters of D_0-D_4 state-space configurations, that some of ratios involving off diagonal components of metric tensor are also positive.

Interestingly, the related correlations weaken as the concerned parameters are increased. In particular, we find an accordance for two-charge extremal black holes or an excited string with two state-space variables, viz., brane numbers or brane charges and Kaluza-Klein momentum or three-charge D_1-D_5-P extremal solutions having n_1 number of D_1-branes, n_5 number of D_5-branes, n_p number of Kaluza-Klein momentum. Then, we find for a pair of distinct state-space variables $\{X_i, X_j\}$ that the state-space pair correlations of both such extremal configurations scale as

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{X_j}{X_i}\right)^2, \qquad \frac{g_{ij}}{g_{ii}} = -\frac{X_i}{X_j}. \tag{101}$$

Furthermore, the particular behavior of generic statistical pair correlation functions, characterizing state-space configurations of four- and six-charge nonextremal black holes in string theory, satisfies inverse like scaling properties, with integer or half-integer exponents. It may thus be envisaged that the generic state-state correlations of string theory and M-theory black holes, with or without rotation, decrease as an increment is introduced in the parameters of the concerned solution.

In order to appreciate definite global properties of the concerned systems, we have explained in this article that one is required to determine the nature of stabilities along each direction, each plane, and each hyper plane, as well as on all the intrinsic state-space configurations. Our analysis has in effect demonstrated that the determinant of the metric tensor is negative definite as well for the configurations having large numbers of branes and/ or antibrane. It has however been known from the Ruppeiner geometry that only the classical fluctuations having definite thermal origin deal with the probability distribution, which has a positive definite invariant intrinsic Riemannian metric tensor over an equilibrium statistical configuration. This signals that the system becomes highly quantum in nature when all the parameters fluctuate. In fact, our state-space construction, for the string theory and M-theory black holes dealing with the parameters of microscopic CFTs, illustrates that the local stabilities, degeneracy, and global signature of a state-space manifold can as well be indefinite, and in effect these notions are sensitive to the location chosen in the moduli space geometry of the black branes.

Importantly, the sign of principle minors and determinant of the state-space metric tensor implies whether the chosen black-brane solution is thermodynamically stable or not. In contrast, the vacuum phase transitions may rather be characterized via the scalar curvature of the concerned state-space configuration. The present investigation thus serves as a prelude to the state-space geometry of an arbitrary parameter black-brane configuration in string theory and M theory. Moreover, it has been explicated that the explored examples have an interesting set up of intrinsic state-space geometric characterizations, which are based on the general nature of the quadratic Gaussian fluctuations of the chosen black brane's statistical configuration. In this concern, we find, in general, that these configurations are categorized as

(1) The underlying subconfigurations turn out to be well-defined over possible domains whenever there exists a respective set of nonzero state-space principle minors.

(2) The underlying full configuration turns out to be everywhere well-defined whenever there exists a nonzero state-space determinant.

(3) The underlying configuration corresponds to an interacting statistical system whenever there exists a nonzero state-space scalar curvature.

The main line of thought, which has been followed here, has first been to develop an intrinsic Riemannian geometric conception of the underlying state-space geometry, arising from leading order statistical interactions, which exist among various CFT microstates of (rotating) black-brane configurations in string theory and M theory. The perspective notions indicate that novel scaling aspects of the state-space pair correlation and state-space stability, in effect, arise from the negative Hessian matrix of the coarse graining entropy, defined over an ensemble of a large number of brane microstates, characterizing the considered black-brane attractor configurations. Intimately, we have investigated whether the associated state-space geometries are nondegenerate and possess nonvanishing scalar curvature, implying an interacting statistical basis for these configurations, like the one above. For instance, we have described the state-space configuration of the multicentered D_6-D_4-D_2-D_0 solutions. In particular, we have presented the complete list of the corresponding related state-space correlation functions. Thus, the present investigation unifies the thermodynamic properties of extremal and nonextremal solutions in the string theory and M theory. Moreover, a good test for the thermodynamic stability of the underlying configurations is to calculate the signs of the principle minors of the Ruppeiner metric and check if all of them are positive. Interestingly, we find that the behavior of statistical pair correlations between equilibrium microstates is governed by a set of consistent parameters defining underlying CFT vacuum configurations, and, then, the same has been anticipated to remain valid for the other associated intrinsic geometric quantities on the concerned state-space manifold as well. In this context, the state-space investigation of the other string theory and M-theory black hole or black-brane configurations has been further considered in the Refs. [59–61].

Finally, the higher order α' corrections, when taken into account in the underlying effective theory, are envisaged to offer diverse well-defined state-space configurations. Generically, the α'-corrected state-space configurations are at least expected to be nondegenerate, rather than an ill-defined intrinsic Riemannian geometry arising from the leading order entropy configuration. Such a notion has been offered for entropy solutions in the two-charge D_0-D_4 black holes or in an excited string to leading order. Similar motivations along these directions have been obtained in previous state-space investigations [14–16,59]. Herewith, we have contemplated that the state-space geometry of black branes in string theory and M theory would ascribe definite well-defined, nondegenerate and curved intrinsic Riemannian manifolds, whose pair correlation functions scale as inverse functions of the parameters.

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APPENDIX

In this appendix, we provide explicit forms of the state-space relative correlation functions at the first and second centers of the multicentered $D_6D_4D_2D_0$ black holes describing the family of the four-charged configurations. Our analysis illustrates that the physical properties of the specific state-space correlations may exactly be exploited in general. The definite behavior of state-space correlations, as accounted in the concerned section, suggests that the various intriguing single center and multicenter state-space examples of black-brane solutions include the nice feature that they do have definite stability properties, except for the fact that the determinant may be nonpositive definite in some cases. As mentioned in the main sections, these $D_6D_4D_2D_0$ configurations are generically well-defined and indicate an interacting statistical basis. We discover here that their state-space geometries indicate the possible nature of general two-center equilibrium thermodynamic configurations. Significantly, we notice from the very definition of the intrinsic metric tensor that the related state-space correlations may be analyzed as follows.

1. State-space relative correlations at the first center

Here, we shall explicitly provide the exact expressions for the four-parameter multicentered solutions at the first center of the double centered black holes. It turns out that the functional nature of a large number of branes, within a small neighborhood of statistical fluctuations, introduced in an equilibrium ensemble of brane configurations, may precisely be divulged. Surprisingly, we can expose in this framework that the related state-space correlations at the first center with charges $p0 := 1$; $p := 3\Lambda$; $q := 6\Lambda^2$; and $q0 := -6\Lambda$ take an exact and simple set of expressions

$$c_{1112} = -2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{7\Lambda^2 + 16\Lambda^4 + 12\Lambda^6 + 1}, \qquad c_{1113} = 2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{4\Lambda^2 + 4\Lambda^4 + 1}$$

$$c_{1114} = -108\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{18\Lambda^4 + 27\Lambda^6 - 1}, \qquad c_{1122} = 6\Lambda^2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6}$$

$$c_{1123} = -36\Lambda^3\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}, \qquad c_{1124} = 12\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{1 + 2\Lambda^2}$$

$$c_{1133} = 36\Lambda^2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{2 + 9\Lambda^2 + 12\Lambda^4}, \qquad c_{1134} = -72\Lambda\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{1 + 3\Lambda^2}$$

$$c_{1144} = 1296\Lambda^2 + 2592\Lambda^4 + 1728\Lambda^6 + 216, \qquad c_{1213} = -\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda(4\Lambda^2 + 1 + 4\Lambda^4)}$$

$$c_{1222} = -3\Lambda\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6}, \qquad c_{1223} = 18\Lambda^2\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}$$

$$c_{1224} = -6\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda(1 + 2\Lambda^2)}, \qquad c_{1233} = -18\Lambda\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{9\Lambda^2 + 12\Lambda^4 + 2}$$

$$c_{1234} = 36\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{3\Lambda^2 + 1}, \qquad c_{1244} = -108\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda}$$

$$c_{1314} = -54\Lambda^3\frac{4\Lambda^2 + 1 + 4\Lambda^4}{18\Lambda^4 + 27\Lambda^6 - 1}, \qquad c_{1322} = 3\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6}$$

$$c_{1323} = -18\Lambda^3\frac{4\Lambda^2 + 1 + 4\Lambda^4}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}, \qquad c_{1324} = 6\frac{4\Lambda^2 + 1 + 4\Lambda^4}{1 + 2\Lambda^2} \qquad c_{1333} = 18\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{9\Lambda^2 + 12\Lambda^4 + 2},$$

$$c_{1334} = -36\Lambda\frac{4\Lambda^2 + 1 + 4\Lambda^4}{1 + 3\Lambda^2} \qquad c_{1344} = 432\Lambda^2 + 108 + 432\Lambda^4, \qquad c_{1422} = -\frac{1}{18\Lambda}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6}$$

$$c_{1423} = \frac{1}{3}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}, \qquad c_{1424} = -\frac{1}{9\Lambda^3}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{1 + 2\Lambda^2} \qquad c_{1433} = -\frac{1}{3\Lambda}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{9\Lambda^2 + 12\Lambda^4 + 2},$$

$$c_{1434} = \frac{2}{3\Lambda^2}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{1 + 3\Lambda^2} \qquad c_{1444} = -\frac{2}{\Lambda^3}(18\Lambda^4 + 27\Lambda^6 - 1), \qquad c_{2223} = -6\Lambda\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}$$

$$c_{2224} = \frac{2}{\Lambda^2}\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{1 + 2\Lambda^2}, \qquad c_{2233} = 6\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{9\Lambda^2 + 12\Lambda^4 + 2}$$

$$c_{2234} = -12\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{\Lambda(1 + 3\Lambda^2)}, \qquad c_{2244} = 36\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{\Lambda^2}$$

$$c_{2324} = -\frac{1}{3\Lambda^3}\frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{1 + 2\Lambda^2}, \qquad c_{2333} = -\frac{1}{\Lambda}\frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{9\Lambda^2 + 12\Lambda^4 + 2}$$

$$c_{2334} = \frac{2}{\Lambda^2}\frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{1 + 3\Lambda^2}, \qquad c_{2344} = -\frac{6}{\Lambda^2}(42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1)$$

$$c_{2433} = 3\Lambda^2\frac{1 + 2\Lambda^2}{9\Lambda^2 + 12\Lambda^4 + 2}, \qquad c_{2434} = -6\Lambda\frac{1 + 2\Lambda^2}{1 + 3\Lambda^2} c_{2444} = 18 + 36\Lambda^2,$$

$$c_{3334} = -\frac{2}{\Lambda}\frac{9\Lambda^2 + 12\Lambda^4 + 2}{3\Lambda^2 + 1} \qquad c_{3344} = \frac{6}{\Lambda^2}(9\Lambda^2 + 12\Lambda^4 + 2), \qquad c_{3444} = -\frac{3}{\Lambda}(3\Lambda^2 + 1). \tag{A1}$$

2. State-space relative correlations at the second center

As stated earlier, the state-space metric, in the multicentered and single black-brane configurations, is given by the negative Hessian matrix of the concerned entropy. Here, the charges on the branes, in a given configuration, are respected to be extensive variables. In this case, we find that the four distinct large charges characterize the intrinsic state-space correlation functions. In fact, our computation shows that the exact set of correlations at the second center are given by employing the previously defined notations. We have similarly presented, for the second center of the D_6-D_4-D_2-D_0 system, that the relative state-space correlations simplify

$$c_{1112} = -2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{7\Lambda^2 + 16\Lambda^4 + 12\Lambda^6 + 1}, \qquad c_{1113} = 2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{4\Lambda^2 + 4\Lambda^4 + 1}$$
$$c_{1114} = 108\Lambda^3\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{18\Lambda^4 + 27\Lambda^6 - 1}, \qquad c_{1122} = 6\Lambda^2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6}$$
$$c_{1123} = 36\Lambda^3\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}, \qquad c_{1124} = 12\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{1 + 2\Lambda^2}$$
$$c_{1133} = 36\Lambda^2\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{2 + 9\Lambda^2 + 12\Lambda^4}, \qquad c_{1134} = -72\Lambda\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{1 + 3\Lambda^2}$$
$$c_{1144} = 1296\Lambda^2 + 2592\Lambda^4 + 1728\Lambda^6 + 216, \qquad c_{1213} = -\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda(4\Lambda^2 + 1 + 4\Lambda^4)}$$
$$c_{1222} = -3\Lambda\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{13\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}, \qquad c_{1223} = -18\Lambda^2\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{42\Lambda^2 + 12\Lambda^2 + 45\Lambda^6 + 1}$$
$$c_{1224} = -6\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda(1 + 2\Lambda^2)}, \qquad c_{1233} = -18\Lambda\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{9\Lambda^2 + 12\Lambda^4 + 2}$$
$$c_{1234} = 36\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{3\Lambda^2 + 1}, \qquad c_{1244} = -108\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda} \qquad c_{1314} = 54\Lambda^3\frac{4\Lambda^2 + 1 + 4\Lambda^4}{18\Lambda^4 + 27\Lambda^6 - 1},$$
$$c_{1322} = 3\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6} c_{1323} = 18\Lambda^3\frac{4\Lambda^2 + 1 + 4\Lambda^4}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}, \qquad c_{1324} = 6\frac{4\Lambda^2 + 1 + 4\Lambda^4}{1 + 2\Lambda^2}$$
$$c_{1333} = 18\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{9\Lambda^2 + 12\Lambda^4 + 2}, \qquad c_{1334} = -36\Lambda\frac{4\Lambda^2 + 1 + 4\Lambda^4}{1 + 3\Lambda^2} \qquad c_{1344} = 432\Lambda^2 + 108 + 432\Lambda^4,$$
$$c_{1422} = \frac{1}{18\Lambda}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6} c_{1423} = \frac{1}{3}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}, \qquad c_{1424} = \frac{1}{9\Lambda^3}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{1 + 2\Lambda^2}$$
$$c_{1433} = \frac{1}{3\Lambda}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{9\Lambda^2 + 12\Lambda^4 + 2}, \qquad c_{1434} = -\frac{2}{3\Lambda^2}\frac{18\Lambda^4 + 27\Lambda^6 - 1}{1 + 3\Lambda^2} \qquad c_{1444} = \frac{2}{\Lambda^3}(18\Lambda^4 + 27\Lambda^6 - 1),$$
$$c_{2223} = 6\Lambda\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1} c_{2224} = \frac{2}{\Lambda^2}\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{1 + 2\Lambda^2}, \qquad c_{2233} = 6\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{9\Lambda^2 + 12\Lambda^4 + 2}$$
$$c_{2234} = -12\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{\Lambda(1 + 3\Lambda^2)}, \qquad c_{2244} = 36\frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{\Lambda^2}$$
$$c_{2324} = \frac{1}{3\Lambda^3}\frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{1 + 2\Lambda^2}, \qquad c_{2333} = \frac{1}{\Lambda}\frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{9\Lambda^2 + 12\Lambda^4 + 2}$$
$$c_{2334} = -\frac{2}{\Lambda^2}\frac{42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1}{1 + 3\Lambda^2}, \qquad c_{2344} = \frac{6}{\Lambda^2}(42\Lambda^4 + 12\Lambda^2 + 45\Lambda^6 + 1)$$
$$c_{2433} = 3\Lambda^2\frac{1 + 2\Lambda^2}{9\Lambda^2 + 12\Lambda^4 + 2}, \qquad c_{2434} = -6\Lambda\frac{1 + 2\Lambda^2}{1 + 3\Lambda^2} c_{2444} = 18 + 36\Lambda^2, \qquad c_{3334} = -\frac{2}{\Lambda}\frac{9\Lambda^2 + 12\Lambda^4 + 2}{3\Lambda^2 + 1}$$
$$c_{3344} = \frac{6}{\Lambda^2}(9\Lambda^2 + 12\Lambda^4 + 2), \qquad c_{3444} = -\frac{3}{\Lambda}(3\Lambda^2 + 1). \tag{A2}$$

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