PHYSICAL REVIEW D **82,** 073010 (2010)

Universal mass matrix for quarks and leptons and *CP* violation

J. Barranco*

Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Am Mühlenberg 1, D-14476 Golm, Germany

F. González Canales† and A. Mondragón‡

Instituto de Física, Universidad Nacional Autónoma de México, 04510, México D.F., Mexico

(Received 24 May 2010; published 20 October 2010)

The measurements of the neutrino and quark mixing angles satisfy the empirical relations called quark-lepton complementarity. These empirical relations suggest the existence of a correlation between the mixing matrices of leptons and quarks. In this work, we examine the possibility that this correlation between the mixing angles of quarks and leptons originates in the similar hierarchy of quarks and charged lepton masses and the seesaw mechanism type I, that gives mass to the Majorana neutrinos. We assume that the similar mass hierarchies of charged lepton and quark masses allows us to represent all the mass matrices of Dirac fermions in terms of a universal form with four texture zeroes.

DOI: 10.1103/PhysRevD.82.073010 PACS numbers: 12.15.Ff, 14.60.Pq, 12.15.Hh, 14.60.St

I. INTRODUCTION

The neutrino oscillations between different flavour states were measured in a series of experiments with atmospheric neutrinos [1], solar neutrinos [2], and neutrinos produced in nuclear reactors [3] and accelerators [4]. As a result of the global combined analysis including all dominant and subdominant oscillation effects, the difference of the squared neutrino masses and the mixing angles in the lepton mixing matrix, U_{PMNS}, were determined at 1σ (3σ) confidence level [5]:

$$\Delta m_{21}^2 = 7.67_{-0.21}^{+0.22}\begin{pmatrix}+0.67\\-0.61\end{pmatrix}\times 10^{-5}\ \text{eV}^2, \qquad \Delta m_{31}^2 = \begin{cases}-2.37 \pm 0.15(^{+0.43}_{-0.46}) \times 10^{-3}\ \text{eV}^2, (m_{\nu_2} > m_{\nu_1} > m_{\nu_3}).\\ +2.46 \pm 0.15(^{+0.47}_{-0.42}) \times 10^{-3}\ \text{eV}^2, (m_{\nu_3} > m_{\nu_2} > m_{\nu_1}).\end{cases} \tag{1}$$

$$\theta_{12}^l = 34.5^\circ \pm 1.4\begin{pmatrix}+4.8\\-4.0\end{pmatrix}, \qquad \theta_{23}^l = 42.3^{o}{}_{-3.3}^{+5.1}\begin{pmatrix}+11.3\\-7.7\end{pmatrix}, \qquad \theta_{13}^l = 0.0^{o}{}_{-0.0}^{+7.9}\begin{pmatrix}+12.9\\-0.0\end{pmatrix}. \tag{2}$$

Thus, values of the magnitudes of all nine elements of the lepton mixing matrix, U_{PMNS}, at 90% C.L., are

$$U_{PMNS} = \begin{pmatrix} 0.80 \to 0.84 & 0.53 \to 0.60 & 0.00 \to 0.17\\ 0.29 \to 0.52 & 0.51 \to 0.69 & 0.61 \to 0.76\\ 0.26 \to 0.50 & 0.46 \to 0.66 & 0.64 \to 0.79 \end{pmatrix}. \tag{3}$$

The CHOOZ experiment determined an upper bound for the θ_{13}^l mixing angle [6]. The latest analyses give the following best values [7,8]:

$$\theta_{13}^l = -0.07_{-0.11}^{+0.18} \tag{4}$$

and [at 1σ (3σ)]

$$\theta_{13}^l = 5.6_{-2.7}^{+3.0}(\leq 12.5)^\circ, \qquad \theta_{13}^l = 5.1_{-3.3}^{+3.0}(\leq 12.0)^\circ; \tag{5}$$

see also [9]. On the other hand, in the last years extensive research has been done in the precise determination of the values of the V_{CKM} quark mixing matrix elements. The most precise fit results for the values of the magnitudes of all nine Cabibbo-Kobayashi-Maskawa (CKM) elements are [10]

*jbarranc@aei.mpg.de
†ffelix@fisica.unam.mx
‡mondra@fisica.unam.mx

$$V_{CKM} = \begin{pmatrix} 0.974\,19 \pm .000\,22 & 0.2257 \pm .0010 & 0.00359 \pm .000\,16 \\ 0.2256 \pm .0010 & 0.973\,34 \pm .00023 & 0.0415^{+.0010}_{-.0011} \\ 0.008\,74^{+.000\,26}_{-.000\,37} & 0.0407 \pm .0010 & 0.999\,133^{+.000\,044}_{-.000\,043} \end{pmatrix} \tag{6}$$

and the Jarlskog invariant is

$$J^q = (3.05^{+0.19}_{-0.20}) \times 10^{-5}. \tag{7}$$

We also have the three angles of the unitarity triangle with the following reported best values [10]:

$$\alpha = (88^{+6}_{-5})^\circ, \qquad \beta = (21.46 \pm 0.71)^\circ, \qquad \gamma = (77^{+30}_{-32})^\circ. \tag{8}$$

Each of the elements of the V_{CKM} matrix can be extracted from a large number of decays and, for the purpose of our analysis, will be considered as independent. Hence, current knowledge of the mixing angles for the quark sector can be summarized at 1σ as [10]

$$\begin{aligned} \sin\theta^q_{12} &= 0.2257 \pm 0.001, \\ \sin\theta^q_{23} &= 0.0415^{+0.001}_{-0.0011}, \\ \sin\theta^q_{13} &= 0.003\,59 \pm 0.000\,16. \end{aligned} \tag{9}$$

The solar mixing angle θ^l_{12} and the corresponding mixing angle in the quark sector, the Cabibbo angle θ^q_{12}, satisfy an interesting and intriguing numerical relation (at 90% confidence level) [11],

$$\theta^l_{12} + \theta^q_{12} \approx 45^\circ + 2.5^\circ \pm 1.5^\circ; \tag{10}$$

see also [12]. Equation (10) relates the 1–2 mixing angles in the quark and lepton sectors, it is commonly known as quark-lepton complementarity relation (QLC) and, if not accidental, it could imply a quark-lepton symmetry. A second QLC relation between the atmospheric and 2–3 mixing angles is also satisfied [13],

$$\theta^l_{23} + \theta^q_{23} = (44.67^{+5.1}_{-3.3})^\circ. \tag{11}$$

However, this is not as interesting as (10) because θ^q_{23} is only about 2°, and the corresponding QLC relation would be satisfied, within the errors, even if the angle θ^q_{23} had been zero, as long as θ^l_{23} is close to the maximal value $\pi/4$. A third possible QLC relation is not realized at all, or at least not realized in the same way, since it is less than 10° [13]:

$$\theta^l_{13} + \theta^q_{13} < 8.1^\circ. \tag{12}$$

Equations (10)–(12) are known as the extended quark-lepton complementarity; for a review see [14]. The extended QLC relations could imply a quark-lepton symmetry [14] or a quark-lepton unification [15]. A systematic numerical exploration of all CP conserving textures of the neutrino mass matrix compatible to the QLC relations and the experimental information on neutrino mixings is given in [16].

The neutrino oscillations do not provide information about either the absolute mass scale or if neutrinos are Dirac or Majorana particles [17]. Thus, one of the most fundamental problems of the neutrinos physics is the question of the nature of massive neutrinos. A direct way to reveal the nature of massive neutrinos is to investigate processes in which the total lepton number is not conserved [18]. The matrix elements for these processes are proportional to the effective Majorana neutrino masses, which are defined as

$$\langle m_{ll} \rangle \equiv \sum_{j=1}^{3} m_{\nu_j} U^2_{lj}, \qquad l = e, \mu, \tau, \tag{13}$$

where m_{ν_j} are the neutrino Majorana masses and U_{lj} are the elements of the lepton mixing matrix.

In this work, we will focus our attention on understanding the nature of the QLC relation and finding possible values for the effective Majorana neutrino masses. Thus, we made a unified treatment of quarks and leptons, where we assumed that the charged lepton and quark mass matrices have the same generic form with four texture zeroes from a universal S_3 flavor symmetry and its sequential explicit breaking.

II. UNIVERSAL MASS MATRIX WITH A FOUR ZEROES TEXTURE

In particle physics, the imposition of a flavor symmetry has been successful in reducing the number of parameters of the standard model. Recent flavor symmetry models are reviewed in [19]; see also the references therein. In particular, a permutational S_3 flavor symmetry and its sequential explicit breaking allows us to take the same generic form for the mass matrices of all Dirac fermions, conventionally called the generalized Fritzsch ansatz with four texture zeroes [20,21]:

$$\mathbf{M_i} = \begin{pmatrix} 0 & A_i & 0 \\ A_i^* & B_i & C_i \\ 0 & C_i & D_i \end{pmatrix}, \qquad i = u, d, l, \nu_D, \tag{14}$$

where B_i, C_i, and D_i are real, while $A_i = |A_i|e^{i\phi_i}$ with $\phi_i = \arg\{A_i\}$.

In the most general case, all entries in the Hermitian mass matrix M_i are complex and nonvanishing. However, without loss of generality, by means of a common unitary transformation of the Dirac fields Ψ_{u,ν_D} and $\Psi_{d,l}$, it is always possible to change to a new flavor basis where the off-diagonal elements $(M_i)_{13} = (M_i)_{31}$ vanish [21]. The vanishing of the diagonal elements $(M_{u,\nu_D})_{11}$ and $(M_{d,l})_{11}$, constrains the physics and allows for the predictions of the

Cabibbo angle as function of the u and d-type quark masses in the quark sector and the solar angle as function of the charged leptons and Majorana neutrinos masses in the leptonic sector in good agreement with the experimental values.

Then, in the quark sector, M_u and M_d totally have four texture zeroes and, in the leptonic sector, M_{ν_D} and M_e, totally have four texture zeroes (here a pair of off-diagonal texture zeroes are counted as one zero, due to the Hermiticity of M_i) [21]. Hence, following a common convention we will refer to M_i as a generalized Fritzsch ansatz with four texture zeros.

Some reasons to propose the validity of a generalized Fritzsch ansatz with four texture zeros as a universal form for the mass matrix of all Dirac fermions in the theory are the following:

(1) The idea of S_3 flavor symmetry and its explicit breaking has been successfully realized as a mass matrix with four texture zeroes in the quark sector to interpret the strong mass hierarchy of up and down type quarks [22].

(2) The quark mixing angles and the CP violating phase, appearing in the V_{CKM} mixing matrix, were computed as explicit, exact functions of the four quark mass ratios $(m_u/m_t, m_c/m_t, m_d/m_b, m_s/m_b)$, one symmetry breaking parameter defined as $Z^{1/2} \equiv \frac{C_i}{B_i}$, and one CP violating phase $\phi_{u-d} = \phi_u - \phi_d$. Assuming that $Z_u = Z_d = Z$, a χ^2 fit of the theoretical expression for $V_{CKM}^{\rm th}$ to the experimentally determined $V_{CKM}^{\rm exp}$ gave $Z^{1/2} = (\frac{81}{32})^{1/2}$ and $\phi_{u-d} = 90°$, in good agreement with the experimental data [20]. This agreement has improved as the precision of the experimental data has improved and, now, it is very good [10].

(3) Since the mass spectrum of the charged leptons exhibits a hierarchy similar to the quark's one, it would be natural to consider the same S_3 symmetry and its explicit breaking to justify the use of the same generic form with four texture zeroes for the charged lepton mass matrix.

(4) As for the Dirac neutrinos, we have no direct information about the absolute values or the relative values of the neutrino masses, but the mass matrix with four texture zeroes can be obtained from an $SO(10)$ neutrino model which describes the data on neutrino masses and mixings well [23]. Furthermore, from supersymmetry arguments, it would be sensible to assume that the Dirac neutrinos have a mass hierarchy similar to that of the u-quarks and it would be natural to take for the Dirac neutrino mass matrix also a matrix with four texture zeroes.

The Hermitian mass matrix (14) may be written in terms of a real symmetric matrix $\bar{M}_i$ and a diagonal matrix of phases $P_i \equiv \mathrm{diag}[1, e^{i\phi_i}, e^{i\phi_i}]$ as follows:

$$M_i = P_i^\dagger \bar{M}_i P_i. \tag{15}$$

The real symmetric matrix $\bar{M}_i$ may be brought to diagonal form by means of an orthogonal transformation,

$$\bar{M}_i = \mathbf{O}_i \mathrm{diag}\{m_{i1}, m_{i2}, m_{i3}\} \mathbf{O}_i^T, \tag{16}$$

where the m_i's are the eigenvalues of M_i and $\mathbf{O}_i$ is a real orthogonal matrix. Now computing the invariants of the real symmetric matrix $\bar{M}_i$, $\mathrm{tr}\{\bar{M}_i\}$, $\mathrm{tr}\{\bar{M}_i^2\}$, and $\det\{\bar{M}_i\}$, we may express the parameters A_i, B_i, C_i, and D_i occurring in (14) in terms of the mass eigenvalues. In this way, we get that the $\bar{M}_i$ matrix ($i = u, d, l, \nu_D$), reparametrized in terms of its eigenvalues and the parameter $D_i \equiv 1 - \delta_i$ is

$$\bar{M}_i = \begin{pmatrix} 0 & \sqrt{\frac{\tilde{m}_{i1}\tilde{m}_{i2}}{1-\delta_i}} & 0 \\ \sqrt{\frac{\tilde{m}_{i1}\tilde{m}_{i2}}{1-\delta_i}} & \tilde{m}_{i1} - \tilde{m}_{i2} + \delta_i & \sqrt{\frac{\delta_i}{(1-\delta_i)} f_{i1} f_{i2}} \\ 0 & \sqrt{\frac{\delta_i}{(1-\delta_i)} f_{i1} f_{i2}} & 1 - \delta_i \end{pmatrix}, \tag{17}$$

where $\tilde{m}_{i1} = \frac{m_{i1}}{m_{i3}}$, $\tilde{m}_{i2} = \frac{|m_{i2}|}{m_{i3}}$,

$$f_{i1} = 1 - \tilde{m}_{i1} - \delta_i, \qquad f_{i2} = 1 + \tilde{m}_{i2} - \delta_i. \tag{18}$$

The small parameters δ_i are also functions of the mass ratios and the flavor symmetry breaking parameter $Z_i^{1/2}$ [20]. The flavor symmetry breaking parameter $Z_i^{1/2}$, which measures the mixing of singlet and doublet irreducible representations of S_3, is defined as the ratio

$$Z_i^{1/2} = \frac{(M_i)_{23}}{(M_i)_{22}}. \tag{19}$$

It is related with the parameters δ_i by the following cubic equation [20]:

$$\begin{aligned} \delta_i^3 &- \frac{1}{Z_i + 1}(2 + \tilde{m}_{i2} - \tilde{m}_{i1} + (1 + 2(\tilde{m}_{i2} - \tilde{m}_{i1}))Z_i)\delta_i^2 \\ &+ \frac{1}{Z_i + 1}(Z_i(\tilde{m}_{i2} - \tilde{m}_{i1})(2 + \tilde{m}_{i2} - \tilde{m}_{i1}) \\ &+ (1 + \tilde{m}_{i2})(1 - \tilde{m}_{i1}))\delta_i + \frac{Z_i(\tilde{m}_{i2} - \tilde{m}_{i1})^2}{Z_i + 1} = 0. \end{aligned} \tag{20}$$

Thus, the small parameter δ_i is obtained as the solution of the cubic equation (20), which vanishes when Z_i vanishes. The last term in the left-hand side of (20) is equal to the product of the three roots of (20). Therefore, the root that vanishes when Z_i vanishes may be written as

$$\delta_i = \frac{Z_i}{Z_i + 1} \frac{(\tilde{m}_{i2} - \tilde{m}_{i1})^2}{W_i(Z)}, \tag{21}$$

where $W_i(Z)$ is the product of the two roots of (20) which do not vanish when Z_i vanishes. The explicit form of $W_i(Z)$ is [20]

$$W_i(Z) = [p_i^3 + 2q_i^2 + 2q\sqrt{p_i^3 + q_i^2}]^{1/3} - |p_i| + [p_i^3 + 2q_i^2 - 2q_i\sqrt{p_i^3 + q_i^2}]^{1/3} + \frac{1}{9}(Z_i(2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1}) + 2)^2 - \frac{1}{3}([q_i + \sqrt{p_i^3 + q_i^2}]^{1/3} + [q_i - \sqrt{p_i^3 + q_i^2}]^{1/3}) \times (Z_i(2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1}) + 2) \tag{22}$$

with

$$p_i = -\frac{1}{3}\frac{Z_i}{Z_i + 1}(Z_i(2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + \tilde{m}_{i2} - \tilde{m}_{i1} + 2)^2 + \frac{1}{Z_i + 1}[Z_i(\tilde{m}_{i2} - \tilde{m}_{i1})(\tilde{m}_{i2} - \tilde{m}_{i1} + 2) \times (1 + \tilde{m}_{i2})(1 - \tilde{m}_{i1})], \tag{23}$$

$$q_i = -\frac{1}{27}\frac{1}{(Z_i + 1)^3}(Z_i(2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + \tilde{m}_{i2} - \tilde{m}_{i1} + 2)^3 + \frac{1}{6}\frac{1}{(Z_i + 1)^2}[Z_i(\tilde{m}_{i2} - \tilde{m}_{i1}) \times (\tilde{m}_{i2} - \tilde{m}_{i1} + 2)(1 + \tilde{m}_{i2})(1 - \tilde{m}_{i1})] \times (Z_i(2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + \tilde{m}_{i2} - \tilde{m}_{i1} + 2). \tag{24}$$

Also, the values allowed for the parameters δ_i are in the following range: $0 < \delta_i < 1 - \tilde{m}_{i1}$.

Now, the entries in the real orthogonal matrix $\mathbf{O}$, Eq. (16), may also be expressed in terms of the eigenvalues of the mass matrix (14) as

$$\mathbf{O_i} = \begin{pmatrix} \left[\frac{\tilde{m}_{i2} f_{i1}}{\mathcal{D}_{i1}}\right]^{1/2} & -\left[\frac{\tilde{m}_{i1} f_{i2}}{\mathcal{D}_{i2}}\right]^{1/2} & \left[\frac{\tilde{m}_{i1}\tilde{m}_{i2}\delta_i}{\mathcal{D}_{i3}}\right]^{1/2} \\ \left[\frac{\tilde{m}_{i1}(1-\delta_i) f_{i1}}{\mathcal{D}_{i1}}\right]^{1/2} & \left[\frac{\tilde{m}_{i2}(1-\delta_i) f_{i2}}{\mathcal{D}_{i2}}\right]^{1/2} & \left[\frac{(1-\delta_i)\delta_i}{\mathcal{D}_{i3}}\right]^{1/2} \\ -\left[\frac{\tilde{m}_{i1} f_{i2}\delta_i}{\mathcal{D}_{i1}}\right]^{1/2} & -\left[\frac{\tilde{m}_{i2} f_{i1}\delta_i}{\mathcal{D}_{i2}}\right]^{1/2} & \left[\frac{f_{i1} f_{i2}}{\mathcal{D}_{i3}}\right]^{1/2} \end{pmatrix}, \tag{25}$$

where,

$$\begin{aligned} \mathcal{D}_{i1} &= (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 - \tilde{m}_{i1}), \\ \mathcal{D}_{i2} &= (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 + \tilde{m}_{i2}), \\ \mathcal{D}_{i3} &= (1 - \delta_i)(1 - \tilde{m}_{i1})(1 + \tilde{m}_{i2}). \end{aligned} \tag{26}$$

III. SEESAW MECHANISM AND PHASES OF THE LEFT-HANDED NEUTRINO MASS MATRIX

The left-handed Majorana neutrinos naturally acquire their small masses through an effective type-I seesaw mechanism of the form

$$M_{\nu_L} = M_{\nu_D} M_{\nu_R}^{-1} M_{\nu_D}^T, \tag{27}$$

where M_{ν_D} and M_{ν_R} denote the Dirac and right-handed Majorana neutrino mass matrices, respectively. The symmetry of the mass matrix of the left-handed Majorana neutrinos, $M_{\nu_L} = M_{\nu_L}^T$, and the seesaw mechanism of type I, Eq. (27), fix the form of the right-handed Majorana neutrinos mass matrix, M_{ν_R}, which has to be nonsingular and symmetric. Further restrictions on M_{ν_R} follow from requiring that M_{ν_L} also has a texture with four zeroes, as will be shown below. With this purpose in mind, the seesaw mechanism, Eq. (27), may be written in a more explicit form as

$$M_{\nu_L} = \frac{1}{\det(M_{\nu_R})} M_{\nu_D} \operatorname{adj}(M_{\nu_R}) M_{\nu_D}^T, \tag{28}$$

where $\det(M_{\nu_R})$ and $\operatorname{adj}(M_{\nu_R})$ are the determinant and adjugate matrix of M_{ν_R}, respectively. Now, if we consider the more general form of a complex symmetric matrix of 3×3

$$M_{\nu_R} = \begin{pmatrix} g_{\nu_R} & a_{\nu_R} & e_{\nu_R} \\ a_{\nu_R} & b_{\nu_R} & c_{\nu_R} \\ e_{\nu_R} & c_{\nu_R} & d_{\nu_R} \end{pmatrix} \tag{29}$$

to represent the right-handed Majorana neutrinos mass matrix, we may write Eq. (28) in a more explicit form if we express $\det(M_{\nu_R})$ and $\operatorname{adj}(M_{\nu_R})$ in terms of the cofactors of the elements of the matrix M_{ν_R}. Then,

$$\det(M_{\nu_R}) = g_{\nu_R} X_{11} - a_{\nu_R} X_{12} + e_{\nu_R} X_{13} \tag{30}$$

and

$$M_{\nu_L} = \frac{1}{\det(M_{\nu_R})} \begin{pmatrix} G_{\nu_L} & A_{\nu_L} & E_{\nu_L} \\ A_{\nu_L} & B_{\nu_L} & C_{\nu_L} \\ E_{\nu_L} & C_{\nu_L} & D_{\nu_L} \end{pmatrix}, \tag{31}$$

where

$$\begin{aligned} G_{\nu_L} &= X_{22} A_{\nu_D}^2, \\ A_{\nu_L} &= -X_{12}|A_{\nu_D}|^2 + X_{22} A_{\nu_D} B_{\nu_D} - X_{23} A_{\nu_D} C_{\nu_D}, \\ B_{\nu_L} &= X_{11} A_{\nu_D}^{*2} + X_{22} B_{\nu_D}^2 + X_{33} C_{\nu_D}^2 - 2X_{12} A_{\nu_D}^* B_{\nu_D} \\ &\quad + 2X_{13} A_{\nu_D}^* C_{\nu_D} - 2X_{23} B_{\nu_D} C_{\nu_D}, \\ E_{\nu_L} &= X_{22} A_{\nu_D} C_{\nu_D} - X_{23} A_{\nu_D} D_{\nu_D}, \\ C_{\nu_L} &= X_{13} A_{\nu_D}^* D_{\nu_D} - X_{12} A_{\nu_D}^* C_{\nu_D} + X_{22} B_{\nu_D} C_{\nu_D} \\ &\quad - X_{23}(B_{\nu_D} D_{\nu_D} + C_{\nu_D}^2) + X_{33} C_{\nu_D} D_{\nu_D}, \\ D_{\nu_L} &= X_{22} C_{\nu_D}^2 - 2X_{23} C_{\nu_D} D_{\nu_D} + X_{33} D_{\nu_D}^2. \end{aligned} \tag{32}$$

In these expressions, the X_{nm} (m, $n = 1, 2, 3$) are the cofactors of the corresponding elements of the $\operatorname{adj}(M_{\nu_R})$ matrix.[1]

[1]The cofactors of the elements of M_{ν_R} matrix, are defined as $X_{nm} = (-1)^{n+m} \det(H_{nm})$, where H_{nm} is obtained by deleting the n row and the m column of M_{ν_R} matrix.

From Eqs. (31) and (32), when conditions $X_{22} = X_{23} = 0$ are satisfied, the mass matrix of the left-handed Majorana neutrinos will have the same universal form with four texture zeroes as the Dirac mass matrices. These conditions are equivalent to

$$g_{\nu_R} d_{\nu_R} = e_{\nu_R}^2, \qquad g_{\nu_R} c_{\nu_R} = a_{\nu_R} e_{\nu_R}. \tag{33}$$

Thus, we obtain the relation

$$\frac{a_{\nu_R}}{c_{\nu_R}} = \frac{e_{\nu_R}}{d_{\nu_R}}. \tag{34}$$

For nonvanishing $\det(M_{\nu_R})$, these conditions (33) are satisfied, if

$$g_{\nu_R} = 0 \quad \text{and} \quad e_{\nu_R} = 0. \tag{35}$$

If we extend the meaning of a mass matrix with four texture zeroes, defined in (14), to include the symmetric mass matrix of the right-handed Majorana neutrinos, M_{ν_R} [24], which is non-Hermitian, we could say that the matrix with four zeroes texture is invariant under the action of the seesaw mechanism of type I [13,21,24]. It may also be noticed that, if we set $b_{\nu_R} = 0$ or/and $c_{\nu_R} = 0$, the resulting expression for M_{ν_L} still has four texture zeroes. Therefore, M_{ν_L} may also have four texture zeroes when M_{ν_R} has four, three, or two texture zeroes (the two last cases are called Fritzsch textures).

Let us further assume that the phases in the entries of the M_{ν_R} may be factorized out as

$$M_{\nu_R} = R\bar{M}_{\nu_R}R, \tag{36}$$

where

$$\bar{M}_{\nu_R} = \begin{pmatrix} 0 & a_{\nu_R} & 0 \\ a_{\nu_R} & |b_{\nu_R}| & |c_{\nu_R}| \\ 0 & |c_{\nu_R}| & d_{\nu_R} \end{pmatrix}, \tag{37}$$

and $R \equiv \mathrm{diag}[e^{-i\phi_c}, e^{i\phi_c}, 1]$ with $\phi_c \equiv \arg\{c_{\nu_R}\}$. Then, the type-I seesaw mechanism takes the form

$$M_{\nu_L} = P_D^\dagger \bar{M}_{\nu_D} P_D R^\dagger \bar{M}_{\nu_R}^{-1} R^\dagger P_D \bar{M}_{\nu_D} P_D^\dagger, \tag{38}$$

and the mass matrix of the left-handed neutrinos has the following form with four texture zeroes[2]:

$$M_{\nu_L} = \begin{pmatrix} 0 & a_{\nu_L} & 0 \\ a_{\nu_L} & b_{\nu_L} & c_{\nu_L} \\ 0 & c_{\nu_L} & d_{\nu_L} \end{pmatrix}, \tag{39}$$

where

$$\begin{aligned} a_{\nu_L} &= \frac{|a_{\nu_D}|^2}{a_{\nu_R}}, \\ b_{\nu_L} &= \frac{c_{\nu_D}^2}{d_{\nu_R}} + \frac{|c_{\nu_R}|^2 - |b_{\nu_R}| d_{\nu_R}}{d_{\nu_R}} \frac{|a_{\nu_D}|^2}{a_{\nu_R}^2} e^{i2(\phi_c - \phi_{\nu_D})} \\ &\quad + 2\frac{|a_{\nu_D}|}{|a_{\nu_R}|}\left(b_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{c_{\nu_D}|c_{\nu_R}|}{d_{\nu_R}} e^{i(\phi_c - \phi_{\nu_D})}\right), \\ c_{\nu_L} &= \frac{c_{\nu_D} d_{\nu_D}}{d_{\nu_R}} + \frac{|a_{\nu_D}|}{|a_{\nu_R}|}\left(c_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{|c_{\nu_R}| d_{\nu_D}}{d_{\nu_R}} e^{i(\phi_c - \phi_{\nu_D})}\right), \\ d_{\nu_L} &= \frac{d_{\nu_D}^2}{d_{\nu_R}}. \end{aligned} \tag{40}$$

The elements a_{ν_L} and d_{ν_L} are real, while b_{ν_L} and c_{ν_L} are complex. Notice that the phase factors appearing in Eqs. (38) and (40) are fully determined by the seesaw mechanism and our choice of a generalized Fritzsch ansatz with four texture zeroes for the mass matrices of all Dirac fermions and the complex symmetric, but non-Hermitian, mass matrix of the right-handed Majorana neutrinos.

Now, to diagonalize the left-handed Majorana neutrino mass matrix M_{ν_L} by means of a unitary matrix, we need to construct the Hermitian matrices $M_{\nu_L} M_{\nu_L}^\dagger$ and $M_{\nu_L}^\dagger M_{\nu_L}$, which can be diagonalized with unitary matrices through the following transformations:

$$\begin{aligned} U_R^\dagger M_{\nu_L}^\dagger M_{\nu_L} U_R &= \mathrm{diag}[|m_{\nu_1}^s|^2, |m_{\nu_2}^s|^2, |m_{\nu_3}^s|^2], \\ U_L^\dagger M_{\nu_L} M_{\nu_L}^\dagger U_L &= \mathrm{diag}[|m_{\nu_1}^s|^2, |m_{\nu_2}^s|^2, |m_{\nu_3}^s|^2], \end{aligned} \tag{41}$$

where the $m_{\nu_j}^s$ ($j = 1, 2, 3$) are the singular values of the M_{ν_L} matrix. Thus, with the help of the symmetry of the matrix (39) and the transformations (41), the left-handed Majorana neutrino mass matrix, M_{ν_L}, is diagonalized by a unitary matrix

$$U_\nu^\dagger M_{\nu_L} U_\nu^* = \mathrm{diag}[|m_{\nu_1}^s|, |m_{\nu_2}^s|, |m_{\nu_3}^s|], \tag{42}$$

where $U_\nu \equiv U_L \mathcal{K}$ and $\mathcal{K} \equiv \mathrm{diag}[e^{i\eta_1/2}, e^{i\eta_2/2}, e^{i\eta_3/2}]$ is the diagonal matrix of the Majorana phases. From the previous analysis, the matrix M_{ν_L} has two nonignorable phases which are

$$\phi_1 \equiv \arg\{b_{\nu_L}\} \quad \text{and} \quad \phi_2 \equiv \arg\{c_{\nu_L}\}. \tag{43}$$

However, to describe the phenomenology of neutrino masses and mixing, only one phase in M_{ν_L} is required. Therefore, without loss of generality, we may chose $\phi_1 = 2\phi_2 = 2\varphi$ and the following relationship is fulfilled[3]:

$$\tan\phi_1 = \frac{2\Im m c_{\nu_L} \Re e c_{\nu_L}}{(\Re e c_{\nu_L})^2 - (\Im m c_{\nu_L})^2}. \tag{44}$$

[2]The seesaw invariance of the four zeroes mass matrix of the Majorana neutrino is also derived in [24]. However, these authors ignored the phases in the elements of mass matrices in their discussion.

[3]The general case, when $\phi_1 \neq 2\phi_2$ is slightly more complicated. This case will be treated in detail in a later paper.

In this case, the analysis simplifies since the phases in M_{ν_L} may be factorized out as

$$M_{\nu_L} = Q\bar{M}_{\nu_L}Q, \tag{45}$$

where Q is a diagonal matrix of phases $Q \equiv \mathrm{diag}[e^{-i\varphi}, e^{i\varphi}, 1]$ and $\bar{M}_{\nu_L}$ is a real symmetric matrix. Then, the matrix M_{ν_L}, can be diagonalized by a unitary matrix through the transformation

$$U_\nu^\dagger M_{\nu_L} U_\nu^* = \mathrm{diag}[m_{\nu_1}, m_{\nu_2}, m_{\nu_3}]; \tag{46}$$

where m_{ν_j} ($j = 1, 2, 3$) are the eigenvalues of the matrix M_{ν_L}, and the unitary matrix is $U_\nu \equiv Q\mathbf{O}_\nu\mathcal{K}$ where $\mathbf{O}_\nu$ is the orthogonal real matrix (25), that diagonalizes the real symmetric matrix $\bar{M}_{\nu_L}$.

It is also important to mention that when the Hermitian matrix with four texture zeroes defined in Eq. (14) is taken as a universal mass matrix for all Dirac fermions and right-handed Majorana neutrinos [13], the phases of all entries in the right-handed Majorana neutrino mass matrix are fixed at the numerical value of $\phi_{\nu_R} = n\pi$. Thus, the right-handed Majorana neutrino mass matrix is real and symmetric and has the form with four texture zeroes shown in (14). In the more general case in which the Dirac fermions and right-handed neutrino mass matrices are represented by Hermitian matrices, that can be written in polar form as $A = P^\dagger \bar{A} P$, where P is a diagonal matrix of phases and $\bar{A}$ is a real symmetric matrix; the symmetry of the left-handed Majorana neutrino mass matrix also fixes all phases in the mass matrix of the right-handed neutrinos at the numerical value $\phi_{\nu_R} = n\pi$. Hence, the only undetermined phases in the mass matrix of the left-handed Majorana neutrinos M_{ν_L} are the phases ϕ_{ν_D}, coming from the mass matrix of the Dirac neutrinos.

IV. MIXING MATRICES

The quark and lepton flavor mixing matrices, U_{PMNS} and V_{CKM}, arise from the mismatch between diagonalization of the mass matrices of u and d type quarks [10] and the diagonalization of the mass matrices of charged leptons and left-handed neutrinos [25], respectively,

$$U_{PMNS} = U_l^\dagger U_\nu, \qquad V_{CKM} = U_u U_d^\dagger. \tag{47}$$

Therefore, in order to obtain the unitary matrices appearing in (47) and get predictions for the flavor mixing angles and CP violating phases, we should specify the mass matrices. In the quark sector, the unitarity of V_{CKM} leads to the relations $\sum_i V_{ij}V_{ik}^* = \delta_{jk}$ and $\sum_j V_{ij}V_{kj}^* = \delta_{ik}$. The vanishing combinations can be represented as triangles in a complex plane. The area of all triangles is equal to half of the Jarlskog invariant, J_q [26], which is a rephasing invariant measure of CP violation. The term unitarity triangle is usually reserved for the triangle obtained from the relation $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$. In this case the Jarlskog invariant is

$$J_q = \Im m[V_{us}V_{cs}^*V_{ub}^*V_{cb}], \tag{48}$$

and the inner angles of the unitarity triangle are

$$\alpha \equiv \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \quad \beta \equiv \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \quad \gamma \equiv \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right). \tag{49}$$

For the lepton sector, when the left-handed neutrinos are Majorana particles, the mixing matrix is defined as [27] $U_{PMNS} = U_l^\dagger U_L K$ where $K \equiv \mathrm{diag}[1, e^{i\beta_1}, e^{i\beta_2}]$ is the diagonal matrix of the Majorana CP violating phases. Also in the case of three neutrino mixing there are three CP violation rephasing invariants [25], associated with the three CP violating phases present in the U_{PMNS} matrix. The rephasing invariant related to the Dirac phase, analogous to the Jarlskog invariant in the quark sector, is given by

$$J_l \equiv \Im m[U_{e1}^*U_{\mu3}^*U_{e3}U_{\mu1}]. \tag{50}$$

The rephasing invariant J_l controls the magnitude of CP violation effects in neutrino oscillations and is a directly observable quantity. The other two rephasing invariants associated with the two Majorana phases in the U_{PMNS} matrix can be chosen as

$$S_1 \equiv \Im m[U_{e1}U_{e3}^*], \qquad S_2 \equiv \Im m[U_{e2}U_{e3}^*]. \tag{51}$$

These rephasing invariants are not uniquely defined, but the ones shown in Eqs. (50) and (51) are relevant for the definition of the effective Majorana neutrino mass, m_{ee}, in the neutrinoless double beta decay.

A. Mixing matrices as functions of the fermion masses

The unitary matrices $U_{u,d}$ occurring in the definition of V_{CKM}, Eq. (47), may be written in polar form as $U_{u,d} = \mathbf{O}_{u,d}^T P_{u,d}$. In this expression, $P_{u,d}$ is the diagonal matrix of phases appearing in the four texture zeroes mass matrix (15). Then, from (47), the quark mixing matrix takes the form

$$V_{CKM}^{\mathrm{th}} = \mathbf{O}_u^T P^{(u-d)} \mathbf{O}_d, \tag{52}$$

where $P^{(u-d)} = \mathrm{diag}[1, e^{i\phi}, e^{i\phi}]$ with $\phi = \phi_u - \phi_d$, and $\mathbf{O}_{u,d}$, are the real orthogonal matrices (25) that diagonalize the real symmetric mass matrices $\bar{M}_i$. A similar analysis shows that U_{PMNS} may also be written as $U_{PMNS} = U_l^\dagger U_\nu$, with $U_{\nu,l} = P_{\nu,l}\mathbf{O}_{\nu,l}$. This matrix takes the form

$$U_{PMNS}^{\mathrm{th}} = \mathbf{O}_l^T P^{(\nu-l)} \mathbf{O}_\nu K, \tag{53}$$

where $P^{(\nu-l)} = \mathrm{diag}[1, e^{i\Phi_1}, e^{i\Phi_2}]$ is the diagonal matrix of the Dirac phases, with $\Phi_1 = 2\varphi - \phi_l$ and $\Phi_2 = \varphi - \phi_l$. The real orthogonal matrices $\mathbf{O}_{\nu,l}$ are defined in Eq. (25). Substitution of the expressions (18) and (26) in the unitary matrices (52) and (53) allows us to express the mixing matrices V^{th}_{CKM} and U^{th}_{PMNS} as explicit functions of the masses of quarks and leptons. For the elements of the V^{th}_{CKM} mixing matrix, we obtained the same theoretical expressions given by Mondragón and Rodríguez-Jauregui [20]:

$$V^{\mathrm{th}}_{CKM} = \begin{pmatrix} V^{\mathrm{th}}_{ud} & V^{\mathrm{th}}_{us} & V^{\mathrm{th}}_{ub} \\ V^{\mathrm{th}}_{cd} & V^{\mathrm{th}}_{cs} & V^{\mathrm{th}}_{cb} \\ V^{\mathrm{th}}_{td} & V^{\mathrm{th}}_{ts} & V^{\mathrm{th}}_{tb} \end{pmatrix}, \tag{54}$$

where

$$\begin{aligned}
V^{\mathrm{th}}_{ud} &= \sqrt{\frac{\tilde{m}_c\tilde{m}_s f_{u1} f_{d1}}{\mathcal{D}_{u1}\mathcal{D}_{d1}}} + \sqrt{\frac{\tilde{m}_u\tilde{m}_d}{\mathcal{D}_{u1}\mathcal{D}_{d1}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u1} f_{d1}} + \sqrt{\delta_u\delta_d f_{u2} f_{d2}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{us} &= -\sqrt{\frac{\tilde{m}_c\tilde{m}_d f_{u1} f_{d2}}{\mathcal{D}_{u1}\mathcal{D}_{d2}}} + \sqrt{\frac{\tilde{m}_u\tilde{m}_s}{\mathcal{D}_{u1}\mathcal{D}_{d2}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u1} f_{d2}} + \sqrt{\delta_u\delta_d f_{u2} f_{d1}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{ub} &= \sqrt{\frac{\tilde{m}_c\tilde{m}_d\tilde{m}_s\delta_d f_{u1}}{\mathcal{D}_{u1}\mathcal{D}_{d3}}} + \sqrt{\frac{\tilde{m}_u}{\mathcal{D}_{u1}\mathcal{D}_{d3}}}\left(\sqrt{(1-\delta_u)(1-\delta_d)\delta_d f_{u1}} - \sqrt{\delta_u f_{u2} f_{d1} f_{d2}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{cd} &= -\sqrt{\frac{\tilde{m}_u\tilde{m}_s f_{u2} f_{d1}}{\mathcal{D}_{u2}\mathcal{D}_{d1}}} + \sqrt{\frac{\tilde{m}_c\tilde{m}_d}{\mathcal{D}_{u2}\mathcal{D}_{d1}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u2} f_{d1}} + \sqrt{\delta_u\delta_d f_{u1} f_{d2}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{cs} &= \sqrt{\frac{\tilde{m}_u\tilde{m}_d f_{u2} f_{d2}}{\mathcal{D}_{u2}\mathcal{D}_{d2}}} + \sqrt{\frac{\tilde{m}_c\tilde{m}_s}{\mathcal{D}_{u2}\mathcal{D}_{d2}}}\left(\sqrt{(1-\delta_u)(1-\delta_d) f_{u2} f_{d2}} + \sqrt{\delta_u\delta_d f_{u1} f_{d1}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{cb} &= -\sqrt{\frac{\tilde{m}_u\tilde{m}_d\tilde{m}_s\delta_d f_{u2}}{\mathcal{D}_{u2}\mathcal{D}_{d3}}} + \sqrt{\frac{\tilde{m}_c}{\mathcal{D}_{u2}\mathcal{D}_{d3}}}\left(\sqrt{(1-\delta_u)(1-\delta_d)\delta_d f_{u2}} - \sqrt{\delta_u f_{u1} f_{d1} f_{d2}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{td} &= \sqrt{\frac{\tilde{m}_u\tilde{m}_c\tilde{m}_s\delta_u f_{d1}}{\mathcal{D}_{u3}\mathcal{D}_{d1}}} + \sqrt{\frac{\tilde{m}_d}{\mathcal{D}_{u3}\mathcal{D}_{d1}}}\left(\sqrt{\delta_u(1-\delta_u)(1-\delta_d) f_{d1}} - \sqrt{\delta_d f_{u1} f_{u2} f_{d2}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{ts} &= -\sqrt{\frac{\tilde{m}_u\tilde{m}_c\tilde{m}_d\delta_u f_{d2}}{\mathcal{D}_{u3}\mathcal{D}_{d2}}} + \sqrt{\frac{\tilde{m}_s}{\mathcal{D}_{u3}\mathcal{D}_{d2}}}\left(\sqrt{\delta_u(1-\delta_u)(1-\delta_d) f_{d2}} - \sqrt{\delta_d f_{u1} f_{u2} f_{d1}}\right)e^{i\phi},\\
V^{\mathrm{th}}_{tb} &= \sqrt{\frac{\tilde{m}_u\tilde{m}_c\tilde{m}_d\tilde{m}_s\delta_u\delta_d}{\mathcal{D}_{u3}\mathcal{D}_{d3}}} + \left(\sqrt{\frac{f_{u1} f_{u2} f_{d1} f_{d2}}{\mathcal{D}_{u3}\mathcal{D}_{d3}}} + \sqrt{\frac{\delta_u\delta_d(1-\delta_u)(1-\delta_d)}{\mathcal{D}_{u3}\mathcal{D}_{d3}}}\right)e^{i\phi}.
\end{aligned} \tag{55}$$

Here, the m's, f's, and $\mathcal{D}$'s are defined in (18) and (26), respectively. These take the form

$$\begin{aligned}
\tilde{m}_{u(d)} &= \frac{m_{u(d)}}{m_{t(b)}},\\
\tilde{m}_{c(s)} &= \frac{m_{c(s)}}{m_{t(b)}},\\
f_{u(d)1} &= (1 - \tilde{m}_{u(d)} - \delta_{u(d)}),\\
f_{u(d)2} &= (1 + \tilde{m}_{c(s)} - \delta_{u(d)}),\\
\mathcal{D}_{u(d)1} &= (1-\delta_{u(d)})(\tilde{m}_{u(d)} + \tilde{m}_{c(s)})(1 - \tilde{m}_{u(d)}),\\
\mathcal{D}_{u(d)2} &= (1-\delta_{u(d)})(\tilde{m}_{u(d)} + \tilde{m}_{c(s)})(1 + \tilde{m}_{u(d)}),\\
\mathcal{D}_{u(d)3} &= (1-\delta_{u(d)})(1 - \tilde{m}_{u(d)})(1 + \tilde{m}_{c(s)}).
\end{aligned} \tag{56}$$

Now, with the help of Eqs. (25) and (53), we obtain the theoretical expression of the elements of the lepton mixing matrix, U^{th}_{PMNS}. This expression has the following form:

$$U^{\mathrm{th}}_{PMNS} = \begin{pmatrix} U^{\mathrm{th}}_{e1} & U^{\mathrm{th}}_{e2}e^{i\beta_1} & U^{\mathrm{th}}_{e3}e^{i\beta_2} \\ U^{\mathrm{th}}_{\mu1} & U^{\mathrm{th}}_{\mu2}e^{i\beta_1} & U^{\mathrm{th}}_{\mu3}e^{i\beta_2} \\ U^{\mathrm{th}}_{\tau1} & U^{\mathrm{th}}_{\tau2}e^{i\beta_1} & U^{\mathrm{th}}_{\tau3}e^{i\beta_2} \end{pmatrix} \tag{57}$$

where

$$
\begin{aligned}
U_{e1}^{\rm th} &= \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_2} f_{l1} f_{\nu 1}}{\mathcal{D}_{l1}\mathcal{D}_{\nu 1}}} + \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_1}}{\mathcal{D}_{l1}\mathcal{D}_{\nu 1}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l1} f_{\nu 1}}\, e^{i\Phi_1} + \sqrt{\delta_l \delta_\nu f_{l2} f_{\nu 2}}\, e^{i\Phi_2}\right),\\
U_{e2}^{\rm th} &= -\sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_1} f_{l1} f_{\nu 2}}{\mathcal{D}_{l1}\mathcal{D}_{\nu 2}}} + \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_2}}{\mathcal{D}_{l1}\mathcal{D}_{\nu 2}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l1} f_{\nu 2}}\, e^{i\Phi_1} + \sqrt{\delta_l \delta_\nu f_{l2} f_{\nu 1}}\, e^{i\Phi_2}\right),\\
U_{e3}^{\rm th} &= \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_1} \tilde{m}_{\nu_2} \delta_\nu f_{l1}}{\mathcal{D}_{l1}\mathcal{D}_{\nu 3}}} + \sqrt{\frac{\tilde{m}_e}{\mathcal{D}_{l1}\mathcal{D}_{\nu 3}}}\left(\sqrt{\delta_\nu (1-\delta_l)(1-\delta_\nu) f_{l1}}\, e^{i\Phi_1} - \sqrt{\delta_e f_{l2} f_{\nu 1} f_{\nu 2}}\, e^{i\Phi_2}\right),\\
U_{\mu 1}^{\rm th} &= -\sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_2} f_{l2} f_{\nu 1}}{\mathcal{D}_{l2}\mathcal{D}_{\nu 1}}} + \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_1}}{\mathcal{D}_{l2}\mathcal{D}_{\nu 1}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l2} f_{\nu 1}}\, e^{i\Phi_1} + \sqrt{\delta_l \delta_\nu f_{l1} f_{\nu 2}}\, e^{i\Phi_2}\right),\\
U_{\mu 2}^{\rm th} &= \sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_1} f_{l2} f_{\nu 2}}{\mathcal{D}_{l2}\mathcal{D}_{\nu 2}}} + \sqrt{\frac{\tilde{m}_\mu \tilde{m}_{\nu_2}}{\mathcal{D}_{l2}\mathcal{D}_{\nu 2}}}\left(\sqrt{(1-\delta_l)(1-\delta_\nu) f_{l2} f_{\nu 2}}\, e^{i\Phi_1} + \sqrt{\delta_l \delta_\nu f_{l1} f_{\nu 1}}\, e^{i\Phi_2}\right),\\
U_{\mu 3}^{\rm th} &= -\sqrt{\frac{\tilde{m}_e \tilde{m}_{\nu_1} \tilde{m}_{\nu_2} \delta_\nu f_{l2}}{\mathcal{D}_{l2}\mathcal{D}_{\nu 3}}} + \sqrt{\frac{\tilde{m}_\mu}{\mathcal{D}_{l2}\mathcal{D}_{\nu 3}}}\left(\sqrt{\delta_\nu (1-\delta_l)(1-\delta_\nu) f_{l2}}\, e^{i\Phi_1} - \sqrt{\delta_l f_{l1} f_{\nu 1} f_{\nu 2}}\, e^{i\Phi_2}\right),\\
U_{\tau 1}^{\rm th} &= \sqrt{\frac{\tilde{m}_e \tilde{m}_\mu \tilde{m}_{\nu_2} \delta_l f_{\nu 1}}{\mathcal{D}_{l3}\mathcal{D}_{\nu 1}}} + \sqrt{\frac{\tilde{m}_{\nu_1}}{\mathcal{D}_{l3}\mathcal{D}_{\nu 1}}}\left(\sqrt{\delta_l (1-\delta_l)(1-\delta_\nu) f_{\nu 1}}\, e^{i\Phi_1} - \sqrt{\delta_\nu f_{l1} f_{l2} f_{\nu 2}}\, e^{i\Phi_2}\right),\\
U_{\tau 2}^{\rm th} &= -\sqrt{\frac{\tilde{m}_e \tilde{m}_\mu \tilde{m}_{\nu_1} \delta_l f_{\nu 2}}{\mathcal{D}_{l3}\mathcal{D}_{\nu 2}}} + \sqrt{\frac{\tilde{m}_{\nu_2}}{\mathcal{D}_{l3}\mathcal{D}_{\nu 2}}}\left(\sqrt{\delta_l (1-\delta_l)(1-\delta_\nu) f_{\nu 2}}\, e^{i\Phi_1} - \sqrt{\delta_\nu f_{l1} f_{l2} f_{\nu 1}}\, e^{i\Phi_2}\right),\\
U_{\tau 3}^{\rm th} &= \sqrt{\frac{\tilde{m}_e \tilde{m}_\mu \tilde{m}_{\nu_1} \tilde{m}_{\nu_2} \delta_l \delta_\nu}{\mathcal{D}_{l3}\mathcal{D}_{\nu 3}}} + \sqrt{\frac{\delta_l \delta_\nu (1-\delta_l)(1-\delta_\nu)}{\mathcal{D}_{l3}\mathcal{D}_{\nu 3}}}\, e^{i\Phi_1} + \sqrt{\frac{f_{l1} f_{l2} f_{\nu 1} f_{\nu 2}}{\mathcal{D}_{l3}\mathcal{D}_{\nu 3}}}\, e^{i\Phi_2}.
\end{aligned}
\tag{58}
$$

In these expressions the $\tilde{m}$'s, f's, and $\mathcal{D}$'s are defined in (18) and (26), respectively. These take the form

$$
\begin{aligned}
\tilde{m}_{\nu_1(e)} &= \frac{m_{\nu_1(e)}}{m_{\nu_3(\tau)}},\\
\tilde{m}_{\nu_2(\mu)} &= \frac{m_{\nu_2(\mu)}}{m_{\nu_3(\tau)}},\\
f_{\nu(l)1} &= (1 - \tilde{m}_{\nu_1(e)} - \delta_{\nu(l)}),\\
f_{\nu(l)2} &= (1 + \tilde{m}_{\nu_2(\mu)} - \delta_{\nu(l)}),\\
\mathcal{D}_{\nu(l)1} &= (1-\delta_{\nu(l)})(\tilde{m}_{\nu_1(e)} + \tilde{m}_{\nu_2(\mu)})(1 - \tilde{m}_{\nu_1(e)}),\\
\mathcal{D}_{\nu(l)2} &= (1-\delta_{\nu(l)})(\tilde{m}_{\nu_1(e)} + \tilde{m}_{\nu_2(\mu)})(1 + \tilde{m}_{\nu_2(\mu)}),\\
\mathcal{D}_{\nu(l)3} &= (1-\delta_{\nu(l)})(1 - \tilde{m}_{\nu_1(e)})(1 + \tilde{m}_{\nu_2(\mu)}).
\end{aligned}
\tag{59}
$$

B. The χ^2 fit for the quark mixing matrix

We made a χ^2 fit of the exact theoretical expressions for the moduli of the entries of the quark mixing matrix $|(V_{CKM}^{\rm th})_{ij}|$ and the inner angles of the unitarity triangle $\alpha^{\rm th}$, $\beta^{\rm th}$, and $\gamma^{\rm th}$ to the experimental values given by Amsler [10]. In this fit, we computed the moduli of the entries of the quark mixing matrix and the inner angles of the unitarity triangle from the theoretical expression (55) with the following numerical values of the quark mass ratios [10]:

$$
\begin{aligned}
\tilde{m}_u &= 2.5469 \times 10^{-5}, & \tilde{m}_c &= 3.9918 \times 10^{-3},\\
\tilde{m}_d &= 1.5261 \times 10^{-3}, & \tilde{m}_s &= 3.2319 \times 10^{-2}.
\end{aligned}
\tag{60}
$$

The numerical values of the mass ratios were left fixed at the values given in Eq. (60) and the parameters δ_u and δ_d were left as free parameters to be varied. Hence, in the χ^2 fit we have 6 degrees of freedom, namely, the nine observable moduli of the entries in the V_{CKM} matrix less the three free parameters to be varied. Once the best values of the parameters δ_u, δ_d, and ϕ were determined, we computed the three inner angles of the unitary triangle from Eq. (49) and the Jarlskog invariant from Eq. (48).

The resulting best values of the parameters δ_u and δ_d are

$$
\delta_u = 3.829 \times 10^{-3}, \qquad \delta_d = 4.08 \times 10^{-4} \tag{61}
$$

and the Dirac CP violating phase is $\phi = 90°$. The best values for the moduli of the entries of the CKM mixing matrix are given in the following expression:

$$
|V_{CKM}^{\rm th}| = \begin{pmatrix} 0.974\,21 & 0.225\,60 & 0.003\,369 \\ 0.225\,45 & 0.973\,35 & 0.041\,736 \\ 0.008\,754 & 0.040\,94 & 0.999\,12 \end{pmatrix} \tag{62}
$$

and inner angles of the unitary triangle

$$
\alpha^{\rm th} = 91.24°, \qquad \beta^{\rm th} = 20.41°, \qquad \gamma^{\rm th} = 68.33°. \tag{63}
$$

The Jarlskog invariant takes the value

$$J_q^{\rm th} = 2.9 \times 10^{-5}. \tag{64}$$

All these results are in good agreement with the experimental values. The minimum value of χ^2 obtained in this fit is 4.6 and the resulting value of χ^2 for degree of freedom is $\frac{\chi^2_{\rm min}}{\rm d.o.f.} = 0.77$.

C. The χ^2 fit for the lepton mixing matrix

In the case of the lepton mixing matrix, we made a χ^2 fit of the theoretical expressions for the moduli of the entries of the lepton mixing matrix $|(U^{\rm th}_{PMNS})_{ij}|$ given in Eq. (58) to the values extracted from experiment as given by Gonzalez-Garcia [5] and quoted in Eq. (3). The computation was made using the following values for the charged lepton masses [10]:

$$\begin{aligned} m_e &= 0.5109 \text{ MeV}, \\ m_\mu &= 105.685 \text{ MeV}, \\ m_\tau &= 1776.99 \text{ MeV}. \end{aligned} \tag{65}$$

We took for the masses of the left-handed Majorana neutrinos a normal hierarchy. This allows us to write the left-handed Majorana neutrino mass ratios in terms of the neutrino squared mass differences and the neutrino mass m_{ν_3} in the following form:

$$\tilde{m}_{\nu_1} = \sqrt{1 - \frac{(\Delta m^2_{32} + \Delta m^2_{21})}{m^2_{\nu_3}}}, \qquad \tilde{m}_{\nu_2} = \sqrt{1 - \frac{\Delta m^2_{32}}{m^2_{\nu_3}}}. \tag{66}$$

The neutrino squared mass differences were obtained from the experimental data on neutrino oscillations given in Gonzalez-Garcia [5] and we left the mass m_{ν_3} as a free parameter of the χ^2 fit. Also, the parameters δ_e, δ_ν, Φ_1, and Φ_2 were left as free parameters to be varied. Hence, in this χ^2 fit we have 4 degrees of freedom. From the best values obtained for m_{ν_3} and the experimental values of Δm^2_{32} and Δm^2_{21}, we obtained the following best values for the neutrino masses:

$$\begin{aligned} m_{\nu_1} &= 2.7 \times 10^{-3} \text{ eV}, \\ m_{\nu_2} &= 9.1 \times 10^{-3} \text{ eV}, \\ m_{\nu_3} &= 4.7 \times 10^{-2} \text{ eV}. \end{aligned} \tag{67}$$

The resulting best values of the parameters δ_e and δ_ν are

$$\delta_l = 0.06, \qquad \delta_\nu = 0.522, \tag{68}$$

and the best values of the Dirac CP violating phases are $\Phi_1 = \pi$ and $\Phi_2 = 3\pi/2$. The best values for the moduli of the entries of the *PMNS* mixing matrix are given in the following expression:

$$|U^{\rm th}_{PMNS}| = \begin{pmatrix} 0.820\,421 & 0.568\,408 & 0.061\,817 \\ 0.385\,027 & 0.613\,436 & 0.689\,529 \\ 0.422\,689 & 0.548\,277 & 0.721\,615 \end{pmatrix}. \tag{69}$$

The value of the rephasing invariant related to the Dirac phase is

$$J_l^{\rm th} = 8.8 \times 10^{-3}. \tag{70}$$

In the absence of experimental information about the Majorana phases β_1 and β_2, the two rephasing invariants S_1 and S_2, Eq. (51), associated with the two Majorana phases in the U_{PMNS} matrix, could not be determined from experimental values. Therefore, in order to make a numerical estimate of Majorana phases, we maximized the rephasing invariants S_1 and S_2, thus obtaining a numerical value for the Majorana phases β_1 and β_2. Then, the maximum values of the rephasing invariants, Eq. (51), are

$$S_1^{\rm max} = -4.9 \times 10^{-2}, \qquad S_2^{\rm max} = 3.4 \times 10^{-2}, \tag{71}$$

with $\beta_1 = -1.4°$ and $\beta_2 = 77°$. In this numerical analysis, the minimum value of the χ^2, corresponding to the best fit, is $\chi^2 = 0.288$ and the resulting value of χ^2 for degree of freedom is $\frac{\chi^2_{\rm min}}{\rm d.o.f.} = 0.075$. All numerical results of the fit are in very good agreement with the values of the moduli of the entries in the matrix U_{PMNS} as given in Gonzalez-Garcia [5].

V. THE MIXING ANGLES

In the standard Particle Data Group parametrization, the entries in the quark and lepton mixing matrices are parametrized in terms of the mixing angles and phases. Thus, the mixing angles are related to the observable moduli of quark (lepton) $V_{CKM}(U_{PMNS})$ through the relations:

$$\begin{aligned} \sin^2\theta_{12}^{q(l)} &= \frac{|V_{us}(U_{e2})|^2}{1 - |V_{ub}(U_{e3})|^2}, \\ \sin^2\theta_{23}^{q(l)} &= \frac{|V_{cb}(U_{\mu 3})|^2}{1 - |V_{ub}(U_{e3})|^2}, \\ \sin^2\theta_{13}^{q(l)} &= |V_{ub}(U_{e3})|^2. \end{aligned} \tag{72}$$

Then, theoretical expression for the quark mixing angles as functions of the quark mass ratios are readily obtained when the theoretical expressions for the moduli of the entries in the CKM mixing matrix, given in Eqs. (55) and (26), are substituted for $|V_{ij}|$ in the right-hand side of Eqs. (72). In this way, and keeping only the leading order terms, we get

$$\sin^2\theta_{12}^{q^{\rm th}} \approx \frac{\frac{\tilde{m}_d}{\tilde{m}_s} + \frac{\tilde{m}_u}{\tilde{m}_c} - 2\sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}\frac{\tilde{m}_d}{\tilde{m}_s}}\cos\phi}{(1 + \frac{\tilde{m}_u}{\tilde{m}_c})(1 + \frac{\tilde{m}_d}{\tilde{m}_s})}, \tag{73}$$

$$\sin^2\theta_{23}^{q^{\rm th}} \approx \frac{(\sqrt{\delta_u}-\sqrt{\delta_d})^2}{(1+\frac{\tilde{m}_u}{\tilde{m}_c})}, \tag{74}$$

$$\sin^2\theta_{13}^{q^{\rm th}} \approx \frac{\frac{\tilde{m}_u}{\tilde{m}_c}(\sqrt{\delta_u}-\sqrt{\delta_d})^2}{(1+\frac{\tilde{m}_u}{\tilde{m}_c})}. \tag{75}$$

Now, the numerical values of the quark mixing angles may be computed from Eq. (55) and the numerical values of the parameters δ_u and δ_d, Eq. (61), and the *CP* violating phase $\phi = 90°$ obtained from the χ^2 fit of $|V_{CKM}^{\rm th}|$ to the experimentally determined values $|V_{CKM}^{\rm exp}|$. In this way we obtain

$$\theta_{12}^{q^{\rm th}} = 13°, \qquad \theta_{23}^{q^{\rm th}} = 2.38°, \qquad \theta_{13}^{q^{\rm th}} = 0.19°, \tag{76}$$

in very good agreement with the latest analysis of the experimental data [28], see (9). The numerical values of the leptonic mixing angles are computed in a similar fashion. The theoretical expressions for the lepton mixing angles as function of the charged lepton and neutrino mass ratios are obtained from Eqs. (72) when the theoretical expressions for the moduli of the entries in the *PMNS* mixing matrix, given in Eqs. (58) and (26), are substituted for $|U_{ij}|$ in the right-hand side of Eqs. (72). If we keep only the leading-orders terms, we obtain

$$\sin^2\theta_{12}^{l^{\rm th}} \approx \frac{1+\tilde{m}_{\nu_2}-\delta_\nu}{(1+\tilde{m}_{\nu_2})(1-\delta_\nu)(1+\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}})(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})}\left\{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}} + \frac{\tilde{m}_e}{\tilde{m}_\mu}(1-\delta_\nu) + 2\sqrt{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\frac{\tilde{m}_e}{\tilde{m}_\mu}(1-\delta_\nu)}\cos\Phi_1\right\}, \tag{77}$$

$$\sin^2\theta_{23}^{l^{\rm th}} \approx \frac{\delta_\nu+\delta_e f_{\nu 2}-\sqrt{\delta_\nu\delta_e f_{\nu 2}}\cos(\Phi_1-\Phi_2)}{(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})(1+\tilde{m}_{\nu_2})}, \tag{78}$$

$$\sin^2\theta_{13}^{l^{\rm th}} \approx \frac{\delta_\nu}{(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})(1+\tilde{m}_{\nu_2})}\left\{\frac{\tilde{m}_e}{\tilde{m}_\mu} + \frac{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}{(1-\delta_\nu)} - 2\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}{(1-\delta_\nu)}}\cos\Phi_1\right\}. \tag{79}$$

From Eqs. (59) we have that $f_{\nu 2} = 1+\tilde{m}_{\nu_2}-\delta_\nu$. The expressions quoted above are written in terms of the ratios of the lepton masses. When the well-known values of the charged lepton masses, the values of the neutrino masses, Eq. (67), the values of the delta parameters Eq. (68), and the values of the Dirac *CP* violating phases obtained from the χ^2 fit in the lepton sector are inserted in Eqs. (77)–(79), we obtain the following numerical values for the mixing angles:

$$\theta_{12}^{l^{\rm th}} = 34.7°, \qquad \theta_{23}^{l^{\rm th}} = 43.6°, \qquad \theta_{13}^{l^{\rm th}} = 3.5°, \tag{80}$$

which are in very good agreement with the latest experimental data [5,8].

VI. QUARK-LEPTON COMPLEMENTARITY

The relations between mixing angles and the moduli of the entries of the mixing matrices given in Eqs. (72) allow us to write the following identities:

$$\tan(\theta_{12}^q + \theta_{12}^l) = 1 + \Delta_{12}, \tag{81}$$

where

$$\Delta_{12} = \frac{|V_{us}|(|U_{e1}|+|U_{e2}|) - |V_{ud}|(|U_{e1}|-|U_{e2}|)}{|U_{e1}||V_{ud}| - |U_{e2}||V_{us}|}, \tag{82}$$

and

$$\tan(\theta_{23}^q + \theta_{23}^l) = 1 + \Delta_{23}, \tag{83}$$

where

$$\Delta_{23} = \frac{|V_{cb}|(|U_{\tau 3}|+|U_{\mu 3}|) - |V_{tb}|(|U_{\tau 3}|-|U_{\mu 3}|)}{|U_{\tau 3}||V_{tb}| - |U_{\mu 3}||V_{cb}|}, \tag{84}$$

and

$$\tan(\theta_{13}^q + \theta_{13}^l) = \frac{|V_{ub}|\sqrt{1-|U_{e3}|^2} + |U_{e3}|\sqrt{1-|V_{ub}|^2}}{\sqrt{1-|V_{ub}|^2}\sqrt{1-|U_{e3}|^2} - |U_{e3}||V_{ub}|}. \tag{85}$$

We notice that numerical values of Δ_{12} and Δ_{23} obtained from the experimentally determined $|V_{CKM}|$ and $|U_{PMNS}|$ are much smaller than 1,

$$\Delta_{12} \ll 1 \quad \text{and} \quad \Delta_{23} \ll 1,$$

for this reason, the identities (81)–(85) are sometimes called quark-lepton complementarity relations.

The substitution of expressions (55) and (58) for the moduli of the elements of the mixing matrices $V_{CKM}^{\rm th}$ and $U_{PMNS}^{\rm th}$, allows us to express the small terms Δ_{12} and Δ_{23} as functions of the mass ratios of quarks and leptons. Then, Eqs. (81)–(85) take the following form:

$$\tan(\theta_{12}^{q^{\rm th}} + \theta_{12}^{l^{\rm th}}) = 1 + \Delta_{12}^{\rm th}\left(\frac{\tilde{m}_u}{\tilde{m}_c}, \frac{\tilde{m}_d}{\tilde{m}_s}, \frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}, \frac{\tilde{m}_e}{\tilde{m}_\mu}\right), \tag{86}$$

where

$$\Delta_{12}^{\rm th} \approx \frac{\sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}+\frac{\tilde{m}_u}{\tilde{m}_c}}\left[\sqrt{\frac{\tilde{m}_{\nu 1}}{\tilde{m}_{\nu_2}}f_{\nu 2}}\left(1+\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu 2}}{\tilde{m}_{\nu_1}}(1-\delta_\nu)}\right)+\sqrt{(1+\tilde{m}_{\nu_2})(1-\delta_\nu)}\right]-\left[\sqrt{(1+\tilde{m}_{\nu_2})f_{\nu 1}}-\sqrt{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}f_{\nu 2}}\left(1+\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_2}}{\tilde{m}_{\nu_1}}(1-\delta_\nu)}\right)\right]}{\sqrt{(1+\tilde{m}_{\nu_2})(1-\delta_\nu)}-\sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}+\frac{\tilde{m}_u}{\tilde{m}_c}}\left(1+\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_2}}{\tilde{m}_{\nu_1}}(1-\delta_\nu)}\right)} \quad (87)$$

Here, rather than writing a lengthy but not very illuminating exact expression, we give an approximate expression for $\Delta_{12}^{\rm th}$, whose numerical value differs from the one obtained using the exact expression by 12%. In the derivation of Eq. (87) from (82) we used the following approximations:

$$\frac{|V_{us}^{\rm th}|}{|V_{us}^{\rm th}|} \approx \sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}+\frac{\tilde{m}_u}{\tilde{m}_c}} \approx 0.231\,52, \quad (88)$$

which differs from the exact value in less than 1%, and

$$\frac{|U_{e2}^{\rm th}|}{|U_{e1}^{\rm th}|} \approx \sqrt{\frac{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}}{1+\tilde{m}_{\nu_2}}}\sqrt{\frac{1+\tilde{m}_{\nu_2}-\delta_\nu}{1-\tilde{m}_{\nu_1}-\delta_\nu}}\left\{1+\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_2}}{\tilde{m}_{\nu_1}}(1-\delta_\nu)}\right\} \approx 0.688, \quad (89)$$

which differs from the exact value in less than 1%. The identity (86) that defines $\Delta_{12}^{\rm th}(\frac{\tilde{m}_u}{\tilde{m}_c},\frac{\tilde{m}_d}{\tilde{m}_s},\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}},\frac{\tilde{m}_e}{\tilde{m}_\mu})$ is frequently written in terms of the angle $\varepsilon_{12}^{\rm th}$ that measures the deviation of $(\theta_{12}^{q^{\rm th}}+\theta_{12}^{l^{\rm th}})$ from $\frac{\pi}{4}$. Then, Eq. (86) may also be written as

$$\tan(\theta_{12}^{q^{\rm th}}+\theta_{12}^{l^{\rm th}}) = \tan\left(\frac{\pi}{4}+\varepsilon_{12}^{\rm th}\right) = 1+\Delta_{12}^{\rm th}. \quad (90)$$

From this expression, we get

$$\varepsilon_{12}^{\rm th} = \arctan\left\{\frac{\Delta_{12}^{\rm th}}{2+\Delta_{12}^{\rm th}}\right\}, \qquad |\varepsilon_{12}^{\rm th}| < \frac{\pi}{2} \quad (91)$$

which gives $\varepsilon_{12}^{\rm th}$ as function of the mass ratios of quarks and leptons. Similarly,

$$\tan(\theta_{23}^{q^{\rm th}}+\theta_{23}^{l^{\rm th}}) = 1+\Delta_{23}^{\rm th}\left(\frac{\tilde{m}_u}{\tilde{m}_c},\frac{\tilde{m}_d}{\tilde{m}_s},\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}},\frac{\tilde{m}_e}{\tilde{m}_\mu}\right), \quad (92)$$

where

$$\Delta_{23}^{\rm th} \approx \frac{([(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})(1+\tilde{m}_{\nu_2})-\delta_\nu-\delta_e f_{\nu 2}]^{1/2}+\sqrt{\delta_\nu+\delta_e f_{\nu 2}})\left(\sqrt{1+\frac{\tilde{m}_u}{\tilde{m}_c}-(\sqrt{\delta_u}-\sqrt{\delta_d})^2}+(\sqrt{\delta_u}-\sqrt{\delta_d})\right)}{[(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})(1+\tilde{m}_{\nu_2})-\delta_\nu-\delta_e f_{\nu 2}]^{1/2}\sqrt{1+\frac{\tilde{m}_u}{\tilde{m}_c}-(\sqrt{\delta_u}-\sqrt{\delta_d})^2}-(\sqrt{\delta_u}-\sqrt{\delta_d})\sqrt{\delta_\nu+\delta_e f_{\nu 2}}} \quad (93)$$

Also,

$$\tan(\theta_{13}^{q^{\rm th}}+\theta_{13}^{l^{\rm th}}) \approx \frac{\sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}}(\sqrt{\delta_u}-\sqrt{\delta_d})\left[(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})(1+\tilde{m}_{\nu_2})-\delta_\nu\left(\sqrt{\frac{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}\right)^2\right]^{1/2}+}{\sqrt{1+\frac{\tilde{m}_u}{\tilde{m}_c}-\frac{\tilde{m}_u}{\tilde{m}_c}(\sqrt{\delta_u}-\sqrt{\delta_d})^2}\left[(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})(1+\tilde{m}_{\nu_2})-\delta_\nu\left(\sqrt{\frac{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}\right)^2\right]^{1/2}-} \times \frac{+\sqrt{\delta_\nu}\left(\sqrt{\frac{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}\right)\sqrt{1+\frac{\tilde{m}_u}{\tilde{m}_c}-\frac{\tilde{m}_u}{\tilde{m}_c}(\sqrt{\delta_u}-\sqrt{\delta_d})^2}}{-\sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}}(\sqrt{\delta_u}-\sqrt{\delta_d})\sqrt{\delta_\nu}\left(\sqrt{\frac{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}\right)}. \quad (94)$$

After substitution of the numerical values of the mass ratios of quarks and leptons in Eqs. (87)–(94), we obtain

$$\Delta_{12}^{\rm th} = 0.1, \qquad \Delta_{23}^{\rm th} = 3.23\times 10^{-2}, \qquad \tan(\theta_{23}^{q^{\rm th}}+\theta_{23}^{l^{\rm th}}) = 6.53\times 10^{-2}. \quad (95)$$

Hence,

$$\theta_{12}^{q^{\rm th}}+\theta_{12}^{l^{\rm th}} = 45^\circ + 2.7^\circ, \quad (96)$$

$$\theta_{23}^{q^{\rm th}}+\theta_{23}^{l^{\rm th}} = 45^\circ + 1^\circ, \quad (97)$$

$$\theta_{13}^{q^{\rm th}}+\theta_{13}^{l^{\rm th}} = 3.7^\circ. \quad (98)$$

Equations (86) and (87) are obtained from an exact analytical expression for $\tan(\theta_{12}^{q^{\rm th}}+\theta_{12}^{l^{\rm th}})$ as a function of the absolute values of the entries in the mixing matrices $V_{CKM}^{\rm th}$ and $U_{PMNS}^{\rm th}$, Eqs. (81) and (82). In Eqs. (55) and (58), the elements of the mixing matrices $V_{CKM}^{\rm th}$ and $U_{PMNS}^{\rm th}$ are given as exact, explicit analytical functions of the quark and lepton mass ratios. Let us stress that these expressions are exact and valid for any possible values of the quark and lepton mass ratios. From (87), it becomes evident that the small numerical value of $\Delta_{12}^{\rm th}$ is due to the partial cancellation of two large terms of almost the same magnitude but opposite sign appearing in the numerator of the expression in the right-hand side of Eq. (87), namely,

$$\sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}+\frac{\tilde{m}_u}{\tilde{m}_c}}\left[\sqrt{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}f_{\nu 2}}\left(1+\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_2}}{\tilde{m}_{\nu_1}}(1-\delta_\nu)}\right)+\sqrt{(1+\tilde{m}_{\nu_2})(1-\delta_\nu)}\right]=0.287, \tag{99}$$

and

$$\sqrt{(1+\tilde{m}_{\nu_2})f_{\nu 1}}-\sqrt{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}f_{\nu 2}}\left(1+\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_2}}{\tilde{m}_{\nu_1}}(1-\delta_\nu)}\right)=0.22. \tag{100}$$

The approximate numerical equality of these two expressions has its origin in the combined effect of the strong hierarchy of charged leptons and u- and d-type quarks which yields small and very small mass ratios, and the seesaw mechanism type I which gives very small neutrino masses but relatively large neutrino mass ratios.

We may conclude that the so-called quark-lepton complementarity as expressed in (86) and (87) is more than a numerical coincidence—it is the result of the combined effect of two factors:

(1) The strong mass hierarchy of the Dirac fermions which produces small and very small mass ratios of u- and d-type quarks and charged leptons. The quark mass hierarchy is then reflected in a similar hierarchy of small and very small quark mixing angles.

(2) The normal seesaw mechanism type I which gives very small masses to the left-handed Majorana neutrinos with relatively large values of the neutrino mass ratio m_{ν_1}/m_{ν_2} and allows for large θ_{12}^l and θ_{23}^l mixing angles [see Eqs. (77)–(79)].

The two factors just mentioned contribute to the numerator of $\Delta_{12}^{q^{\rm th}}$ with two terms of almost equal magnitude but opposite sign. Hence, the small numerical value of $\Delta_{12}^{q^{\rm th}}$ occurring by partial cancellation of these two terms.

VII. THE EFFECTIVE MAJORANA MASSES

The square of the magnitudes of the effective Majorana neutrino masses, Eq. (13), are

$$|\langle m_{ll}\rangle|^2=\sum_{j=1}^{3}m_{\nu_j}^2|U_{lj}|^4+2\sum_{j<k}^{3}m_{\nu_j}m_{\nu_k}\times|U_{lj}|^2|U_{lk}|^2\cos 2(w_{lj}-w_{lk}), \tag{101}$$

where $w_{lj}=\arg\{U_{lj}\}$; this term includes phases of both types, Dirac and Majorana.

The theoretical expression for the squared magnitude of the effective Majorana neutrino mass of electron neutrino, written in terms of the ratios of the lepton masses, is

$$\begin{aligned}|\langle m_{ee}\rangle|^2\approx{}&\frac{1}{(1+\frac{\tilde{m}_e}{\tilde{m}_\mu})^2(1+\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}})^2}\Bigg\{m_{\nu_1}^2\left(1-4\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}(1-\delta_\nu)}\right)+\frac{m_{\nu_2}^2f_{\nu 2}^2}{(1+\tilde{m}_{\nu_2})^2(1-\delta_\nu)^2}\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\left(\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}+4\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}(1-\delta_\nu)}\right.\\&\left.+6\frac{\tilde{m}_e}{\tilde{m}_\mu}(1-\delta_\nu)\right)+2\frac{m_{\nu_1}m_{\nu_3}\delta_\nu}{(1+\tilde{m}_{\nu_2})}\left(1+\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\right)\left(\sqrt{\frac{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}{(1-\delta_\nu)}}-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}\right)^2\times\cos 2(w_{e1}-w_{e3})+2\frac{m_{\nu_1}m_{\nu_2}f_{\nu 2}}{(1+\tilde{m}_{\nu_2})(1-\delta_\nu)}\\&\times\left(\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}+2\left(1-\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\right)\sqrt{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\frac{\tilde{m}_e}{\tilde{m}_\mu}(1-\delta_\nu)}\right)\cos 2(w_{e1}-w_{e2})+2\frac{m_{\nu_2}m_{\nu_3}f_{\nu 2}\delta_\nu}{(1+\tilde{m}_{\nu_2})^2(1-\delta_\nu)^2}\left(1+\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\right)\left(2\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}\right.\\&\left.+\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}(1-\delta_\nu)}\right)\cos 2(w_{e2}-w_{e3})\Bigg\}\end{aligned} \tag{102}$$

where $w_{e2}\approx\beta_1$ and

$$w_{e1}=\arctan\left\{-\frac{\sqrt{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\frac{\tilde{m}_e}{\tilde{m}_\mu}}\delta_e\delta_\nu f_{\nu 2}}{\sqrt{(1-\delta_\nu)}+\sqrt{\frac{\tilde{m}_{\nu_1}}{\tilde{m}_{\nu_2}}\frac{\tilde{m}_e}{\tilde{m}_\mu}}(1-\delta_\nu)}\right\}, \tag{103}$$

$$w_{e3}\approx\arctan\left\{\frac{\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}\delta_e f_{\nu 2}(1-\delta_\nu)+}{-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}\delta_e f_{\nu 2}(1-\delta_\nu)\tan\beta_2+}\times\frac{+\sqrt{\delta_\nu}\left(\sqrt{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}(1-\delta_\nu)\right)\tan\beta_2}{+\sqrt{\delta_\nu}\left(\sqrt{\tilde{m}_{\nu_1}\tilde{m}_{\nu_2}}-\sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}}(1-\delta_\nu)\right)}\right\}. \tag{104}$$

In a similar way, the theoretical expression for the squared magnitude of the effective Majorana neutrino mass of the muon neutrino is

$$|\langle m_{\mu\mu}\rangle|^2 \approx \frac{1}{(1+\frac{\tilde m_e}{\tilde m_\mu})^2(1+\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}})^2(1+\tilde m_{\nu_2})}\Bigg\{\frac{m_{\nu_3}^2}{(1+\tilde m_{\nu_2})}\bigg(1+\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\bigg)^2(\delta_\nu+2\delta_e f_{\nu 2})+\frac{m_{\nu_2}^2}{(1+\tilde m_{\nu_2})(1-\delta_\nu)}\bigg(1-\delta_\nu - 4\sqrt{\frac{\tilde m_e}{\tilde m_\mu}\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}(1-\delta_\nu)}+6\frac{\tilde m_e}{\tilde m_\mu}\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\bigg)+2m_{\nu_1}m_{\nu_2}f_{\nu 2}\bigg(\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}(1-\delta_\nu)+2\sqrt{\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\frac{\tilde m_e}{\tilde m_\mu}(1-\delta_\nu)}\bigg(1-\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\bigg)\bigg) \times \cos 2(w_{\mu 1}-w_{\mu 2})+2m_{\nu_1}m_{\nu_3}\bigg(1+\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\bigg)\bigg(2\delta_\nu\times\sqrt{\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\frac{\tilde m_e}{\tilde m_\mu}(1-\delta_\nu)}+\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}(1-\delta_\nu)(\delta_\nu+\delta_e f_{\nu 2})\bigg) \times \cos 2(w_{\mu 1}-w_{\mu 3})+2\frac{m_{\nu_2}m_{\nu_3}f_{\nu 2}}{(1+\tilde m_{\nu_2})(1-\delta_\nu)}\bigg(1+\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\bigg)\bigg((1-\delta_\nu)(\delta_\nu+\delta_e f_{\nu 2})-2\delta_\nu\sqrt{\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}\frac{\tilde m_e}{\tilde m_\mu}(1-\delta_\nu)}\bigg) \times \cos 2(w_{\mu 2}-w_{\mu 3})\Bigg\}, \tag{105}$$

where

$$w_{\mu 1} \approx \arctan\left\{\frac{\sqrt{\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}}\delta_e\delta_\nu f_{\nu 2}}{\sqrt{\frac{\tilde m_e}{\tilde m_\mu}(1-\delta_\nu)}+\sqrt{\frac{\tilde m_{\nu_1}}{\tilde m_{\nu_2}}(1-\delta_\nu)}}\right\}, \tag{106}$$

and

$$w_{\mu 2} \approx \arctan\left\{\frac{\sqrt{f_{\nu 2}}\tan\beta_1+\sqrt{\delta_e\delta_\nu}}{\sqrt{f_{\nu 2}}-\sqrt{\delta_e\delta_\nu}\tan\beta_1}\right\}, \tag{107}$$

$$w_{\mu 3} \approx \arctan\left\{\frac{\tan\beta_2-\sqrt{f_{\nu 2}}}{1+\sqrt{f_{\nu 2}}\tan\beta_2}\right\}. \tag{108}$$

From these expressions and the numerical values of the neutrinos masses given in Eq. (67), we obtain the following expressions for effective Majorana masses with the phases as free parameters:

$$|\langle m_{ee}\rangle|^2 \approx \{9.41+8.29\cos(1^\circ-2\beta_1)+4.3\cos(1^\circ-2w_{e3})+4.31\cos 2(\beta_1-w_{e3})\}\times 10^{-6}\ \mathrm{eV}^2, \tag{109}$$

where

$$w_{e3} = \arctan\left\{\frac{0.15\tan\beta_2-0.013}{0.15+0.013\tan\beta_2}\right\}. \tag{110}$$

Similarly,

$$|\langle m_{\mu\mu}\rangle|^2 \approx \{4.8+0.17\cos 2(44^\circ-w_{\mu 2})+1.8\cos 2(w_{\mu 2}-w_{\mu 3})\}\times 10^{-4}\ \mathrm{eV}^2, \tag{111}$$

where

$$w_{\mu 2} \approx \arctan\left\{\frac{0.65\tan\beta_1+0.13}{0.65-0.13\tan\beta_1}\right\}, \tag{112}$$

$$w_{\mu 3} \approx \arctan\left\{\frac{\tan\beta_2-0.13}{1+0.13\tan\beta_2}\right\}. \tag{113}$$

In order to make a numerical estimate of the effective Majorana neutrinos masses $|\langle m_{ee}\rangle|$ and $|\langle m_{\mu\mu}\rangle|$, we used the following values for the Majorana phases $\beta_1 = -1.4^\circ$ and $\beta_2 = 77^\circ$ obtained by maximizing the rephasing invariants S_1 and S_2, Eq. (71). Then, the numerical value of the effective Majorana neutrino masses are

$$|\langle m_{ee}\rangle| \approx 4.6\times 10^{-3}\ \mathrm{eV}, \qquad |\langle m_{\mu\mu}\rangle| \approx 2.1\times 10^{-2}\ \mathrm{eV}. \tag{114}$$

These numerical values are consistent with the very small experimentally determined upper bounds for the reactor neutrino mixing angle θ_{13}^l [29].

VIII. CONCLUSIONS

In this communication, we outlined a unified treatment of masses and mixings of quarks and leptons in which the left-handed Majorana neutrinos acquire their masses via the type-I seesaw mechanism, and the mass matrices of all Dirac fermions have a similar form with four texture zeroes and a normal hierarchy. Then, the mass matrix of the left-handed Majorana neutrinos also has a texture with four zeros. In this scheme, we derived exact, explicit expressions for the Cabibbo (θ_{12}^q) and solar (θ_{12}^l) mixing angles as functions of the quark and lepton masses, respectively. The so-called quark-lepton complementarity relation takes the form

$$\theta_{12}^{q^{\mathrm{th}}}+\theta_{12}^{l^{\mathrm{th}}} = 45^\circ+\varepsilon_{12}^{\mathrm{th}}. \tag{115}$$

The correction term, $\varepsilon_{12}^{\mathrm{th}}$, is an explicit function of the ratios of quark and lepton masses, given in Eq. (91), which reproduces the experimentally determined value,

$$\varepsilon_{12}^{\mathrm{exp}} \approx 2.7^o, \tag{116}$$

when the numerical values of the quark and lepton masses are substituted in (91).

Three essential ingredients are needed to explain the correlations implicit in the small numerical value of $\varepsilon_{12}^{\rm th}$:

(1) The strong hierarchy in the mass spectra of the quarks and charged leptons, realized in our scheme through the explicit breaking of the S_3 flavor symmetry in the mass matrices with four texture zeroes, explains the resulting small or very small quark mixing angles; the very small charged lepton mass ratios explain the very small value of θ_{13}^{l}.

(2) The normal seesaw mechanism that gives very small masses to the left-handed Majorana neutrinos with relatively large values of the neutrino mass ratio m_{ν_1}/m_{ν_2} and allows for large θ_{12}^{l} and θ_{23}^{l} mixing angles.

The assumption of a normal hierarchy for the masses of the Majorana neutrinos.

ACKNOWLEDGMENTS

We thank Dr. Myriam Mondragón for many inspiring discussions on this exciting problem. This work was partially supported by CONACyT Mexico under Contracts No. 51554-F and No. 82291, and DGAPA-UNAM Contract No. PAPIIT IN112709.

[1] Y. Fukuda *et al.* (Super-Kamiokande Collaboration), Phys. Rev. Lett. **81**, 1562 (1998); Phys. Lett. B **539**, 179 (2002); M. Sanchez *et al.*, Phys. Rev. D **68**, 113004 (2003); Y. Ashie *et al.* (Super-Kamiokande), Phys. Rev. D **71**, 112005 (2005).

[2] B. T. Cleveland *et al.*, Astrophys. J. **496**, 505 (1998); S. N. Ahmed *et al.* (SNO Collaboration), Phys. Rev. Lett. **92**, 181301 (2004); M. Altmann *et al.* (GNO Collaboration), Phys. Lett. B **616**, 174 (2005); B. Aharmim *et al.* (SNO Collaboration), Phys. Rev. C **72**, 055502 (2005).

[3] K. Eguchi *et al.* (KamLAND Collaboration), Phys. Rev. Lett. **90**, 021802 (2003); T. Araki *et al.* (KamLAND Collaboration), Phys. Rev. Lett. **94**, 081801 (2005).

[4] M. H. Ahn *et al.* (K2K), Phys. Rev. Lett. **90**, 041801 (2003).

[5] M. C. Gonzalez-Garcia and M. Maltoni, Phys. Rep. **460**, 1 (2008).

[6] M. Apollonio *et al.* (CHOOZ Collaboration), Phys. Lett. B **466**, 415 (1999).

[7] J. E. Roa, D. C. Latimer, and D. J. Ernst, Phys. Rev. Lett. **103**, 061804 (2009); Phys. Rev. C **81**, 015501 (2010).

[8] M. C. Gonzalez-Garcia, M. Maltoni, and J. Salvado, J. High Energy Phys. 04 (2010) 056.

[9] M. Maltoni and T. Schwetz, Proc. Sci., IDM2008 (2008) 072 [arXiv:0812.3161]; G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A. M. Rotunno, arXiv:0905.3549; H. L. Ge, C. Giunti, and Q. Y. Liu, Phys. Rev. D **80**, 053009 (2009); A. Palazzo and J. W. F. Valle, Phys. Rev. D **80**, 091301 (2009).

[10] C. Amsler *et al.* (Particle Data Group), Phys. Lett. B **667**, 1 (2008).

[11] A. Y. Smirnov, arXiv:hep-ph/0402264.

[12] A. Y. Smirnov, arXiv:0910.1778.

[13] F. Gonzalez Canales and A. Mondragon, J. Phys. Conf. Ser. **171**, 012063 (2009); AIP Conf. Proc. **857**, 287 (2006).

[14] H. Minakata, arXiv:hep-ph/0505262.

[15] P. H. Frampton, S. T. Petcov, and W. Rodejohann, Nucl. Phys. **B687**, 31 (2004); M. Raidal, Phys. Rev. Lett. **93**, 161801 (2004); H. Minakata and A. Y. Smirnov, Phys. Rev. D 70, 073009 (2004); P. H. Frampton and R. N. Mohapatra, J. High Energy Phys. 01 (2005) 025.

[16] F. Plentinger, G. Seidl, and W. Winter, Nucl. Phys. **B791**, 60 (2008); Phys. Rev. D **76**, 113003 (2007).

[17] L. Camilleri, E. Lisi, and J. F. Wilkerson, Annu. Rev. Nucl. Part. Sci. **58**, 343 (2008).

[18] C. Barbero, L.-F. Li, G. LopezCastro, and A. Mariano, Phys. Rev. D **76**, 116008 (2007).

[19] H. Ishimori *et al.*, Prog. Theor. Phys. Suppl. **183**, 1 (2010); G. Altarelli, F. Feruglio, and L. Merlo, J. High Energy Phys. 05 (2009) 020; R. d. A. Toorop, F. Bazzocchi, and L. Merlo, J. High Energy Phys. 08 (2010) 001; F. Plentinger, G. Seidl, and W. Winter, J. High Energy Phys. 04 (2008) 077; F. Plentinger and G. Seidl, Phys. Rev. D **78**, 045004 (2008).

[20] A. Mondragón and E. Rodríguez-Jáuregui, Phys. Rev. D **61**, 113002 (2000); **59**, 093009 (1999).

[21] H. Fritzsch and Z.-z. Xing, Prog. Part. Nucl. Phys. **45**, 1 (2000).

[22] H. Fritzsch, Phys. Lett. **70B**, 436 (1977); Nucl. Phys. **B155**, 189 (1979); S. Pakvasa and H. Sugawara, Phys. Lett. **73B**, 61 (1978); H. Harari, H. Haut, and J. Weyers, Phys. Lett. **78B**, 459 (1978).

[23] W. Buchmuller and D. Wyler, Phys. Lett. B **521**, 291 (2001); M. Bando and M. Obara, Prog. Theor. Phys. **109**, 995 (2003); M. Bando, S. Kaneko, M. Obara, and M. Tanimoto, arXiv:hep-ph/0405071.

[24] Z.-z. Xing and H. Zhang, Phys. Lett. B **569**, 30 (2003).

[25] K. A. Hochmuth, S. T. Petcov, and W. Rodejohann, Phys. Lett. B **654**, 177 (2007).

[26] C. Jarlskog, Z. Phys. C **29**, 491 (1985); Phys. Rev. Lett. **55**, 1039 (1985).

[27] R. N. Mohapatra and A. Y. Smirnov, Annu. Rev. Nucl. Part. Sci. **56**, 569 (2006).

[28] V. Mateu and A. Pich, J. High Energy Phys. 10 (2005) 041.

[29] G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A. M. Rotunno, Phys. Rev. Lett. **101**, 141801 (2008); S. Pascoli and S. T. Petcov, Phys. Rev. D **77**, 113003 (2008).