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Stability of the aether

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The requirements for stability of a Lorentz violating theory are analyzed. In particular we conclude that Einstein-aether theory can be stable when its modes have any phase velocity, rather than only the speed of light as was argued in a recent paper.

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The purpose of this paper is to argue for the appropriate notion of stability in a theory with broken Lorentz symmetry that supports modes with phase velocities different from the speed of light. In particular we are motivated by the example of Einstein-aether theory, but our considerations are quite general. More specifically, we shall argue that the stability criteria imposed on this theory in Ref. [1] are overly restrictive. The conclusion is that the theory is actually stable for an open set in the four dimensional coupling parameter space rather than for only a one dimensional subspace. The issue of stability in Lorentz violating theories was also addressed in Refs. [2,3], which include arguments closely related to those advanced here.

Einstein-aether theory is an example of a theory where Lorentz symmetry is dynamically broken. Aside from matter, the fundamental fields are the spacetime metric g_{ab} and a timelike unit vector field u^a, the "aether" (see Ref. [4] for a review). Flat spacetime with a constant aether is a solution to the theory, and linearized perturbations of this solution satisfy second order hyperbolic equations. There are modes with five different polarizations: two spin-2, two spin-1, and a single spin-0 mode. For all these modes, the frequency ω and spatial wave vector k defined relative to the rest frame of the aether satisfy a gapless dispersion relation, $\omega^2 = v_i^2 k^2$, where i labels the spin. The squared velocities v_i^2 depend on the coupling parameters in the Lagrangian, and are generally different from each other and different from the "speed of light" c defined by the null cone of the metric g_{ab}. The conditions $v_i^2 > 0$ impose inequalities on the coupling parameters, guaranteeing that the frequency is real, so the perturbations do not grow exponentially in time if the spatial wave vector is real [5]. Another set of inequalities implies that the energy carried by these modes is positive [6–8], and these inequalities can be satisfied simultaneously with the former stability inequalities. (As of yet, no nonlinear extension of this positive energy result is known, except in the special case of static spherical symmetry [9].)

It was recently argued in Ref. [1] that one should require the above stability criteria, i.e., real frequency and positive energy, not only for modes with a real wave vector in the aether frame, but more generally for modes with a real wave vector in any Lorentz frame defined with respect to the metric g_{ab}, and for energy defined in any Lorentz frame. In a non-Lorentz invariant theory this is obviously a much stronger requirement, and in fact it was concluded that Einstein-aether theories are unstable except for a small number of special cases in which all modes propagate at exactly the speed of light.[1] We shall now argue, however, that these stronger conditions are not required by the stability of the theory, and are not justified given the structure of the theory.

In fact the reasoning of Ref. [1] applies to any linear theory with modes propagating at different speeds, not only to Einstein-aether theory. Also, the dynamics of the metric and aether themselves play no essential role except to define those modes. Hence we will discuss the simpler setting of fields on a spacetime with a fixed Minkowski metric η_{ab} [with signature $(+---)$] and a fixed timelike unit vector u^a. A free scalar field φ that propagates at speed v with respect to the rest frame of u^a is minimally coupled to the effective (inverse) metric

$$g^{ab}_{(v)} = u^a u^b + v^2(\eta^{ab} - u^a u^b), \tag{1}$$

with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\sqrt{-\eta}\, g^{ab}_{(v)} \partial_a\varphi \partial_b\varphi = \tfrac{1}{2}(\partial_t^2\varphi - v^2 \partial_i^2\varphi), \tag{2}$$

where the second expression is written in the Minkowski coordinate system (t, x^i) of the metric η_{ab}, adapted to the rest frame of u^a. We consider this model with arbitrary positive values of v.

In the context of Einstein-aether theory, there is good reason to allow v to be greater than c. If the coupling constants are chosen so that the post-Newtonian preferred frame parameters of the theory are in agreement with observational constraints, then the positivity of the energy (in the aether frame) requires $v \geq c$. Also, to satisfy the vacuum Cherenkov constraint for ultra high-energy cosmic

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[1] Actually, in Ref. [1] the decoupling limit was taken. That is, the metric was fixed to the Minkowski metric and not varied in finding the field equations.

rays, all $v > c$ are allowed, but any v less than c must be extremely close to c [4].

The dispersion relation for a scalar field with Lagrangian density (2) is $\omega^2 = v^2 k^2$ when expressed in terms of components of the wave 4-covector k_a in the aether frame. More precisely, the wave phase is $k_\mu x^\mu = \omega t + k_i x^i$, and $k^2 = \sum_i k_i k_i$. For real spatial wave vectors k_i, the stability requirement that the frequency be real amounts to the condition $v^2 > 0$. This condition guarantees that any solution that is a superposition of plane waves on a constant t surface is stable.

One of the further stability criteria of Ref. [1] is the demand that the frequency be real also for plane waves on any Lorentz-boosted constant time surface. To determine what that implies, we may reexpress the dispersion relation in terms of the components of the wave 4-vector in the boosted frame as follows.

The metric η_{ab} can be used to define a set of frames, related in the usual way by Lorentz transformations. With respect to such a frame moving with velocity β, the new time and space coordinates are given by

$$t' = \gamma(t - \beta x_\parallel), \tag{3}$$

$$x'_\parallel = \gamma(x_\parallel - \beta t), \tag{4}$$

$$x'_\perp = x_\perp, \tag{5}$$

where $\parallel$ and $\perp$ refer to the components parallel and perpendicular to the boost direction, and we use units with the metric speed of light equal to unity, $c = 1$. The frame velocity will be taken to be positive, and is assumed to be less than the speed of light, $0 \le \beta < 1$.

The covariant (as opposed to contravariant) frequency and wave 4-vector components in the boosted frame are given by

$$\omega' = \gamma(\omega + \beta k_\parallel), \tag{6}$$

$$k'_\parallel = \gamma(k_\parallel + \beta\omega), \tag{7}$$

$$k'_\perp = k_\perp. \tag{8}$$

The dispersion relation in terms of these boosted components takes the form

$$(1 - v^2\beta^2)\omega'^2 + 2\beta k'_\parallel(1 - v^2)\omega' + (\beta^2 - v^2)k'^2_\parallel - v^2(1 - \beta^2)k'^2_\perp = 0, \tag{9}$$

where we have multiplied by a factor $(1 - \beta^2) = \gamma^{-2}$ for convenience. This is a quadratic equation for ω', so the roots are real for real k' if and only if the discriminant is positive,

$$v^2(1 - \beta^2)^2 k'^2_\parallel + v^2(1 - v^2\beta^2)(1 - \beta^2)k'^2_\perp \ge 0. \tag{10}$$

Since $\beta < 1$, this can be negative only if the term $(1 - v^2\beta^2)$ is negative, which occurs only if $v > 1$ and $\beta > 1/v$. We note that this is just the condition for the constant t' surfaces to be timelike with respect to $g_{(v)ab}$ (1). The frequency ω' then has a nonzero imaginary part when $k'_\perp/k'_\parallel$ is sufficiently large.

Thus for $v > 1$ and $\beta > 1/v$ there exist solutions with real wave vectors and complex frequencies in the boosted frame. Such modes grow exponentially in the time coordinate t' of that frame. Whether or not this indicates an instability comes down to the question of whether or not these solutions are part of the physical phase space of the theory.

As pointed out in Ref. [1], the wave vector $k_\parallel = \gamma(k'_\parallel - \beta\omega')$ in the rest frame of the aether will be complex for such modes, so on a constant t surface the solution will blow up exponentially at spatial infinity. These solutions therefore do not satisfy the usual boundary conditions that define the phase space of the theory on the constant t slices. A consistent theory can be defined by adopting a regular boundary condition on the constant t slices, excluding these solutions. This is what is ordinarily done in a Lorentz invariant theory. For example, one could require that the solutions have compact support, or that they be Fourier transformable on those slices. Moreover, since these boundary conditions are preserved by t evolution, the theory so defined preserves the time translation symmetry of the background. Also, one would define the same phase space imposing these boundary conditions on any other surface that is spacelike with respect to $g_{(v)ab}$.

One may ask whether a consistent theory could instead be defined by adopting a regular boundary condition on the constant t' slices. If so, this would raise the question of which is the correct phase space. But it appears that this can not be done in a natural way. For certain regular initial data on a given constant t' surface the corresponding solution grows exponentially with t'. In any other frame this solution will contain complex wave vectors and will therefore diverge asymptotically on the constant time slices of that frame. This means that, unlike the case for surfaces which are spacelike with respect to $g_{(v)ab}$, the phase space defined by regular data on constant t' surfaces is different for every value of β greater than $1/v$.

Moreover, for β greater than $1/v$, the phase space obtained by requiring regular initial data on a fixed t' slice depends not only on the particular value of β, but on the particular choice of slice. For example, suppose the initial data on a particular surface $t' = t'_0$ possesses a well-defined Fourier transform. In Fourier space, the wave equation then reduces to an infinite number of uncoupled ordinary differential equations that may be solved to obtain the Fourier transform of the solution on a different slice $t' = t'_1$. For modes with sufficiently large $k_\perp$, the solutions to the differential equation grow exponentially with $k_\perp$. The solution at t'_1 in Fourier space therefore does not in general have a convergent inverse Fourier transform. Such initial data on the t'_0 surface do not correspond to any choice of initial data on the t'_1 surface. This means that the phase space defined in this way depends not only on the choice of time coordinate t' but also on the arbitrary value t'_0 of that coordinate, breaking the time translation symmetry of the theory.

Another reason to reject a "t'-phase space" formulation is that allowing for arbitrary initial data at $t' = t'_0$ is unjustified in the context of a causal theory in which the φ field interacts with other degrees of freedom. A simple way to see the problem is to allow for an external source term in the field equation for φ. One can then ask whether the source could generate data at t'_0 that would lead to an exponentially growing solution. The answer is no unless (perhaps) if the source is turned on in the infinite past. As explained above, any such solution will blow up exponentially at spatial infinity on *all* constant t surfaces. If the source is turned on at a finite time, its effects cannot propagate any faster than v in the aether frame, and so the solution can *not* blow up at spatial infinity at any finite time.

The preceding argument depends on a choice of boundary condition for the solution generated by the source, which is equivalent to the choice of Green's function for the wave equation. We implicitly adopted the retarded Green's function, which vanishes for $t < 0$. One might ask whether the argument would continue to hold using a t'-retarded Green's function that would vanish for $t' < 0$. It appears, however, that no such Green's function exists for $\beta > 1/v$. This can be seen as follows. A standard method for constructing Green's functions is via the Fourier transform

$$G(t', x') \propto \int d^3k' d\omega' \frac{e^{ik'\cdot x'} e^{-i\omega' t'}}{(\omega' - \omega'_-)(\omega' - \omega'_+)}, \qquad (11)$$

where $\omega'_\pm$ are the roots of the dispersion relation (9). This integral can be performed along any contour that begins at $-\infty$ and ends at $+\infty$ along the real axis. If the integral converges, then the wave operator acting on $G(t', x')$ can be moved inside the integral, canceling the denominator. The remaining integrand has no poles, so the contour can be freely deformed to lie along the real axis, yielding a representation of the Dirac delta function. To obtain the retarded Green's function, the ω' integral is performed along a contour that passes above all the poles, so that for $t' < 0$ the integral vanishes. For $t' > 0$, both poles are enclosed by the contour that can be closed in the lower half plane, and the ω' integral yields

$$G(t', x') \propto \int d^3k' e^{ik'\cdot x'} \frac{e^{-i\omega'_+ t} - e^{-i\omega'_- t}}{\omega'_+ - \omega'_-}. \qquad (12)$$

For large $k'_\perp$, the roots behave as $\omega'_\pm \sim \pm i k'_\perp$, so that the integrand in (12) grows exponentially with $k'_\perp$ and the integral does not converge. A t'-retarded Green's function therefore cannot be found by this standard method, which strongly suggests that such a Green's function does not exist.[2]

A second sign of possible instability discussed in Ref. [1] is that the Hamiltonians generating t' translations can be unbounded below whenever $v \neq 1$. In particular the Hamiltonian of linear perturbations is unbounded below precisely when $\beta > v$ or $\beta > 1/v$ (the condition in Ref. [1] was expressed in terms of the coupling constants of the theory rather than the mode speeds, nevertheless the two conditions are equivalent). This can be understood in a simple way as follows.

Let ξ^a denote the t' translation 4-vector. The Hamiltonian generating ξ^a translations can be written as an integral over an initial data surface Σ,

$$H_\xi = \int_\Sigma T^b{}_a \xi^a n_b d^3\Sigma, \qquad (13)$$

where $T^b{}_a$ is the canonical energy-momentum tensor

$$\sqrt{-\eta} T^b{}_a = \frac{\partial \mathcal{L}}{\partial(\partial_b \varphi)} \partial_a \varphi - \mathcal{L} \delta^b_a, \qquad (14)$$

n_b is the unit normal covector, and $d^3\Sigma$ is the surface volume element, both normalized with respect to η_{ab}. Positivity of H_ξ is ensured when $T^b{}_a \xi^a n_b$ is positive. In the case $v < 1$ the φ field is "subluminal" relative to η_{ab}, so for $\beta > v$ the vector ξ^a can be timelike relative to η_{ab} but *spacelike* relative to the effective metric $g_{(v)ab}$ for the φ field. In this case H_ξ is in effect a component of the momentum, not the energy of φ, which is clearly not bounded below. In the case $v > 1$, for $\beta > 1/v$ the t' translation vector ξ^a remains timelike with respect to $g_{(v)ab}$, but the constant t' surface becomes timelike with respect to $g_{(v)ab}$. In this case H_ξ is the flux of energy through a timelike surface, and is no longer expected to be bounded below. Moreover, if the surface Σ is timelike with respect to $g_{(v)ab}$ there is no guarantee that H_ξ is conserved under t' translation, because the current $T^b{}_a \xi^a$ can flow out through the boundaries.

In conclusion, while we take no issue with the computations of Ref. [1], the inference of instabilities in Einstein-aether theory when the mode velocities differ from c is unwarranted. A proper identification of the phase space of the theory eliminates the exponentially growing solutions. The Hamiltonians that were found to be unbounded below actually correspond either to momenta or to energy fluxes across timelike surfaces. The Hamiltonian generating time translations in the aether frame is bounded below and plays the usual role of the energy in governing stability. It is therefore sufficient for stability to impose the

[2]Note that the exponential instability alone does not account for the absence of a retarded Green's function. For example, in the case of a tachyonic scalar field with a negative m^2, the instability occurs only at low k, so the convergence of the Green's function is not spoiled, and a retarded Green's function exists.

conditions of real frequencies and positive energy in the aether frame. The opposite conclusion was reached in Ref. [1] by considering the Lorentz symmetry of the background metric to be a physical symmetry of the phase space of linear perturbations. Since the background aether breaks this symmetry, that viewpoint is untenable.

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