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Holographic QCD integrated back to hidden local symmetry

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We develop a previously proposed gauge-invariant method to integrate out an infinite tower of Kaluza-Klein (KK) modes of vector and axial-vector mesons in a class of models of holographic QCD (HQCD). The HQCD is reduced by our method to chiral perturbation theory with hidden local symmetry (HLS), having only the lowest KK mode identified as the HLS gauge boson. We take the Sakai-Sugimoto model as a concrete HQCD, and completely determine the $\mathcal{O}(p^4)$ terms as well as the $\mathcal{O}(p^2)$ terms from the Dirac-Born-Infeld part and the anomaly-related (intrinsic-parity odd) gauge-invariant terms from the Chern-Simons part. Effects of higher KK modes are fully included in these terms. To demonstrate the power of our method, we compute momentum dependences of several form factors, such as the pion electromagnetic form factors, and the π^0-γ and ω-π^0 transition form factors, compared with experiment, which was not achieved before due to the complication of handling infinite sums. We also study other anomaly-related quantities like γ^*-π^0-π^+-π^- and ω-π^0-π^+-π^- vertex functions.

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I. INTRODUCTION

Holography, based on gauge/gravity duality [1,2], has been the lastest trend in revealing features in strongly coupled gauge theories. Application to QCD, which is called holographic QCD (HQCD), is useful to check the validity of the holographic correspondence. In some models [3–6] which realize the chiral symmetry breaking of QCD, it has been shown in the large N_c limit that some observables of low-energy QCD are consistent with experiment. There are two types of holographic approaches: One is called the "top-down" approach, starting with a stringy setting; the other is called the "bottom-up" approach, beginning with a five-dimensional gauge theory defined on an anti–de Sitter space (AdS) background. A key point to notice is that, whichever approach is used, one eventually employs a five-dimensional gauge model with a characteristic induced metric and some boundary conditions on a certain brane configuration.

The holographic recipe tells us that classical solutions for boundary values of bulk fields serve as sources coupled to currents in the dual four-dimensional QCD. Green functions in QCD-like current correlators are thus evaluated straightforwardly from the boundary action as a generating functional in the large N_c limit [5,6]. Equivalently, one can show that those things are calculable from the five-dimensional action by performing a Kaluza-Klein (KK) decomposition of the bulk gauge fields and identifying KK fields themselves as vector and axial-vector fields of a low-energy effective model dual to QCD [3,4]. In this sense, one can say that, in the low-energy region, any model of HQCD is reduced to a certain effective hadron model in four dimensions. Such effective models include vector and axial-vector mesons as an infinite tower of KK modes, together with the Nambu-Goldstone bosons (NGBs) associated with the spontaneous chiral symmetry breaking. An infinite tower of KK modes (vector and axial-vector mesons) then contributes to Green functions such as current correlators and form factors. Here we follow the latter approach [3,4], dealing with the bulk action as a functional of the gauge fields.

It was pointed out [3,4,7–9] that the infinite tower of KK modes is interpreted as a set of gauge bosons of hidden local symmetries (HLSs) [10–12]. Note that since the KK modes as the gauge bosons of HLSs are not necessarily mass eigenstates, we should distinguish them ["HLS-KK modes," $V_\mu^{(n)}$ in Eq. (2.12)] from the conventional KK modes [$B_\mu^{(n)}$ in Eq. (2.14)]. Hereafter we shall call the HLS-KK modes simply the KK modes. Solving away higher KK modes through the equations of motion derived from the five-dimensional effective action, which is equivalent to integrating out KK modes in terms of a functional integral, we showed [13] that, in the low-energy region, any holographic model can be formulated in the HLS notion to be reduced to the HLS model having a finite set of HLS gauge bosons with the lowest one identified as the ρ meson and its flavor partners.

Instead of dealing with the infinite tower of KK modes, we demonstrated [13] that effects from the higher KK modes are *fully* incorporated into coefficients of the

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$\mathcal{O}(p^4)$ terms in the HLS field theory extended from the conventional chiral perturbation theory (ChPT) [14]; we call this the HLS-ChPT [12,15]. (A similar method of integrating out was also considered in Ref. [16].) Furthermore, since it is a manifestly HLS gauge-invariant formulation, one can calculate any Green function order by order in the derivative expansion, or loop expansion, in which higher order corrections may be identified with the $1/N_c$-subleading effects which are not easily figured out in HQCD. In fact, we calculated meson-loop corrections as $1/N_c$-subleading effects in terms of the HLS-ChPT for the Dirac-Born-Infeld (DBI) part in the Sakai-Sugimoto (SS) model [3,4].

In this paper, our method will be developed in detail, including external gauge fields such as photons in a class of HQCD models including the SS model. By construction our method is manifestly invariant under HLS and chiral symmetry, including external gauge symmetry. It will be shown that, on the contrary, a naive truncation simply neglecting higher KK modes of the HLS gauge bosons violates HLS and chiral symmetry, including external gauge symmetry. We further extend our method to the Chern-Simons (CS) part. In the case of the SS model we present a full set of the $\mathcal{O}(p^4)$ terms of the HLS Lagrangian computed from the DBI part at the leading order of the $1/N_c$ expansion, which was partially reported in the previous work [13]. In addition, the anomaly-related [intrinsic-parity-odd (IP-odd)] gauge-invariant terms introduced in Refs. [11,12,17] are completely determined from the CS part. Once the $\mathcal{O}(p^4)$ terms are determined, calculation of meson-loop corrections of subleading order in the $1/N_c$ expansion in terms of the HLS-ChPT can be performed.

Throughout this paper, we will confine ourselves to the large N_c limit, leaving calculations of $1/N_c$-subleading order to future works. Even in the large N_c limit, our method is useful, especially for studying the momentum dependences of several form factors, which was not achieved due to the complication of dealing with the infinite sum. Actually, given a concrete holographic model not restricted to the SS model, our method enables us to deduce definite predictions of the model for any physical quantity to be compared with experimental data.

Here we demonstrate the power of our method in the case of the SS model. The form factors are calculable in the general framework of the HLS model with its parameters determined by the SS model for IP-odd processes as well as IP-even ones. The electromagnetic (EM) gauge invariance and the chiral invariance are automatically maintained since our method is manifestly invariant under the external gauge symmetry as well as the HLS. As to IP-even processes, an explicit form of the pion EM form factor is given to be compared with the experimental data. As to IP-odd processes, we also give explicit forms of the π^0-γ and ω-π^0 transition form factors and the related quantities such as γ^*-π^0-π^+-π^- and ω-π^0-π^+-π^- vertex functions. To show that our formulation correctly includes contributions from an infinite set of higher KK modes, we further derive the same results by a different method, dealing with the infinite sum explicitly without using the general HLS Lagrangian. This reveals the fact that an infinite sum is crucial for the gauge invariance. Actually, the EM gauge symmetry and chiral symmetry (low-energy theorem) are obviously violated by a naive truncation, simply neglecting higher KK modes instead of taking the infinite sum. Note that the importance of the higher KK modes is visible only in the HLS basis: The higher KK modes in the mass eigenstates basis (KK modes in the usual sense) do not contribute at all, since our method is equivalent to setting the higher mass-eigenstate fields to zero, $B_\mu^{(n)} = 0 (n \neq 1)$, via the equation of motion of $B_\mu^{(n)}$ [see Eq. (2.34)]. This is in accordance with the fact that the SS model may not be reliable beyond the scale of $M_{\rm KK} \sim 1$ GeV.

This paper is organized as follows:

In Sec. II we start with a class of models of HQCD, including the SS model [3,4], and explain our formulation by integrating out arbitrary parts of an infinite tower of vector and axial-vector mesons in a manner manifestly invariant under HLS and external gauge symmetry. We demonstrate that low-energy effective models of HQCD can be formulated by the HLS with $\mathcal{O}(p^4)$ terms. In Sec. III we calculate the parameters of the HLS Lagrangian from the SS model for the IP-even and IP-odd $\mathcal{O}(p^4)$ terms. In Sec. IV we present several applications of our method, including the pion EM form factor and IP-odd form factors such as π^0-γ and ω-π^0 transition form factors. Section V is devoted to a summary and discussion. Appendix A is a proof that Γ_3 defined in the text as a part of the CS action of the SS model is HLS invariant (and thereby provides the HLS-invariant terms of the IP-odd part of the HLS Lagrangian). In Appendix B DBI and CS terms are expanded in terms of the HLS building blocks. Appendix C demonstrates that, as done for the IP-even processes in the text, the same result as that of our integrating-out method for the IP-odd form factors is obtained by an alternative method explicitly using sum rules of the infinite tower of the HLS-KK modes.

II. A GAUGE-INVARIANT WAY TO INTEGRATE OUT HQCD

In this section, we develop a detailed formulation of our method [13]: Starting with a class of HQCD models, including the SS model [3,4], we introduce a way to obtain a low-energy effective model in four dimensions, described only by the lightest vector meson identified as the ρ meson, based on HLS, together with the NGBs. Although most of the notations adopted here follow the SS model [3,4], our methodology is applicable to other types of HQCD.

A. Reducing 5d models to 4d models with an infinite tower of vector and axial-vector mesons

Suppose that the fifth direction, spanned by the coordinate z, extends from minus infinity to plus infinity $(-\infty < z < \infty)$.[1] The parity is introduced by imposing a reflection symmetry under an interchange $z \leftrightarrow -z$ along the fifth direction. We employ a five-dimensional gauge theory which has a vectorial $U(N)$ gauge symmetry defined on a certain background associated with the gauge/gravity duality.

The five-dimensional gauge field, $A_M(x^\mu, z)$ with $M = (\mu, z)$, transforms inhomogeneously under $U(N)$ gauge symmetry as

$$A_M(x^\mu, z) \to g(x^\mu, z) A_M(x^\mu, z) g^\dagger(x^\mu, z) - i\partial_M g(x^\mu, z) g^\dagger(x^\mu, z), \tag{2.1}$$

where $g(x^\mu, z)$ is the transformation matrix of gauge symmetry. As far as a gauge-invariant sector such as the Dirac-Born-Infeld part of the SS model [3,4] is concerned, the five-dimensional action in the large N_c limit can be written as[2]

$$S_5 = N_c \int d^4x dz \Big(-\frac{1}{2} K_1(z)\, \mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] + K_2(z) M_{\mathrm{KK}}^2\, \mathrm{tr}[F_{\mu z}F^{\mu z}] \Big), \tag{2.2}$$

where $K_{1,2}(z)$ denote a set of metric functions of z constrained by the gauge/gravity duality. M_{KK} is a typical mass scale of the KK modes of the gauge field A_M.

We choose the same boundary condition for the five-dimensional gauge field A_M as done in Refs. [3,4]:

$$A_M(x^\mu, z = \pm\infty) = 0. \tag{2.3}$$

A transformation which does not change this boundary condition satisfies $\partial_M g(x^\mu, z)|_{z=\pm\infty} = 0$. This implies an emergence of global chiral $U(N)_L \times U(N)_R$ symmetry in four dimensions characterized by the transformation matrices $g_{R,L} = g(z = \pm\infty)$. With the boundary condition (2.3) imposed, the zero mode of A_z is identified with the NGB associated with the spontaneous breaking of chiral symmetry. The chiral field

$$U(x^\mu) = P \exp\Big[i \int_{-\infty}^{\infty} dz' A_z(x^\mu, z') \Big] \tag{2.4}$$

is parametrized by the NGB field π as

$$U(x^\mu) = e^{2i\pi(x^\mu)/F_\pi}, \tag{2.5}$$

where F_π denotes the decay constant of π. U is divided as

$$U(x^\mu) = \xi_L^\dagger(x^\mu) \cdot \xi_R(x^\mu), \tag{2.6}$$

such that $\xi_{R,L}$ transform as

$$\xi_{R,L} \to h(x^\mu) \cdot \xi_{R,L} \cdot g_{R,L}^\dagger, \tag{2.7}$$

with $h(x^\mu)$ being the transformation of HLS [10–12]. Here we note that we can introduce an infinite number of HLSs by dividing U into a product of an infinite number of ξ fields [10,11].

Chiral $U(N_f)_L \times U(N_f)_R$ symmetry can be gauged by the external fields $\mathcal{L}_\mu$ and $\mathcal{R}_\mu$, including the photon field, through the boundary condition [4]

$$\begin{aligned} A_\mu(x^\mu, z = +\infty) &= \mathcal{R}_\mu(x^\mu) = \mathcal{V}_\mu(x^\mu) + \mathcal{A}_\mu(x^\mu), \\ A_\mu(x^\mu, z = -\infty) &= \mathcal{L}_\mu(x^\mu) = \mathcal{V}_\mu(x^\mu) - \mathcal{A}_\mu(x^\mu), \end{aligned} \tag{2.8}$$

instead of Eq. (2.3).

Following Refs. [3,4,13], we work in $A_z = 0$ gauge. There still exists a four-dimensional gauge symmetry under which $A_\mu(x^\mu, z)$ transforms as

$$A_\mu(x^\mu, z) \to h(x^\mu) \cdot A_\mu(x^\mu, z) \cdot h^\dagger(x^\mu) - i\partial_\mu h(x^\mu) \cdot h^\dagger(x^\mu). \tag{2.9}$$

This gauge symmetry is identified [3,4,13] with the above HLS. In this gauge the NGB fields reside in the boundary condition for the five-dimensional gauge field A_μ as

$$\begin{aligned} A_\mu(x^\mu, z = +\infty) &= \alpha_\mu^R(x^\mu), \\ A_\mu(x^\mu, z = -\infty) &= \alpha_\mu^L(x^\mu), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \alpha_\mu^R(x^\mu) &= i\xi_R(x^\mu) \mathcal{D}_\mu \xi_R^\dagger(x^\mu) \\ &= i\xi_R(x^\mu)(\partial_\mu - i\mathcal{R}_\mu)\xi_R^\dagger(x^\mu), \\ \alpha_\mu^L(x^\mu) &= i\xi_L(x^\mu) \mathcal{D}_\mu \xi_L^\dagger(x^\mu) \\ &= i\xi_L(x^\mu)(\partial_\mu - i\mathcal{L}_\mu)\xi_L^\dagger(x^\mu), \end{aligned} \tag{2.11}$$

which transform under the HLS in the same way as in Eq. (2.9). Note that, in this gauge, we explicitize a single HLS among an infinite number of HLSs while the chiral symmetry is "hidden."

We introduce an infinite tower of the massive KK modes of the vector $[V_\mu^{(n)}(x^\mu)]$ and the axial-vector $[A_\mu^{(n)}(x^\mu)]$ meson fields. The vector-meson fields $V_\mu^{(n)}(x^\mu)$ transform as the HLS gauge boson:

$$V_\mu^{(n)}(x^\mu) \to h(x^\mu) \cdot V_\mu^{(n)}(x^\mu) \cdot h^\dagger(x^\mu) - i\partial_\mu h(x^\mu) \cdot h^\dagger(x^\mu), \tag{2.12}$$

while the axial-vector-meson fields $A_\mu^{(n)}(x^\mu)$ transform as the matter fields:

[1] In an application to another type of HQCD [6], the z coordinate is defined on a finite interval, which is different from the z coordinate used here. They are related by an appropriate coordinate transformation as done in Refs. [3,4].

[2] Models of HQCD having the left- and right-bulk fields such as F_L, F_R [5,6] can be described by the same action as in Eq. (2.2) with a suitable z-coordinate transformation prescribed.

$$A_\mu^{(n)}(x^\mu) \to h(x^\mu)\cdot A_\mu^{(n)}(x^\mu)\cdot h^\dagger(x^\mu). \tag{2.13}$$

It should be noted that the vector-meson fields $V_\mu^{(n)}(x^\mu)$ are different from the mass-eigenstate fields $B_\mu^{(n)}(x^\mu)$ in Refs. [3,4] which transform as matter fields,

$$B_\mu^{(n)}(x^\mu) \to h(x^\mu)\cdot B_\mu^{(n)}(x^\mu)\cdot h^\dagger(x^\mu). \tag{2.14}$$

The five-dimensional gauge field $A_\mu(x^\mu, z)$ is now expanded as[3]

$$A_\mu(x^\mu, z) = \alpha_\mu^R(x^\mu)\phi^R(z) + \alpha_\mu^L(x^\mu)\phi^L(z) + \sum_{n=1}^\infty (A_\mu^{(n)}(x^\mu)\psi_{2n}(z) - V_\mu^{(n)}(x^\mu)\psi_{2n-1}(z)). \tag{2.15}$$

The functions $\{\psi_{2n-1}(z)\}$ and $\{\psi_{2n}(z)\}$ are the eigenfunctions[4] satisfying the eigenvalue equation obtained from the action (2.2):

$$-K_1^{-1}(z)\partial_z(K_2(z)\partial_z\psi_n(z)) = \lambda_n\psi_n(z) \qquad (n = 0, 1, 2, \ldots), \tag{2.16}$$

where λ_n denotes the nth eigenvalue. On the other hand, the gauge invariance requires the functions $\phi^{R,L}(z)$ to be different from the eigenfunctions: From the transformation properties in Eqs. (2.9), (2.11), (2.12), and (2.13), we see that the functions $\phi^{R,L}(z)$, $\{\psi_{2n-1}(z)\}$, and $\{\psi_{2n}(z)\}$ are constrained as

$$\phi^R(z) + \phi^L(z) - \sum_{n=1}^\infty \psi_{2n-1}(z) = 1. \tag{2.17}$$

Using this, we may rewrite Eq. (2.15) to obtain

$$A_\mu(x^\mu, z) = \alpha_{\mu\parallel}(x^\mu) + \alpha_{\mu\perp}(x^\mu)(\phi^R(z) - \phi^L(z)) + \sum_{n=1}^\infty A_\mu^{(n)}(x^\mu)\psi_{2n}(z) + \sum_{n=1}^\infty (\alpha_{\mu\parallel}(x^\mu) - V_\mu^{(n)}(x^\mu))\psi_{2n-1}(z), \tag{2.18}$$

where

$$\alpha_{\mu\parallel,\perp}(x^\mu) = \frac{\alpha_\mu^R(x^\mu) \pm \alpha_\mu^L(x^\mu)}{2}, \tag{2.19}$$

respectively, transform under the HLS as

$$\alpha_{\mu\parallel}(x^\mu) \to h(x^\mu)\cdot\alpha_{\mu\parallel}(x^\mu)\cdot h^\dagger(x^\mu) - i\partial_\mu h(x^\mu)\cdot h^\dagger(x^\mu), \tag{2.20}$$

$$\alpha_{\mu\perp}(x^\mu) \to h(x^\mu)\cdot\alpha_{\mu\perp}(x^\mu)\cdot h^\dagger(x^\mu). \tag{2.21}$$

Note that $\alpha_{\mu\perp}$ includes the NGB fields as $\alpha_{\mu\perp} = \frac{1}{F_\pi}\partial_\mu\pi + \cdots$. The corresponding wave function $(\phi^R - \phi^L)$ should therefore be the eigenfunction for the zero mode [$n = 0$ in Eq. (2.16)], ψ_0:

$$\phi^R(z) - \phi^L(z) = \psi_0(z). \tag{2.22}$$

Thus we see from Eqs. (2.17) and (2.22) that the wave functions ϕ^R and ϕ^L are not the eigenfunctions but are given as

$$\phi^{R,L}(z) = \frac{1}{2}\left[1 + \sum_{n=1}^\infty \psi_{2n-1}(z) \pm \psi_0(z)\right]. \tag{2.23}$$

By substituting Eq. (2.18) into the action (2.2), with Eq. (2.22) taken into account, the five-dimensional theory is now described by the NGB fields along with an infinite tower of vector and axial-vector-meson fields in four dimensions: The action (2.2) is expressed as

$$S_5 = N_c M_{\rm KK}^2 \int dz d^4x \Big\{K_2(z)\dot\psi_0^2(z)\,{\rm tr}[\alpha_{\mu\perp}(x^\mu)]^2 + K_2(z)\sum_{n=1}^\infty \lambda_{2n}\psi_{2n}^2(z)\,{\rm tr}[A_\mu^{(n)}(x^\mu)]^2 + K_2(z)\sum_{n=1}^\infty \lambda_{2n-1}\psi_{2n-1}^2(z)\,{\rm tr}[\alpha_{\mu\parallel}(x^\mu) - V_\mu^{(n)}(x^\mu)]^2\Big\} - \frac{1}{2}N_c\int dz d^4x K_1(z)\,{\rm tr}[F_{\mu\nu}F^{\mu\nu}], \tag{2.24}$$

where we have used the eigenvalue equation (2.16) and the orthogonality relation among the eigenfunctions. In the last term of Eq. (2.24) the five-dimensional field strength $F_{\mu\nu}(x^\mu, z)$ can be decomposed into three parts:

$$F_{\mu\nu}(x^\mu, z) = F_{\mu\nu}^{(0)}(x^\mu, z) + \sum_{n=1}^\infty F_{\mu\nu}^{(n)}(x^\mu, z) + \sum_{n=1}^\infty\sum_{m=1}^\infty F_{\mu\nu}^{(n,m)}(x^\mu, z), \tag{2.25}$$

where

[3]In Eq. (2.15) we put a minus sign in front of $V_\mu^{(n)}$ for a convention.

[4]The eigenfunction for $n = 2k$ ($k = 1, 2, \ldots$) is an odd function of z, while that for $n = (2k - 1)$ is an even function.

$$F_{\mu\nu}^{(0)}(x^\mu,z) = F_{\mu\nu}(\alpha_\parallel) + \psi_0(z)(D_\mu\alpha_{\nu\perp}(x^\mu) - D_\nu\alpha_{\mu\perp}(x^\mu)) - i\psi_0^2(z)[\alpha_{\mu\perp}(x^\mu),\alpha_{\nu\perp}(x^\mu)],$$
$$F_{\mu\nu}^{(n)}(x^\mu,z) = \psi_{2n}(z)(D_\mu A_\nu^{(n)} - D_\nu A_\mu^{(n)}) - \psi_{2n-1}(z)(D_\mu\tilde V_\nu^{(n)} - D_\nu\tilde V_\mu^{(n)}) - i\psi_{2n}^2(z)[A_\mu^{(n)},A_\nu^{(n)}] - i\psi_{2n-1}^2(z)[\tilde V_\mu^{(n)},\tilde V_\nu^{(n)}]$$
$$- i\psi_0(z)\psi_{2n}(z)([\alpha_{\mu\perp},A_\nu^{(n)}] - [\alpha_{\nu\perp},A_\mu^{(n)}]) + i\psi_0(z)\psi_{2n-1}(z)([\alpha_{\mu\perp},\tilde V_\nu^{(n)}] - [\alpha_{\nu\perp},\tilde V_\mu^{(n)}]),$$
$$F_{\mu\nu}^{(n,m)}(x^\mu,z) = i\psi_{2n-1}(z)\psi_{2m}(z)([A_\mu^{(m)},\tilde V_\nu^{(n)}] - [A_\nu^{(m)},\tilde V_\mu^{(n)}]), \tag{2.26}$$

with

$$F_{\mu\nu}(\alpha_\parallel) \equiv \partial_\mu\alpha_{\nu\parallel}(x^\mu) - \partial_\nu\alpha_{\mu\parallel}(x^\mu) - i[\alpha_{\mu\parallel}(x^\mu),\alpha_{\nu\parallel}(x^\mu)], \qquad D_\mu \equiv \partial_\mu - i[\alpha_{\mu\parallel},], \qquad \tilde V_\mu^{(n)} \equiv V_\mu^{(n)} - \alpha_{\mu\parallel}. \tag{2.27}$$

B. Integrating out KK modes of vector and axial-vector mesons

We are interested in constructing a low-energy effective theory of HQCD written in terms of the meson fields with their masses lower than a certain energy scale. Suppose that those mesons are given by the KK modes of the HQCD at the level of $m \le M$ for the axial-vector mesons $A_\mu^{(m)}(x^\mu)$ and the level of $n \le N$ for the vector mesons $V_\mu^{(n)}(x^\mu)$.

Let us first discuss the naive truncation of the KK modes of the HLS gauge bosons as the vector and the axial-vector mesons simply by putting $A_\mu^{(m)}(x^\mu) = 0$ for $m > M$ and $V_\mu^{(n)}(x^\mu) = 0$ for $n > N$ in Eq. (2.15):

$$A_\mu^{\rm trun}(x^\mu,z) = \alpha_\mu^R(x^\mu)\phi^R(z) + \alpha_\mu^L(x^\mu)\phi^L(z) + \sum_{m=1}^{M} A_\mu^{(m)}(x^\mu)\psi_{2m}(z) - \sum_{n=1}^{N} V_\mu^{(n)}(x^\mu)\psi_{2n-1}(z) \tag{2.28}$$

with the constraint in Eqs. (2.17) and (2.22) unchanged:

$$\phi^R(z) + \phi^L(z) - \sum_{n=1}^{\infty}\psi_{2n-1}(z) = 1, \tag{2.29}$$

$$\phi^R(z) - \phi^L(z) = \psi_0(z). \tag{2.30}$$

As a result, $A_\mu^{\rm trun}(x^\mu,z)$ transforms under the HLS as

$$A_\mu^{\rm trun}(x^\mu,z) \to h(x^\mu)\cdot A_\mu^{\rm trun}(x^\mu,z)\cdot h^\dagger(x^\mu) - iC^{\rm trun}(z)\partial_\mu h(x^\mu)\cdot h^\dagger(x^\mu), \tag{2.31}$$

where

$$C^{\rm trun}(z) \equiv \phi^R(z) + \phi^L(z) - \sum_{n=1}^{N}\psi_{2n-1}(z) = 1 + \sum_{n=N+1}^{\infty}\psi_{2n-1}(z) \neq 1. \tag{2.32}$$

Then $A_\mu^{\rm trun}(x^\mu,z)$ no longer transforms as the gauge field. Since the action in Eq. (2.2) is invariant under the transformation in Eq. (2.9) but not in Eq. (2.31), then this truncation violates gauge symmetry (HLS) and hence chiral symmetry.[5]

The violation of HLS/chiral symmetry can also be seen in the expression of Eq. (2.24) with a nave truncation, $A_\mu^{(m)}(x^\mu) = 0$ for $m > M$ and $V_\mu^{(n)}(x^\mu) = 0$ for $n > N$:

$$S_5^{\rm trun} = N_c M_{\rm KK}^2 \int dz d^4x \Big\{ K_2(z)\dot\psi_0^2(z)\,{\rm tr}[\alpha_{\perp\mu}(x^\mu)]^2 + K_2(z)\sum_{m=1}^{M}\lambda_{2m}\psi_{2m}^2(z)\,{\rm tr}[A_\mu^{(m)}(x^\mu)]^2 + K_2(z)\sum_{n=1}^{N}\lambda_{2n-1}\psi_{2n-1}^2(z)\,{\rm tr}[\alpha_{\mu\parallel}(x^\mu) - V_\mu^{(n)}(x^\mu)]^2 + K_2(z)\sum_{n=N+1}^{\infty}\lambda_{2n-1}\psi_{2n-1}^2(z)\,{\rm tr}[\alpha_{\mu\parallel}(x^\mu)]^2\Big\} - \frac{1}{2}N_c\int dz d^4x K_1(z)\,{\rm tr}[(F_{\mu\nu}^{\rm trun})^2], \tag{2.33}$$

where $F_{\mu\nu}^{\rm trun} = \partial_\mu A_\nu^{\rm trun} - \partial_\nu A_\mu^{\rm trun} - i[A_\mu^{\rm trun}, A_\nu^{\rm trun}]$. It is obvious that the last line is not invariant under HLS/chiral symmetry since $F_{\mu\nu}^{\rm trun} \not\to h\cdot F_{\mu\nu}^{\rm trun}\cdot h^\dagger$ under HLS. Similarly, one can easily see from Eq. (2.20) that the fourth line also violates chiral symmetry as well as HLS.

Now we shall discuss a method [13] to integrate out KK modes of the HLS gauge bosons, or to solve them away through the equations of motion in an HLS/chiral-invariant manner. Equivalently, our method [13] involves nothing but eliminating the mass-eigenstate fields $B_\mu^{(n)}$ in Eq. (2.14) through the equations of motion, $B_\mu^{(n)} = 0$, which may be phrased as "neglecting the higher mass excitation modes" [18]. Consider a low-energy effective theory below the axial-vector-meson mass of the $m = M+1$ level and the vector-meson mass of the $n = N+1$ level, where the higher dimensional terms such as the kinetic terms may be ignored. Then the equations of motion for $B_\mu^{(2m)} = A_\mu^{(m)}$ with $m > M$ and $B_\mu^{(2n-1)} = (V_\mu^{(n)} - \alpha_{\mu\parallel})$ with $n > N$ read

[5] Some reflections of the violation of the HLS/chiral symmetry will be discussed in Sec. III.

$$B_\mu^{(2m)}(x^\mu) = A_\mu^{(m)}(x^\mu) = 0$$
$$(m = M+1, M+2, \cdots, \infty),$$
$$B_\mu^{(2n-1)}(x^\mu) = (V_\mu^{(n)}(x^\mu) - \alpha_{\mu\parallel}(x^\mu)) = 0$$
$$(n = N+1, N+2, \cdots, \infty). \tag{2.34}$$

Note that the naive truncation is meant to eliminate the HLS fields $V_\mu^{(n)} = A_\mu^{(m)} = 0$, in contrast to integrating them out as $B_\mu^{(n)} = 0$. Putting these solutions into Eq. (2.15), we obtain

$$A_\mu^{\rm integ}(x^\mu, z) = \alpha_\mu^R(x^\mu)\phi^R(z) + \alpha_\mu^L(x^\mu)\phi^L(z) + \sum_{m=1}^{M} A_\mu^{(m)}(x^\mu)\psi_{2m}(z) - \sum_{n=1}^{N} V_\mu^{(n)}(x^\mu)\psi_{2n-1}(z) - \sum_{n=N+1}^{\infty} \alpha_{\mu\parallel}(x^\mu)\psi_{2n-1}(z). \tag{2.35}$$

This $A_\mu^{\rm integ}$ transforms under HLS as

$$A_\mu^{\rm integ}(x^\mu, z) \to h(x^\mu)\cdot A_\mu^{\rm integ}(x^\mu, z)\cdot h^\dagger(x^\mu) - iC^{\rm integ}(z)\partial_\mu h(x^\mu)\cdot h^\dagger(x^\mu), \tag{2.36}$$

where $C^{\rm integ}(z)$ is identically unity from Eq. (2.29):

$$C^{\rm integ}(z) \equiv \phi^R(z) + \phi^L(z) - \sum_{n=1}^{\infty}\psi_{2n-1}(z) = 1, \tag{2.37}$$

in comparison with $C^{\rm trun}(z) \neq 1$ in Eq. (2.32). This implies that $A_\mu^{\rm integ}$ transforms as the gauge field, in contrast to $A_\mu^{\rm trun}$ in the naive truncation, and hence the action (2.2) remains invariant under the HLS/chiral transformation after higher KK modes are integrated out. The reason why $A_\mu^{\rm integ}$ transforms correctly is that the presence of the last term of Eq. (2.35), consisting of $\alpha_{\mu\parallel}$ as a result of equations of motion (2.34), keeps the transformation property of the original higher KK fields, in contrast to $A_\mu^{\rm trun}$ which lacks the corresponding term. It is convenient to rewrite the expression in Eq. (2.35) as [13]

$$A_\mu^{\rm integ}(x^\mu, z) = \alpha_\mu^R(x^\mu)\varphi^R(z) + \alpha_\mu^L(x^\mu)\varphi^L(z) + \sum_{m=1}^{M} A_\mu^{(m)}(x^\mu)\psi_{2m}(z) - \sum_{n=1}^{N} V_\mu^{(n)}(x^\mu)\psi_{2n-1}(z), \tag{2.38}$$

where

$$\varphi^R(z) + \varphi^L(z) - \sum_{n=1}^{N}\psi_{2n-1}(z) = 1, \tag{2.39}$$

$$\varphi^R(z) - \varphi^L(z) = \psi_0(z). \tag{2.40}$$

Note the crucial difference between the finite sum in Eq. (2.39) and the infinite sum in Eq. (2.29). This point will be discussed in Sec. IV and will be important for the HLS/chiral invariance which includes the electromagnetic gauge invariance when the system is coupled to the photon as in the pion form factor.

The invariance can also be seen by the action (2.24), with the condition of integrating out KK modes in Eq. (2.34):

$$S_5^{\rm integ} = N_c M_{\rm KK}^2 \int dz d^4x \Big\{ K_2(z)\dot\psi_0^2(z)\,{\rm tr}[\alpha_{\perp\mu}(x^\mu)]^2 + K_2(z)\sum_{m=1}^{M}\lambda_{2m}\psi_{2m}^2(z)\,{\rm tr}[A_\mu^{(m)}(x^\mu)]^2 + K_2(z)\sum_{n=1}^{N}\lambda_{2n-1}\psi_{2n-1}^2(z) \times {\rm tr}[\alpha_{\mu\parallel}(x^\mu) - V_\mu^{(n)}(x^\mu)]^2\Big\} - \frac{1}{2}N_c\int dz d^4x K_1(z)\,{\rm tr}[(F_{\mu\nu}^{\rm integ})^2], \tag{2.41}$$

where

$$F_{\mu\nu}^{\rm integ}(x^\mu, z) = \partial_\mu A_\nu^{\rm integ} - \partial_\nu A_\mu^{\rm integ} - i[A_\mu^{\rm integ}, A_\nu^{\rm integ}]. \tag{2.42}$$

It is obvious that each term in Eq. (2.41) is invariant under HLS.

C. Integrating out HQCD back to HLS

Let us next consider a low-energy effective model obtained by integrating out all the higher vector and axial-vector-meson fields in HQCD except the lowest vector-meson field $V_\mu^{(1)}(x^\mu) \equiv V_\mu(x^\mu)$, i.e. $M = 0$ and $N = 1$ in Eqs. (2.38), (2.39), and (2.40). Such an effective model can be described by the HLS model having only the NGBs and the lightest vector mesons denoted by ρ (ρ meson and its flavor partners) plus the $\mathcal{O}(p^4)$ terms coming from the last term in Eq. (2.41). Given a particular HQCD we can compute all the coefficients of $\mathcal{O}(p^4)$ terms [13]: The $\mathcal{O}(p^4)$ terms include the effects from an infinite tower of higher KK modes and are completely determined, as will be explicitly seen in the next sections.

Substituting Eqs. (2.39) and (2.40) into Eq. (2.38) with $M = 0$ and $N = 1$, we obtain

$$A_\mu^{\rm integ}(x^\mu, z) = \hat\alpha_{\mu\perp}(x^\mu)\psi_0(z) + (\hat\alpha_{\mu\parallel}(x^\mu) + V_\mu(x^\mu)) + \hat\alpha_{\mu\parallel}(x^\mu)\psi_1(z), \tag{2.43}$$

where

$$\hat\alpha_{\mu\perp} = \frac{i}{2}(\xi_R D_\mu \xi_R^\dagger - \xi_L D_\mu \xi_L^\dagger) = \alpha_{\mu\perp}, \tag{2.44}$$

$$\hat{\alpha}_{\mu\parallel} = \frac{i}{2}(\xi_R D_\mu \xi_R^\dagger + \xi_L D_\mu \xi_L^\dagger) = -V_\mu + \alpha_{\mu\parallel}, \quad (2.45)$$

with

$$D_\mu \xi_R^\dagger = \partial_\mu \xi_R^\dagger - i\mathcal{R}_\mu \xi_R^\dagger + i\xi_R^\dagger V_\mu, \quad (2.46)$$

$$D_\mu \xi_L^\dagger = \partial_\mu \xi_L^\dagger - i\mathcal{L}_\mu \xi_L^\dagger + i\xi_L^\dagger V_\mu. \quad (2.47)$$

The resultant low-energy effective theory is given by putting Eq. (2.43) into Eq. (2.41) with $M = 0$ and $N = 1$ through Eq. (2.42).

III. APPLICATION TO THE SAKAI-SUGIMOTO MODEL

In this section, we apply our integrating-out method to the SS model based on D8/$\bar{\text{D}}$8/D4-brane configuration [3,4]. As a result of integrating out higher KK modes, other than the lowest one (ρ and its flavor partners), we give a complete list of the $\mathcal{O}(p^4)$ terms of HLS at the large N_c limit, which was partially reported in the previous work [13]. The anomaly-related (IP-odd) gauge-invariant terms introduced in Ref. [17] are also completely determined by integrating out higher KK modes in the CS term.

We shall first summarize the action of the SS model relevant to our discussion, following Refs. [3,4]. The model consists of two parts, the DBI part and the CS part.

The DBI part is given by

$$S_{\rm SS}^{\rm DBI} = N_c G \int d^4x dz \Big(-\frac{1}{2} K^{-1/3}(z)\, {\rm tr}[F_{\mu\nu}F^{\mu\nu}] + K(z) M_{\rm KK}^2\, {\rm tr}[F_{\mu z}F^{\mu z}] \Big), \quad (3.1)$$

where $K(z) = 1 + z^2$ is the induced metric of the five-dimensional space-time. The overall coupling G is the rescaled 't Hooft coupling expressed as $G = N_c g_{\rm YM}^2/(108\pi^3)$, with $g_{\rm YM}$ being the gauge coupling of the $U(N_c)$ gauge symmetry on the N_c D4-branes. The mass scale $M_{\rm KK}$ is related to the scale of the compactification of the N_c D4-branes onto the S^1. Comparing Eq. (3.1) with Eq. (2.2), we read off

$$K_1(z) = GK^{-1/3}(z), \qquad K_2(z) = GK(z). \quad (3.2)$$

Referring to Eq. (2.16), furthermore, we can easily see that Eq. (3.1) yields the eigenvalue equation

$$-K^{1/3}(z)\partial_z(K(z)\partial_z\psi_n) = \lambda_n \psi_n, \quad (3.3)$$

with the eigenvalues λ_n and the eigenfunctions ψ_n of the KK modes of the five-dimensional gauge field $A_\mu(x^\mu, z)$.

The CS action in the SS model is given by

$$S_{\rm SS}^{\rm CS}(A) = \frac{N_c}{24\pi^2}\int_{M^4\times R} w_5(A), \quad (3.4)$$

where M^4 and R represent the four-dimensional Minkowski space-time and the z-coordinate space, respectively. In terms of five-dimensional differential forms, the gauge field and the field strength are written as $A = A_M dx^M = A_\mu dx^\mu + A_z dz$, $F = dA - iA^2$, where we use the Hermitian gauge field instead of the anti-Hermitian one used in Refs. [3,4]. Then the CS five-form $w_5(A)$ is expressed in terms of these five-dimensional differential forms as

$$w_5(A) = {\rm tr}\Big[AF^2 + \frac{1}{2}iA^3F - \frac{1}{10}A^5\Big]. \quad (3.5)$$

It is crucial to notice that the CS action (3.4) is not gauge invariant under the five-dimensional gauge symmetry: Once the $A_z \equiv 0$ gauge is realized by the gauge transformation $A \to A^g = gAg^\dagger + igdg^\dagger$, the CS five-form $w_5(A)$ in Eq. (3.5) no longer takes the same form as in Eq. (3.5) but is modified as

$$w_5(A) = w_5(A^g) - \frac{1}{10}{\rm tr}[gdg^\dagger]^5 - d\alpha_4(idg^\dagger g, A), \quad (3.6)$$

where α_4 is the four-form function given by

$$\alpha_4(V, A) = -\tfrac{1}{2}{\rm tr}[V(iAdA + idAA + A^3) - \tfrac{1}{2}VAVA - V^3A], \quad (3.7)$$

and the modified CS five-form $w_5(A^g)$ becomes

$$w_5(A^g) = {\rm tr}[A^g dA^g dA^g - \tfrac{3}{2}i(A^g)^3 dA^g]. \quad (3.8)$$

Putting Eq. (3.6) along with Eq. (3.8) into the CS action (3.4), we have

$$\begin{aligned} S_{\rm SS}^{\rm CS} = {}& \frac{N_c}{24\pi^2}\int_{M^4}\{\alpha_4(idg^\dagger(+\infty)g(+\infty), \mathcal{R}) \\ & - \alpha_4(idg^\dagger(-\infty)g(-\infty), \mathcal{L})\} \\ & + \frac{N_c}{240\pi^2}\int_{M^4\times R}{\rm tr}[gdg^\dagger]^5 \\ & - \frac{N_c}{24\pi^2}\int_{M^4\times R}{\rm tr}\Big[A^g dA^g dA^g - \frac{3}{2}i(A^g)^3 dA^g\Big] \\ \equiv {}& \Gamma_1 + \Gamma_2 + \Gamma_3, \end{aligned} \quad (3.9)$$

where we have introduced external gauge fields $\mathcal{R}$ and $\mathcal{L}$ at the boundaries $z = \pm\infty$, and

$$g(\pm\infty) \equiv g(x^\mu, \pm\infty). \quad (3.10)$$

Γ_1 and Γ_2 in Eq. (3.9) exactly reproduce the covariantized Wess-Zumino-Witten (WZW) term [19,20], while Γ_3 is HLS gauge invariant, as shown in Appendix A, and provides the IP-odd interactions involving vector and axial-vector mesons.

A. Dirac-Born-Infeld part

In this subsection, we shall integrate out higher KK modes in the DBI part of the SS model given in Eq. (3.1):

$$S_{\rm DBI}^{\rm SS}(A) \to S_{\rm DBI\,integ}^{\rm SS}(A^{\rm integ}), \quad (3.11)$$

where A^{integ} is the integrated-out five-dimensional gauge field given in Eq. (2.43). The integrated-out action $S^{\text{SS}}_{\text{DBI integ}} = \int d^4x \mathcal{L}$ is expanded in terms of derivatives: The leading order terms counted as $\mathcal{O}(p^2)$ arise from the $(F^{\text{integ}}_{\mu z})^2$ term, together with the kinetic term of the HLS gauge field V_μ from the $(F^{\text{integ}}_{\mu\nu})^2$ term. On the other hand, the $\mathcal{O}(p^4)$ terms come from the remainder of the $(F^{\text{integ}}_{\mu\nu})^2$ term. The Lagrangian $\mathcal{L}$ thus takes the form of the HLS Lagrangian [10–12]:

$$\mathcal{L} = F_\pi^2 \,\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}^\mu_\perp] + aF_\pi^2\,\text{tr}[\hat{\alpha}_{\mu\parallel}\hat{\alpha}^\mu_\parallel] - \frac{1}{2g^2}\text{tr}[V_{\mu\nu}V^{\mu\nu}] + \mathcal{L}_{(4)}, \tag{3.12}$$

where a is a parameter and $V_{\mu\nu}$ is the field strength of the HLS gauge field defined as $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]$. $\mathcal{L}_{(4)}$ includes the $\mathcal{O}(p^4)$ terms [12,15] given by

$$\begin{aligned}\mathcal{L}_{(4)} = {}& y_1\,\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}^\mu_\perp\hat{\alpha}_{\nu\perp}\hat{\alpha}^\nu_\perp] + y_2\,\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{\nu\perp}\hat{\alpha}^\mu_\perp\hat{\alpha}^\nu_\perp] + y_3\,\text{tr}[\hat{\alpha}_{\mu\parallel}\hat{\alpha}^\mu_\parallel\hat{\alpha}_{\nu\parallel}\hat{\alpha}^\nu_\parallel] + y_4\,\text{tr}[\hat{\alpha}_{\mu\parallel}\hat{\alpha}_{\nu\parallel}\hat{\alpha}^\mu_\parallel\hat{\alpha}^\nu_\parallel]\\
&+ y_5\,\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}^\mu_\perp\hat{\alpha}_{\nu\parallel}\hat{\alpha}^\nu_\parallel] + y_6\,\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{\nu\perp}\hat{\alpha}^\mu_\parallel\hat{\alpha}^\nu_\parallel] + y_7\,\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{\nu\perp}\hat{\alpha}^\nu_\parallel\hat{\alpha}^\mu_\parallel] + y_8\{\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}^\mu_\parallel\hat{\alpha}_{\nu\perp}\hat{\alpha}^\nu_\parallel]\\
&+ \text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}^\nu_\parallel\hat{\alpha}_{\nu\perp}\hat{\alpha}^\mu_\parallel]\} + y_9\,\text{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}^\nu_\parallel\hat{\alpha}_{\mu\perp}\hat{\alpha}^\nu_\parallel] + \sum_{i=10}^{18} y_i\mathcal{L}_i + z_1\,\text{tr}[\hat{\mathcal{V}}_{\mu\nu}\hat{\mathcal{V}}^{\mu\nu}] + z_2\,\text{tr}[\hat{\mathcal{A}}_{\mu\nu}\hat{\mathcal{A}}^{\mu\nu}]\\
&+ z_3\,\text{tr}[\hat{\mathcal{V}}_{\mu\nu}V^{\mu\nu}] + iz_4\,\text{tr}[V_{\mu\nu}\hat{\alpha}^\mu_\perp\hat{\alpha}^\nu_\perp] + iz_5\,\text{tr}[V_{\mu\nu}\hat{\alpha}^\mu_\parallel\hat{\alpha}^\nu_\parallel] + iz_6\,\text{tr}[\hat{\mathcal{V}}_{\mu\nu}\hat{\alpha}^\mu_\perp\hat{\alpha}^\nu_\perp] + iz_7\,\text{tr}[\hat{\mathcal{V}}_{\mu\nu}\hat{\alpha}^\mu_\parallel\hat{\alpha}^\nu_\parallel]\\
&- iz_8\,\text{tr}[\hat{\mathcal{A}}_{\mu\nu}(\hat{\alpha}^\mu_\perp\hat{\alpha}^\nu_\parallel + \hat{\alpha}^\mu_\parallel\hat{\alpha}^\nu_\perp)],\end{aligned} \tag{3.13}$$

where the explicit form [12,15] of $\mathcal{L}_i$ ($i = 10$–18) is irrelevant to the discussions here, and

$$\begin{aligned}\hat{\mathcal{V}}_{\mu\nu} &= \tfrac{1}{2}(\xi_R\mathcal{R}_{\mu\nu}\xi_R^\dagger + \xi_L\mathcal{L}_{\mu\nu}\xi_L^\dagger),\\ \hat{\mathcal{A}}_{\mu\nu} &= \tfrac{1}{2}(\xi_R\mathcal{R}_{\mu\nu}\xi_R^\dagger - \xi_L\mathcal{L}_{\mu\nu}\xi_L^\dagger).\end{aligned} \tag{3.14}$$

In the SS model all the HLS parameters in $\mathcal{L}$ are calculated as[6] (for details, see Appendix B)

$$F_\pi^2 = N_c G M_{\text{KK}}^2 \int dz K(z)[\dot{\psi}_0(z)]^2, \tag{3.15}$$

$$aF_\pi^2 = N_c G M_{\text{KK}}^2 \lambda_1 \langle\psi_1^2\rangle \qquad (\lambda_1 \simeq 0.669), \tag{3.16}$$

$$\frac{1}{g^2} = N_c G\langle\psi_1^2\rangle, \tag{3.17}$$

$$y_1 = -y_2 = -N_c G\cdot\langle(1+\psi_1-\psi_0^2)^2\rangle, \tag{3.18}$$

$$y_3 = -y_4 = -N_c G\cdot\langle\psi_1^2(1+\psi_1)^2\rangle, \tag{3.19}$$

$$y_5 = 2y_8 = -y_9 = -2N_c G\cdot\langle\psi_1^2\psi_0^2\rangle, \tag{3.20}$$

$$y_6 = -y_5 - y_7, \tag{3.21}$$

$$y_7 = 2N_c G\cdot\langle\psi_1(1+\psi_1)(1+\psi_1-\psi_0^2)\rangle, \tag{3.22}$$

$$y_i = 0 \qquad (i = 10\text{–}18), \tag{3.23}$$

$$z_1 = -\tfrac{1}{2}N_c G\langle(1+\psi_1)^2\rangle, \tag{3.24}$$

$$z_2 = -\tfrac{1}{2}N_c G\langle\psi_0^2\rangle, \tag{3.25}$$

$$z_3 = N_c G\langle\psi_1(1+\psi_1)\rangle, \tag{3.26}$$

$$z_4 = 2N_c G\langle\psi_1(1+\psi_1-\psi_0^2)\rangle, \tag{3.27}$$

$$z_5 = -2N_c G\langle\psi_1^2(1+\psi_1)\rangle, \tag{3.28}$$

$$z_6 = -2N_c G\langle(1+\psi_1-\psi_0^2)(1+\psi_1)\rangle, \tag{3.29}$$

$$z_7 = 2N_c G\langle\psi_1(1+\psi_1)^2\rangle, \tag{3.30}$$

$$z_8 = -2N_c G\langle\psi_1\psi_0^2\rangle, \tag{3.31}$$

where $\lambda_1 \simeq 0.669$ is the eigenvalue obtained from Eq. (3.3) [3,4] and we defined

$$\langle A\rangle \equiv \int_{-\infty}^{\infty} dz K^{-1/3}(z)A(z) \tag{3.32}$$

for a function $A(z)$. In obtaining Eq. (3.16) we used the following identity:

$$\int dz K(z)\dot{\psi}_1^2(z) = \lambda_1\int dz K^{-1/3}(z)\psi_1^2(z). \tag{3.33}$$

Note that the result $y_i = 0$ ($i = 10$–18) reflects the fact that the SS model picks up only the large N_c limit, since $\mathcal{L}_i$ such as $\mathcal{L}_{10} = \text{tr}[\hat{\alpha}_{\perp\mu}\hat{\alpha}^\mu_\perp]\text{tr}[\hat{\alpha}_{\perp\nu}\hat{\alpha}^\nu_\perp]$ are of $1/N_c$-subleading order.

The 't Hooft coupling G and the mass scale M_{KK} are free parameters of the SS model to be fixed by physical inputs, e.g., experimental values of F_π and m_ρ. In the holographic

[6] In Ref. [13], the overall sign of z_4 and the expression of y_1 should be corrected. In Eqs. (3.18) and (3.27) these corrections have been made properly.

QCD the normalization of the eigenfunction ψ_1 is usually taken to be $N_c G\langle\psi_1^2\rangle = 1$, corresponding to the canonical normalization of the kinetic term of the HLS gauge field V_μ in Eq. (3.12). In that case, the corresponding HLS gauge coupling g is moved over to the expression $\hat{\alpha}_{\mu\parallel}$, etc., such as $\hat{\alpha}_{\mu\parallel} = -V_\mu + \alpha_{\mu\parallel} \to \hat{\alpha}_{\mu\parallel} = -gV_\mu + \alpha_{\mu\parallel}$. Thus the HLS gauge coupling g is not determined by the holography. However, as far as the tree-level computation including $\mathcal{O}(p^4)$ terms is concerned, it turns out that physical quantities are independent of g, and thus of the normalization of ψ_1.[7]

B. Chern-Simons part

We shall next turn to the CS part in the SS model. In this subsection we integrate out higher KK modes of the HLS gauge bosons in the CS part of the SS model to determine the anomaly-related IP-odd terms in the HLS model [10–12,17]. In Refs. [3,4] it was shown that Γ_1 and Γ_2 in Eq. (3.9) exactly reproduce the covariantized WZW term [19,20]. On the other hand, the HLS gauge-invariant portion Γ_3 produces the IP-odd interactions involving vector and axial-vector mesons. After integrating out higher KK modes in Γ_3, we obtain the four HLS gauge-invariant IP-odd terms introduced in Refs. [11,12,17][8]:

$$\Gamma^{\rm HLS}_{\rm IP\text{-}odd} = \frac{N_c}{16\pi^2}\int_{M^4}\{c_1 i\,{\rm tr}[\hat{\alpha}_L^3\hat{\alpha}_R - \hat{\alpha}_R^3\hat{\alpha}_L] + c_2 i\,{\rm tr}[\hat{\alpha}_L\hat{\alpha}_R\hat{\alpha}_L\hat{\alpha}_R] + c_3\,{\rm tr}[F_V(\hat{\alpha}_L\hat{\alpha}_R - \hat{\alpha}_R\hat{\alpha}_L)] + c_4\,{\rm tr}[\hat{F}_V(\hat{\alpha}_L\hat{\alpha}_R - \hat{\alpha}_R\hat{\alpha}_L)]\}, \tag{3.34}$$

where the normalization of c_1-c_4 terms followed Ref. [12], and

$$\hat{\alpha}_{R,L} = \hat{\alpha}_\parallel \pm \hat{\alpha}_\perp, \qquad F_V = dV - iV^2,$$
$$\hat{F}_V = \frac{\hat{F}_L + \hat{F}_R}{2}, \qquad \hat{F}_{L,R} = \xi^\dagger_{L,R}\cdot F_{L,R}\cdot \xi_{L,R}, \tag{3.35}$$
$$F_{L,R} = d\mathcal{L}(\mathcal{R}) - i\mathcal{L}^2(\mathcal{R}^2).$$

The coefficients of the IP-odd terms are determined as (details of the derivation are given in Appendix B)

$$c_1 = \langle\langle\dot{\psi}_0\psi_1(\tfrac{1}{2}\psi_0^2 + \tfrac{1}{6}\psi_1^2 - \tfrac{1}{2})\rangle\rangle, \tag{3.36}$$

$$c_2 = \langle\langle\dot{\psi}_0\psi_1(-\tfrac{1}{2}\psi_0^2 + \tfrac{1}{6}\psi_1^2 + \tfrac{1}{2}\psi_1 + \tfrac{1}{2})\rangle\rangle, \tag{3.37}$$

$$c_3 = \langle\langle\dot{\psi}_0\psi_1(\tfrac{1}{2}\psi_1)\rangle\rangle, \tag{3.38}$$

$$c_4 = \langle\langle\dot{\psi}_0\psi_1(-\tfrac{1}{2}\psi_1 - 1)\rangle\rangle, \tag{3.39}$$

where we have introduced, for a function $A(z)$,

$$\langle\langle A\rangle\rangle \equiv \int_{-\infty}^{\infty} dz A(z). \tag{3.40}$$

Thus, after integrating out higher KK modes, the CS action (3.9) is reduced to the covariantized WZW term $\Gamma^{\rm WZW}_{\rm cov}$ and the IP-odd HLS gauge-invariant terms:

$$S^{\rm SS}_{\rm CS} \to S^{\rm SS}_{\rm CS\ integ} = \Gamma^{\rm WZW}_{\rm cov}(U, \mathcal{R}, \mathcal{L}) + \Gamma^{\rm HLS}_{\rm IP\text{-}odd}(\hat{\alpha}_\perp, \hat{\alpha}_\parallel, V). \tag{3.41}$$

IV. APPLICATIONS

Given a concrete holographic model, our method presented in Sec. II enables us to deduce definite predictions of the model for any physical quantity to be compared with experimental data. In this section we demonstrate the power of our method in the case of the SS model [3,4]. Physical quantities are written in terms of the generic HLS model with $\mathcal{O}(p^4)$ terms, with the Lagrangian parameters being determined by the SS model. Since we have integrated out the higher KK modes of the HLS gauge bosons, keeping only the lowest one (the ρ meson and its flavor partners), the applicable momentum range should be restricted to $0 \le Q^2 \ll \{m^2_{\rho'}, m^2_{\rho''}, \cdots\}$. We compute the momentum dependence of several form factors in the low-energy region ($\lesssim 1$ GeV), including the pion EM form factor (Sec. IV A) and IP-odd form factors such as the π^0-γ and ω-π^0 transition form factors (Sec. IV B). In Sec. IV B we also calculate anomaly-related vertex functions such as the γ^*-π^0-π^+-π^- vertex function. Such results were not obtained in the original formulation of the SS model [4] due to the complication of handling the infinite sum. We further confirm that our method correctly incorporates contributions from higher KK modes of the HLS gauge bosons in a different way, starting with the original expressions of the form factors in the SS model written in terms of the infinite sum of KK modes. We perform a low-energy expansion of those form factors in a way consistent with our formalism, which integrates out higher KK modes into $\mathcal{O}(p^4)$ terms of the HLS Lagrangian. This reproduces the same results as those obtained from our integrating-out method.

Hereafter, we will take $N_f = 3$, in which case the vector-meson fields ($\rho_\mu^{\pm,0}, \ldots$) and the photon (A_μ) field are embedded as follows:

[7] It has been shown [12,21] that some of the $\mathcal{O}(p^4)$ terms can be absorbed into the redundancy of g through the redefinition of the HLS gauge field. This redundancy corresponds to the fact that physical quantities in the holography do not depend on the normalization of ψ_1. This redundancy is no longer true at loop level; i.e., physical quantities should depend on g, or on the normalization of ψ_1 [13].

[8] The same result follows in a different approach at tree level; see Ref. [22].

$$V_\mu = g\rho_\mu = \frac{g}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}(\rho^0_\mu + \omega_\mu) & \rho^+_\mu & K^{*+}_\mu \\ \rho^-_\mu & -\frac{1}{\sqrt{2}}(\rho^0_\mu - \omega_\mu) & K^{*0}_\mu \\ K^{*-}_\mu & \bar{K}^{*0}_\mu & \phi_\mu \end{pmatrix},$$

$$\mathcal{V}_\mu = eA_\mu \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{4.1}$$

A. Pion electromagnetic form factor

We shall first derive an expression of the pion EM form factor from the general HLS Lagrangian given in Eq. (3.12). Taking the unitary gauge of HLS, we expand $\hat{\alpha}_{\parallel\mu}$ and $\hat{\alpha}_{\perp\mu}$, defined in Eqs. (2.44) and (2.45), in terms of the pion fields π as [12]

$$\hat{\alpha}_{\parallel\mu} = \frac{1}{F_\pi}\partial_\mu \pi + \mathcal{A}_\mu - \frac{i}{F_\pi}[\mathcal{V}_\mu, \pi] + \cdots, \tag{4.2}$$

$$\hat{\alpha}_{\perp\mu} = -V_\mu + \mathcal{V}_\mu - \frac{i}{2F_\pi^2}[\partial_\mu\pi, \pi] + \cdots. \tag{4.3}$$

Substituting these expansion forms into the Lagrangian (3.12), we have

$$\begin{aligned}
\mathcal{L}_{\rho\rho} &= -\frac{1}{2}\,\mathrm{tr}[(\partial_\mu\rho_\nu - \partial_\nu\rho_\mu)^2] + ag^2F_\pi^2\,\mathrm{tr}[\rho_\mu\rho^\mu], \\
\mathcal{L}_{\gamma\pi\pi} &= 2ie\left(1-\frac{a}{2}\right)\mathrm{tr}[A_\mu[\partial^\mu\pi, \pi]] \\
&\quad + \frac{iez_6}{F_\pi^2}\,\mathrm{tr}[\partial_\mu A_\nu[\partial^\mu\pi, \partial^\nu\pi]], \\
\mathcal{L}_{\rho\pi\pi} &= iag\,\mathrm{tr}[\rho_\mu[\partial^\mu\pi, \pi]] + \frac{iz_4}{F_\pi^2}\,\mathrm{tr}[\partial_\mu\rho_\nu[\partial^\mu\pi, \partial^\nu\pi]], \\
\mathcal{L}_{\gamma\rho} &= -2eagF_\pi^2\,\mathrm{tr}[\rho_\mu A^\mu] \\
&\quad + 2eagz_3\,\mathrm{tr}[\partial_\mu A_\nu(\partial^\mu\rho^\nu - \partial^\nu\rho^\mu)].
\end{aligned} \tag{4.4}$$

From these, we read off the ρ mass m_ρ, the direct γ-π-π vertex $g_{\gamma\pi\pi}(Q^2)$, the ρ-π-π vertex $g_{\rho\pi\pi}(Q^2)$, and the ρ-γ mixing strength $g_\rho(Q^2)$:

$$m_\rho^2 = ag^2F_\pi^2, \tag{4.5}$$

$$g_{\gamma\pi\pi}(Q^2) = \left(1-\frac{a}{2}\right) + \frac{ag^2z_6}{4}\frac{Q^2}{m_\rho^2}, \tag{4.6}$$

$$g_\rho(Q^2) = \frac{m_\rho^2}{g}\left(1 + g^2z_3\frac{Q^2}{m_\rho^2}\right), \tag{4.7}$$

$$g_{\rho\pi\pi}(Q^2) = \frac{1}{2}ag\left(1 + \frac{g^2z_4}{2}\frac{Q^2}{m_\rho^2}\right). \tag{4.8}$$

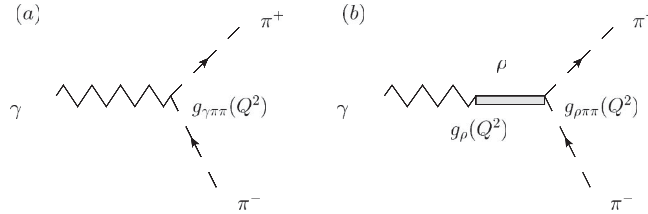

FIG. 1. Diagrams relevant to the pion EM form factor $F_V^{\pi^\pm}$ in the general HLS Lagrangian (3.12).

The on-shell g_ρ and $g_{\rho\pi\pi}$ couplings[9] are

$$g_\rho \equiv g_\rho(Q^2 = -m_\rho^2) = agF_\pi^2(1 - g^2z_3), \tag{4.9}$$

$$g_{\rho\pi\pi} \equiv g_{\rho\pi\pi}(Q^2 = -m_\rho^2) = \tfrac{1}{2}ag(1 - \tfrac{1}{2}g^2z_4), \tag{4.10}$$

where $Q^2 = -p^2$ is spacelike momentum squared.

The pion EM form factor $F_V^{\pi^\pm}$ is thus constructed from two contributions, as illustrated in Fig. 1: one from the ρ-mediated diagram [graph (b)], and the rest [graph (a)]. By using quantities in Eqs. (4.6), (4.7), and (4.8), $F_V^{\pi^\pm}(Q^2)$ can be written as

$$F_V^{\pi^\pm}(Q^2) = g_{\gamma\pi\pi}(Q^2) + \frac{g_\rho(Q^2)g_{\rho\pi\pi}(Q^2)}{m_\rho^2 + Q^2}. \tag{4.11}$$

We may rewrite this expression as

$$F_V^{\pi^\pm}(Q^2) = \left(1 - \frac{1}{2}\tilde{a}\right) + \tilde{z}\frac{Q^2}{m_\rho^2} + \frac{\tilde{a}}{2}\frac{m_\rho^2}{m_\rho^2 + Q^2}, \tag{4.12}$$

where

$$\tilde{a} = a\left(1 - \frac{g^2z_4}{2} - g^2z_3 + \frac{(g^2z_3)(g^2z_4)}{2}\right), \tag{4.13}$$

$$\tilde{z} = \tfrac{1}{4}a(g^2z_6 + (g^2z_3)(g^2z_4)). \tag{4.14}$$

Note that our form factor (4.12) automatically ensures the EM gauge invariance no matter what values $\tilde{a}$ and $\tilde{z}$ may take,

$$F_V^{\pi^\pm}(0) = \left(1 - \frac{\tilde{a}}{2}\right) + \frac{\tilde{a}}{2} = 1. \tag{4.15}$$

Here we note that the "ρ-meson dominance" is defined as

$$F_V^{\pi^\pm}(Q^2) = \frac{m_\rho^2}{m_\rho^2 + Q^2}, \tag{4.16}$$

which is equivalent to taking

$$\tilde{a} = 2, \qquad \tilde{z} = 0, \tag{4.17}$$

and is different from taking $g_{\gamma\pi\pi}(Q^2) = 0$ in Eq. (4.6) [the

[9] In Ref. [13] the plus sign in front of the g^2z_4 term in the expression of $g_{\rho\pi\pi}$ should be a minus sign as in Eq. (4.10).

absence of graph (a) of Fig. 1]. Were it not for the $\mathcal{O}(p^4)$ terms, the definition of the ρ-meson dominance would be the same as $g_{\gamma\pi\pi}(Q^2) = (1 - a/2) = 0$.

Let us now evaluate the form (4.12) in the SS model. By using Eqs. (3.17), (3.26), (3.27), and (3.29), with Eqs. (3.15) and (3.16), $\tilde{a}$ and $\tilde{z}$ are determined in the SS model as

$$\tilde{a}_{\rm SS} = \frac{\pi}{4}\lambda_1 \frac{\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle} \simeq 2.62, \tag{4.18}$$

$$\tilde{z}_{\rm SS} = \frac{\pi}{8}\lambda_1\left(\frac{\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle} - \langle 1-\psi_0^2\rangle\right) = \frac{\tilde{a}}{2} - \frac{\pi}{8}\lambda_1\langle 1-\psi_0^2\rangle \simeq 0.08, \tag{4.19}$$

where we have used $\lambda_1 \simeq 0.669$ in Eq. (3.16). Substituting the values in Eqs. (4.18) and (4.19) into Eq. (4.12), we evaluate the momentum dependence of $F_V^{\pi^\pm}$ as a definite prediction of the SS model for the pion EM form factor. See Fig. 2 (black solid curve). Here we used the experimental input of m_ρ, $m_\rho = 775$ MeV [23]. (We do not need the experimental input of F_π for this quantity.) The experimental data from Refs. [24–27] are also shown. The χ^2 fit results in good agreement with the data ($\chi^2/\text{d.o.f} = 147/53 = 2.8$).

For comparison, we have also shown the best fit curve (denoted by a red dotted line) resulting from fitting the parameters $(\tilde{a}, \tilde{z})$ in the general HLS model (4.12) to the experimental data, which yields the best fit values of $\tilde{a}$ and $\tilde{z}$, $\tilde{a}|_{\rm best} = 2.44$, $\tilde{z}|_{\rm best} = 0.08$ ($\chi^2/\text{d.o.f} = 81/51 = 1.6$). It is interesting to note that the best fit values of $\tilde{a}$ and $\tilde{z}$ are quite close to those in the predicted curve, which reflects the fact that the predicted curve fits the experimental data well. Comparison with the ρ-meson dominance with $\tilde{a} = 2$ and $\tilde{z} = 0$ in Eq. (4.12) is also shown by a blue dashed curve ($\chi^2/\text{d.o.f} = 226/53 = 4.3$).

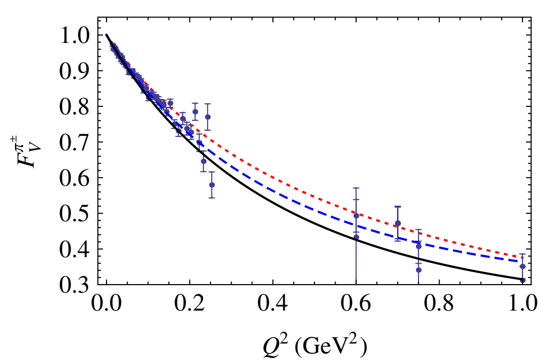

FIG. 2 (color online). The prediction (black solid curve) of the pion EM form factor $F_V^{\pi^\pm}$ fitting the experimental data [24–27] with $\chi^2/\text{d.o.f} = 147/53 = 2.8$. The red dotted curve corresponds to the form factor in the ρ-meson dominance hypothesis with $\tilde{a} = 2$ and $\tilde{z} = 0$ ($\chi^2/\text{d.o.f} = 226/53 = 4.3$). The blue dashed curve is the best fit to experimental data with $\tilde{a}|_{\rm best} = 2.44$, $\tilde{z}|_{\rm best} = 0.08$ ($\chi^2/\text{d.o.f} = 81/51 = 1.6$).

Given any holographic model, our method can make a definite prediction of the model for the pion EM form factor in terms of two parameters, $\tilde{a}$ and $\tilde{z}$, of the general HLS model which are determined by integrating out higher KK modes of the HLS gauge bosons. The best fit values in the above examples are the reference values to be compared with those of the holographic models. In comparison with the ρ-meson dominance, the deviation from $\tilde{a} = 2$ and $\tilde{z} = 0$ represents the contributions of higher KK modes in the generic holographic model, as will be shown below.

We shall next discuss an implication of our method from a different point of view. We start with the expression of the form factor originally studied in the SS model, which takes the form of the infinite sum of KK modes of the HLS gauge fields:

$$F_V^{\pi^\pm}(Q^2)|_{\rm SS} = \sum_{k=1}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2 + Q^2}. \tag{4.20}$$

Our method is to integrate out the higher KK mode effects into the $\mathcal{O}(p^4)$ terms of the HLS Lagrangian having only the ρ meson as a dynamical degree of freedom. To be consistent with our method, we expand this form factor as

$$\begin{aligned} F_V^{\pi^\pm}(Q^2)|_{\rm SS} &= \frac{g_\rho g_{\rho\pi\pi}}{m_\rho^2 + Q^2} + \sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2 + Q^2} \\ &= \frac{g_\rho g_{\rho\pi\pi}}{m_\rho^2 + Q^2} + \left(\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2}\right) \\ &\quad + \left(-\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2}\frac{Q^2}{m_{\rho_k}^2}\right), \end{aligned} \tag{4.21}$$

up to $\mathcal{O}(Q^4/m_{\rho_k}^4)$ $(k \geq 2)$. Note that the $\mathcal{O}(p^4)$ terms of the Lagrangian correspond to the $\mathcal{O}(Q^2)$ terms in the expansion of the pion EM form factor. Using the sum rules [4],

$$\sum_{k=1}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2} = 1, \tag{4.22}$$

$$\sum_{k=1}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^4} = \frac{\pi}{8m_\rho^2}\lambda_1\langle 1-\psi_0^2\rangle, \tag{4.23}$$

we have

$$\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2} = 1 - \frac{g_\rho g_{\rho\pi\pi}}{m_\rho^2}, \tag{4.24}$$

$$-\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^4} = \frac{g_\rho g_{\rho\pi\pi}}{m_\rho^4} - \frac{\pi}{8m_\rho^2}\lambda_1\langle 1-\psi_0^2\rangle. \tag{4.25}$$

From Ref. [4], we read off g_ρ and $g_{\rho\pi\pi}$ as well as m_ρ,[10]

[10]In Ref. [13] the expression corresponding to Eq. (4.28) has a typo.

$$m_\rho^2 = \lambda_1 M_{\rm KK}^2, \tag{4.26}$$

$$g_\rho = \sqrt{N_c G}\lambda_1 M_{\rm KK}^2 \sqrt{\frac{\langle\psi_1\rangle^2}{\langle\psi_1^2\rangle}}, \tag{4.27}$$

$$g_{\rho\pi\pi} = \frac{\pi}{8}\frac{\lambda_1}{\sqrt{N_c G}}\sqrt{\frac{\langle\psi_1(1-\psi_0^2)\rangle^2}{\langle\psi_1^2\rangle}}, \tag{4.28}$$

where we retained $\langle\psi_1^2\rangle$ to make explicit the ambiguity of the normalization of ψ_1 in contrast to Ref. [4] where the normalization of ψ_1 is fixed as $N_c G\langle\psi_1^2\rangle = 1$. Substituting these into the right-hand sides of Eqs. (4.24) and (4.25), we find

$$\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2} = 1 - \frac{\pi}{8}\lambda_1\frac{\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle},$$
$$-\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^4} = \frac{\pi}{8m_\rho^2}\lambda_1\left(\frac{\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle} - \langle 1-\psi_0^2\rangle\right). \tag{4.29}$$

Comparing Eqs. (4.18) and (4.19), we arrive at

$$\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2} = 1 - \frac{\tilde a_{\rm SS}}{2}, \qquad -\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^4} = \frac{\tilde z_{\rm SS}}{m_\rho^2}, \tag{4.30}$$

and hence at the same result as that obtained by our method of integrating out higher KK modes [Eq. (4.15) with Eqs. (4.18) and (4.19)]:

$$F_V^{\pi^\pm}(Q^2)|_{\rm SS} = \left(1 - \frac{1}{2}\tilde a_{\rm SS}\right) + \tilde z_{\rm SS}\frac{Q^2}{m_\rho^2} + \frac{\tilde a_{\rm SS}}{2}\frac{m_\rho^2}{m_\rho^2 + Q^2}. \tag{4.31}$$

The deviation from the ρ-meson dominance is parametrized as

$$\Delta\tilde a \equiv \left(\frac{\tilde a - 2}{2}\right), \tag{4.32}$$

$$\Delta\tilde z \equiv (\tilde z - 0). \tag{4.33}$$

From Eq. (4.30) and referring to Ref. [4], we may numerically read off these quantities in the SS model as

$$\Delta\tilde a_{\rm SS} = \frac{\tilde a_{\rm SS}}{2} - 1 \simeq 0.31 = -\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^2}$$
$$= (0.346)_{\rho'} + (-0.0505)_{\rho''} + (0.0128)_{\rho'''} + \cdots, \tag{4.34}$$

$$\Delta\tilde z_{\rm SS} = -m_\rho^2\sum_{k=2}^{\infty}\frac{g_{\rho_k}g_{\rho_k\pi\pi}}{m_{\rho_k}^4}$$
$$= (0.0806)_{\rho'} + (-0.0051)_{\rho''} + (0.0007)_{\rho'''} + \cdots. \tag{4.35}$$

This implies that the deviation from the ρ-meson dominance ($\tilde a \simeq 2.62$, $\tilde z \simeq 0.08$) comes dominantly from the ρ'-meson contribution.

Since the result is identical to that of our method, which is manifestly EM gauge invariant by construction [see Eqs. (2.36), (2.37), and (2.41)], the resultant form factor (4.31) should be EM gauge invariant. In fact, we have

$$F_V^{\pi^\pm}(0)|_{\rm SS} = \left(1 - \frac{\tilde a_{\rm SS}}{2}\right) + \frac{\tilde a_{\rm SS}}{2} = 1. \tag{4.36}$$

In contrast, a naive truncation corresponding to the Lagrangian in Eq. (2.33) would read

$$F_V^{\pi^\pm}(Q^2)|_{\rm SS}^{\rm trun} = \frac{g_\rho g_{\rho\pi\pi}}{m_\rho^2 + Q^2}, \tag{4.37}$$

which corresponds to ignoring the last two terms in Eq. (4.21) coming from higher KK modes. Since the Lagrangian (2.33) is not gauge invariant, so is the form factor above,

$$F_V^{\pi^\pm}(0)|_{\rm SS}^{\rm trun} = \frac{g_\rho g_{\rho\pi\pi}}{m_\rho^2} = \frac{\tilde a_{\rm SS}}{2} \simeq 1.31 \neq 1, \tag{4.38}$$

where we used Eq. (4.18) together with Eqs. (4.26), (4.27), and (4.28). Note that the truncation (4.37) is different from the ρ-meson dominance (4.16), which is gauge invariant.

B. Anomaly-related intrinsic-parity-odd processes

In this subsection we calculate the momentum dependence of the IP-odd form factors, and the π^0-γ and ω-π^0 transition form factors. We also study several IP-odd vertex functions such as π^0-γ^*-γ^* (Sec. IV B 1), ω-π^0-γ^* (Sec. IV B 2), γ^*-π^0-π^+-π^- (Sec. IV B 3), and $\omega \to \pi^0\pi^+\pi^-$ decay (Sec. IV B 4), and discuss the ρ/ω-meson dominance. In Appendix C we will discuss an alternative method as shown in Sec. IV A [see Eq. (4.21)] which leads to the same results as those obtained in this subsection.

The IP-odd interactions in the general HLS model are read off from Eq. (3.34), together with the WZW term, as [12]

$$\mathcal{L}_{VV\pi} = -\frac{g^2 N_c}{4\pi^2 F_\pi} c_3 \epsilon^{\mu\nu\lambda\sigma} \mathrm{tr}[\partial_\mu \rho_\nu \partial_\lambda \rho_\sigma \pi],$$
$$\mathcal{L}_{VA\pi} = -\frac{egN_c}{8\pi^2 F\pi}(c_4 - c_3)\epsilon^{\mu\nu\lambda\sigma} \mathrm{tr}[\{\partial_\mu \rho_\nu, \partial_\lambda A_\sigma\}\pi],$$
$$\mathcal{L}_{AA\pi} = -\frac{e^2 N_c}{4\pi^2 F\pi}(1 - c_4)\epsilon^{\mu\nu\lambda\sigma} \mathrm{tr}[\partial_\mu A_\nu \partial_\lambda A_\sigma \pi],$$
$$\mathcal{L}_{A\pi^3} = -i\frac{eN_c}{3\pi^2 F_\pi^3}\left(1 - \frac{3(c_1 - c_2 + c_3)}{4}\right)\epsilon^{\mu\nu\lambda\sigma} \times \mathrm{tr}[A_\mu \partial_\nu \pi \partial_\lambda \pi \partial_\sigma \pi],\tag{4.39}$$
$$\mathcal{L}_{V\pi^3} = -i\frac{gN_c}{4\pi^2 F_\pi^3}(c_1 - c_2 - c_3)\epsilon^{\mu\nu\lambda\sigma} \times \mathrm{tr}[\rho_\mu \partial_\nu \pi \partial_\lambda \pi \partial_\sigma \pi].$$

For relevant Feynman graphs for each IP-odd process, see Figs. 4, 5, 6, and 7 in Ref. [12].

1. The π^0-γ^*-γ^* vertex function and π^0-γ transition form factor

We start with an expression of the effective π^0-γ^*-γ^* vertex function from the general HLS Lagrangian [Eqs. (3.12) and (4.39)]:

$$\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)] = \frac{e^2 N_c}{12\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta}\Bigg[(1 - c_4) + \frac{(c_4 - c_3)}{4}\left\{D_\rho(q_1^2)\left(1 - g^2 z_3\frac{q_1^2}{m_\rho^2}\right) + D_\omega(q_1^2)\left(1 - g^2 z_3\frac{q_1^2}{m_\omega^2}\right) + (q_1^2 \to q_2^2)\right\} + \frac{c_3}{2}\left\{D_\rho(q_1^2)\left(1 - g^2 z_3\frac{q_1^2}{m_\rho^2}\right)D_\omega(q_2^2)\left(1 - g^2 z_3\frac{q_2^2}{m_\omega^2}\right) + (q_1^2 \leftrightarrow q_2^2)\right\}\Bigg],\tag{4.40}$$

where $q_{1,2}$ represent outgoing four-momenta of virtual photons γ^* and

$$D_{\rho,\omega}(p^2) \equiv \frac{m_{\rho,\omega}^2}{m_{\rho,\omega}^2 - p^2},\tag{4.41}$$

$$g_{\rho,\omega}(p^2) \equiv \frac{m_{\rho,\omega}^2}{g}\left(1 - g^2 z_3\frac{p^2}{m_{\rho,\omega}^2}\right).\tag{4.42}$$

We further rewrite Eq. (4.40) as follows:

$$\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)] = \frac{e^2 N_c}{12\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta}\Bigg[A^{\pi2\gamma} + \frac{B^{\pi2\gamma}}{4}\{D_\rho(q_1^2) + D_\omega(q_1^2) + (q_1^2 \to q_2^2)\} + \frac{C^{\pi2\gamma}}{2}\{D_\rho(q_1^2)\cdot D_\omega(q_2^2) + (q_1^2 \leftrightarrow q_2^2)\}\Bigg],\tag{4.43}$$

where we have used an identity for an arbitrary coefficient C,

$$D_{\rho,\omega}(p^2)\left(1 - C\cdot\frac{p^2}{m_{\rho,\omega}^2}\right) = (1 - C)D_{\rho,\omega}(p^2) + C,\tag{4.44}$$

and defined

$$A^{\pi2\gamma} = 1 - (1 - g^2 z_3)(c_4 + c_3\cdot g^2 z_3),\tag{4.45}$$

$$B^{\pi2\gamma} = (1 - g^2 z_3)[(c_4 + c_3\cdot g^2 z_3) - c_3(1 - g^2 z_3)],\tag{4.46}$$

$$C^{\pi2\gamma} = c_3(1 - g^2 z_3)^2.\tag{4.47}$$

Note that these parameters satisfy

$$A^{\pi2\gamma} + B^{\pi2\gamma} + C^{\pi2\gamma} = 1,\tag{4.48}$$

which reproduces the low-energy theorem:

$$\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)] \overset{q_1^2, q_2^2 \to 0}{\longrightarrow} \frac{e^2 N_c}{12\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta}.\tag{4.49}$$

The ρ/ω-meson dominance [28] for this process is defined in a way similar to Eq. (4.16) by taking $A^{\pi2\gamma} = B^{\pi2\gamma} = 0$ ($C^{\pi2\gamma} = 1$),

$$\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)] = \frac{e^2 N_c}{24\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta}(D_\rho(q_1^2)\cdot D_\omega(q_2^2) + (q_1^2 \leftrightarrow q_2^2)).\tag{4.50}$$

The π^0-γ transition form factor $F_{\pi^0\gamma}$ is obtained from Eq. (4.43) by setting the photon-momentum squared (q_1^2 or q_2^2) to zero:

$$F_{\pi^0\gamma}(Q^2) = (1-\tilde{c}) + \frac{\tilde{c}}{2}[D_\rho(Q^2) + D_\omega(Q^2)], \quad (4.51)$$

where $Q^2 = -q_1^2$ (or $-q_2^2$), and we defined

$$\tilde{c} \equiv 1 - (A^{\pi 2\gamma} + B^{\pi 2\gamma}). \quad (4.52)$$

The ρ/ω-meson dominance defined by Eq. (4.50) reads

$$F_{\pi^0\gamma}(Q^2) = \frac{1}{2}\left(\frac{m_\rho^2}{m_\rho^2 + Q^2} + \frac{m_\omega^2}{m_\omega^2 + Q^2}\right), \qquad \tilde{c} = 1. \quad (4.53)$$

We shall now evaluate the parameters in Eqs. (4.43) and (4.51) in the SS model. Using Eqs. (3.17), (3.26), (3.38), and (3.39), we calculate $A^{\pi 2\gamma}$, $B^{\pi 2\gamma}$, and $C^{\pi 2\gamma}$ to get

$$A_{\rm SS}^{\pi 2\gamma} = 1 - \left[\frac{\langle\psi_1\rangle\langle\langle\dot{\psi}_0\psi_1\rangle\rangle}{\langle\psi_1^2\rangle} - \frac{1}{2}\frac{\langle\psi_1\rangle^2\langle\langle\dot{\psi}_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^2}\right] \simeq 1 - (0.61) = 0.39, \quad (4.54)$$

$$B_{\rm SS}^{\pi 2\gamma} = \frac{\langle\psi_1\rangle\langle\langle\dot{\psi}_0\psi_1\rangle\rangle}{\langle\psi_1^2\rangle} - \frac{\langle\psi_1\rangle^2\langle\langle\dot{\psi}_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^2} \simeq -0.09, \quad (4.55)$$

$$C_{\rm SS}^{\pi 2\gamma} = \frac{1}{2}\left[\frac{\langle\psi_1\rangle^2\langle\langle\dot{\psi}_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^2}\right] \simeq 0.50. \quad (4.56)$$

By using these values and Eq. (4.52), the value of $\tilde{c}$ of the SS model is calculated as

$$\tilde{c}_{\rm SS} \simeq 1.31, \quad (4.57)$$

which implies that the form factor $F_{\pi^0\gamma}$ in the SS model violates (by about 30%) the ρ/ω-meson dominance. Putting the value in Eq. (4.57) into Eq. (4.51), we evaluate the momentum dependence of $F_{\pi^0\gamma}$ as a definite prediction of the SS model for the π^0-γ transition form factor. See Fig. 3 (black solid curve). Here we use the experimental inputs of m_ρ and m_ω, $m_\rho = 775$ MeV and $m_\omega = 783$ MeV [23]. The experimental data are from Ref. [29], which is the only experiment in the spacelike region.[11] Figure 3 shows that the momentum dependence of $F_{\pi^0\gamma}$ in the SS model disagrees with the experiment ($\chi^2/\text{d.o.f} = 63/5 = 13$).

For a comparison, in Fig. 3 we plot a curve, drawn by a blue dashed line, obtained by fitting the parameter $\tilde{c}$ of the general HLS model to the experimental data, which yields the best fit value of $\tilde{c}$, $\tilde{c}|_{\rm best} = 1.03$ ($\chi^2/\text{d.o.f} = 3/4 = 0.7$). This best fit value is very close to $\tilde{c} = 1$ of the ρ/ω-meson dominance (red dotted curve in the figure).

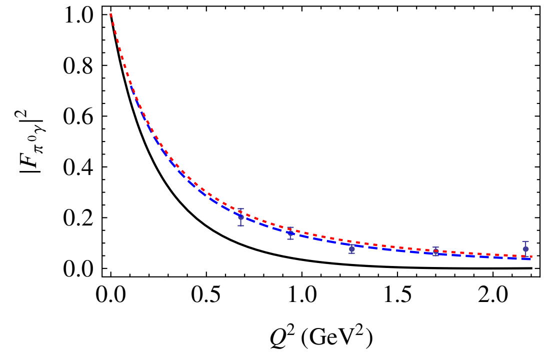

FIG. 3 (color online). The prediction (black solid curve) of the π^0-γ transition form factor $F_{\pi^0\gamma}$ with respect to the spacelike momentum squared Q^2. Comparison with the experimental data [29] yields $\chi^2/\text{d.o.f} = 63/5 = 13$. The blue dashed and red dotted curves, respectively, correspond to the best fit curve with $\tilde{c}_{\rm best} = 1.03$ ($\chi^2/\text{d.o.f} = 3.0/4 = 0.7$) and the ρ/ω-meson dominance with $\tilde{c} = 1$ ($\chi^2/\text{d.o.f} = 4.8/5 = 1.0$).

2. The ω-π^0-γ^* vertex function and ω-π^0 transition form factor

We begin with an expression of the effective ω-π^0-γ^* vertex function obtained from the general HLS Lagrangian [Eqs. (3.12) and (4.39)]:

$$\Gamma^{\mu\nu}[\omega_\mu(p), \pi^0, \gamma_\nu^*(k)] = \frac{eN_c}{8\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}p_\alpha k_\beta \cdot g\left[\left(\frac{c_4 - c_3}{2}\right) + c_3 \cdot D_\rho(k^2) \times \left(1 - g^2 z_3 \frac{k^2}{m_\rho^2}\right)\right], \quad (4.58)$$

where p and k, respectively, denote the incoming four-momentum of ω and the outgoing momentum of γ^*. Using the identity (4.44), we rewrite Eq. (4.58) into the following form:

$$\Gamma^{\mu\nu}[\omega_\mu(p), \pi^0, \gamma_\nu^*(k)] = \frac{eN_c}{8\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}p_\alpha k_\beta \cdot [A^{\omega\pi\gamma} + B^{\omega\pi\gamma}D_\rho(k^2)], \quad (4.59)$$

where

$$A^{\omega\pi\gamma} = \tfrac{1}{2}g[(c_4 + c_3 \cdot g^2 z_3) - c_3(1 - g^2 z_3)], \quad (4.60)$$

$$B^{\omega\pi\gamma} = g c_3(1 - g^2 z_3). \quad (4.61)$$

Note that $(A^{\omega\pi\gamma} + B^{\omega\pi\gamma})$ is not constrained, which is consistent with the fact that there is no low-energy theorem for Eq. (4.59) in the low-energy limit:

[11] The experiment of Ref. [29] yields the linear coefficient of $F_{\pi^0\gamma}$ consistent with the current average value [23], $a|_{\rm Ave} = 0.032 \pm 0.004$, within 1σ error.

$$\Gamma^{\mu\nu}[\omega_\mu(p), \pi^0, \gamma^*_\nu(k)] \xrightarrow{k^2\to 0} \frac{eN_c}{8\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}p_\alpha k_\beta \cdot [A^{\omega\pi\gamma} + B^{\omega\pi\gamma}], \tag{4.62}$$

in contrast to Eq. (4.48) for the π^0-γ^*-γ^* vertex function.

The ω-π^0 transition form factor $F_{\omega\pi^0}$ can be extracted from the $\omega \to \pi^0 l^+ l^-$ decay width $\Gamma(\omega \to \pi^0 l^+ l^-)$ ($l^\pm = e^\pm$, $\mu^\pm$), which is calculated through the effective ω-π^0-γ^* vertex function as

$$\Gamma(\omega \to \pi^0 l^+ l^-) = \int_{4m_l^2}^{(m_\omega - m_{\pi^0})^2} dq^2 \frac{\alpha}{3\pi}\frac{\Gamma(\omega\to\pi^0\gamma)}{q^2}\left(1+\frac{2m_l^2}{q^2}\right) \times \sqrt{\frac{q^2-4m_l^2}{q^2}}\left[\left(1+\frac{q^2}{m_\omega^2-m_{\pi^0}^2}\right)^2 - \frac{4m_\omega^2 q^2}{(m_\omega^2-m_{\pi^0}^2)^2}\right]^{3/2} \cdot |F_{\omega\pi^0}(q^2)|^2, \tag{4.63}$$

where $\Gamma(\omega \to \pi^0\gamma)$ denotes the $\omega \to \pi^0\gamma$ decay width,[12]

$$\Gamma(\omega\to\pi^0\gamma) = \frac{3\alpha}{64\pi^4 F_\pi^2} g^2_{\omega\pi\gamma}\left(\frac{m_\omega^2 - m_{\pi^0}^2}{2m_\omega}\right)^3, \tag{4.64}$$

with

$$g_{\omega\pi\gamma} = A^{\omega\pi\gamma} + B^{\omega\pi\gamma} = \frac{g(c_3+c_4)}{2}. \tag{4.65}$$

The transition form factor $F_{\omega\pi^0}$ is then expressed as

$$F_{\omega\pi^0}(q^2) = (1-\tilde{r}) + \tilde{r}D_\rho(q^2), \tag{4.66}$$

where

$$\tilde{r} = \frac{B^{\omega\pi\gamma}}{A^{\omega\pi\gamma}+B^{\omega\pi\gamma}}. \tag{4.67}$$

The ρ-meson dominance [28] for this transition form factor is defined in a way similar to Eq. (4.16) by taking $\tilde{r} = 1$ as

$$F_{\omega\pi^0}(q^2) = \frac{m_\rho^2}{m_\rho^2 - q^2}, \qquad \tilde{r} = 1. \tag{4.68}$$

Let us now evaluate the parameter $\tilde{r}$ in Eq. (4.66) in the SS model. Using Eqs. (3.17), (3.26), (3.38), and (3.39), we have

[12]The SS model predicts [4] $g_{\omega\pi\gamma} = g_{\rho\pi\pi}$. This leads to $\frac{\Gamma(\omega\to\pi^0\gamma)}{\Gamma(\rho^0\to\pi^+\pi^-)}|_{\rm SS} = (5.37 \pm 0.03)\times 10^{-3}$, which is compared with the experimental value [23] $\frac{\Gamma(\omega\to\pi^0\gamma)}{\Gamma(\rho^0\to\pi^+\pi^-)}|_{\rm exp} = (5.07\pm 0.12)\times 10^{-3}$, although values for each decay width deviate (by about 40%) from the experimental values.

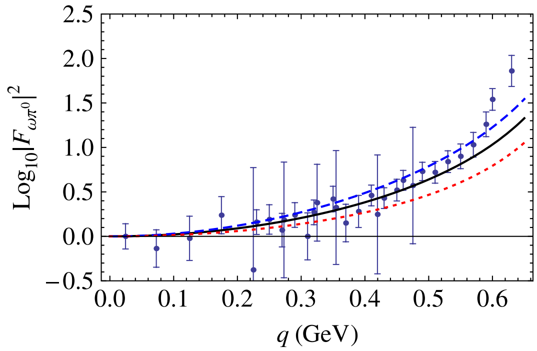

FIG. 4 (color online). The prediction (black solid curve) of the ω-π^0 transition form factor $F_{\omega\pi^0}(q^2)$ with respect to the timelike momentum q. Comparison with the experimental data [30,31] yields χ^2/d.o.f $= 45/31 = 1.5$. The best fit curve with $\tilde{r}_{\rm best} = 2.08$ (χ^2/d.o.f $= 24/30 = 0.8$) and the curve corresponding to the ρ-meson dominance with $\tilde{r} = 1$ (χ^2/d.o.f $= 124/31 = 4.0$) are drawn by blue dashed and red dotted lines, respectively.

$$A_{\rm SS}^{\omega\pi\gamma} = -\frac{1}{2\sqrt{N_c G}}\left[\frac{\langle\langle\dot{\psi}_0\psi_1\rangle\rangle}{\sqrt{\langle\psi_1^2\rangle}} - \frac{\langle\psi_1\rangle\langle\langle\dot{\psi}_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^{3/2}}\right], \tag{4.69}$$

$$B_{\rm SS}^{\omega\pi\gamma} = -\frac{1}{2\sqrt{N_c G}}\left[\frac{\langle\psi_1\rangle\langle\langle\dot{\psi}_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^{3/2}}\right], \tag{4.70}$$

and calculate $\tilde{r}$ to get

$$\tilde{r}_{\rm SS} \simeq 1.53, \tag{4.71}$$

which implies that the form factor $F_{\omega\pi^0}$ in the SS model violates (by about 50%) the ρ-meson dominance with $\tilde{r} = 1$. The predicted curve in the timelike momentum region is shown in Fig. 4 as a black solid line, together with the experimental data [30,31]. Figure 4 shows that the momentum dependence of $F_{\omega\pi^0}$ in the SS model is consistent with the experimental data (χ^2/d.o.f $= 45/31 = 1.5$).

The predicted curve is compared with a blue dashed curve obtained by fitting the parameter $\tilde{r}$ of the general HLS model to the experimental data, which gives the best fit value of $\tilde{r}$, $\tilde{r}_{\rm best} = 2.08$ (χ^2/d.o.f $= 24/30 = 0.8$). Comparison with the ρ-meson dominance with $\tilde{r} = 1$ (red dotted curve) is also given (χ^2/d.o.f $= 124/31 = 4.0$).

3. The γ^*-π^0-π^+-π^- vertex function

We start with an expression of the effective γ^*-π^0-π^+-π^- vertex function obtained from the general HLS Lagrangian [Eqs. (3.12) and (4.39)]:

$$\begin{aligned}
&\Gamma_\mu[\gamma^*_\mu(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)] \\
&= -\frac{eN_c}{12\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta \Big[1 - \frac{3}{4}(c_1 - c_2 + c_4) + \frac{3}{4}(c_1 - c_2 - c_3) D_\omega(p^2)\Big(1 - g^2 z_3 \frac{p^2}{m_\omega^2}\Big) \\
&\quad + \Big\{\frac{c_4 - c_3}{4} + \frac{c_3}{2} D_\omega(p^2)\Big(1 - g^2 z_3 \frac{p^2}{m_\omega^2}\Big)\Big\} \times \Big\{D_\rho((q_+ + q_-)^2)\Big(1 - \frac{1}{2} g^2 z_4 \frac{(q_+ + q_-)^2}{m_\rho^2}\Big) \\
&\quad + D_\rho((q_- + q_0)^2)\Big(1 - \frac{1}{2} g^2 z_4 \frac{(q_- + q_0)^2}{m_\rho^2}\Big) + D_\rho((q_0 + q_+)^2)\Big(1 - \frac{1}{2} g^2 z_4 \frac{(q_0 + q_+)^2}{m_\rho^2}\Big)\Big\}\Big], \qquad (4.72)
\end{aligned}$$

where p and $q_{\pm,0}$, respectively, stand for the incoming four-momentum of γ^* and the outgoing four-momenta of $\pi^\pm$ and π^0. By using the identity (4.44), the expression (4.72) may be rewritten as

$$\begin{aligned}
&\Gamma_\mu[\gamma^*_\mu(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)] \\
&= -\frac{eN_c}{12\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta \Big[A^{\gamma 3\pi} + B^{\gamma 3\pi} \cdot D_\omega(p^2) + \frac{C^{\gamma 3\pi}}{3} \cdot \{D_\rho((q_+ + q_-)^2) + D_\rho((q_- + q_0)^2) \\
&\quad + D_\rho((q_0 + q_+)^2)\} + \frac{D^{\gamma 3\pi}}{3} \cdot D_\omega(p^2) \cdot \{D_\rho((q_+ + q_-)^2) + D_\rho((q_- + q_0)^2) + D_\rho((q_0 + q_+)^2)\}\Big], \qquad (4.73)
\end{aligned}$$

where

$$\begin{aligned}
A^{\gamma 3\pi} = 1 - \tfrac{3}{4}[&(1 - g^2 z_3)(c_1 - c_2 + c_3) \\
&- 2c_3(1 - g^2 z_3)(1 - \tfrac{1}{2} g^2 z_4) \\
&+ (1 - \tfrac{1}{2} g^2 z_4)(c_3 + c_4)], \qquad (4.74)
\end{aligned}$$

$$\begin{aligned}
B^{\gamma 3\pi} = \tfrac{3}{4}[&(1 - g^2 z_3)(c_1 - c_2 + c_3) \\
&- 2c_3(1 - g^2 z_3)(1 - \tfrac{1}{2} g^2 z_4)], \qquad (4.75)
\end{aligned}$$

$$\begin{aligned}
C^{\gamma 3\pi} = \tfrac{3}{4}[&-2c_3(1 - g^2 z_3)(1 - \tfrac{1}{2} g^2 z_4) \\
&+ (1 - \tfrac{1}{2} g^2 z_4)(c_3 + c_4)], \qquad (4.76)
\end{aligned}$$

$$D^{\gamma 3\pi} = \tfrac{3}{4}[2c_3(1 - g^2 z_3)(1 - \tfrac{1}{2} g^2 z_4)]. \qquad (4.77)$$

Note that these parameters satisfy

$$A^{\gamma 3\pi} + B^{\gamma 3\pi} + C^{\gamma 3\pi} + D^{\gamma 3\pi} = 1, \qquad (4.78)$$

which reproduces the low-energy theorem:

$$\begin{aligned}
&\Gamma_\mu[\gamma^*_\mu(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)] \\
&\overset{q^2_{\pm,0}\to 0}{\longrightarrow} -\frac{eN_c}{12\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta. \qquad (4.79)
\end{aligned}$$

The ρ/ω-meson dominance for this process is defined in a way similar to Eq. (4.16) by taking $A^{\gamma 3\pi} = B^{\gamma 3\pi} = C^{\gamma 3\pi} = 0$ ($D^{\gamma 3\pi} = 1$) as

$$\begin{aligned}
&\Gamma_\mu[\gamma^*_\mu(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)] \\
&= -\frac{eN_c}{36\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta \frac{m_\omega^2}{m_\omega^2 - p^2} \\
&\quad \times \Big(\frac{m_\rho^2}{m_\rho^2 - (q_+ + q_-)^2} + (q_+ \leftrightarrow q_0) + (q_- \leftrightarrow q_0)\Big). \qquad (4.80)
\end{aligned}$$

We shall now evaluate the parameters in Eq. (4.73) in the SS model. Using Eqs. (3.17), (3.26), (3.27), (3.36), (3.37), (3.38), and (3.39), we have

$$\begin{aligned}
A^{\gamma 3\pi}_{\rm SS} = 1 - \frac{3}{4}\Big[&\frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1(1-\psi_0^2)\rangle\rangle}{\langle\psi_1^2\rangle} \\
&+ \frac{\langle\langle\dot\psi_0\psi_1\rangle\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle} \\
&- \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\Big], \qquad (4.81)
\end{aligned}$$

$$\begin{aligned}
B^{\gamma 3\pi}_{\rm SS} = \frac{3}{4}\Big[&\frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1(1-\psi_0^2)\rangle\rangle}{\langle\psi_1^2\rangle} \\
&- \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\Big], \qquad (4.82)
\end{aligned}$$

$$\begin{aligned}
C^{\gamma 3\pi}_{\rm SS} = \frac{3}{4}\Big[&\frac{\langle\langle\dot\psi_0\psi_1\rangle\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle} \\
&- \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\Big], \qquad (4.83)
\end{aligned}$$

$$D^{\gamma 3\pi}_{\rm SS} = \frac{3}{4}\Big[\frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\Big]. \qquad (4.84)$$

They are calculated in the SS model as

$$A_{\rm SS}^{\gamma 3\pi} \simeq 1 - (1.03) = -0.03, \qquad B_{\rm SS}^{\gamma 3\pi} \simeq 0.04, \qquad C_{\rm SS}^{\gamma 3\pi} \simeq -0.51, \qquad D_{\rm SS}^{\gamma 3\pi} \simeq 1.50, \tag{4.85}$$

in which $D_{\rm SS}^{\gamma 3\pi} \neq 1$ implies that the effective γ^*-π^0-π^+-π^- vertex function in the SS model violates (by about 50%) the ρ/ω-meson dominance.

4. $\omega \to \pi^0\pi^+\pi^-$ decay

The $\omega \to \pi^0\pi^+\pi^-$ decay width is given by

$$\Gamma(\omega \to \pi^0\pi^+\pi^-) = \frac{m_\omega}{192\pi^3}\iint dE_+ dE_- (|\vec{q}_+|^2|\vec{q}_-|^2 - (\vec{q}_+\cdot\vec{q}_-)^2)|F_{\omega\to3\pi}|^2, \tag{4.86}$$

where $E_\pm$ and $\vec{q}_\pm$ are, respectively, the energies and three-momenta of $\pi^\pm$ in the rest frame of ω. We construct the $\omega \to 3\pi$ form factor $F_{\omega\to3\pi}$ from the general HLS Lagrangian [Eqs. (3.12) and (4.39)]:

$$F_{\omega\to3\pi} = -\frac{N_c}{4\pi^2 F_\pi m_\rho^2}\bigg[\frac{3ag}{4}g^2(c_1 - c_2 - c_3) + \frac{ag^3}{2}c_3\bigg\{\bigg(1 - \frac{g^2 z_4}{2}\frac{(q_+ + q_-)^2}{m_\rho^2}\bigg) \cdot D_\rho((q_+ + q_-)^2) + (q_+ \leftrightarrow q_0) + (q_- \leftrightarrow q_0)\bigg\}\bigg], \tag{4.87}$$

where $q_{0,\pm}$ are the four-momenta of $\pi^{0,\pm}$, respectively. Using the identity in Eq. (4.44) we may rewrite $F_{\omega\to3\pi}$ into the following form:

$$F_{\omega\to3\pi} = -\frac{N_c}{4\pi^2 F_\pi m_\rho^2}g_{\rho\pi\pi} \times \bigg[A^{\omega3\pi} + \frac{B^{\omega3\pi}}{3}\{D_\rho((q_+ + q_-)^2) + (q_+ \leftrightarrow q_0) + (q_- \leftrightarrow q_0)\}\bigg], \tag{4.88}$$

where

$$A^{\omega3\pi} = \frac{3}{2}\frac{g^2(c_1 - c_2 + c_3)}{1 - g^2 z_4/2} - 3g^2 c_3, \tag{4.89}$$

$$B^{\omega3\pi} = 3g^2 c_3. \tag{4.90}$$

Note that $(A^{\omega3\pi} + B^{\omega3\pi})$ is not constrained because there is no low-energy theorem for Eq. (4.88) in the low-energy limit:

$$F_{\omega\to3\pi} \overset{q_{\pm,0}^2\to0}{\longrightarrow} -\frac{N_c}{4\pi^2 F_\pi m_\rho^2}g_{\rho\pi\pi}(A^{\omega3\pi} + B^{\omega3\pi}), \tag{4.91}$$

in contrast to Eqs. (4.48) and (4.79) for the π^0-γ^*-γ^* and γ^*-π^0-π^+-π^- vertex functions, respectively. The ρ-meson dominance [28] for this process is defined in a way similar to Eq. (4.16) by taking $A^{\omega3\pi} = 0$ in the form factor $F_{\omega\to3\pi}$ as

$$F_{\omega\to3\pi} = -\frac{N_c}{12\pi^2 F_\pi m_\rho^2}g_{\rho\pi\pi}\cdot B^{\omega3\pi} \cdot\bigg[\frac{m_\rho^2}{m_\rho^2 - (q_+ + q_-)^2} + (q_+ \leftrightarrow q_0) + (q_- \leftrightarrow q_0)\bigg]. \tag{4.92}$$

We shall evaluate the parameters in Eq. (4.88) in the SS model. Using Eqs. (3.17), (3.27), (3.36), (3.37), and (3.38), we have

$$A_{\rm SS}^{\omega3\pi} = \frac{3}{2N_c G}\bigg(\frac{\langle\langle\psi_1\dot\psi_0(1-\psi_0^2)\rangle\rangle}{\langle\psi_1(1-\psi_0^2)\rangle} - \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle}\bigg), \tag{4.93}$$

$$B_{\rm SS}^{\omega3\pi} = \frac{3}{2N_c G}\frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle}, \tag{4.94}$$

which are calculated in the SS model as

$$A_{\rm SS}^{\omega3\pi} \simeq 2.31, \qquad B_{\rm SS}^{\omega3\pi} \simeq 81.01, \tag{4.95}$$

where we used the experimental inputs of m_ρ and F_π, $m_\rho = 775$ MeV and $F_\pi = 92.4$ MeV [23], to determine the value of $N_c G$ ($N_c G \simeq 0.01$). An amount of deviation from the ρ-meson dominance is estimated independently of the value of $N_c G$ as

$$\frac{B_{\rm SS}^{\omega3\pi}}{A_{\rm SS}^{\omega3\pi} + B_{\rm SS}^{\omega3\pi}} \simeq 0.97, \tag{4.96}$$

which implies that the form factor $F_{\omega\to3\pi}$ in the SS model is well approximated by the ρ-meson dominance.

Let us now calculate the decay width $\Gamma(\omega \to \pi^0\pi^+\pi^-)$ in the SS model. To do this, we first estimate the value of $g_{\rho\pi\pi}$ which appears in Eq. (4.88) as the overall coefficient. Using Eq. (C35) and the experimental values of m_ρ and F_π, we get $g_{\rho\pi\pi}|_{\rm SS} \simeq 4.84$. Second, we evaluate the phase space integral using experimental inputs for $m_{\pi^{\pm,0}}$ and m_ω, $m_{\pi^\pm} = 140$ MeV, $m_{\pi^0} = 135$ MeV, and $m_\omega = 783$ MeV [23]. Thus we obtain

$$\Gamma(\omega \to \pi^0\pi^+\pi^-)|_{\rm SS} \simeq 2.78 \text{ MeV}. \tag{4.97}$$

This is the first full result obtained by our method which includes effects from an *infinite* tower of higher KK modes.

The result is compared with the experimental value [23] $\Gamma(\omega \to \pi^0\pi^+\pi^-)|_{\rm exp} = 7.57 \pm 0.09$ MeV.

It is interesting to note that the estimate in Eq. (4.97) is different by about 7% from the value obtained in Ref. [4], where higher KK modes of the HLS gauge bosons are truncated at the level of $n = 4$:

$$\Gamma(\omega \to \pi^0\pi^+\pi^-)|_{\rm SS}^{{\rm trun}(n\leq 4)} \simeq 2.58 \text{ MeV}. \tag{4.98}$$

In order to study this difference, let us discuss the original form of the form factor [4]:

$$F_{\omega\to 3\pi}|_{\rm SS} = -\frac{N_c}{4\pi^2 F_\pi}\sum_{k=1}^{\infty}\left[\frac{g_{\omega\rho_k\pi}g_{\rho_k\pi\pi}}{m_{\rho_k}^2 - (q_+ + q_-)^2} + (q_+ \to q_0) + (q_- \to q_0)\right]. \tag{4.99}$$

To be consistent with our method, which integrates out higher KK modes into $\mathcal{O}(p^4)$ terms of the general HLS Lagrangian, we expand Eq. (4.99) as

$$F_{\omega\to 3\pi}|_{\rm SS} = -\frac{N_c}{4\pi^2 F_\pi m_\rho^2} g_{\rho\pi\pi}\Bigg[\left(3\frac{m_\rho^2}{g_{\rho\pi\pi}}\sum_{k=2}^{\infty}\frac{g_{\omega\rho_k\pi}g_{\rho_k\pi\pi}}{m_{\rho_k}^2}\right) + \frac{1}{3}(3g_{\omega\rho\pi})\{D_\rho((q_+ + q_-)^2) + (q_+ \to q_0) + (q_- \to q_0)\}\Bigg], \tag{4.100}$$

up to $\mathcal{O}(q_{\pm,0}^2/m_{\rho_k}^2)$ $(k \geq 2)$ which corresponds to terms higher than $\mathcal{O}(p^4)$ in the Lagrangian. The coefficient of the second term, $g_{\omega\rho\pi}$, is read off from Ref. [4] as

$$g_{\omega\rho\pi} = \frac{1}{2N_c G}\frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle}. \tag{4.101}$$

Note that $3g_{\omega\rho\pi}$ is exactly the same as $B_{\rm SS}^{\omega 3\pi}$ in Eq. (4.94). We may therefore write

$$F_{\omega\to 3\pi}|_{\rm SS} = -\frac{N_c}{4\pi^2 F_\pi m_\rho^2} g_{\rho\pi\pi}\Bigg[\left(3\frac{m_\rho^2}{g_{\rho\pi\pi}}\sum_{k=2}^{\infty}\frac{g_{\omega\rho_k\pi}g_{\rho_k\pi\pi}}{m_{\rho_k}^2}\right) + \frac{B_{\rm SS}^{\omega 3\pi}}{3}\{D_\rho((q_+ + q_-)^2) + (q_+ \leftrightarrow q_0) + (q_- \leftrightarrow q_0)\}\Bigg]. \tag{4.102}$$

Identifying the first term of Eq. (4.102) with $A_{\rm SS}^{\omega 3\pi}$ in Eq. (4.93),

$$3\frac{m_\rho^2}{g_{\rho\pi\pi}}\sum_{k=2}^{\infty}\frac{g_{\omega\rho_k\pi}g_{\rho_k\pi\pi}}{m_{\rho_k}^2} = \frac{3}{2N_c G}\left(\frac{\langle\langle\psi_1\dot\psi_0(1-\psi_0^2)\rangle\rangle}{\langle\psi_1(1-\psi_0^2)\rangle} - \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle}\right) = \frac{3}{2N_c G}\left(\frac{\langle\langle\psi_1\dot\psi_0(1-\psi_0^2)\rangle\rangle}{\langle\psi_1(1-\psi_0^2)\rangle}\right) - 3g_{\omega\rho\pi}, \tag{4.103}$$

and using the expression of $g_{\rho\pi\pi}$ in Eq. (4.28) and that of $g_{\omega\rho\pi}$ in Eq. (4.101), we may read off

$$\sum_{k=1}^{\infty}\frac{g_{\omega\rho_k\pi}g_{\rho_k\pi\pi}}{m_{\rho_k}^2} = \frac{1}{2m_\rho^2(N_c G)^{3/2}} \times \frac{\langle\langle\psi_0\dot\psi_1\rangle\rangle\langle\langle\psi_1\dot\psi_0(1-\psi_0^2)\rangle\rangle}{\langle\psi_1\rangle^{1/2}\langle\psi_1(1-\psi_0^2)\rangle}. \tag{4.104}$$

This is a new sum rule which was not obtained in Ref. [4]. This sum rule shows that the form factor (4.102) includes effects of the *full set* of the infinite tower of the vector mesons. In contrast, in Ref. [4] some parts of the contributions are examined by naively truncating the infinite tower as in Eq. (2.33).

V. SUMMARY AND DISCUSSION

In this paper, we developed our method of integrating out higher KK modes of the HLS gauge bosons identified as vector and axial-vector mesons in a class of HQCD models including the SS model. Our method is to integrate out higher KK modes through their equations of motion for the HLS gauge bosons $A_\mu^{(m)}$ and $V_\mu^{(n)}$ in Eq. (2.34): $A_\mu^{(m)} = 0$ and $V_\mu^{(n)} = \alpha_{\mu\parallel}$. Thus the higher vector mesons are replaced by $\alpha_{\mu\parallel} = \frac{i}{2F_\pi^2}[\pi, \partial_\mu\pi] + \cdots$ ("pion cloud") which generates the $\mathcal{O}(p^4)$ terms as well as the $\mathcal{O}(p^2)$ terms of the HLS Lagrangian. Since $\alpha_{\mu\parallel}$ keeps the same HLS transformation property as that of the fields of the integrated-out KK modes, our method is manifestly invariant under HLS and chiral symmetry, including external gauge symmetry. On the contrary, a naive truncation corresponds to simply putting fields of higher KK modes to zero, which does not reproduce the correct transformation property as shown in Eq. (2.31); hence, this violates HLS and external gauge symmetry.

Given a concrete HQCD not restricted to the SS model, our method enables us to deduce definite predictions for any physical quantity, which can always be written in terms of the parameters of the general HLS model, and thus can be compared with experimental data once those parameters are determined from the HQCD.

To show the power of our method, we took the SS model as an example. The SS model is thought to be valid only below the $M_{\rm KK}$ scale, so higher KK (mass-eigenstate) fields should not contribute in the low-energy physics. In our integrating-out method this was reflected by setting higher mass-eigenstate fields $B_\mu^{(n)} = 0$ through the equations of motion: In terms of the HLS basis $A_\mu^{(m)}$ and $V_\mu^{(n)}$ are no longer independent degrees of freedom but simply generate $\mathcal{O}(p^4)$ terms and modify $\mathcal{O}(p^2)$ terms as well. We presented a full set of the $\mathcal{O}(p^4)$ terms of the HLS Lagrangian computed from the DBI part and the CS part at the leading order of the $1/N_c$ expansion. Once the parameters of the HLS model are determined by the SS model, we can compute the form factors which are always given in the general framework of the HLS model. The EM gauge invariance and the chiral invariance are automatically maintained since our method is manifestly invariant under external gauge symmetry as well as HLS. The result of the pion EM form factor was compared with the experimental data, together with the best fit within the general HLS model and the result of the ρ-meson dominance (see Fig. 2). It turned out that the SS model agrees with the experiment.

In the same fashion, we evaluated the π^0-γ (Fig. 3) and ω-π^0 (Fig. 4) transition form factors, which were compared with experimental data, together with the best fit within the general HLS model and the result of the ρ/ω-meson dominance. It turned out that in the SS model the π^0-γ transition form factor disagrees with the experimental data, while the ω-π^0 transition form factor is consistent with the data. We also presented the results for the related quantities such as γ^*-π^0-π^+-π^- and ω-π^0-π^+-π^- vertex functions.

We further derived the same form factors by a different method, dealing with the infinite sum explicitly without using the general HLS Lagrangian. This confirms that our formulation correctly includes contributions from an infinite set of higher KK modes and that the infinite sum is crucial for the gauge invariance. Actually, the EM gauge symmetry and chiral symmetry (low-energy theorem) in the form factors are obviously violated by a naive truncation, simply neglecting higher KK modes instead of taking the infinite sum.

Our method was used to deduce predictions of the SS model which were not available before. Below we summarize the SS model prediction compared with the experiment:

(I) The pion EM form factor (Fig. 2) agrees with the experiment ($\chi^2/\text{d.o.f} = 147/53 = 2.8$), compared with the best fit of the general HLS model ($\chi^2/\text{d.o.f} = 81/51 = 1.6$) and the ρ-meson dominance ($\chi^2/\text{d.o.f} = 226/53 = 4.3$).

(II) The π^0-γ transition form factor (Fig. 3) disagrees with the experiment ($\chi^2/\text{d.o.f} = 63/5 = 13$), compared with the best fit of the general HLS model ($\chi^2/\text{d.o.f} = 3/4 = 0.7$) and the ρ/ω-meson dominance ($\chi^2/\text{d.o.f} = 4.8/5 = 1.0$).

(III) The ω-π^0 transition form factor (Fig. 4) is consistent with the experiment ($\chi^2/\text{d.o.f} = 45/31 = 1.5$), compared with the best fit of the general HLS model ($\chi^2/\text{d.o.f} = 24/30 = 0.8$) and the ρ-meson dominance ($\chi^2/\text{d.o.f} = 124/31 = 4.0$).

Item (I) implies that there is no obvious need for corrections as to spacelike momentum region, while in the timelike region the DBI part of the SS model yields for the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relations [4],

$$\frac{g_\rho}{2g_{\rho\pi\pi}F_\pi^2}\bigg|_{\rm SS} \simeq 2.0, \qquad \left(\frac{g_\rho}{2g_{\rho\pi\pi}F_\pi^2}\bigg|_{\rm exp} \simeq 1.0\right)$$
$$\frac{m_\rho^2}{g_{\rho\pi\pi}^2F_\pi^2}\bigg|_{\rm SS} \simeq 3.0, \qquad \left(\frac{m_\rho^2}{g_{\rho\pi\pi}^2F_\pi^2}\bigg|_{\rm exp} \simeq 2.0\right), \tag{5.1}$$

which would need some corrections such as $1/N_c$-subleading corrections [13]. Item (II) implies that the SS model would need corrections for the CS term which may arise as $1/N_c$-subleading terms. Our formulation can also be used to test other HQCD and suggest possible corrections.

Throughout this paper, we confined ourselves to the leading order in the $1/N_c$ expansion. We demonstrated that as far as the $1/N_c$-leading order form factors are concerned, the same results as those of our method can also be obtained by other methods using sum rules for an infinite sum of KK modes instead of the HLS Lagrangian. As far as the tree level is concerned, our method which sets fields of higher mass eigenstates $B_\mu^{(n)}$ to zero as in Eq. (2.34) obviously gives the same results as those obtained from a Lagrangian written in terms of $B_\mu^{(n)}$ [18] without explicit use of the HLS basis. However, in our method based on the HLS formalism, the systematic chiral perturbation can straightforwardly incorporate the $1/N_c$-subleading effects through loop calculations. Further studies along this line will be done in the future.

Our focus in this paper has been on the $1/N_c$-leading action and its derivative expansion. There could be another source, which affects coefficients of $\mathcal{O}(p^4)$ terms, arising from the $1/\lambda$ expansion. Further development of our method incorporating such a source will be pursued in the future.

Finally, we emphasize that our formulation can be applicable to several types of HQCD models [5,6,32] as well as models including baryons [18,33]. It will also be interesting to apply our method to HQCD models in hot and/or dense matter [34], and furthermore, to so-called holographic (walking) technicolor models [35] and Higgsless models [36].

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APPENDIX A: HLS GAUGE INVARIANCE OF Γ_3

In this section we give a proof for the HLS gauge invariance of the Γ_3 term in Eq. (3.9).

We begin by decomposing the five-dimensional gauge field $A = A_M dx^M$ $(M = \mu, z)$[13] in Eq. (2.18) into two parts, including an infinite tower of vector and axial-vector-meson fields:

$$A = v + a, \tag{A1}$$

$$v = v_0 + \tilde{v} = \alpha_\parallel + \sum_n^\infty \hat{\alpha}_\parallel^{(n)} \psi_{2n-1}, \tag{A2}$$

$$a = a_0 + \tilde{a} = \alpha_\perp \psi_0 + \sum_n^\infty A^{(n)} \psi_{2n}, \tag{A3}$$

where $\hat{\alpha}^{(n)} \equiv \alpha_\parallel - V^{(n)}$. They transform under HLS as

$$\alpha_\parallel \to h \cdot \alpha_\parallel \cdot h^\dagger + h \cdot d \cdot h^\dagger, \tag{A4}$$

$$\hat{\alpha}_\parallel^{(n)} \to h \cdot \alpha_\parallel^{(n)} \cdot h^\dagger, \tag{A5}$$

$$\hat{\alpha}_\perp \to h \cdot \hat{\alpha}_\parallel \cdot h^\dagger, \tag{A6}$$

$$A^{(n)} \to h \cdot A^{(n)} \cdot h^\dagger, \tag{A7}$$

so that $v \to h \cdot v \cdot h^\dagger + h \cdot d \cdot h^\dagger$ and $a \to h \cdot a \cdot h^\dagger$.

In $A_z \equiv 0$ gauge, the action Γ_3 in Eq. (3.9) takes the form

$$\Gamma_3 = \frac{N_c}{24\pi^2} \int_{M^4\times R} \mathrm{tr}[3advdv + adada + 3(v^2a + av^2 + a^3)dv + [3avada]_{\mathrm{nonzero}}], \tag{A8}$$

which can be separated into two portions,

$$\Gamma_3 \equiv \Gamma_{31} + \Gamma_{32}, \tag{A9}$$

$$\Gamma_{31} = \int_{M^4\times R} \mathrm{tr}(3advdv + 3(v^2a + av^2)dv), \tag{A10}$$

$$\Gamma_{32} = \int_{M^4\times R} \mathrm{tr}(adada + 3a^3dv + 3[avada]_{\mathrm{nonzero}}). \tag{A11}$$

As to Γ_{31}, we calculate it as

$$\begin{aligned}\Gamma_{31} &= 3\int dz \int_{M^4} \mathrm{tr}(-a(\partial_z v)dv - adv(\partial_z v) - (v^2a + av^2)\partial_z v) \\ &= -3\int dz \int_{M^4} \mathrm{tr}([dva + adv + (v^2a + av^2)]\partial_z v) \\ &= -3\int dz \int_{M^4} \mathrm{tr}(\{dv + v^2\}a + a\{dv + v^2\}\partial_z v).\end{aligned} \tag{A12}$$

From the transformation properties of v and a, and noting that $\partial_z v = \partial_z \tilde{v}$ transforms homogeneously under HLS, we see that Γ_{31} is HLS gauge invariant.

As to Γ_{32}, we first consider the first term

$$\int_{M^4\times R} \mathrm{tr}(adada) = \int_{M^4\times R} \mathrm{tr}(a_0da_0da_0 + [adada]_{\mathrm{nonzero}}). \tag{A13}$$

The first term of Eq. (A13) is calculated to be zero:

$$\begin{aligned}&\int_{M^4\times R} \mathrm{tr}(a_0da_0da_0) \\ &= -\int dz \int_{M^4} \mathrm{tr}(a_0(\partial_z a_0)da_0 + a_0da_0\partial_z a_0) \\ &= -\int dz \int_{M^4} \mathrm{tr}((da_0a_0 + a_0da_0)\partial_z a_0) \\ &= -\frac{1}{8}\int dz \int_{M^4} \mathrm{tr}([(d\hat{\alpha}_\perp)\hat{\alpha}_\perp + \hat{\alpha}_\perp(d\hat{\alpha}_\perp)]\hat{\alpha}_\perp)\psi_0^2\dot{\psi}_0 \\ &= -\frac{1}{24}\int dz \frac{\partial}{\partial z}(\psi_0^3) \int_{M^4} \mathrm{tr}([(d\hat{\alpha}_\perp)\hat{\alpha}_\perp + \hat{\alpha}_\perp(d\hat{\alpha}_\perp)]\hat{\alpha}_\perp) \\ &= -\frac{1}{12}\int_{M^4} \mathrm{tr}([(d\hat{\alpha}_\perp)\hat{\alpha}_\perp + \hat{\alpha}_\perp(d\hat{\alpha}_\perp)]\hat{\alpha}_\perp) \\ &= -\frac{1}{12}\int_{M^4} \mathrm{tr}((d\hat{\alpha}_\perp)(\hat{\alpha}_\perp)^2 - (d\hat{\alpha}_\perp)(\hat{\alpha}_\perp)^2) = 0.\end{aligned} \tag{A14}$$

Thus we have

$$\int_{M^4\times R} \mathrm{tr}(adada) = \int_{M^4\times R} \mathrm{tr}([adada]_{\mathrm{nonzero}}). \tag{A15}$$

From this and noting that the second term in Γ_{32} does not include zero modes since $\partial_z v = \partial_z \tilde{v}$, we may write

$$\Gamma_{32} = \int_{M^4\times R} \mathrm{tr}([adada + 3a^3dv + 3avada]_{\mathrm{nonzero}}). \tag{A16}$$

We further rewrite this Γ_{32} as follows:

[13]In this section we take A_M to be anti-Hermitian.

$$\begin{aligned}\Gamma_{32} &= -\int_{M^4\times R} \mathrm{tr}([ada\partial_z a + a\partial_z ada + 3a^3\partial_z v + 3ava\partial_z a]_{\mathrm{nonzero}})\\ &= -\int_{M^4\times R} \mathrm{tr}([\{ada + daa + 3ava\}\partial_z a + 3a^3\partial_z v]_{\mathrm{nonzero}})\\ &= -\int_{M^4\times R} \mathrm{tr}([\{a(da + va + av) + (da + va + av)a + ava - a^2 v - va^2\}\partial_z a + 3a^3\partial_z v]_{\mathrm{nonzero}})\\ &= -\int_{M^4\times R} \mathrm{tr}([\{a(da + va + av) + (da + va + av)a\}\partial_z a]_{\mathrm{nonzero}})\\ &\quad -\int_{M^4\times R} \mathrm{tr}([a\partial_z aav + \partial_z aa^2 v + a^2\partial_z av + 3a^3\partial_z v]_{\mathrm{nonzero}})\\ &= -\int_{M^4\times R} \mathrm{tr}([\{a(da + va + av) + (da + va + av)a\}\partial_z a + 2a^3\partial_z v]_{\mathrm{nonzero}}) - \int_{M^4\times R} \mathrm{tr}\partial_z([a^3 v]_{\mathrm{nonzero}}). \qquad \text{(A17)}\end{aligned}$$

Note that the last term includes at least one nonzero mode/normalizable mode which vanishes due to the boundary condition at $z = \pm\infty$. Thus we find it goes to zero after integration with respect to z:

$$\int_{M^4\times R} \mathrm{tr}\partial_z([a^3 v]_{\mathrm{nonzero}}) = 0. \qquad \text{(A18)}$$

On the other hand, we can easily see that the remaining terms in the last line of Eq. (A17) are HLS gauge invariant.

Thus it has been proven that the action Γ_3 is HLS gauge invariant.

APPENDIX B: EXPANDING DIRAC-BORN-INFELD AND CHERN-SIMONS PARTS IN TERMS OF HLS BUILDING BLOCKS

In this section we derive Eqs. (3.15)–(3.31) in the DBI part and Eqs. (3.36)–(3.39) in the CS part.

1. Dirac-Born-Infeld part

Taking into account the $A_z \equiv 0$ gauge and substituting Eq. (2.44) into the field strength $F_{\mu z}$, we have

$$\begin{aligned}F_{\mu z} &= \partial_\mu A_z - \partial_z A_\mu - i[A_\mu, A_z]\\ &= -\partial_z(\hat\alpha_{\mu\perp}\psi_0 + V_\mu + \hat\alpha_{\mu\parallel}(1+\psi_1))\\ &= -\hat\alpha_{\mu\perp}\dot\psi_0 - \hat\alpha_{\mu\parallel}\dot\psi_1. \qquad \text{(B1)}\end{aligned}$$

Similarly for $F_{\mu\nu}$, one can calculate it as

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\\ &= \partial_\mu(\hat\alpha_{\nu\perp}\psi_0 + V_\nu + \hat\alpha_{\nu\parallel}(1+\psi_1)) - \partial_\nu(\hat\alpha_{\mu\perp}\psi_0 + V_\mu + \hat\alpha_{\mu\parallel}(1+\psi_1))\\ &\quad - i[\hat\alpha_{\mu\perp}\psi_0 + V_\mu + \hat\alpha_{\mu\parallel}(1+\psi_1), \hat\alpha_{\nu\perp}\psi_0 + V_\nu + \hat\alpha_{\nu\parallel}(1+\psi_1)]\\ &= (D_\mu\hat\alpha_{\nu\perp} - D_\nu\hat\alpha_{\mu\perp})\psi_0 + (D_\mu\hat\alpha_{\nu\parallel} - D_\nu\hat\alpha_{\mu\parallel})(1+\psi_1) + V_{\mu\nu} - i[\hat\alpha_{\mu\perp}, \hat\alpha_{\nu\perp}]\psi_0^2\\ &\quad - i([\hat\alpha_{\mu\perp}, \hat\alpha_{\nu\parallel}] + [\hat\alpha_{\mu\parallel}, \hat\alpha_{\nu\perp}])(1+\psi_1)\psi_0 - i[\hat\alpha_{\mu\parallel}, \hat\alpha_{\nu\parallel}](1+\psi_1)^2, \qquad \text{(B2)}\end{aligned}$$

where we have defined

$$\begin{aligned}D_\mu\hat\alpha_{\nu\perp} &= \partial_\mu\hat\alpha_{\nu\perp} - i[V_\mu, \hat\alpha_{\nu\perp}],\\ D_\mu\hat\alpha_{\nu\parallel} &= \partial_\mu\hat\alpha_{\nu\parallel} - i[V_\mu, \hat\alpha_{\nu\parallel}].\end{aligned} \qquad \text{(B3)}$$

Using the identities

$$\begin{aligned}D_\mu\hat\alpha_{\nu\parallel} - D_\nu\hat\alpha_{\mu\parallel} &= i[\hat\alpha_{\mu\parallel}, \hat\alpha_{\nu\parallel}] + i[\hat\alpha_{\mu\perp}, \hat\alpha_{\nu\perp}]\\ &\quad + \hat{\mathcal{V}}_{\mu\nu} - V_{\mu\nu}, \qquad \text{(B4)}\end{aligned}$$

$$D_\mu\hat\alpha_{\nu\perp} - D_\nu\hat\alpha_{\mu\perp} = i[\hat\alpha_{\mu\parallel}, \hat\alpha_{\nu\perp}] + i[\hat\alpha_{\mu\perp}, \hat\alpha_{\nu\parallel}] + \hat{\mathcal{A}}_{\mu\nu}, \qquad \text{(B5)}$$

we obtain

$$\begin{aligned}F_{\mu\nu} &= -V_{\mu\nu}\psi_1 + \hat{\mathcal{V}}_{\mu\nu}(1+\psi_1) + \hat{\mathcal{A}}_{\mu\nu}\psi_0\\ &\quad - i[\hat\alpha_{\mu\parallel}, \hat\alpha_{\nu\parallel}](1+\psi_1)\psi_1\\ &\quad + i[\hat\alpha_{\mu\perp}, \hat\alpha_{\nu\perp}](1+\psi_1-\psi_0^2)\\ &\quad - i([\hat\alpha_{\mu\parallel}, \hat\alpha_{\nu\perp}] + [\hat\alpha_{\mu\perp}, \hat\alpha_{\nu\parallel}])\psi_1\psi_0. \qquad \text{(B6)}\end{aligned}$$

Substituting the final expressions in Eqs. (B1) and (B6) into the DBI part (3.1), we are readily led to Eqs. (3.15)–(3.31).

2. Chern-Simons part: Γ_3

We start with an expression of Γ_3 given in Eq. (A8):

$$\Gamma_3 = \frac{N_c}{24\pi^2}\int_5 \mathrm{tr}[3advdv + adada + 3(v^2a + av^2 + a^3)dv + [3avada]_{\mathrm{nonzero}}] = -\frac{3N_c}{24\pi^2}\int dz \int_{M^4} \mathrm{tr}[a\partial_z v dv + adv\partial_z v + v^2 a\partial_z v + av^2\partial_z v + a^3\partial_z v + [ava\partial_z a]_{\mathrm{nonzero}}], \tag{B7}$$

where vector (v) and axial-vector (a) fields are taken to be anti-Hermitian, for convenience. After integrating out higher KK modes in the CS part, v and a, respectively, become

$$v = \alpha_\parallel + \hat{\alpha}_\parallel \psi_1(z), \qquad \alpha_\parallel = \hat{\alpha}_\parallel + V, \tag{B8}$$

$$a = \hat{\alpha}_\perp \psi_0(z). \tag{B9}$$

Substituting these expressions into Eq. (B7), we calculate Γ_3 as follows:

$$\begin{aligned}\Gamma_3 &= -\frac{3N_c}{24\pi^2}\int_{M^4}\mathrm{tr}[\langle\psi_0\dot{\psi}_1\rangle(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)(d\alpha_\parallel + \alpha_\parallel^2) + \langle\psi_0\dot{\psi}_1\psi_1\rangle\{2\alpha_\parallel\hat{\alpha}_\parallel\hat{\alpha}_\perp\hat{\alpha}_\parallel + (\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)d\hat{\alpha}_\parallel \\ &\quad + (\hat{\alpha}_\perp\alpha_\parallel - \alpha_\parallel\hat{\alpha}_\perp)\hat{\alpha}_\parallel^2\} + \langle\psi_0\dot{\psi}_1\psi_1^2\rangle(2\hat{\alpha}_\perp\hat{\alpha}_\parallel^3) + \langle\psi_0^2\dot{\psi}_0\psi_1\rangle(2\hat{\alpha}_\parallel\hat{\alpha}_\perp^3)] \\ &= -\frac{3N_c}{24\pi^2}\int_{M^4}\mathrm{tr}[\langle\psi_0\dot{\psi}_1\rangle(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)(\hat{\alpha}_\parallel^2 + D\hat{\alpha}_\parallel + F_V) + \langle\psi_0\dot{\psi}_1\psi_1\rangle\{(D\hat{\alpha}_\parallel + 2\hat{\alpha}_\parallel^2)(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)\} \\ &\quad + \langle\psi_0\dot{\psi}_1\psi_1^2\rangle(2\hat{\alpha}_\perp\hat{\alpha}_\parallel^3) + \langle\psi_0^2\dot{\psi}_0\psi_1\rangle(2\hat{\alpha}_\parallel\hat{\alpha}_\perp^3)], \\ &= -\frac{3N_c}{24\pi^2}\int_{M^4}\mathrm{tr}[\langle\psi_0\dot{\psi}_1\rangle(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)(-\hat{\alpha}_\perp^2 + \hat{F}_V) + \langle\psi_0\dot{\psi}_1\psi_1\rangle\{(\hat{\alpha}_\parallel^2 - \hat{\alpha}_\perp^2 - F_V + \hat{F}_V)(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)\} \\ &\quad + \langle\psi_0\dot{\psi}_1\psi_1^2\rangle(2\hat{\alpha}_\perp\hat{\alpha}_\parallel^3) + \langle\psi_0^2\dot{\psi}_0\psi_1\rangle(2\hat{\alpha}_\parallel\hat{\alpha}_\perp^3)] \\ &= -\frac{3N_c}{24\pi^2}\int_{M^4}\mathrm{tr}\bigg[\bigg\langle -\frac{1}{2}\psi_0\dot{\psi}_1\psi_1\bigg\rangle(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)F_V + \langle 2\psi_0\dot{\psi}_1\psi_1(1+\psi_1)\rangle\hat{\alpha}_\perp\hat{\alpha}_\parallel^3 \\ &\quad + \bigg\langle 2\psi_0\dot{\psi}_1\bigg(1 + \psi_1 - \frac{1}{3}\psi_0^2\bigg)\bigg\rangle\hat{\alpha}_\parallel\hat{\alpha}_\perp^3 + \langle\psi_0\dot{\psi}_1(1+\psi_1)\rangle(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)\hat{F}_V\bigg],\end{aligned} \tag{B10}$$

where $\langle A\rangle = \int dz A(z)$ for an arbitrary function $A(z)$, and we have used the identity

$$D\hat{\alpha}_\parallel \equiv d\hat{\alpha}_\parallel + V\hat{\alpha}_\parallel + \hat{\alpha}_\parallel V = -\hat{\alpha}_\parallel^2 - \hat{\alpha}_\perp^2 - F_V + \hat{F}_V, \tag{B11}$$

and defined

$$F_V = dV + V^2, \tag{B12}$$

$$\hat{F}_V = \frac{\hat{F}_L + \hat{F}_R}{2}, \tag{B13}$$

$$\hat{F}_{L,R} = \xi_{L,R}^{-1}\cdot F_{L,R}\cdot\xi_{L,R}, \tag{B14}$$

$$F_{L,R} = dA_{L,R} + A_{L,R}^2. \tag{B15}$$

Moving on to four-dimensional Minkowski space-time and rewriting one-forms in terms of Hermitian fields, we obtain

$$\Gamma_3 = \frac{N_c}{24\pi^2}\int_4\{x_1\,\mathrm{tr}[(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)F_V] + x_2 i\,\mathrm{tr}[\hat{\alpha}_\perp\hat{\alpha}_\parallel^3] + x_3 i\,\mathrm{tr}[\hat{\alpha}_\parallel\hat{\alpha}_\perp^3] + x_4\,\mathrm{tr}[(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)\hat{F}_V]\}, \tag{B16}$$

where

$$x_1 = \langle 3\psi_1\psi_0\dot{\psi}_1\rangle, \tag{B17}$$

$$x_2 = \langle 6\psi_1\psi_0\dot{\psi}_1(1+\psi_1)\rangle, \tag{B18}$$

$$x_3 = \langle 2\psi_0\dot{\psi}_1(-\psi_0^2 + 3\psi_1 + 3)\rangle, \tag{B19}$$

$$x_4 = \langle -3(\psi_1 + 1)\psi_0\dot{\psi}_1\rangle. \tag{B20}$$

The IP-odd terms in the HLS model are given in terms of $\hat{\alpha}_\parallel$, $\hat{\alpha}_\perp$ as [see Eq. (3.34)]

$$\Gamma_{\text{IP-odd}}^{\text{HLS}} = \frac{N_c}{16\pi^2}\int_{M^4}\{(-4c_1 - 4c_2)i\,\mathrm{tr}[\hat{\alpha}_\perp\hat{\alpha}_\parallel^3] + (4c_1 - 4c_2)i\,\mathrm{tr}[\hat{\alpha}_\parallel\hat{\alpha}_\perp^3] + (-2c_3)\times\mathrm{tr}[(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)F_V] + (-2c_4)\times\mathrm{tr}[(\hat{\alpha}_\perp\hat{\alpha}_\parallel - \hat{\alpha}_\parallel\hat{\alpha}_\perp)\hat{F}_V]\}. \tag{B21}$$

Comparing this form with Γ_3 in Eq. (B16), we find

$$c_1 = -\frac{1}{12}x_2 + \frac{1}{12}x_3, \tag{B22}$$

$$c_2 = -\frac{1}{12}x_2 - \frac{1}{12}x_3, \tag{B23}$$

$$c_3 = -\tfrac{1}{3}x_1, \tag{B24}$$

$$c_4 = -\frac{1}{3}x_4, \tag{B25}$$

which readily lead to Eqs. (3.36)–(3.39).

APPENDIX C: ALTERNATIVE DERIVATION OF OUR RESULTS FOR IP-ODD PROCESSES AND THEIR GAUGE/CHIRAL INVARIANCE

In this appendix, to see that our formalism correctly incorporates contributions from higher KK modes of the HLS gauge bosons in the IP-odd sector, we shall perform a low-energy expansion of the original forms of the vertex functions [4] for π^0-γ^*-γ^*, ω-π^0-γ^*, and γ^*-π^0-π^+-π^- to be consistent with our integrating-out method as was done in Sec. IVA [see Eq. (4.21)]. We also discuss a naive truncation and violation of gauge/chiral invariance.

1. The π^0-γ^*-γ^* vertex function and π^0-γ transition form factor

We begin with the original form of the π^0-γ^*-γ^* vertex function written in terms of an infinite sum of vector-meson exchanges [4]:

$$\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)]|_{\rm SS} = \frac{e^2 N_c}{24\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}\cdot\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\biggl[\frac{g_{\omega_k\rho_l\pi}g_{\rho_l}g_{\omega_k}}{(m^2_{\omega_k}+q_1^2)(m^2_{\rho_l}+q_2^2)} + (q_1^2\leftrightarrow q_2^2)\biggr], \tag{C1}$$

where the coupling $g_{\omega_k\rho_l\pi}$ is defined in the SS model as

$$\mathcal{L}_{\omega_k\rho_l\pi} = -\frac{N_c}{8\pi F_\pi}g_{\omega_k\rho_l\pi}\sum_{a=1}^{3}\epsilon^{\mu\nu\lambda\sigma}\partial_\mu(\omega_k)_\nu\partial_\lambda(\rho_l^a)_\sigma\cdot\pi^a. \tag{C2}$$

Consistently with our method which integrates out higher KK modes into $\mathcal{O}(p^4)$ terms of the HLS Lagrangian, we expand Eq. (C1) as

$$\begin{aligned}\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)]|_{\rm SS} &= \frac{e^2 N_c}{12\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}\cdot\biggl[\biggl(\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\frac{g_{\omega_k\rho_l\pi}g_{\rho_l}g_{\omega_k}}{m^2_{\omega_k}m^2_{\rho_l}}\biggr) \\ &+ \frac{1}{4}\biggl(2\sum_{k=2}^{\infty}\frac{g_{\omega_k\rho\pi}g_\rho g_{\omega_k}}{m^2_{\omega_k}m^2_\rho}\biggr)\{D_\omega(q_1^2)+D_\rho(q_2^2) \\ &+ (q_1^2\leftrightarrow q_2^2)\} + \frac{1}{2}\biggl(\frac{g_{\omega\rho\pi}g_\rho g_\omega}{m^2_\omega m^2_\rho}\biggr)\{D_\omega(q_1^2)\cdot D_\rho(q_2^2) \\ &+ (q_1^2\leftrightarrow q_2^2)\}\biggr],\end{aligned} \tag{C3}$$

up to terms of $\mathcal{O}(q^2_{1,2}/m^2_{\rho_k,\omega_k})$ $(k\geq 2)$, which correspond to terms higher than $\mathcal{O}(p^4)$ in the Lagrangian. Using the sum rules [4]

$$\sum_{k=1}^{\infty}\frac{g_{\omega_l\rho_k\pi}g_{\rho_k}}{m^2_{\rho_k}} = \sum_{k=1}^{\infty}\frac{g_{\omega_k\rho_l\pi}g_{\omega_k}}{m^2_{\omega_k}} = g_{\rho_l\pi\pi}, \qquad \sum_{k=1}^{\infty}\frac{g_{\rho_k\pi\pi}g_{\rho_k}}{m^2_{\rho_k}} = 1, \tag{C4}$$

we have

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\frac{g_{\omega_k\rho_l\pi}g_{\rho_l}g_{\omega_k}}{m^2_{\omega_k}m^2_{\rho_l}} = 1 - \frac{2g_{\rho\pi\pi}g_\rho}{m^2_\rho} + \frac{g_{\omega\rho\pi}g_\rho g_\omega}{m^2_\omega m^2_\rho}, \tag{C5}$$

$$\sum_{k=2}^{\infty}\frac{g_{\omega_k\rho\pi}g_\rho g_{\omega_k}}{m^2_{\omega_k}m^2_\rho} = \frac{g_{\rho\pi\pi}g_\rho}{m^2_\rho} - \frac{g_{\omega\rho\pi}g_\rho g_\omega}{m^2_\omega m^2_\rho}. \tag{C6}$$

From Ref. [4] we have

$$\frac{g_{\rho\pi\pi}g_\rho}{m^2_\rho} = \frac{1}{2}\frac{\langle\langle\psi_1\dot\psi_0\rangle\rangle\langle\psi_1\rangle}{\langle\psi_1^2\rangle}, \tag{C7}$$

$$\frac{g_{\omega\rho\pi}g_\rho g_\omega}{m^2_\rho m^2_\omega} = \frac{1}{2}\frac{\langle\langle\psi_1^2\dot\psi_0\rangle\rangle\langle\psi_1\rangle^2}{\langle\psi_1^2\rangle^2}. \tag{C8}$$

Substituting these into the right-hand sides of Eqs. (C5) and (C6), we have

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\frac{g_{\omega_k\rho_l\pi}g_{\rho_l}g_{\omega_k}}{m^2_{\omega_k}m^2_{\rho_l}} = 1 - \biggl[\frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1\rangle\rangle}{\langle\psi_1^2\rangle} - \frac{1}{2}\frac{\langle\psi_1\rangle^2\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^2}\biggr], \tag{C9}$$

$$2\sum_{k=2}^{\infty}\frac{g_{\omega_k\rho\pi}g_\rho g_{\omega_k}}{m^2_{\omega_k}m^2_\rho} = \frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1\rangle\rangle}{\langle\psi_1^2\rangle} - \frac{\langle\psi_1\rangle^2\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^2}, \tag{C10}$$

$$\frac{g_{\omega\rho\pi}g_\rho g_\omega}{m^2_\omega m^2_\rho} = \frac{1}{2}\biggl[\frac{\langle\psi_1\rangle^2\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^2}\biggr]. \tag{C11}$$

The right-hand sides are identical to $A^{\pi2\gamma}_{\rm SS}$-$C^{\pi2\gamma}_{\rm SS}$ in Eqs. (4.54), (4.55), and (4.56):

$$\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\frac{g_{\omega_k\rho_l\pi}g_{\rho_l}g_{\omega_k}}{m^2_{\omega_k}m^2_{\rho_l}} = A^{\pi2\gamma}_{\rm SS}, \tag{C12}$$

$$2\sum_{k=2}^{\infty}\frac{g_{\omega_k\rho\pi}g_\rho g_{\omega_k}}{m^2_{\omega_k}m^2_\rho} = B^{\pi2\gamma}_{\rm SS}, \tag{C13}$$

$$\frac{g_{\omega\rho\pi}g_\rho g_\omega}{m^2_\omega m^2_\rho} = C^{\pi2\gamma}_{\rm SS}, \tag{C14}$$

and hence we arrive at the same result as that derived from our method of integrating out higher KK modes [Eq. (4.43) with Eqs. (4.54), (4.55), and (4.56)]:

$$\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)]|_{SS} = \frac{e^2 N_c}{12\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta} q_{1\alpha}q_{2\beta}\Big[A^{\pi 2\gamma}_{SS} + \frac{B^{\pi 2\gamma}_{SS}}{4}\{D_\rho(q_1^2) + D_\omega(q_1^2) + (q_1^2 \to q_2^2)\} + \frac{C^{\pi 2\gamma}_{SS}}{2}\{D_\rho(q_1^2)\cdot D_\omega(q_2^2) + (q_1^2 \leftrightarrow q_2^2)\}\Big]. \quad \text{(C15)}$$

For the transition form factor $F_{\pi^0\gamma}$, we have

$$F_{\pi^0\gamma}(Q^2)|_{SS} = \left(A^{\pi 2\gamma}_{SS} + \frac{B^{\pi 2\gamma}_{SS}}{2}\right) + \left(\frac{B^{\pi 2\gamma}_{SS}}{4} + \frac{C^{\pi 2\gamma}_{SS}}{2}\right) \times [D_\rho(Q^2) + D_\omega(Q^2)]. \quad \text{(C16)}$$

Because the resultant form (C15) is the same as that obtained from our method, which is manifestly gauge invariant by construction [see Eqs. (2.36), (2.37), and (2.41)], the low-energy theorem in Eq. (4.48) is actually satisfied:

$$A^{\pi 2\gamma}_{SS} + B^{\pi 2\gamma}_{SS} + C^{\pi 2\gamma}_{SS} = 1. \quad \text{(C17)}$$

For a comparison, let us consider what would happen if one had naively truncated a tower of the HLS gauge bosons at the lowest level, as in Eq. (2.33). From Eqs. (C12) and (C13), one can easily see that such a naive truncation corresponds to simply neglecting higher KK modes, $A^{\pi 2\gamma}_{SS} = B^{\pi 2\gamma}_{SS} = 0$:

$$F_{\pi^0\gamma}(Q^2)|^{\rm trun}_{SS} = \frac{C^{\pi 2\gamma}_{SS}}{2}\left(\frac{m_\rho^2}{m_\rho^2 + Q^2} + \frac{m_\omega^2}{m_\omega^2 + Q^2}\right), \quad \text{(C18)}$$

with $C^{\pi 2\gamma}_{SS} \simeq 0.5$ from Eq. (4.56). At $Q^2 = 0$ we have

$$F_{\pi^0\gamma}(0)|^{\rm trun}_{SS} = C^{\pi 2\gamma}_{SS} \simeq 0.5 \neq 1, \quad \text{(C19)}$$

which breaks the EM gauge symmetry. Note again that the naive truncation (C18) is different from the ρ/ω-meson dominance (4.50) which is gauge invariant. The violation of gauge symmetry can also be seen in the vertex function as

$$\Gamma^{\mu\nu}[\pi^0, \gamma^*_\mu(q_1), \gamma^*_\nu(q_2)]|^{\rm trun}_{SS} \overset{q_1^2, q_2^2 \to 0}{\longrightarrow} \frac{e^2 N_c}{12\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}\cdot(C^{\pi 2\gamma}_{SS}) \neq \frac{e^2 N_c}{12\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}, \quad \text{(C20)}$$

which contradicts the low-energy theorem (4.49).

2. The ω-π^0-γ^* vertex function and ω-π^0 transition form factor

We start with the form [4]

$$\Gamma^{\mu\nu}[\omega_\mu(p), \pi^0, \gamma^*_\nu(k)]|_{SS} = \frac{eN_c}{8\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}p_\alpha k_\beta \cdot \sum_{n=1}^{\infty}\frac{g_{\omega\rho_n\pi}g_{\rho_n}}{m^2_{\rho_n} + k^2}. \quad \text{(C21)}$$

We expand this expression to be consistent with our method of integrating out higher KK modes into $\mathcal{O}(p^4)$ terms of the general HLS Lagrangian:

$$\Gamma^{\mu\nu}[\omega_\mu(p), \pi^0, \gamma^*_\nu(k)]|_{SS} = \frac{eN_c}{8\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}p_\alpha k_\beta \cdot \left[\left(\sum_{n=2}^{\infty}\frac{g_{\omega\rho_n\pi}g_{\rho_n}}{m^2_{\rho_n}}\right) + \left(\frac{g_{\omega\rho\pi}g_\rho}{m_\rho^2}\right)D_\rho(k^2)\right], \quad \text{(C22)}$$

up to terms of $\mathcal{O}(k^2/m^2_{\rho_n})$ $(n \geq 2)$ which correspond to terms higher than $\mathcal{O}(p^4)$ in the Lagrangian. Using the first sum rule displayed in Eq. (C4), we have

$$\sum_{n=2}^{\infty}\frac{g_{\omega\rho_n\pi}g_{\rho_n}}{m^2_{\rho_n}} = g_{\rho\pi\pi} - \frac{g_{\omega\rho\pi}g_\rho}{m_\rho^2}. \quad \text{(C23)}$$

From Ref. [4] we have

$$g_{\rho\pi\pi} = \frac{1}{2\sqrt{N_c G}}\sqrt{\frac{\langle\langle\psi_0\dot\psi_1\rangle\rangle^2}{\langle\psi_1^2\rangle}}, \quad \text{(C24)}$$

$$\frac{g_{\omega\rho\pi}g_\rho}{m_\rho^2} = \frac{1}{2\sqrt{N_c G}}\frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle}{\langle\psi_1^2\rangle^{3/2}}. \quad \text{(C25)}$$

We then have

$$\sum_{n=2}^{\infty}\frac{g_{\omega\rho_n\pi}g_{\rho_n}}{m^2_{\rho_n}} = -\frac{1}{2\sqrt{N_c G}}\left[\frac{\langle\langle\dot\psi_0\psi_1\rangle\rangle}{\sqrt{\langle\psi_1^2\rangle}} - \frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle^{3/2}}\right]. \quad \text{(C26)}$$

Comparing Eqs. (C25) and (C26) with Eqs. (4.70) and (4.69), respectively, we find

$$\sum_{n=2}^{\infty}\frac{g_{\omega\rho_n\pi}g_{\rho_n}}{m^2_{\rho_n}} = A^{\omega\pi\gamma}_{SS}, \quad \text{(C27)}$$

$$\frac{g_{\omega\rho\pi}g_\rho}{m_\rho^2} = B^{\omega\pi\gamma}_{SS}, \quad \text{(C28)}$$

and hence arrive at the same result as that derived from our integrating-out method [Eq. (4.59) with Eqs. (4.69) and (4.70)]:

$$\Gamma^{\mu\nu}[\omega_\mu(p), \pi^0, \gamma^*_\nu(k)]|_{SS} = \frac{eN_c}{8\pi^2 F_\pi}\epsilon^{\mu\nu\alpha\beta}p_\alpha k_\beta \cdot [A^{\omega\pi\gamma}_{SS} + B^{\omega\pi\gamma}_{SS}D_\rho(k^2)]. \quad \text{(C29)}$$

For the transition form factor, we have

$$F_{\omega\pi^0}(q^2)|_{\rm SS} = \left(\frac{A_{\rm SS}^{\omega\pi\gamma}}{A_{\rm SS}^{\omega\pi\gamma}+B_{\rm SS}^{\omega\pi\gamma}}\right) + \left(\frac{B_{\rm SS}^{\omega\pi\gamma}}{A_{\rm SS}^{\omega\pi\gamma}+B_{\rm SS}^{\omega\pi\gamma}}\right) \times \frac{m_\rho^2}{m_\rho^2 - q^2}, = (1-\tilde{r}_{\rm SS}) + \tilde{r}_{\rm SS}\frac{m_\rho^2}{m_\rho^2-q^2}. \tag{C30}$$

A naive truncation as in Eq. (2.33), which corresponds to setting $A_{\rm SS}^{\omega\pi\gamma}=0$, would lead to the same form of $F_{\omega\pi^0}$ as that of the ρ-meson dominance (4.68), although $g_{\omega\pi\gamma}$ in Eq. (4.65) yields a value about $1/(1.53)\simeq 2/3$ times smaller (see footnote 12.). Unlike the case of the pion EM and π^0-γ transition form factors, the violation of gauge symmetry is not manifest in the ω-π^0 transition form factor $F_{\omega\pi^0}$ since there is no low-energy theorem for this process.

3. The γ^*-π^0-π^+-π^- vertex function

The original form [4] can be expanded consistently with our integrating-out method:

$$\begin{aligned}
&\Gamma_\mu[\gamma_\mu^*(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)]|_{\rm SS} \\
&= -\frac{eN_c}{12\pi^2 F_\pi}\epsilon_{\mu\nu\alpha\beta}q_0^\nu q_+^\alpha q_-^\beta \sum_{k=1}^\infty\sum_{l=1}^\infty\left[\frac{g_{\omega_k\rho_l\pi}g_{\rho_l\pi\pi}g_{\omega_k}}{(m_{\omega_k}^2+p^2)(m_{\rho_l}^2+(q_++q_-)^2)} + (q_+\leftrightarrow q_0) + (q_-\leftrightarrow q_0)\right] \\
&= -\frac{eN_c}{12\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta}q_0^\nu q_+^\alpha q_-^\beta\left[\left(3F_\pi^2\sum_{k=2}^\infty\sum_{l=2}^\infty\frac{g_{\omega_k\rho_l\pi}g_{\rho_l\pi\pi}g_{\omega_k}}{m_{\omega_k}^2 m_{\rho_l}^2}\right) + \left(3F_\pi^2\sum_{l=2}^\infty\frac{g_{\omega\rho_l\pi}g_{\rho_l\pi\pi}g_\omega}{m_\omega^2 m_{\rho_l}^2}\right)\cdot D_\omega(p^2)\right. \\
&\quad + \frac{1}{3}\left(3F_\pi^2\sum_{k=2}^\infty\frac{g_{\omega_k\rho\pi}g_{\rho\pi\pi}g_{\omega_k}}{m_{\omega_k}^2 m_\rho^2}\right)\cdot\{D_\rho((q_++q_-)^2) + (q_+\leftrightarrow q_0) + (q_-\leftrightarrow q_0)\} \\
&\quad \left. + \frac{1}{3}\left(3F_\pi^2\frac{g_{\omega\rho\pi}g_{\rho\pi\pi}g_\omega}{m_\omega^2 m_\rho^2}\right)D_\omega(p^2)\cdot\{D_\rho((q_++q_-)^2) + (q_+\leftrightarrow q_0) + (q_-\leftrightarrow q_0)\}\right],
\end{aligned} \tag{C31}$$

where we have neglected terms of $\mathcal{O}(p^2, q_{\pm,0}^2/m_{\rho_k,\omega_k}^2)$ ($k\geq 2$) which correspond to terms higher than $\mathcal{O}(p^4)$ in the Lagrangian. Using the sum rules in Eq. (C4) and [4]

$$\sum_{l=1}^\infty \frac{g_{\rho_l\pi\pi}^2}{m_{\rho_l}^2} = \frac{1}{3F_\pi^2}, \tag{C32}$$

we have

$$3F_\pi^2\sum_{k=2}^\infty\sum_{l=2}^\infty\frac{g_{\omega_k\rho_l\pi}g_{\rho_l\pi\pi}g_{\omega_k}}{m_{\omega_k}^2 m_{\rho_l}^2} = 1 - 3\left(\frac{g_{\rho\pi\pi}^2F_\pi^2}{m_\rho^2}\right)\left(2-\frac{g_{\omega\rho\pi}g_\omega}{g_{\rho\pi\pi}m_\omega^2}\right), \tag{C33}$$

$$3F_\pi^2\sum_{l=2}^\infty\frac{g_{\omega\rho_l\pi}g_{\rho_l\pi\pi}g_\omega}{m_\omega^2 m_{\rho_l}^2} = 3F_\pi^2\sum_{k=2}^\infty\frac{g_{\omega_k\rho\pi}g_{\rho\pi\pi}g_{\omega_k}}{m_{\omega_k}^2 m_\rho^2} = 3\left(\frac{g_{\rho\pi\pi}^2F_\pi^2}{m_\rho^2}\right)\left(1-\frac{g_{\omega\rho\pi}g_\omega}{g_{\rho\pi\pi}m_\omega^2}\right). \tag{C34}$$

From Ref. [4] we read off

$$\frac{g_{\rho\pi\pi}^2F_\pi^2}{m_\rho^2} = \frac{1}{4}\frac{\langle\langle\dot\psi_0\psi_1\rangle\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle}, \tag{C35}$$

$$\frac{g_{\omega\rho\pi}g_\omega}{g_{\rho\pi\pi}m_\omega^2} = \frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1^2\rangle\rangle}{\langle\psi_1^2\rangle\langle\langle\dot\psi_0\psi_1\rangle\rangle}. \tag{C36}$$

Putting these into the right-hand sides of Eqs. (C33) and (C34), we have

$$\begin{aligned}
3F_\pi^2\sum_{k=2}^\infty\sum_{l=2}^\infty\frac{g_{\omega_k\rho_l\pi}g_{\rho_l\pi\pi}g_{\omega_k}}{m_{\omega_k}^2 m_{\rho_l}^2}
&= 1 - \frac{3}{4}\left[\frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1(1-\psi_0^2)\rangle\rangle}{\langle\psi_1^2\rangle}\right. \\
&\quad + \frac{\langle\langle\dot\psi_0\psi_1\rangle\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle} \\
&\quad \left. - \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\right],
\end{aligned} \tag{C37}$$

$$3F_\pi^2\sum_{l=2}^\infty\frac{g_{\omega\rho_l\pi}g_{\rho_l\pi\pi}g_\omega}{m_\omega^2 m_{\rho_l}^2} = \frac{3}{4}\left[\frac{\langle\psi_1\rangle\langle\langle\dot\psi_0\psi_1(1-\psi_0^2)\rangle\rangle}{\langle\psi_1^2\rangle} - \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\right], \tag{C38}$$

$$3F_\pi^2\sum_{k=2}^\infty\frac{g_{\omega_k\rho\pi}g_{\rho\pi\pi}g_{\omega_k}}{m_{\omega_k}^2 m_\rho^2} = \frac{3}{4}\left[\frac{\langle\langle\dot\psi_0\psi_1\rangle\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle} - \frac{\langle\langle\dot\psi_0\psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\right], \tag{C39}$$

$$3F_\pi^2 \frac{g_{\omega\rho\pi} g_{\rho\pi\pi} g_\omega}{m_\omega^2 m_\rho^2} = \frac{3}{4}\left[\frac{\langle\langle\dot\psi_0 \psi_1^2\rangle\rangle\langle\psi_1\rangle\langle\psi_1(1-\psi_0^2)\rangle}{\langle\psi_1^2\rangle^2}\right]. \tag{C40}$$

The right-hand sides are identical to $A_{\rm SS}^{\gamma 3\pi}$-$D_{\rm SS}^{\gamma 3\pi}$ in Eqs. (4.81), (4.82), (4.83), and (4.84):

$$3F_\pi^2 \sum_{k=2}^\infty \sum_{l=2}^\infty \frac{g_{\omega_k\rho_l\pi} g_{\rho_l\pi\pi} g_{\omega_k}}{m_{\omega_k}^2 m_{\rho_l}^2} = A_{\rm SS}^{\gamma 3\pi}, \tag{C41}$$

$$3F_\pi^2 \sum_{l=2}^\infty \frac{g_{\omega\rho_l\pi} g_{\rho_l\pi\pi} g_{\omega}}{m_{\omega}^2 m_{\rho_l}^2} = B_{\rm SS}^{\gamma 3\pi}, \tag{C42}$$

$$3F_\pi^2 \sum_{k=2}^\infty \frac{g_{\omega_k\rho\pi} g_{\rho\pi\pi} g_{\omega_k}}{m_{\omega_k}^2 m_{\rho}^2} = C_{\rm SS}^{\gamma 3\pi}. \tag{C43}$$

$$3F_\pi^2 \frac{g_{\omega\rho\pi} g_{\rho\pi\pi} g_{\omega}}{m_{\omega}^2 m_{\rho}^2} = D_{\rm SS}^{\gamma 3\pi}, \tag{C44}$$

and hence we arrive at the same result as that of our method [Eq. (4.73) with Eqs. (4.81), (4.82), (4.83), and (4.84)]:

$$\begin{aligned}
&\Gamma_\mu[\gamma_\mu^*(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)]|_{\rm SS} \\
&= -\frac{eN_c}{12\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta \Big[A_{\rm SS}^{\gamma 3\pi} + B_{\rm SS}^{\gamma 3\pi}\cdot D_\omega(p^2) \\
&\quad + \frac{C_{\rm SS}^{\gamma 3\pi}}{3}\cdot\{D_\rho((q_+ + q_-)^2) + D_\rho((q_- + q_0)^2) \\
&\quad + D_\rho((q_0 + q_+)^2)\} + \frac{D_{\rm SS}^{\gamma 3\pi}}{3}\cdot D_\omega(p^2) \\
&\quad \cdot\{D_\rho((q_+ + q_-)^2) + D_\rho((q_- + q_0)^2) \\
&\quad + D_\rho((q_0 + q_+)^2)\}\Big].
\end{aligned} \tag{C45}$$

Since the resultant form is equivalent to that obtained from our method, which is manifestly gauge invariant by construction [see Eqs. (2.36), (2.37), and (2.41)], the low-energy theorem (4.79) is actually satisfied:

$$A_{\rm SS}^{\gamma 3\pi} + B_{\rm SS}^{\gamma 3\pi} + C_{\rm SS}^{\gamma 3\pi} + D_{\rm SS}^{\gamma 3\pi} = 1. \tag{C46}$$

In contrast, a naive truncation as in Eq. (2.33), which corresponds to taking $A_{\rm SS}^{\gamma 3\pi} = B_{\rm SS}^{\gamma 3\pi} = C_{\rm SS}^{\gamma 3\pi} = 0$ in Eqs. (C41)–(C43), would provide us with

$$\begin{aligned}
&\Gamma_\mu[\gamma_\mu^*(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)]|_{\rm SS}^{\rm trun} \\
&= -\frac{eN_c}{36\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta D_{\rm SS}^{\gamma 3\pi}\frac{m_\omega^2}{m_\omega^2 - p^2} \\
&\quad \times\left[\frac{m_\rho^2}{m_\rho^2 - (q_+ + q_-)^2} + (q_+ \leftrightarrow q_0) + (q_- \leftrightarrow q_0)\right],
\end{aligned} \tag{C47}$$

with $D_{\rm SS}^{\gamma 3\pi} \simeq 1.5$ from Eq. (4.85). At the low-energy limit $p^2, q_{\pm,0}^2 \to 0$, we have

$$\begin{aligned}
&\Gamma_\mu[\gamma_\mu^*(p), \pi^0(q_0), \pi^+(q_+), \pi^-(q_-)]|_{\rm SS}^{\rm trun} \\
&\xrightarrow{p^2, q_{\pm,0}^2 \to 0} -\frac{eN_c}{12\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta (D_{\rm SS}^{\gamma 3\pi}) \\
&\neq -\frac{eN_c}{12\pi^2 F_\pi^3}\epsilon_{\mu\nu\alpha\beta} q_0^\nu q_+^\alpha q_-^\beta,
\end{aligned} \tag{C48}$$

which contradicts the low-energy theorem (4.79) and hence breaks EM gauge symmetry. It should be noted again that the ρ/ω truncation (C47) is different from the ρ/ω-meson dominance (4.80) which is gauge invariant.

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