

**Quantum stress tensor for a massive vector field in the space-time of a cylindrical black hole**

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The components of the renormalized quantum energy-momentum tensor for a massive vector field coupled to the gravitational field configuration of static 3 + 1 dimensional black strings in anti-de Sitter space are analytically evaluated using the Schwinger-DeWitt approximation. The general results are employed to investigate the pointwise energy conditions for the quantized matter field, and it is shown that they are violated at some regions of the space-time, in particular the horizon of the black hole.

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Quantum theory and general relativity are two beautiful parts of modern physics that, for more than a century, have been developed in such an extent that our knowledge of the universe at short and long scales has increased as never before in the human history. With the help of the quantum theory we can explain microworld phenomena. On the other hand, the general theory of relativity allows us a deep understanding of the large scale structure of the universe. These two major achievements in theoretical physics in the 20th century are still, nearly 100 years later, going separate ways. There is not yet such a thing as a theory of quantum gravity, but in their quest for the theory of everything, the physicists try to bring them together. Quantum gravitation is a tool that would be very important to describe, among other things, the creation of the universe and its later development.

One of the approaches developed to consider quantum effects in gravitation, called semiclassical gravity, considers the quantum dynamics of fields in a gravitational background, which at this level of description is considered as a classical external field. In the absence of a full theory of quantum gravity, semiclassical gravity is a well established physical theory that helps us to know the expected behavior of a gravitational system under the influence of the interaction between it and matter fields that obey the laws of quantum theory [1].

In this approximate theory, fundamental information about the quantum matter fields is contained in the renormalized quantum stress-energy tensor  $\langle T_{\mu}^{\nu} \rangle_{\text{ren}}$ , that can, in principle, be constructed using a variety of mathematical techniques, including analytical, semianalytical, and numerical ones, see [2–17] and references therein.

For the important case of massive fields, one of the developed approaches for determining  $\langle T_{\mu}^{\nu} \rangle_{\text{ren}}$  is based in the calculation of the renormalized quantum effective

action for the quantized matter field, using the Schwinger-DeWitt proper time technique to give an expansion of the effective action in terms of the field inverse square mass. This is the celebrated Schwinger-DeWitt expansion, in which the first three terms renormalize the bare gravitational and cosmological constant, and add some higher order terms to the Einstein gravitational action. The next order term, proportional to  $m^{-2}$ , where  $m$  is the mass of the field, gives us the one-loop effective action  $W_{\text{ren}}$  for the matter field [3–6,14–17].

By functional differentiation of the one-loop effective action, we can obtain the desired quantum stress tensor using the standard formula

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}}{\delta g^{\mu\nu}}. \quad (1)$$

The above method has been applied to a number of space-times of interest, including Schwarzschild [2,3], Reissner-Nordstrom [2,14], charge dilatonic black holes, and nonlinear electrically charged black holes in four dimensions [15]. Also, in two recent papers we developed the Schwinger-DeWitt technique for the calculation of the renormalized stress-energy tensor of massive scalar and spinor fields up to one-loop order in the space-time of static four-dimensional black strings in anti-de Sitter space [16–18]. For this interesting system, the problems of investigating the renormalized stress tensor components for conformally coupled massless scalar fields were studied by DeBenedictis in [19,20], who used the obtained  $\langle T_{\mu}^{\nu} \rangle_{\text{ren}}$  for the calculation of gravitational backreaction of the quantum field. In this work we complete the series of papers [16,17] dedicated to the calculation of  $\langle T_{\mu}^{\nu} \rangle_{\text{ren}}$  for massive fields in the static black string background, determining the components of this tensor for the case of a massive vector field. We also investigate the fulfillment of the pointwise energy conditions for the quantized field in this gravitational background.

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The corresponding metric element for the static black string space-time is

$$ds^2 = -\left(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}\right) dt^2 + \frac{1}{(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho})} d\rho^2 + \rho^2 d\varphi^2 + \alpha^2 \rho^2 dz^2, \quad (2)$$

where  $M$  is the mass per unit length of the string. As we can see from (2), the considered metric has an event horizon located at  $\rho_+ = \frac{\sqrt{4M}}{\alpha}$  and the only true singularity is a polynomial one at the origin.

The action for a single massive vector field  $A_\mu$  with mass  $m$  in some generic curved space-time in four dimensions is

$$S = - \int d^4x \sqrt{-g} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right). \quad (3)$$

The equation of motion for the field have the form

$$\hat{D}_\nu^\mu(\nabla) A_\mu = 0, \quad (4)$$

where the second order operator  $\hat{D}_\nu^\mu(\nabla)$  is given by

$$\hat{D}_\nu^\mu(\nabla) = \delta_\nu^\mu \square - \nabla_\nu \nabla^\mu - R_\nu^\mu - m^2 \delta_\nu^\mu, \quad (5)$$

where  $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$  is the covariant D'Alembert operator,  $\nabla_\mu$  is the covariant derivative.

The usual formalism of quantum field theory gives an expression for the effective action of the quantum field  $A_\beta$  as a perturbation expansion in the number of loops:

$$\Gamma(A_\beta) = S(A_\beta) + \sum_{k \geq 1} \Gamma_{(k)}(A_\beta), \quad (6)$$

where  $S(A_\beta)$  is the classical action of the free field. The one-loop contribution of the field  $A_\beta$  to the effective action is expressed in terms of the operator (5) as

$$\Gamma_{(1)} = \frac{i}{2} \ln(\mathfrak{Det} \hat{D}), \quad (7)$$

where  $\mathfrak{Det} \hat{D} = \exp(\text{Tr} \ln \hat{D})$  is the functional Berezin superdeterminant [6] of the operator  $\hat{D}$ , and  $\text{Tr} \hat{D} = (-1)^i F_i^i = \int d^4x (-1)^A F_A^A(x)$  is the functional supertrace [6]. If the Compton's wavelength of the field is less than the characteristic radius of space-time curvature [3–6, 14–17], we can develop an expansion of the above effective action in powers of the inverse square mass of the field. This approximation is known as the Schwinger-DeWitt one, and before applying this approach to the particular problem considered in this work we make the following remarks. In the first place, we mention that the Schwinger-DeWitt technique is directly applicable to “minimal” second order differential operators that have the general form

$$\hat{K}_\nu^\mu(\nabla) = \delta_\nu^\mu \square - m^2 \delta_\nu^\mu + Q_\nu^\mu, \quad (8)$$

where  $Q_\nu^\mu(x)$  is some arbitrary matrix playing the role of the potential.

As we can see, because of the presence of the non-diagonal term in (5) it becomes a nonminimal operator, and this fact is an obstacle to applying the Schwinger-DeWitt technique. By fortune we can put (5) as a function of some minimal operators, if we note that it satisfies the identity  $\hat{D}_\nu^\mu(\nabla)(m^2 \delta_\nu^\mu - \nabla_\nu \nabla^\mu) = m^2(\delta_\nu^\mu \square - R_\nu^\mu - m^2 \delta_\nu^\mu)$ .

Then the one-loop effective action for the nonminimal operator (5) omitting an inessential constant can be written as

$$\begin{aligned} \frac{i}{2} \text{Tr} \ln \hat{D}_\nu^\mu(\nabla) &= \frac{i}{2} \text{Tr}(\delta_\nu^\mu \square - R_\nu^\mu - m^2 \delta_\nu^\mu) \\ &\quad - \frac{i}{2} \text{Tr}(m^2 \delta_\nu^\mu - \nabla_\nu \nabla^\mu). \end{aligned} \quad (9)$$

We can see in (9) that the first term is the effective action of a minimal second order operator  $K_\nu^\mu(\nabla)$  with potential  $-R_\nu^\mu$ . The second term can be transformed as  $\text{Tr}[\frac{1}{m^2} \nabla^\mu \nabla_\nu]^n = \text{Tr}[\frac{1}{m^2} \nabla^\mu \square^{n-1} \nabla_\nu] = \text{Tr}[\frac{1}{m^2} \square]^n$  and

$$\frac{i}{2} \text{Tr}(m^2 \delta_\nu^\mu - \nabla_\nu \nabla^\mu) = \frac{i}{2} \text{Tr}(m^2 - \square). \quad (10)$$

Then, the effective action for the massive vector field is equal to the effective action of the minimal second order operator  $K_\nu^\mu(\nabla)$  minus the effective action of a minimal operator  $S_\nu^\mu(\nabla)$  corresponding to a massive scalar field minimally coupled to gravity.

Now using the Schwinger-DeWitt representation for the Green's function of the minimal operators, we can obtain for the renormalized one-loop effective action of the quantum massive vector field the expression  $\Gamma_{(1)\text{ren}} = \int d^4x \sqrt{-g} \mathfrak{L}_{\text{ren}}$ , where the renormalized effective Lagrangian reads

$$\mathfrak{L}_{\text{ren}} = \frac{1}{2(4\pi)^2} \sum_{k=3}^{\infty} \frac{(\text{Tra}_k^{(1)}(x, x) - \text{Tra}_k^{(0)}(x, x))}{k(k-1)(k-2)m^{2(k-2)}}. \quad (11)$$

The quantities  $[a_k^{(1)}] = a_k^{(1)}(x, x')$  and  $[a_k^{(0)}] = a_k^{(0)}(x, x')$ , whose coincidence limit appears under the supertrace operation in (11), are the Hadamard-Minakshisundaram-DeWitt-Seeley coefficients for the minimal operators  $K_\nu^\mu(\nabla)$  and  $S_\nu^\mu(\nabla)$ , respectively. As usual, the first three coefficients of the DeWitt-Schwinger expansion,  $a_0$ ,  $a_1$ , and  $a_2$ , contribute to the divergent part of the action and can be absorbed in the classical gravitational action by renormalization of the bare gravitational and cosmological constants.

Restricting ourselves here to the terms proportional to  $m^{-2}$ , using integration by parts and the elementary properties of the Riemann tensor [6, 14–17], we obtain for the renormalized effective Lagrangian in the case of the massive vector field considered in this work

$$\begin{aligned}
\mathcal{Q}_{\text{ren}} = & \frac{1}{192\pi^2 m^2} \left[ \frac{9}{28} R_{\mu\nu} \square R^{\mu\nu} - \frac{27}{280} R \square R - \frac{5}{72} R^3 \right. \\
& + \frac{31}{60} R R_{\mu\nu} R^{\mu\nu} - \frac{52}{63} R_{\nu}^{\mu} R_{\gamma}^{\nu} R_{\mu}^{\gamma} \\
& - \frac{19}{105} R^{\mu\nu} R_{\gamma\rho} R^{\gamma\rho}_{\mu\nu} + \frac{61}{140} R_{\mu\nu} R^{\mu\sigma\gamma\rho} R^{\nu\sigma\gamma\rho} \\
& - \frac{1}{10} R R_{\mu\nu\gamma\rho} R^{\mu\nu\gamma\rho} - \frac{67}{2520} R_{\gamma\rho}{}^{\mu\nu} R_{\mu\nu}{}^{\sigma\tau} R_{\sigma\tau}{}^{\gamma\rho} \\
& \left. + \frac{1}{18} R^{\gamma\rho}{}_{\mu\nu} R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\gamma\rho} \right]. \quad (12)
\end{aligned}$$

As we can see, this final expression of the one loop effective for the massive vector field only differs from that of the massive scalar and spinor fields in the numerical coefficients in front of the purely geometric terms. For  $\langle T_{\mu\nu} \rangle_{\text{ren}}$  we obtain a very cumbersome expression that, as in the case of (12), is different from that obtained for scalar and spinor fields only in the numerical coefficients that appear in front of the purely geometrical terms. For this reason we do not put this very long expression for the stress tensor here and refer the readers to our previous papers [14–17].

It is interesting to mention that in a beautiful paper Decáninis and Folacci [21] have presented irreducible expressions for the metric variations of the gravitational action terms constructed from the 17 curvature invariants of order six in derivatives of the metric tensor, i.e. from the geometrical terms appearing in the diagonal coefficient  $a_3(x, x)$  of the Schwinger-DeWitt approximation, thus providing us with a general method to reduce the inevitable differences in the final expressions obtained for these quantities, due to the different simplification and canonization schemes chosen.

From the general form of the geometric terms conforming the general expression for the constructed  $\langle T_{\mu}^{\nu} \rangle_{\text{ren}}$ , we see that it is covariantly conserved, thus indicating that it is a good candidate for the expected exact one in our large mass approximation.

After a direct calculation, we obtain for  $\langle T_{\mu}^{\nu} \rangle_{\text{ren}}$  in the space-time of a static cylindrical black hole metric

$$\langle T_{\mu}{}^{\nu}(y) \rangle_{\text{ren}} = \frac{1}{3360\pi^2 m^2 \alpha^{-6}} \left( a_{\mu} + \frac{\Lambda_{\mu}}{y^6} + \frac{\Omega_{\mu}}{y^9} \right), \quad (13)$$

TABLE I. Numerical coefficients in the general expression for the quantum stress tensor of massive vector field in the space-time of the cylindrical black hole.

| $\mu$  | $a_{\mu}$ | $\Lambda_{\mu}$ | $\Omega_{\mu}$ |
|--------|-----------|-----------------|----------------|
| $t$    | -25       | 387/4           | -611/4         |
| $\rho$ | -25       | -399/4          | 175/4          |
| $z$    | -25       | 405/4           | -809/4         |

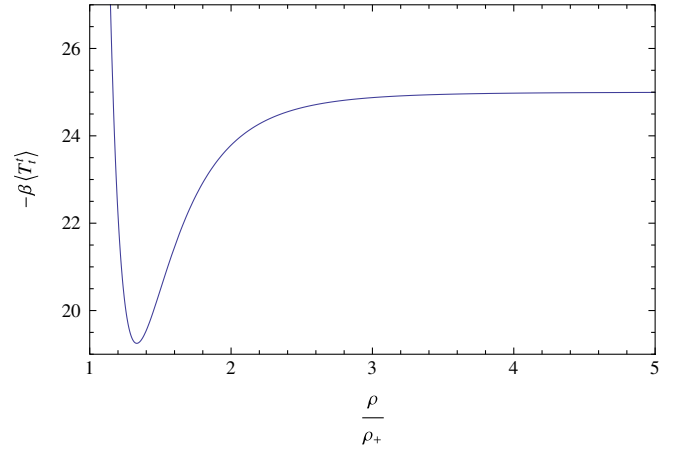


FIG. 1 (color online). Radial dependence of the rescaled component of the energy density  $\mathcal{Q} = -\langle T_t^t \rangle$  of the quantum massive vector field in the geometry of a static black string. The coefficient  $\beta = 3360\pi^2 m^2 \alpha^{-6}$ .

where we have defined the variable  $y = \frac{\rho}{\rho_+}$  and due to the cylindrical symmetry we have  $\langle T_z^z \rangle_{\text{ren}} = \langle T_{\varphi}^{\varphi} \rangle_{\text{ren}}$ . The numerical coefficients are given in Table I for each index  $\mu$ . The dependence of the components of  $\langle T_{\mu}^{\mu} \rangle_{\text{ren}}$  with  $y$  is displayed in Figs. 1–3.

If we consider the general expression (13) at the horizon of the cylindrical black hole, i.e., at  $y = 1$ , we easily found that the energy density  $\mathcal{Q} = -\langle T_t^t \rangle_{\text{ren}}$  for the quantum massive vector field is positive, in contrast with the results found in previous work for the scalar and spinor fields [16,17].

Inspection of Fig. 1 shows that the energy density is positive everywhere. The principal pressures  $p_1 = -\tau = \langle T_{\rho}^{\rho} \rangle_{\text{ren}}$  and  $p_2 = p_3 = p = \langle T_z^z \rangle_{\text{ren}}$  are negative at the horizon. Figures 2 and 3 indicate that the radial pressure is

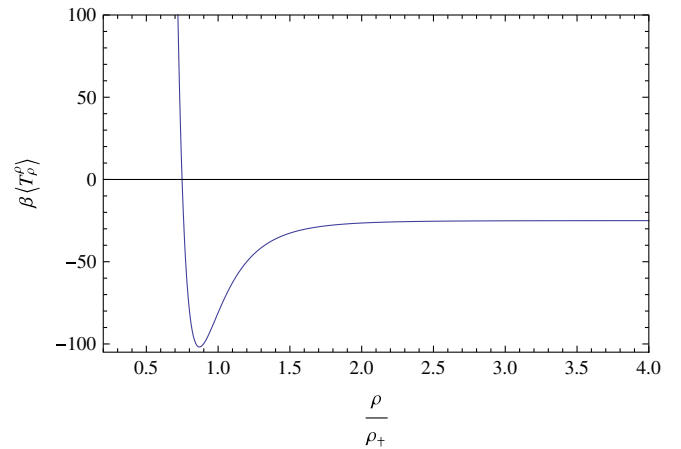


FIG. 2 (color online). Radial dependence of the rescaled component  $\langle T_{\rho}^{\rho} \rangle$  of the quantum massive vector field in the geometry of a static black string. The coefficient  $\beta = 3360\pi^2 m^2 \alpha^{-6}$ .

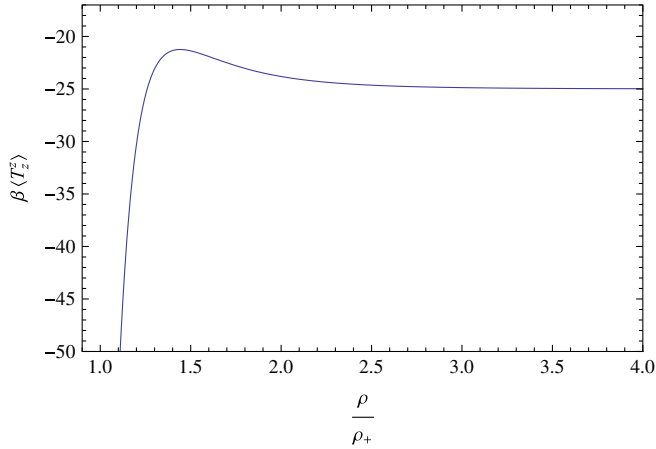


FIG. 3 (color online). Radial dependence of the rescaled component  $\langle T_{\tau\tau}^{\pm} \rangle$  of the quantum massive vector field in the geometry of a static black string. The coefficient  $\beta = 3360\pi^2 m^2 \alpha^{-6}$ .

negative in the region outside the horizon and that the other pressures are negative everywhere. At the event horizon  $\varrho - \tau = 0$ ,  $\varrho + p < 0$ , and  $\varrho - \tau + 2p < 0$ . Also we

have  $p < -\varrho < \varrho$ . The second of the above relations indicates that the null energy condition (NEC) is violated at the event horizon of the static black string. For the weak energy condition to be satisfied, we need that the energy density be positive, as is indeed the case at the horizon, but we require that the NEC be satisfied. Then, in our case, also the weak energy conditions are violated. If the NEC is satisfied and the sum of the principal pressures and the energy density of the field is positive, then the strong energy condition (SEC) is valid. The dominant energy condition (DEC) requires  $-\varrho \leq p_j \leq \varrho$ . As we can see from the above relations between the energy density and the principal pressures at the horizon of the black string, the massive vector field also violates the SEC and DEC.

The results of this work are expected to be employed to investigate the backreaction of the quantum scalar field, on the black string metric. For this purpose, the Einstein equations for the metric should be solved after including in them the calculated stress tensor for the black string solution. Our results of the implementation of this program will be published in the future.

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