

Toward a first-principle derivation of confinement and chiral-symmetry-breaking crossover transitions in QCD

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We give a theoretical framework to obtain a low-energy effective theory of quantum chromodynamics (QCD) towards a first-principle derivation of confinement/deconfinement and chiral-symmetry breaking/restoration crossover transitions. In fact, we demonstrate that an effective theory obtained using simple but nontrivial approximations within this framework enables us to treat both transitions simultaneously on equal footing. A resulting effective theory is regarded as a modified and improved version of nonlocal Polyakov-loop extended Nambu-Jona-Lasinio (nonlocal PNJL) models proposed recently by Hell, Rössner, Cristoforetti, and Weise, and Sasaki, Friman, and Redlich, extending the original (local) PNJL model by Fukushima and others. A novel feature is that the nonlocal NJL coupling depends explicitly on the temperature and Polyakov loop, which affects the entanglement between confinement and chiral-symmetry breaking, together with the cross term introduced through the covariant derivative in the quark sector considered in the conventional PNJL model. The chiral-symmetry breaking/restoration transition is controlled by the nonlocal NJL interaction, while the confinement/deconfinement transition in the pure gluon sector is specified by the nonperturbative effective potential for the Polyakov loop obtained recently by Braun, Gies, Marhauser, and Pawłowski. The basic ingredients are a reformulation of QCD based on new variables and the flow equation of the Wetterich type in the Wilsonian renormalization group. This framework can be applied to investigate the QCD phase diagram at finite temperature and density.

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I. INTRODUCTION

The relation between confinement and chiral-symmetry breaking is one of the long-standing puzzles in theoretical physics. Recently, strong interest on this issue revived in extreme environments especially at high temperatures and baryon densities, stimulated by the heavy-ion programs at GSI, CERN SPS, RHIC, and LHC, see, e.g., [1,2] for a review. Quantum chromodynamics (QCD) for strong interactions is a fundamental theory for solving this problem.

In pure Yang-Mills theory, i.e., in the limit of infinitely heavy quark mass $m_q \rightarrow \infty$ of QCD, the Polyakov-loop average $\langle L \rangle$, i.e., the vacuum expectation value of the Polyakov-loop operator L , can be used as a criterion for quark confinement [3]. The Polyakov-loop operator L is a gauge-invariant operator charged under the center group $Z(N_c)$ of the color gauge group $SU(N_c)$. The Polyakov-loop average $\langle L \rangle$ vanishes $\langle L \rangle = 0$ and quarks are confined at low temperatures $T < T_d$ where the global center symmetry $Z(N_c)$ is intact, while it is nonzero $\langle L \rangle \neq 0$ and quarks are deconfined at high temperature $T > T_d$ where the global center symmetry $Z(N_c)$ is spontaneously broken. Thus, we can define T_d as a critical temperature for confinement/deconfinement phase transition.

When dynamical quarks in the fundamental representation of the gauge group are added to the Yang-Mills theory, the center symmetry is no longer exact. On the other hand,

QCD with massless quarks $m_q \rightarrow 0$ exhibits chiral symmetry $SU(N_f)_L \times SU(N_f)_R$. The chiral condensate $\langle \bar{\psi} \psi \rangle$, i.e., the vacuum expectation value of a gauge-invariant composite operator $\bar{\psi} \psi$, is used as an order parameter for chiral-symmetry breaking. The chiral condensate $\langle \bar{\psi} \psi \rangle$ is nonzero $\langle \bar{\psi} \psi \rangle \neq 0$ at low temperatures $T < T_\chi$ where the chiral symmetry is spontaneously broken, while it vanishes $\langle \bar{\psi} \psi \rangle = 0$ at high temperature $T > T_\chi$ where the chiral symmetry is restored. Thus, we can define T_χ as a critical temperature for chiral-symmetry breaking/restoration phase transition.

For realistic quark mass (with finite and nonzero m_q : $0 < m_q < \infty$), there are no exact symmetries directly related to the phase transitions, since both the center and chiral symmetries are explicitly broken, and $\langle L \rangle$ and $\langle \bar{\psi} \psi \rangle$ are approximate order parameters. In this case, there is no critical temperature T_c in the strict sense, and the transition can be a crossover transition for which the pseudocritical temperature T_c^* is defined such that the susceptibility takes the maximal value at $T = T_c^*$. If quarks are in the fundamental representation, deconfinement (a rise in the Polyakov-loop average) happens at the temperature where the chiral symmetry is restored (chiral condensate decreases rapidly). The chiral and deconfinement transitions seem to coincide, $T_d^* = T_\chi^* \simeq T_c$ [4,5], although the property of the phase transition, e.g., the critical temperature and the order of the transition depend on the numbers of color N_c and flavor N_f . Whereas, for quarks in the adjoint

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representation, deconfinement and chiral-symmetry restoration do not happen at the same temperature, rather $T_\chi^* \gg T_d^*$ [6]. Although there exist theoretical considerations on the interplay between chiral-symmetry breaking and confinement at zero baryon density [7], the underlying reasons for the coincidence are still unknown and uncertain at nonzero baryon density [8–10].

The hadronic properties, especially chiral dynamics at low energy have been successfully described by chiral effective models such as the linear sigma model [11], the Nambu-Jona-Lasinio (NJL) model [12–14], the chiral random matrix model [15], chiral perturbation theory [16], and so on. However, those models based on chiral symmetry lack any dynamics coming from confinement dictated by the Polyakov loop, although there are some efforts to clarify the interplay between chiral dynamics and the Polyakov loop [17,18].

Recent chiral effective models with the Polyakov-loop degrees of freedom augmented called the Polyakov-loop extended NJL (PNJL) model or quark-meson model [19–26] are successful from a phenomenological point of view to incorporate a coupling between the chiral condensate and the Polyakov-loop. However, these PNJL/quark-meson models are still far from treating the chiral condensate and the Polyakov loop on an equal footing, except for the work [26] where the back coupling of the matter sector to the glue sector was discussed by changing the phase transition parameter. In fact, the gluonic part in these models has several fitting parameters which are determined only from lattice QCD data.

Here we must mention a preceding work for a first-principle derivation of confinement/deconfinement and chiral-symmetry breaking/restoration crossover phase transition based on the flow equation [27] of the functional renormalization group [28,29] given by Braun, Haas, Marhauser, and Pawłowski [30] for the full dynamical QCD with 2 massless flavors (at zero and imaginary chemical potential). In this work, the Yang-Mills theory is fully coupled to the matter sector by taking into account the Polyakov-loop effective potential [31] obtained in a nonperturbative way put forward by Braun, Gies, and Pawłowski [32] and Marhauser and Pawłowski [33].

The main purpose of this paper is to provide a theoretical framework (a reformulation of QCD) which enables one to describe in a unified way the chiral dynamics and confinement signaled by the Polyakov loop. We give an important step towards a first-principle derivation of confinement/deconfinement and chiral-symmetry breaking/restoration crossover transition. In fact, we demonstrate that a low-energy effective theory of QCD obtained in simple but nontrivial approximations within this framework enables one to treat both transitions simultaneously on equal footing.

The basic ingredients in this paper are a reformulation of QCD based on new variables [34–51] and the flow equation

of the Wetterich type in the Wilsonian renormalization group [27–29]. The reformulation was used to confirm quark confinement in pure Yang-Mills theory at zero temperature and zero density based on a dual superconductor picture [52]. In this paper, it is extended to QCD at finite temperature and density. In principle, our framework can be applied to any color gauge group and arbitrary number of flavors. For technical reasons, however, we study two color QCD with two flavors in this paper. The three color and/or three flavor case will be studied in a subsequent paper. In future publications, this framework will be applied to investigate the QCD phase diagram at finite density. We hope that this paper will give an insight into this issue complementary to other works, e.g., [30].

In Sec. II, we give a reformulation of QCD written in terms of new variables and explain why the reformulated QCD is efficient to study the interplay between confinement and chiral-symmetry breaking.

In Sec. III, we choose a specific gauge (modified Polyakov gauge) to simplify the representation of the Polyakov loop. We can choose any gauge to calculate the Polyakov-loop average and the chiral condensate, since both are gauge-invariant quantities and should not depend on the gauge chosen.

In Sec. IV, we give a definition of the Polyakov-loop operator and examine how the Polyakov-loop average is related to the average of the time component of the gauge field.

In Secs. V and VI, we study the confinement/deconfinement phase transition in pure $SU(2)$ Yang-Mills (YM) theory at finite temperature. We exploit the Wilsonian renormalization group in our framework to obtain the effective potential V_{eff} of the Polyakov loop L , whose minimum gives the Polyakov-loop average $\langle L \rangle$. It is known that the Weiss potential V_W [31] calculated in the perturbation theory to one loop exhibits spontaneous center-symmetry breaking, i.e., deconfinement, irrespective of the temperature T . This result can be used at high temperature where the perturbation theory will be trustworthy due to asymptotic freedom, while the nonperturbative approach is necessary to treat the low-temperature case. The Weiss potential can be improved according to the Wilsonian renormalization group to obtain a nonperturbative effective potential which is valid even at low temperature.

In Sec. V, we write down the flow equation of the Wetterich type for the effective potential of the Polyakov loop in our framework. In fact, the effective potential obtained by solving the flow equation in a numerical way shows the existence of confinement phase below a certain temperature T_d . This solution was shown for the first time by Marhauser and Pawłowski [33] and by Braun, Gies, and Pawłowski [32], see [53] for the previous works. In this sense, this section is nothing but the translation of their results [32,33] into our framework.

In Sec. VI, we give a qualitative understanding for the confinement/deconfinement transition given in Sec. V based on the Landau-Ginzburg argument. We answer a question why the center-symmetry restoration occurs as the temperature is decreased, by observing the flow equation for the coefficient of the effective potential.

In Sec. VII, we describe the low-energy effective interaction among quarks by a nonlocal version of the (gauged) NJL model in which the effect of confinement is explicitly incorporated through the Polyakov-loop dependent nonlocal interaction. The resulting effective theory is regarded as a modified and improved version of nonlocal PNJL models proposed recently by Hell, Rössner, Cristoforetti, and Weise [22], Sasaki, Friman, and Redlich [23], and Blaschke, Buballa, Radzhabov, and Volkov [24], extending the original (local) PNJL model by Fukushima [19]. The nonlocal (gauged) NJL model can be converted to the nonlocal (gauged) Yukawa model to be bosonized to study the chiral dynamics.

In Sec. VIII, we show that the nonlocal NJL interaction among quarks becomes temperature dependent through the coupling to the Polyakov loop. This is a first nontrivial indication for the entanglement between the chiral-symmetry breaking and confinement. This feature was overlooked in conventional PNJL models.

In Sec. IX, we consider how to understand the entanglement between confinement and chiral-symmetry breaking in our framework. This is just a short sketch for our strategy following the line given in the preceding sections.

The final section is used to summarize the results and give some perspective in the future works. Some technical materials are collected in the Appendices.

II. REFORMULATION OF QCD

To fix the notation, we write the action of QCD in terms of the gluon field \mathcal{A}_μ and the quark field ψ

$$S_{\text{QCD}} = S_q + S_{\text{YM}},$$

$$S_q := \int d^D x \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu[\mathcal{A}] - \hat{m}_q + \mu_q \gamma^0) \psi, \quad (1)$$

$$S_{\text{YM}} := \int d^D x \frac{-1}{2} \text{tr}(\mathcal{F}_{\mu\nu}[\mathcal{A}] \mathcal{F}^{\mu\nu}[\mathcal{A}]),$$

where ψ is the quark field, $\mathcal{A}_\mu = \mathcal{A}_\mu^A T_A$ is the gluon field with $su(N_c)$ generators T_A for the gauge group $G = SU(N_c)$ ($A = 1, \dots, \dim SU(N_c) = N_c^2 - 1$), \hat{m}_q is the quark mass matrix, μ_q is the quark chemical potential, γ^μ are the Dirac gamma matrices ($\mu = 0, \dots, D - 1$), $\mathcal{D}_\mu[\mathcal{A}] := \partial_\mu - ig\mathcal{A}_\mu$ is the covariant derivative in the fundamental representation, $\mathcal{F}_{\mu\nu}[\mathcal{A}] := \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu]$ is the field strength, and g is the QCD coupling constant. In what follows, we suppress the spinor, color, and flavor indices.

The main purpose of this paper is to give a theoretical framework for extracting a low-energy effective theory

which enables one to discuss the confinement/deconfinement and chiral-symmetry breaking/restoration (crossover) transition simultaneously on an equal footing. We reformulate QCD in terms of new variables which are efficient for this purpose. We start with decomposing the original $SU(N)$ Yang-Mills field $\mathcal{A}_\mu(x) = \mathcal{A}_\mu^A(x)T_A$ into two pieces $\mathcal{V}_\mu = \mathcal{V}_\mu^A(x)T_A$ and $\mathcal{X}_\mu = \mathcal{X}_\mu^A(x)T_A$

$$\mathcal{A}_\mu(x) = \mathcal{V}_\mu(x) + \mathcal{X}_\mu(x), \quad (2)$$

to rewrite the original QCD action into a new form

$$S_q = \int d^D x \{ \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu[\mathcal{V}] - \hat{m}_q + \mu_q \gamma^0) \psi + g \mathcal{J}^\mu \cdot \mathcal{X}_\mu \},$$

$$S_{\text{YM}} = \int d^D x \left\{ \frac{-1}{4} (\mathcal{F}_{\mu\nu}^A[\mathcal{V}])^2 - \frac{1}{2} \mathcal{X}^{\mu A} Q_{\mu\nu}^{AB} \mathcal{X}^{\nu B} - \frac{1}{4} (ig[\mathcal{X}_\mu, \mathcal{X}_\nu])^2 \right\} + S_{\text{FP}}, \quad (3)$$

where $\mathcal{J}^{\mu A} := g \bar{\psi} \gamma^\mu T_A \psi$ is the color current, $D_\mu[\mathcal{V}] := \partial_\mu - ig[\mathcal{V}_\mu, \cdot]$ is the covariant derivative in the adjoint representation, and

$$Q_{\mu\nu}^{AB}[\mathcal{V}] := G^{AB}[\mathcal{V}] g_{\mu\nu} + 2g f^{ABC} \mathcal{F}_{\mu\nu}^C[\mathcal{V}],$$

$$G^{AB}[\mathcal{V}] := -(D_\rho[\mathcal{V}] D^\rho[\mathcal{V}])^{AB}$$

$$= -(\partial_\rho \delta^{AC} + g f^{AEC} \mathcal{V}_\rho^E) (\partial^\rho \delta^{CB} + g f^{CFB} \mathcal{V}^{\rho F})$$

$$= -\partial_\rho^2 \delta^{AB} + g^2 f^{AEC} f^{BFC} \mathcal{V}_\rho^E \mathcal{V}^{\rho F}$$

$$+ 2g f^{ABE} \mathcal{V}_\rho^E \partial^\rho + g f^{ABE} \partial^\rho \mathcal{V}_\rho^E. \quad (4)$$

In what follows we use the notation $\mathcal{A} \cdot \mathcal{B}$ for two Lie-algebra valued functions $\mathcal{A} = \mathcal{A}^A T_A$ and $\mathcal{B} = \mathcal{B}^A T_A$ in the sense that $\mathcal{A} \cdot \mathcal{B} := \mathcal{A}^A \mathcal{B}^A = 2 \text{tr}(\mathcal{A} \mathcal{B})$ and especially $\mathcal{A}^2 := \mathcal{A} \cdot \mathcal{A} = \mathcal{A}^A \mathcal{A}^A$.

Historically, the decomposition of Yang-Mills theory into new variables has been proposed by Cho [34] and Duan and Ge [35] independently, and readdressed later by Faddeev and Niemi [36]. The decomposition was further developed by Shabanov [37].

The decomposition (2) is performed such that \mathcal{V}_μ transforms under the gauge transformation just like the original gauge field \mathcal{A}_μ

$$\mathcal{V}_\mu(x) \rightarrow \mathcal{V}'_\mu(x) = \Omega(x) (\mathcal{V}_\mu(x) + ig^{-1} \partial_\mu) \Omega^{-1}(x), \quad (5)$$

while \mathcal{X}_μ transforms like an adjoint matter field

$$\mathcal{X}_\mu(x) \rightarrow \mathcal{X}'_\mu(x) = \Omega(x) \mathcal{X}_\mu(x) \Omega^{-1}(x). \quad (6)$$

In the decomposition (2), we introduce a new field

$$\mathbf{n}(x) = n^A(x) T_A, \quad (7)$$

with a unit length in the sense that $n^A(x) n^A(x) = 1$, which we call the *color field*. In the decomposition (2), the color

field $\mathbf{n}(x)$ plays a crucial role as follows. The color field is defined by the following property. It must be a functional or composite operator of the original Yang-Mills field $\mathcal{A}_\mu(x)$ such that it transforms according to the adjoint representation under the gauge transformation

$$\mathbf{n}(x) \rightarrow \mathbf{n}'(x) = \Omega(x)\mathbf{n}(x)\Omega^{-1}(x). \quad (8)$$

The color field plays the key role in the reformulation. Once a color field is given, the decomposition is uniquely determined by solving a set of defining equations and hence $\mathcal{V}_\mu(x)$ and $\mathcal{X}_\mu(x)$ are written in terms of $\mathcal{A}_\mu(x)$ and $\mathbf{n}(x)$. For $G = SU(2)$, the defining equations are given by

(I) covariant constantness of color field $\mathbf{n}(x)$ in $\mathcal{V}_\mu(x)$

$$0 = D_\mu[\mathcal{V}]\mathbf{n}(x), \quad (9)$$

(II) orthogonality of $\mathcal{X}_\mu(x)$ to $\mathbf{n}(x)$

$$0 = \mathcal{X}_\mu(x) \cdot \mathbf{n}(x). \quad (10)$$

Then the decomposition for $G = SU(2)$ is uniquely determined as

$$\begin{aligned} \mathcal{V}_\mu(x) &= c_\mu(x)\mathbf{n}(x) + ig^{-1}[\mathbf{n}(x), \partial_\mu\mathbf{n}(x)], \\ c_\mu(x) &:= \mathcal{A}_\mu(x) \cdot \mathbf{n}(x), \\ \mathcal{X}_\mu(x) &= ig^{-1}[D_\mu[\mathcal{A}]\mathbf{n}(x), \mathbf{n}(x)]. \end{aligned} \quad (11)$$

To arrive at the result (3), we have used the following facts. See Appendix A for the details.

- (i) The $O(\mathcal{X})$ terms vanish, $\frac{1}{2}\mathcal{F}^{\mu\nu}[\mathcal{V}] \cdot (D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu) = 0$, from the property of the new variables as shown using the *defining equations of the decomposition* (2). This is somewhat similar to the usual background field method in which $O(\mathcal{X})$ terms in the quantum fluctuation field \mathcal{X}_μ are eliminated by requiring that the background field \mathcal{V}_μ satisfies the classical Yang-Mills equation of motion, i.e., $D_\mu[\mathcal{V}]\mathcal{F}^{\mu\nu}[\mathcal{V}] = 0$. In our case, however, \mathcal{V}_μ does not necessarily satisfy the classical equation of motion.
- (ii) To obtain $Q_{\mu\nu}^{AB}[\mathcal{V}]$ in (4), an $O(\mathcal{X}^2)$ term is eliminated, $-\frac{1}{2}\mathcal{X}^{\mu A}D_\mu^{AC}[\mathcal{V}]D_\nu^{CB}[\mathcal{V}]\mathcal{X}^{\nu B} = 0$, by imposing the condition

$$D_\mu[\mathcal{V}]\mathcal{X}^\mu = 0. \quad (12)$$

For the reformulated QCD to be equivalent to the original QCD, we must impose such a constraint to avoid mismatch in the independent degrees of freedom, which is called the *reduction condition* [38,41].

- (iii) The $O(\mathcal{X}^3)$ term vanishes, $\frac{1}{2}(D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu) \cdot ig[\mathcal{X}^\mu, \mathcal{X}^\nu] = 0$, since $D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu$ is orthogonal to $[\mathcal{X}^\mu, \mathcal{X}^\nu]$.

For $G = SU(2)$, \mathcal{V} can be chosen in such a way that the field strength $\mathcal{F}[\mathcal{V}]$ of the field \mathcal{V} is proportional to \mathbf{n}

$$\begin{aligned} \mathcal{F}_{\mu\nu}[\mathcal{V}](x) &:= \partial_\mu\mathcal{V}_\nu(x) - \partial_\nu\mathcal{V}_\mu(x) - ig[\mathcal{V}_\mu(x), \mathcal{V}_\nu(x)] \\ &= \mathbf{n}(x)G_{\mu\nu}(x), \end{aligned} \quad (13)$$

where $G_{\mu\nu}$ is a gauge-invariant antisymmetric tensor of rank 2, i.e., $\mathcal{F}'_{\mu\nu}[\mathcal{V}](x) = \mathcal{F}_{\mu\nu}[\mathcal{V}'] \times (x) = \Omega(x)\mathcal{F}_{\mu\nu}[\mathcal{V}](x)\Omega^{-1}(x) = \mathbf{n}'(x)G_{\mu\nu}(x)$. The explicit form of $G_{\mu\nu}$ is written in terms of $\mathcal{A}_\mu(x)$ and $\mathbf{n}(x)$ as

$$\begin{aligned} G_{\mu\nu}(x) &= \partial_\mu[\mathbf{n}(x) \cdot \mathcal{A}_\nu(x)] - \partial_\nu[\mathbf{n}(x) \cdot \mathcal{A}_\mu(x)] \\ &\quad + ig^{-1}\mathbf{n}(x) \cdot [\partial_\mu\mathbf{n}(x), \partial_\nu\mathbf{n}(x)]. \end{aligned} \quad (14)$$

In the present approach, we wish to regard the field decomposition as a change of variable from the original gluon field to new variables describing a reformulated Yang-Mills theory in the quantum level [38,40,41] (see, [42–47,54,55] for the corresponding lattice gauge formulation). To achieve this goal, first of all, $\mathbf{n}(x)$ must be written as a functional of $\mathcal{A}_\mu(x)$ and thereby all new fields are written in terms of the original gluon field $\mathcal{A}_\mu(x)$. Such a required relationship between $\mathcal{A}_\mu(x)$ and $\mathbf{n}(x)$ is given by the reduction condition, which is given as a variational problem of obtaining an absolute minimum of a given functional. The condition for local minima is given in the form of a differential equation. For $G = SU(2)$,

$$[\mathbf{n}(x), D_\mu[\mathcal{A}]D_\mu[\mathcal{A}]\mathbf{n}(x)] = 0. \quad (15)$$

This is another form of (12). See [38] in the $SU(2)$ case and [41] in the $SU(N)$ case for the full details.

Remarkable properties of new variables are as follows. First, we remind you of the role played by the field \mathcal{V} .

- (i) The variable \mathcal{V}_μ alone is responsible for the Wilson loop operator $W_C[\mathcal{A}]$ and the Polyakov-loop operator $L[\mathcal{A}]$ in the sense that

$$W_C[\mathcal{A}] = W_C[\mathcal{V}], \quad L[\mathcal{A}] = L[\mathcal{V}], \quad (16)$$

where the Wilson loop operator is defined by

$$W_C[A] := \mathcal{N}^{-1} \text{tr} \left[\mathcal{P} \exp \left\{ ig \oint_C dx^\mu \mathcal{A}_\mu(x) \right\} \right], \quad (17)$$

where \mathcal{P} denotes the path ordering and the normalization factor \mathcal{N} is the dimension of the representation R , in which the Wilson loop is considered, i.e., $\mathcal{N} := d_R = \dim(\mathbf{1}_R) = \text{tr}(\mathbf{1}_R)$. The Polyakov-loop operator will be defined later. In other words, \mathcal{X}_μ does not contribute to the Wilson loop and the Polyakov loop in the operator level. This is because the defining equation for the decomposition is a (necessary and) sufficient condition for a *gauge-invariant Abelian dominance* (or \mathcal{V} dominance) in the operator level. This proposition was first proved in [48] for $SU(2)$ and for $SU(N)$ in the

continuum [49] and for $SU(N)$ on a lattice [50]. On the lattice, the equality does not exactly hold due to nonzero lattice spacing ϵ , but the deviation vanishes in the continuum limit of the lattice spacing ϵ going to zero, $\epsilon \rightarrow 0$. It should be remarked that both the Wilson loop operator and the Polyakov-loop operator are gauge-invariant quantities and that their average does not depend on the gauge-fixing condition adopted in the calculation.

- (ii) We can introduce a gauge-invariant magnetic monopole current k in Yang-Mills theory (without matter fields) where k is the $(D - 3)$ form. For $D = 4$ and $G = SU(2)$,

$$\begin{aligned} k_\mu(x) &:= \partial_\nu {}^*G_{\mu\nu}(x), \\ G_{\mu\nu} &:= \mathbf{n} \cdot \mathcal{F}_{\mu\nu}[\mathcal{V}] \\ &= \partial_\mu c_\nu - \partial_\nu c_\mu + ig^{-1} \mathbf{n} \cdot [\partial_\mu \mathbf{n}, \partial_\nu \mathbf{n}], \end{aligned} \quad (18)$$

where $f_{\mu\nu}$ is gauge-invariant field strength. This is because the field strength $\mathcal{F}_{\mu\nu}[\mathcal{V}] := \partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu - ig[\mathcal{V}_\mu, \mathcal{V}_\nu]$ is proportional to \mathbf{n}

$$\mathcal{F}_{\mu\nu}[\mathcal{V}] = \mathbf{n} \{ \partial_\mu c_\nu - \partial_\nu c_\mu + ig^{-1} \mathbf{n} \cdot [\partial_\mu \mathbf{n}, \partial_\nu \mathbf{n}] \}. \quad (19)$$

- (iii) The gauge-invariant ‘‘Abelian’’ dominance [56,57] (or \mathcal{V} dominance) and magnetic monopole dominance [58] (constructed from \mathcal{V}) in quark confinement have been confirmed at $T = 0$ (and $\mu_q = 0$) by comparing string tensions calculated from the Wilson loop average by numerical simulations by [43] for $SU(2)$ and by [45] for $SU(3)$. Here it should be remarked that the Abelian dominance is the dominance for the vacuum expectation value (or average)

$$\langle W_C[\mathcal{A}] \rangle \simeq \langle W_C[\mathcal{V}] \rangle, \quad \langle L[\mathcal{A}] \rangle \simeq \langle L[\mathcal{V}] \rangle. \quad (20)$$

Next, we pay attention to the role played by the remaining field \mathcal{X} . In the absence of dynamical quarks (corresponding to the limit $m_q = \infty$ of QCD, i.e., gluodynamics), \mathcal{X}_μ^A decouples in the low-energy regime as the correlator $\langle \mathcal{X}_\mu^A(x) \mathcal{X}_\mu^A(y) \rangle$ behaves like a massive correlator with mass M_X . In fact, numerical simulations demonstrate for $G = SU(2)$ and $D = 4$ [44]

$$M_X = 1.2 \sim 1.3 \text{ GeV}. \quad (21)$$

We can understand this result as follows. The field \mathcal{X}_μ can acquire the (gauge-invariant) mass dynamically. This comes from a fact that, in sharp contrast to the field \mathcal{A}_μ , a ‘‘gauge-invariant mass term’’ for \mathcal{X}_μ can be introduced

$$\frac{1}{2} M_X^2 \mathcal{X}_\mu^A(x) \mathcal{X}_\mu^A(x), \quad (22)$$

since $\mathcal{X}_\mu^A(x) \mathcal{X}_\mu^A(x)$ is a gauge-invariant operator. Moreover, this mass term can originate from a vacuum condensation of ‘‘mass dimension 2,’’ $\langle \mathcal{X}_\nu^B(x) \mathcal{X}_\nu^B(x) \rangle \neq 0$

as proposed in [59]. In fact, this condensation can be generated through self-interactions $O(\mathcal{X}^4)$ among \mathcal{X}_μ gluons, $M_X^2 \simeq \langle \mathcal{X}_\nu^B(x) \mathcal{X}_\nu^B(x) \rangle$, as examined in [40,54]. It is instructive to remark that the value (21) agrees with the earlier result of the off-diagonal ‘‘gluon mass’’ M_A in the maximally Abelian gauge [60] for the $SU(2)$ case, $M_A \simeq 1.2 \text{ GeV}$. See [61] for the $SU(3)$ case, $M_A \simeq 1.1 \text{ GeV}$. In maximally Abelian gauge, it was shown that even at finite temperatures Abelian dominance (diagonal part dominance and off-diagonal part suppression) holds for the spatial propagation of gluons in the long distance greater than 0.4 fm. It was observed that the diagonal gluon correlator largely changes between the confinement and the deconfinement phase, while the off-diagonal gluon correlator is almost the same even in the deconfinement phase [62]. Although the similar results are expected to hold in our formulation, this observation must be checked directly, as will be confirmed in [63].

In the presence of dynamical quarks ($m_q < \infty$), \mathcal{X}_μ^A is responsible for chiral-symmetry breaking in the following sense. We consider to integrate out the field \mathcal{X}_μ^A in a naive way. This helps us to obtain an intuitive and qualitative understanding for the interplay between the chiral-symmetry breaking and confinement. Later, this integration procedure will be reconsidered from the viewpoint of the renormalization group to obtain a systematic improvement of the result.

Here we neglect $O(\mathcal{X}^4)$ terms, which will be taken into account later. Then the integration over \mathcal{X}_μ^A can be achieved by the Gaussian integration according to [40]. Consequently, a nonlocal 4 fermion-interaction is generated

$$\begin{aligned} S_{\text{eff}}^{\text{QCD}} &= S_{\text{eff}}^{\text{glue}} + S_{\text{eff}}^{\text{gNJL}}, \\ S_{\text{eff}}^{\text{glue}} &:= \int d^D x \frac{-1}{4} (\mathcal{F}_{\mu\nu}[\mathcal{V}])^2 + \frac{i}{2} \text{Indet} Q[\mathcal{V}]_{\mu\nu}^{AB} \\ &\quad - i \text{Indet} G[\mathcal{V}]^{AB}, \\ S_{\text{eff}}^{\text{gNJL}} &:= \int d^D x \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu[\mathcal{V}] - \hat{m}_q + i\gamma^0 \mu) \psi \\ &\quad + \int d^D x \int d^D y \frac{g^2}{2} \mathcal{J}_A^\mu(x) Q^{-1}[\mathcal{V}]_{\mu\nu}^{AB}(x, y) \mathcal{J}_B^\nu(y), \end{aligned} \quad (23)$$

where the last term $-\text{Indet} G^{AB}$ in $S_{\text{eff}}^{\text{glue}}$ comes from the Faddeev-Popov determinant associated with the reduction condition (12) (see, [39] for the precise form).

This is a nonlocal version of a gauged NJL model (realized after Fierz transformation). The chiral-symmetry breaking/restoration transition and the phase structure of a local version of gauge NJL models were first studied by solving the Schwinger-Dyson equation in the ladder approximation for QED-like [64–66] (see, [67] for a review) and QCD-like [68] running gauge coupling constants. They are confirmed later by a systematic approach of the renormalization group [69].

The range of the nonlocality is determined by the correlation length ξ , which is characteristic of the color exchange through gluon fields. Therefore, this correlation length ξ is identified with the inverse of the effective mass M_X , i.e., $\xi \simeq M_X^{-1}$. In fact, $(Q^{-1})_{\mu\nu}^{AB}(x, y)$ is the X field correlator; see (3).

In other words, M_X is identified with the ultraviolet cutoff Λ below which the effective NJL model appears and works well. Interesting enough, M_X is nearly equal to the ultraviolet cutoff adopted in the NJL model

$$\sqrt{p^2} \simeq \Lambda_4 = 1.4 \text{ GeV}, \quad |p| \simeq \Lambda_3 = 0.6 \text{ GeV}; \quad (24)$$

see [14].

We can decompose the gauge field \mathcal{A}_μ into the low-energy (light) mode $p < M_X$ and high-energy (heavy) mode $p > M_X$

$$\mathcal{A}_\mu(p) = \mathcal{A}_\mu(p)\theta(M_X^2 - p^2) + \mathcal{A}_\mu(x)\theta(p^2 - M_X^2). \quad (25)$$

In the above treatment, $\mathcal{X}_\mu(p)$ is supposed to have only the high-energy mode. The low-energy mode, if any, will be responsible for the vacuum condensation [40]. For the precise understanding, we need the renormalization group treatment as given later and the implications for the nonlocal NJL model will be discussed there.

III. GLUON SECTOR AND GAUGE FIXING

The Polyakov-loop operator L and the chiral operator $\bar{\psi}\psi$ are gauge-invariant quantities. Therefore, their average does not depend on the gauge-fixing procedure adopted in the calculation. We can choose a gauge in which the actual calculation becomes easier than other gauges.

In what follows, we treat the time component \mathcal{V}_0 and space component \mathcal{V}_j of \mathcal{V}_μ differently to consider the finite-temperature case. We consider the following Polyakov gauge modified for new variables in our reformulation. If the color field $n^A(x)$ is uniform in time,

$$\partial_0 n^A(x) = 0 \Leftrightarrow n^A(x) = n^A(\mathbf{x}), \quad (26)$$

then \mathcal{V}_0 reduces to

$$\mathcal{V}_0^A(x) = c_0(x)n^A(\mathbf{x}) \quad (A = 1, 2, 3). \quad (27)$$

Moreover, if $c_0(x)$ is uniform in time,

$$\partial_0 c_0(x) = 0 \Leftrightarrow c_0(x) = c_0(\mathbf{x}), \quad (28)$$

then \mathcal{V}_0 reduces to

$$\mathcal{V}_0^A(x) = c_0(\mathbf{x})n^A(\mathbf{x}) \quad (A = 1, 2, 3), \quad (29)$$

which satisfies

$$\partial_0 \mathcal{V}_0^A(x) = 0 \quad (A = 1, 2, 3). \quad (30)$$

In this setting, \mathcal{V}_j is given by

$$\mathcal{V}_j^A(x) = c_j(x)n^A(\mathbf{x}) + g^{-1}\epsilon^{ABC}\partial_j n^B(\mathbf{x})n^C(\mathbf{x}). \quad (31)$$

- (1) In order to simplify the calculation of the Polyakov line, we adopt the Polyakov gauge in which the gauge field is diagonal and time independent: for the background field $\mathcal{V}_0^A(x)$,

$$\mathcal{V}_0^A(x) = c_0(\mathbf{x})\delta^{A3}, \quad (32)$$

which leads to

$$\partial_0 \mathcal{V}_0^A(x) = 0. \quad (33)$$

This is realized, if we take the gauge ¹

$$n^A(\mathbf{x}) = \delta^{A3}. \quad (34)$$

In this gauge, the space component reads

$$\mathcal{V}_j^A(x) = c_j(x)\delta^{A3}, \quad (35)$$

which is not time independent, $\partial_0 \mathcal{V}_j^A(x) = 0$.

- (2) We expand the theory around the nontrivial uniform background $g^{-1}T\varphi\delta^{A3}$ for the time component \mathcal{V}_0^A , while the trivial background for space components \mathcal{V}_j^{A2}

$$\begin{aligned} \mathcal{V}_0^A(x) &= c_0(\mathbf{x})\delta^{A3}, \\ c_0(\mathbf{x}) &= g^{-1}T\varphi + v_0(\mathbf{x}), \\ \mathcal{V}_j^A(x) &= 0 + v_j^A(x), \end{aligned} \quad (36)$$

such that $\langle c_0(\mathbf{x}) \rangle = g^{-1}T\langle \varphi \rangle + \langle v_0(\mathbf{x}) \rangle = g^{-1}T\langle \varphi \rangle$ with $\langle v_0(\mathbf{x}) \rangle = 0$ and $\langle v_j^A(x) \rangle = 0$. Here the prefactor $g^{-1}T = (g\beta)^{-1}$ was introduced just for the purpose of simplifying the expression of the Polyakov loop; see (41).

- (3) We take into account the expansion up to quadratic in the fluctuation fields v_0 and v_j , which we call the quadratic approximation.

In the calculation of $Q_{\mu\nu}^{AB}$, if we neglect all fluctuation fields v_0 and v_j , namely, $\mathcal{V}_0^A(x) = g^{-1}T\varphi\delta^{A3}$ and

¹This is an oversimplified choice for the color field $\mathbf{n}(x)$. By this choice, we cannot separate the nonperturbative contribution coming from topological configurations such as the magnetic monopole. It is desirable to take into account color field degrees of freedom explicitly to see the effect of the magnetic monopole in the confinement/deconfinement transition [70].

²I have assumed that the spatial component of \mathcal{V}_μ has a trivial background. In view of logical consistency, one must expand the spatial and temporal components around nontrivial backgrounds, and then one must search for the minima of the effective potential calculated as a function of two variables, i.e., the temporal and spatial backgrounds. In this paper, it is assumed that a minimum is realized at vanishing spatial background and that the neglect of the spatial background does not so much affect the confinement/deconfinement transition temperature. Indeed, it must be checked whether this assumption is good or not.

$\mathcal{V}_j^A(x) = 0$, then we can put $\mathcal{F}_{\mu\nu}^C[\mathcal{V}] = 0$ in $Q_{\mu\nu}^{AB}$, and $Q_{\mu\nu}^{AB}$ is diagonal in the Lorentz indices

$$\begin{aligned} Q_{\mu\nu}^{AB} &= G^{AB} g_{\mu\nu}, \\ G^{AB} &= -\delta^{AB} \partial_\mu^2 + (\delta^{AB} - \delta^{A3} \delta^{B3})(T\varphi)^2 + 2\epsilon^{AB3} T\varphi \partial_0. \end{aligned} \quad (37)$$

In this approximation, we have

$$G^{AB} = -\delta^{AB} \partial_\ell^2 - D_0^{AC}[\mathcal{V}] D_0^{CB}[\mathcal{V}], \quad (38)$$

with

$$\begin{aligned} -D_0^{AC}[\mathcal{V}] D_0^{CB}[\mathcal{V}] &= -\delta^{AB} \partial_0^2 + 2\epsilon^{AB3} T\varphi \partial_0 \\ &\quad + (\delta^{AB} - \delta^{A3} \delta^{B3})(T\varphi)^2. \end{aligned} \quad (39)$$

Thus we rewrite the gluon part $S_{\text{eff}}^{\text{glue}}[\mathcal{V}]$ as

$$\begin{aligned} S_{\text{eff}}^{\text{glue}}[\mathcal{V}] &= \frac{1}{2} \beta \int d^3x \mathcal{V}_0(x) (-\partial_j \partial_j) \mathcal{V}_0(x) \\ &\quad + \frac{1}{2} \int d^4x \mathcal{V}_T^A(x) \{-\delta^{AB} \partial_\ell^2 - \delta^{AB} \partial_0^2\} \mathcal{V}_T^B(x) \\ &\quad + \frac{1}{2} \int d^4x \mathcal{V}_L^A(x) \{-\delta^{AB} \partial_0^2\} \mathcal{V}_L^B(x), \\ &\quad + \frac{i}{2} \text{Indet} Q[\mathcal{V}]_{\mu\nu}^{AB} - i \text{Indet} G[\mathcal{V}]^{AB}, \end{aligned} \quad (40)$$

where β is the inverse temperature $\beta := 1/T$, and \mathcal{V}_T and \mathcal{V}_L denote the transverse and longitudinal components of \mathcal{V}_μ , respectively.

IV. POLYAKOV LOOP

For $G = SU(2)$, the Polyakov-loop operator $L(\mathbf{x}) = L[\mathcal{V}_0(\mathbf{x}, \cdot)]$ is defined by

$$\begin{aligned} L(\mathbf{x}) &:= \frac{1}{2} \text{tr}(P), \\ P(\mathbf{x}) &:= \mathcal{P} \exp \left[ig \int_0^{\beta=1/T} dx_0 \mathcal{V}_0^A(\mathbf{x}, x_0) \frac{\sigma_A}{2} \right], \end{aligned} \quad (41)$$

where $PP^\dagger = \mathbf{1}$ and $\det P = 1$. In the above gauge choice,

$$P(\mathbf{x}) = \exp \left[ig \beta c_0(\mathbf{x}) n^A(\mathbf{x}) \frac{\sigma_A}{2} \right].$$

After a suitable (t -independent) gauge transformation, the color field $n^A(\mathbf{x})$ is eliminated

$$L(\mathbf{x}) = \frac{1}{2} \text{tr} \left(\exp \left[ig \beta c_0(\mathbf{x}) \frac{\sigma_3}{2} \right] \right) = \cos \left(\frac{g \beta c_0(\mathbf{x})}{2} \right). \quad (42)$$

Owing to periodicity and center symmetry, we can restrict the Polyakov-loop average to $\langle L \rangle \geq 0$ for $G = SU(2)$. Then the Polyakov-loop average $\langle L[\mathcal{V}] \rangle$ is bound from above by $L[\langle \mathcal{V}_0(\mathbf{x}, \cdot) \rangle]$

$$0 \leq \langle L[\mathcal{V}_0(\mathbf{x}, \cdot)] \rangle \leq L[\langle \mathcal{V}_0(\mathbf{x}, \cdot) \rangle] = \cos \left(\frac{\langle \varphi \rangle}{2} \right), \quad (43)$$

where we have only to consider the range $0 \leq \varphi \leq \pi$. This inequality follows from the Jensen inequality, since $\cos(x)$ is concave for $0 \leq x \leq \pi/2$; see [33].

In the case of $m_q = \infty$, if the center symmetry is broken $\langle L \rangle > 0$, namely, deconfinement takes place, then the vacuum [as a minimum of the effective potential $V_{\text{eff}}(\varphi)$] is realized at $\langle \varphi \rangle < \pi$. If the vacuum is realized at $\langle \varphi \rangle = \pi$, then the center symmetry is restored $\langle L \rangle = 0$, namely, confinement occurs. The relation (42) yields the relationship for the average between the gauge field and the Polyakov-loop operator

$$\langle \arccos L(\mathbf{x}) \rangle = \frac{g \beta \langle c_0(\mathbf{x}) \rangle}{2} = \frac{\langle \varphi \rangle}{2}, \quad (44)$$

where the left-hand side is the average of an gauge-invariant object (since L is gauge invariant) and happens to agree with the average $\langle \mathcal{V}_0^3 \rangle$ of the gauge field in the Polyakov gauge. It is also shown [33] that the converse is true: In the center-symmetry-restored phase, $\langle \varphi \rangle = \pi$, since

$$\frac{\langle \varphi \rangle}{2} = \langle \arccos L(\mathbf{x}) \rangle = \arccos \langle L(\mathbf{x}) \rangle = \frac{\pi}{2}. \quad (45)$$

Therefore, $\langle \mathcal{V}_0 \rangle$ or $\langle \varphi \rangle$ in the Polyakov gauge gives a direct physical interpretation as an order parameter for the confinement/deconfinement (order-disorder) phase transition. The effective potential $U_{\text{eff}}(\langle L \rangle)$ of the Polyakov-loop average $\langle L \rangle$ could be different from the effective potential $V_{\text{eff}}(\langle \mathcal{V}_0 \rangle)$ of the gauge field average $\langle \mathcal{V}_0 \rangle$ in the following sense. Although both potentials give the same critical temperature T_d as a boundary between $\langle L \rangle = 0$ and $\langle L \rangle \neq 0$, the value of the effective potential $U_{\text{eff}}(\langle L[\mathcal{V}_0] \rangle)$ does not necessarily agree with $U_{\text{eff}}(L[\langle \mathcal{V}_0 \rangle]) = V_{\text{eff}}(\langle \mathcal{V}_0 \rangle)$ at a given temperature T , since we have only an inequality $\langle L[\mathcal{V}_0] \rangle \leq L[\langle \mathcal{V}_0 \rangle]$. This difference could affect the critical exponent and other physical quantities of interest. Therefore, the result obtained from $V_{\text{eff}}(\langle \mathcal{V}_0 \rangle)$ must be carefully examined.

V. DERIVING THE CONFINEMENT/ DECONFINEMENT TRANSITION

In this section, we restrict our consideration to the pure glue case. We show that the pure gluon part $S_{\text{eff}}^{\text{glue}}$ can describe confinement/deconfinement transition signaled by the Polyakov-loop average $\langle L \rangle$. In this section, we completely follow two remarkable papers by Marhauser and Pawłowski [33] and by Braun, Gies, and Pawłowski [32], which succeeded to show the transition for the first time based on the functional renormalization group. In the next section, we explain how these results are understood from the Landau-Ginzburg argument.

We consider the flow equation called the Wetterich equation [27] for the k (RG scale)-dependent effective action Γ_k

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \left\{ \left[\frac{\overleftarrow{\delta}}{\delta \Phi^\dagger} \Gamma_k[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi} + R_{\Phi,k} \right]^{-1} \cdot \partial_t R_{\Phi,k} \right\}, \quad (46)$$

where t is the renormalization group (RG) time $t := \ln \frac{k}{\Lambda}$, $\partial_t := \frac{d}{dt} = k \frac{d}{dk}$ for some reference scale (UV cutoff) Λ and $R_{\Phi,k}$ is the regulator function for the field Φ . Here STr denotes the supertrace introduced to include both the commutative field (gluon) and the anticommutative field (quark, ghost). See [28,29] for reviews of the functional renormalization group.

If we restrict our consideration to the pure glue case S_{YM} under the gauge $n^A(x) = \delta^{A3}$, then the relevant fields Φ are $\mathcal{V}_\mu^A(x)$, $\mathcal{X}_\mu^A(x)$ and Faddeev-Popov ghosts (ghost and anti-ghost) $\mathcal{C}^A(x)$, $\bar{\mathcal{C}}^A(x)$, i.e., $\Phi^\dagger = (\mathcal{V}_\mu^A, \mathcal{X}_\mu^A, \mathcal{C}^A, \bar{\mathcal{C}}^A)$. In this section, we use the Euclidean formulation. In the modified Polyakov gauge and within the quadratic approximation adopted in Sec. III,

$$\begin{aligned} \partial_t \Gamma_k &= \frac{1}{2} \text{Tr} \left\{ \left[\frac{\overleftarrow{\delta}}{\delta \mathcal{V}^\dagger} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \mathcal{V}} + R_k \right]^{-1} \cdot \partial_t R_k \right\} \\ &+ \partial_t \frac{1}{2} \text{Tr} \{ \ln [Q_{\mu\nu}^{AB} + \delta^{AB} \delta_{\mu\nu} R_k] \} \\ &- \partial_t \text{Tr} \{ \ln [G^{AB} + \delta^{AB} R_k] \}, \end{aligned} \quad (47)$$

where the second contribution in the right-hand side comes from the \mathcal{X} field and the last one from the ghosts fields [39], and we have used the same regulator function R_k for the gluon and ghost up to the difference due to the tensor structure.

We neglect backreactions of the \mathcal{V}_0 potential on the other gauge fields \mathcal{V}_j , as in the treatment [33]. Assuming an expansion around $\mathcal{V}_j = 0$, $\Gamma_k^{(2)} := \frac{\overleftarrow{\delta}}{\delta \mathcal{V}^\dagger} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \mathcal{V}}$ is block diagonal like the regulators, and the flow equation can be decomposed into a sum of two contributions: under the approximation (37),

$$\begin{aligned} \partial_t \Gamma_k &= \frac{1}{2} \text{Tr} \left[\left(\frac{1}{\Gamma_k^{(2)} + R_k} \right)_{\mu\nu} \cdot \partial_t R_{k,\mu\nu} \right] \\ &+ \partial_t \text{Tr} \{ \ln [G^{AB} + \delta^{AB} R_k] \}, \end{aligned} \quad (48)$$

where the gluon regulator $R_{k,\mu\nu}$ is a block-diagonal matrix in field space,

$$\begin{aligned} R_{k,00} &= R_{0,k} = Z_0 R_{\text{opt},k}(\mathbf{p}^2), \\ R_{k,0j} &= 0 = R_{k,j0}, \\ R_{k,j\ell} &= R_{T,k} T_{j\ell}(\mathbf{p}) = Z_j T_{j\ell}(\mathbf{p}) R_{\text{opt},k_T}(\mathbf{p}^2), \end{aligned} \quad (49)$$

where $T_{j\ell} := \delta_{j\ell} - \frac{p_j p_\ell}{p_m^2}$ is the transverse projection operator and $R_{\text{opt},k}(\mathbf{p}^2)$ is the (3 dimensional) optimized choice [71]

$$R_{\text{opt},k}(\mathbf{p}^2) = (k^2 - \mathbf{p}^2) \theta(k^2 - \mathbf{p}^2). \quad (50)$$

The first term in the right-hand side encodes the quantum fluctuations of \mathcal{V}_0 , while the second one encodes those of the other components of the gauge field and ghosts. In the present truncation, the second term is a total derivative with respect to t , and does not receive contributions from the first term. Therefore, we can evaluate the flow of the second contribution, and use its output $V_{T,k}(\mathcal{V}_0)$ as an input for the remaining flow

$$\partial_t \Gamma_k = \frac{1}{2} \beta \int \frac{d^3 p}{(2\pi)^3} \left[\left(\frac{1}{\Gamma_k^{(2)} + R_\gamma} \right)_{00} \partial_t R_{0,k} \right] + \partial_t V_{T,k}, \quad (51)$$

where for $\omega = 2\pi T n$

$$\begin{aligned} V_{T,k} &:= \text{Tr} \{ \ln [G^{AB} + \delta^{AB} R_k] \} \\ &= T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \ln [\tilde{G}^{AB}(\omega, \mathbf{p}) + \delta^{AB} (k_T^2 - \mathbf{p}^2) \\ &\quad \times \theta(k_T^2 - \mathbf{p}^2)] \\ &= T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \ln [\delta^{AB} \mathbf{p}^2 - D_0^2 + \delta^{AB} (k_T^2 - \mathbf{p}^2) \\ &\quad \times \theta(k_T^2 - \mathbf{p}^2)] \\ &= T \sum_{n \in \mathbb{Z}} 4\pi \int_0^{k_T} \frac{dp p^2}{(2\pi)^3} \text{tr} \ln [\delta^{AB} k_T^2 - D_0^2] \\ &\quad - T \sum_{n \in \mathbb{Z}} 4\pi \int_0^{k_T} \frac{dp p^2}{(2\pi)^3} \text{tr} \ln [\delta^{AB} \mathbf{p}^2 - D_0^2] + V_W. \end{aligned} \quad (52)$$

Here we have introduced the Weiss potential V_W which was obtained by a one-loop calculation [31]

$$\begin{aligned} V_W &= \text{Tr} \ln [G^{AB}] \\ &= T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \ln [\tilde{G}^{AB}(p_0 = \omega, \mathbf{p})] \\ &= T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \ln [\mathbf{p}^2 + (\omega + T\varphi)^2] \\ &\quad + T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \ln [\mathbf{p}^2 + (\omega - T\varphi)^2], \end{aligned} \quad (53)$$

where we have neglected the φ -independent (or \mathcal{V}_0 -independent) contributions.

The closed form of the Weiss potential is obtained after summing up the Matsubara frequencies

$$\begin{aligned} V_W(\varphi) &= T^4 \left[-\frac{1}{6} (\varphi - \pi)^2 + \frac{1}{12\pi^2} (\varphi - \pi)^4 + \frac{\pi^2}{12} \right] \\ &\pmod{2\pi}. \end{aligned} \quad (54)$$

The Weiss potential V_W is g^2 independent and the overall curve scales as T^4 . V_W has the symmetries $V_W(-\varphi) = V_W(\varphi)$ and $V_W(\varphi + 2\pi n) = V_W(\varphi)$. $V_W(\varphi)$ has the minima at $\varphi = 2\pi n$, and the Polyakov loop has the

nonvanishing value $L = \cos\frac{\varphi}{2} = (-1)^n$, implying deconfinement; see Fig. 1. Therefore, $V_W(\varphi)$ is considered to be valid at very high temperature where the perturbation theory is trustworthy. In Fig. 2, we observe

$$\lim_{k \downarrow 0} V_{T,k} = V_{T,0} = V_W, \quad \lim_{k \uparrow \infty} V_{T,k} = 0. \quad (55)$$

After integrating over the fields other than \mathcal{V}_0 , we are lead to the effective action of \mathcal{V}_0 ,

$$\Gamma_k[\mathcal{V}_0] = \beta \int d^3x \left\{ -\frac{1}{2} Z_0 \mathcal{V}_0(\mathbf{x}) \partial_j \partial_j \mathcal{V}_0(\mathbf{x}) + V_{\text{eff},k}^{\text{glue}}[\mathcal{V}_0] \right\},$$

$$V_{\text{eff},k}^{\text{glue}}[\mathcal{V}_0] = V_{T,k}[\mathcal{V}_0] + \Delta V_k[\mathcal{V}_0]. \quad (56)$$

Then the flow equation is reformulated for ΔV_k with the external input $V_{T,k}$

$$\beta \partial_t (\Delta V_k[\mathcal{V}_0]) = \frac{1}{2} \beta \int \frac{d^3 p}{(2\pi)^3} \left[\left(\frac{1}{\Gamma_k^{(2)} + R_k} \right)_{00} \partial_t R_{0,k} \right], \quad (57)$$

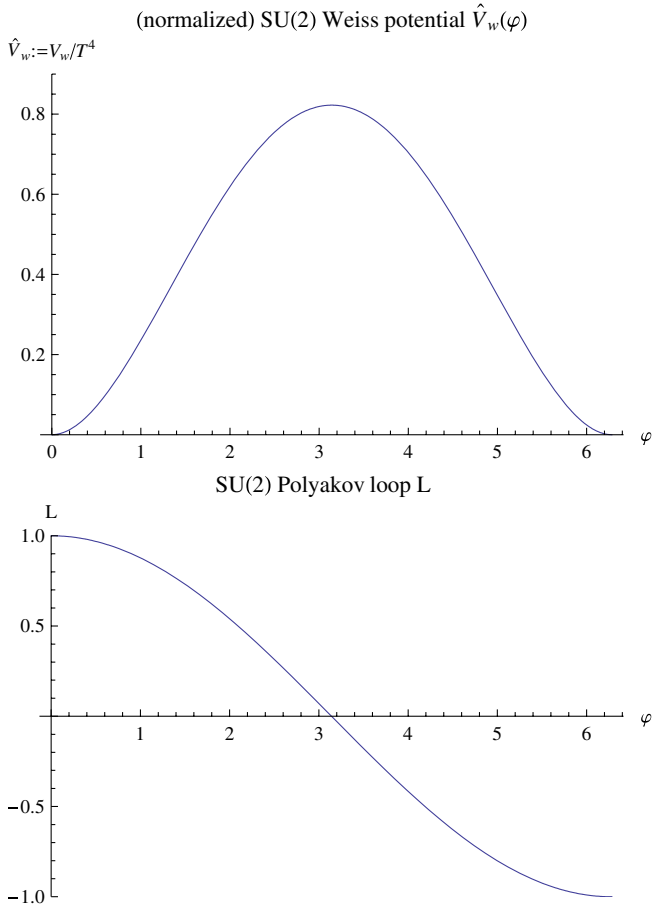


FIG. 1 (color online). (Upper panel) (Normalized) $SU(2)$ Weiss potential \hat{V}_W as a function of φ . (Lower panel) $SU(2)$ Polyakov loop L as a function of φ , $L = \cos(\frac{\varphi}{2})$.

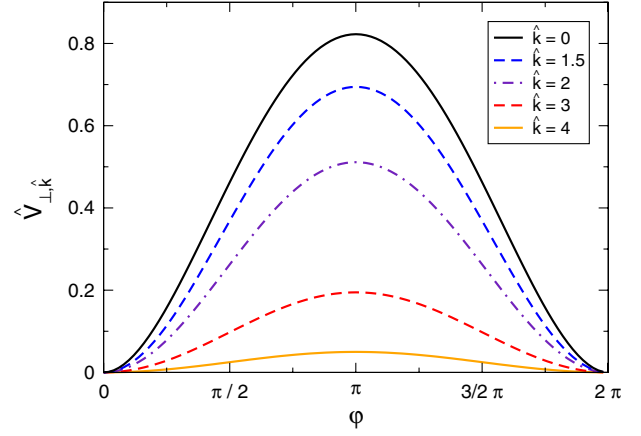


FIG. 2 (color online). $\hat{V}_{T,k}$ for different values of \hat{k} (reprinted from [33]).

where

$$\Gamma_k^{(2)}[\mathcal{V}_0] = \beta \{ Z_0 \mathbf{p}^2 + \partial_{\mathcal{V}_0}^2 V_k[\mathcal{V}_0] \}. \quad (58)$$

Using the specific regulator, $R_{0,k} = Z_0(k^2 - \mathbf{p}^2)\theta(k^2 - \mathbf{p}^2)$, which yields

$$\partial_t R_{0,k} = [\partial_t Z_0(k^2 - \mathbf{p}^2) + 2Z_0 k^2] \theta(k^2 - \mathbf{p}^2), \quad (59)$$

we can perform the momentum integration analytically.

$$\begin{aligned} \beta \partial_t (\Delta V_k[\mathcal{V}_0]) &= \frac{2}{3} \frac{1}{(2\pi)^2} \frac{(\eta_k/5 + 1)k^5}{Z_k^{-1} g^2 \beta^2 \partial_\varphi^2 (V_{T,k}[\mathcal{V}_0] + \Delta V_k[\mathcal{V}_0]) + k^2}, \end{aligned} \quad (60)$$

where we have introduced the running coupling α_k defined by

$$g_k^2 := Z_k^{-1} g^2, \quad \alpha_k := \frac{g_k^2}{4\pi} = Z_k^{-1} \frac{g^2}{4\pi}, \quad (61)$$

and the anomalous dimension η_k defined by

$$\eta_k := \partial_t \ln Z_k = -\partial_t \ln \alpha_k. \quad (62)$$

By introducing the dimensionless RG scale \hat{k} and the dimensionless effective potential \hat{V} defined by

$$\hat{k} := \beta k = k/T, \quad \hat{V} := \beta^4 V = V/T^4, \quad (63)$$

the flow equation is simplified as

$$\partial_{\hat{k}} \Delta \hat{V}_{\hat{k}}[\mathcal{V}_0] = \frac{1}{6\pi^2} \frac{(1 + \eta_k/5)\hat{k}^2}{1 + \frac{4\pi\alpha_k}{\hat{k}^2} \partial_\varphi^2 (\hat{V}_{T,\hat{k}}[\mathcal{V}_0] + \Delta \hat{V}_{\hat{k}}[\mathcal{V}_0])}, \quad (64)$$

where all scales are measured in units of temperature. It turns out that the input in solving the flow equation is just a running gauge coupling constant α_k . A specific choice for the running gauge coupling constant is given in Fig. 3. For the derivation from the renormalization group, see [72].

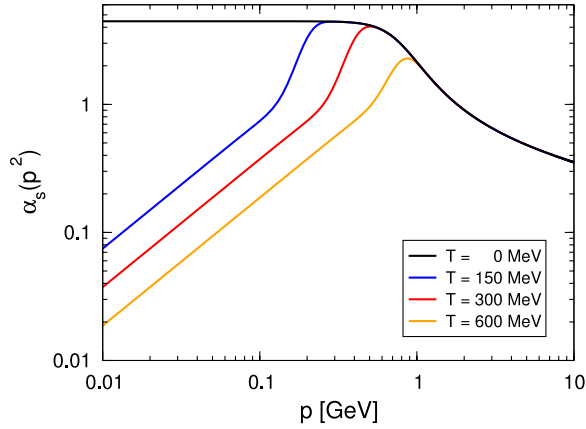


FIG. 3 (color online). The running gauge coupling constant α_s for temperatures $T = 0, 150, 300, 600$ MeV (reprinted from [33]).

The flow is initialized in the broken phase at any temperature. By solving the flow equation in a numerical way with an input for the running gauge coupling given in Fig. 3, the full effective potential $\hat{V}_{\text{eff}}^{\text{glue}}$ (normalized to 0 at $\varphi = 0$) is obtained in Fig. 4 for various temperature.

According to [33], a second order phase transition occurs at a critical temperature

$$T_d = 305_{-55}^{+40} \text{ MeV}, \quad T_d/\sqrt{\sigma} = 0.69_{-0.12}^{+0.04}, \quad (65)$$

with the string tension $\sqrt{\sigma} = 440$ MeV. This agrees within errors with the lattice result $T_d/\sqrt{\sigma} = 0.709$. Moreover, these results were confirmed by considering another gauge [32].

VI. UNDERSTANDING THE EXISTENCE OF CONFINEMENT TRANSITION ACCORDING TO THE LANDAU-GINZBURG ARGUMENT

In this section, we show that some qualitative aspects of the deconfinement/confinement transition found in the pre-

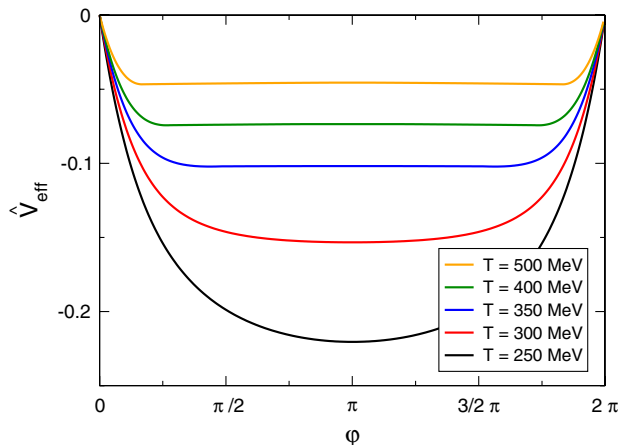


FIG. 4 (color online). Full effective potential $\hat{V}_{\text{eff}}^{\text{glue}}$, normalized to 0 at $\varphi = 0$ (reprinted from [33]).

vious section can be understood without detailed numerical works, although the precise value of the transition temperature T_d cannot be determined without them.

For $G = SU(2)$ in the pure Yang-Mills limit $m_q \rightarrow \infty$, the effective potential $V_{\text{glue}}(L)$ for the Polyakov loop L must be invariant under the center symmetry $Z(2)$. Therefore, $V_{\text{glue}}(L)$ is an even function of L , i.e., $V_{\text{glue}}(L) = V_{\text{glue}}(-L)$ where L is real valued $L = L^*$. Thus the Landau-Ginzburg argument suggests that the effective potential $V_{\text{eff}}^{\text{glue}}(L)$ for $G = SU(2)$ has the power-series expansion in L near the transition point $L = 0$

$$V_{\text{eff}}^{\text{glue}}(L) = c_0 + \frac{c_2}{2}L^2 + \frac{c_4}{4}L^4 + O(L^6). \quad (66)$$

As the vacuum is specified as the minima of the effective potential, the confinement/deconfinement transition temperature T_d is determined from the condition $c_2(T_d) = 0$ so that the low-temperature ($T < T_d$) confinement phase $\langle L \rangle = 0$ is realized for $c_2(T) > 0$, while the high-temperature ($T > T_d$) deconfinement phase $\langle L \rangle \neq 0$ is realized for $c_2(T) < 0$, provided that the positivity $c_4(T) > 0$ is maintained across the transition temperature. Consequently, the transition is of the 2nd order.

Indeed, we confirm that the Landau-Ginzburg description is correct and valid for the confinement/deconfinement transition, by making use of the flow equation given in the previous section. This is a microscopic justification of the Landau-Ginzburg argument for the confinement/deconfinement transition. In our treatment, however, it is more convenient to write the effective potential $V_{\text{eff}}^{\text{glue}}$ in terms of the angle variable φ (rather than L) around the transition point $\varphi = \pi$ (instead of $L = 0$). Defining $\tilde{\varphi} := \varphi - \pi$, we find that $V_{\text{eff}}^{\text{glue}}(\tilde{\varphi})$ must be an even function $V_{\text{eff}}^{\text{glue}}(\tilde{\varphi}) = V_{\text{eff}}^{\text{glue}}(-\tilde{\varphi})$ due to the center symmetry and hence odd terms (e.g., $\tilde{\varphi}, \tilde{\varphi}^3$) do not appear

$$V_{\text{eff}}^{\text{glue}}(\tilde{\varphi}) = C_0 + \frac{C_2}{2}\tilde{\varphi}^2 + \frac{C_4}{4!}\tilde{\varphi}^4 + O(\tilde{\varphi}^6). \quad (67)$$

At sufficiently high temperature, we observe that $C_2(T) < 0$ and hence $V_{\text{eff}}^{\text{glue}}$ has the minimum at $\tilde{\varphi} \neq 0$ ($\Leftrightarrow L \neq 0$) leading to deconfinement. In order to show the existence of the confinement/deconfinement transition at $T = T_d$, $C_2(T)$ must change the signature $C_2(T) > 0$ below this temperature $T < T_d$ and hence the minimum occurs at $\tilde{\varphi} = 0$ ($\Leftrightarrow L = 0$) leading to confinement. Therefore, the confinement/deconfinement temperature T_d is determined by $C_2(T_d) = 0$, provided that the positivity $C_4(T) > 0$ is maintained.

For this purpose, we study the scale dependent effective potential $V_{\text{eff},k}^{\text{glue}}$ at $k > 0$

$$V_{\text{eff},k}^{\text{glue}} = C_{0,k} + \frac{C_{2,k}}{2}\tilde{\varphi}^2 + \frac{C_{4,k}}{4!}\tilde{\varphi}^4 + O(\tilde{\varphi}^6), \quad (68)$$

and see how it evolves towards the limit $k \rightarrow 0$ according to the flow equation to obtain the physical effective potential $V_{\text{eff}}^{\text{glue}} := V_{\text{eff},k=0}^{\text{glue}}$.

As in (56), $V_{\text{eff},k}^{\text{glue}}$ is decomposed into two pieces

$$V_{\text{eff},k}^{\text{glue}} = \hat{V}_{T,\hat{k}} + \Delta \hat{V}_{\hat{k}}, \quad (69)$$

where we have defined the dimensionless potential according to the rescaling (63). The first part $\hat{V}_{T,\hat{k}}$ is the (k -dependent) perturbative part (52) obtained essentially by the one-loop calculation with the regulator function R_k being included. For this part, the closed analytical form can be obtained; see Appendix B. While the second part $\Delta \hat{V}_{\hat{k}}$ represents the nonperturbative part which is initially zero $\Delta \hat{V}_{\hat{k}}|_{k=\Lambda} = 0$ and is generated in the evolution of the renormalization group. This part is obtained only by solving the flow Eq. (64) and its analytical form is not available (at this moment).

We expand $\hat{V}_{T,\hat{k}}$ in powers of $\tilde{\varphi} = \varphi - \pi$

$$\hat{V}_{T,\hat{k}} = A_{0,k} + \frac{A_{2,k}}{2} \tilde{\varphi}^2 + \frac{A_{4,k}}{4!} \tilde{\varphi}^4 + O(\tilde{\varphi}^6), \quad (70)$$

where coefficients are drawn as functions of k in Fig. 5; see Appendix B for their closed analytical forms.

Suppose that $\Delta \hat{V}_{\hat{k}}$ is of the form

$$\Delta \hat{V}_{\hat{k}} = a_{0,k} + \frac{a_{2,k}}{2} \tilde{\varphi}^2 + \frac{a_{4,k}}{4!} \tilde{\varphi}^4 + O(\tilde{\varphi}^6). \quad (71)$$

A flow equation (64) for the effective potential (56) is reduced to a set of coupled flow equations for coefficients in the effective potential (69) with (70) and (71)

$$\begin{aligned} \partial_{\hat{k}} a_{2,k} &= - \frac{(1 + \frac{1}{5} \eta_k) \hat{k}^2}{6\pi^2} \frac{\frac{4\pi\alpha_k}{\hat{k}^2} (A_{4,k} + a_{4,k})}{[1 + \frac{4\pi\alpha_k}{\hat{k}^2} (A_{2,k} + a_{2,k})]^2}, \\ \partial_{\hat{k}} a_{4,k} &= + \frac{(1 + \frac{1}{5} \eta_k) \hat{k}^2}{6\pi^2} \frac{6[\frac{4\pi\alpha_k}{\hat{k}^2} (A_{4,k} + a_{4,k})]^2}{[1 + \frac{4\pi\alpha_k}{\hat{k}^2} (A_{2,k} + a_{2,k})]^3}, \\ &\vdots \end{aligned} \quad (72)$$

which are coupled first-order ordinary but nonlinear differential equations for coefficients. In Appendix C, we see that this form (71) is justified as a solution of the flow equation. In fact, it is easy to see that $\partial_{\hat{k}} a_{1,k} = 0$ and $\partial_{\hat{k}} a_{3,k} = 0$ are guaranteed from the flow equation, if the effective potential has no odd terms at arbitrary k . Therefore, if an initial condition, $a_{1,k} = 0 = a_{3,k}$ at $k = \Lambda$ is imposed, then $a_{1,k} \equiv 0$ and $a_{3,k} \equiv 0$ are maintained for $0 \leq k \leq \Lambda$ by solving the flow equation. In performing numerical calculations, however, one must truncate the infinite series of differential equations up to some finite order to obtain a manageable set of equations.

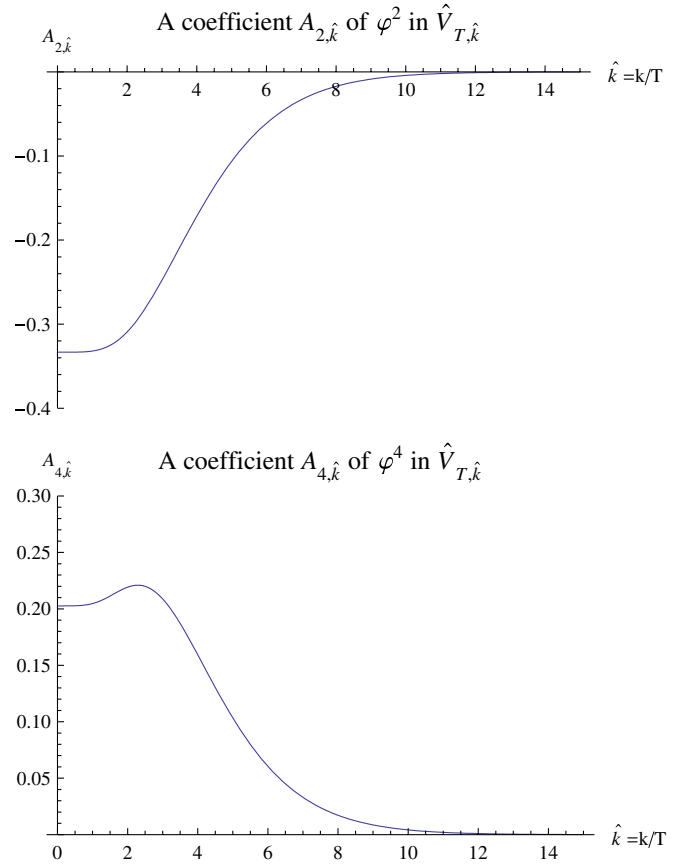


FIG. 5 (color online). $A_{2,k}$ and $A_{4,k}$ as functions of \hat{k} .

We can understand qualitatively why a 2nd order phase transition from the deconfinement phase to the confinement phase can occur by lowering the temperature.

The flow starts from $a_{2,k} = 0$ and hence $C_{2,k} = A_{2,k} + a_{2,k} < 0$ (because of $A_{2,k} < 0$) at $k = \Lambda \gg 1$. We assume $C_{4,k} = A_{4,k} + a_{4,k} > 0$ for $0 \leq k \leq \Lambda$, as a necessary condition for realizing a 2nd order transition. Otherwise, we must consider the higher-order terms, e.g., $O(\varphi^6)$. (This assumption is assured to be true by numerical calculations of the full effective potential [32,33], as reproduced in the previous section.) This assumption allows us to analyze just one differential equation for obtaining qualitative understanding

$$\partial_{\hat{k}} a_{2,k} = - \frac{(1 + \frac{1}{5} \eta_k)}{6\pi^2} \frac{4\pi\alpha_k (A_{4,k} + a_{4,k})}{[1 + \frac{4\pi\alpha_k}{\hat{k}^2} (A_{2,k} + a_{2,k})]^2}. \quad (73)$$

Then the right-hand side of (73) is negative, since the running coupling constant α_k is positive and $1 + \frac{1}{5} \eta_k$ is positive; see Figs. 6 and 7. Consequently, $a_{2,k}$ started at zero becomes positive $a_{2,k} > 0$ just below Λ and increases (monotonically) as k decreases; see Fig. 8.

Note that the denominator can vanish $1 + \frac{4\pi\alpha_k}{\hat{k}^2} (A_{2,k} + a_{2,k}) = 0$ at some k^* (since $C_{2,k} = A_{2,k} + a_{2,k} < 0$ or $0 < a_{2,k} < -A_{2,k}$) where the right-hand side of (73) becomes negative infinity and $a_{2,k}$ blows up there. To avoid this

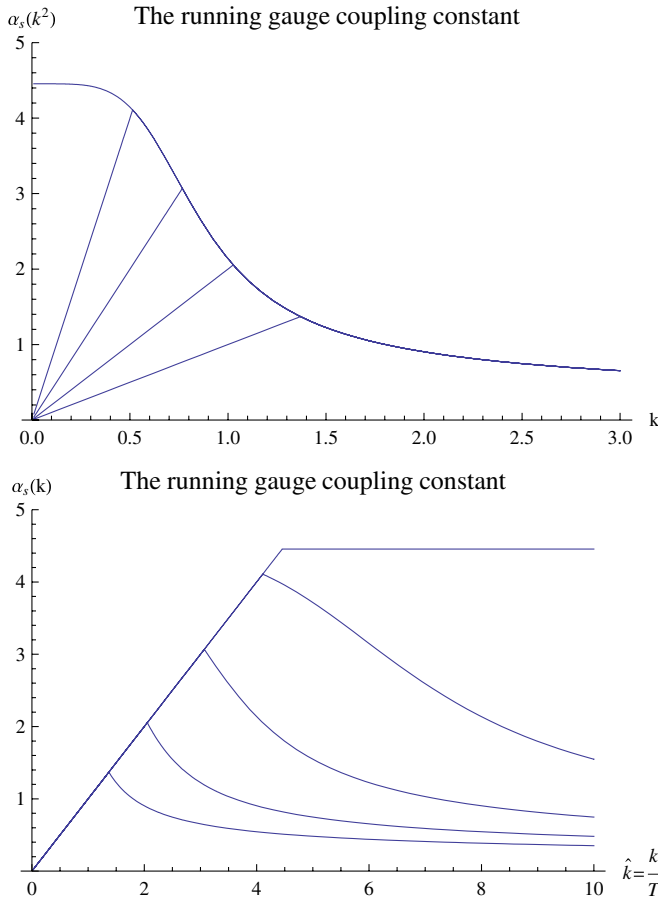


FIG. 6 (color online). The running gauge coupling constant α_k at $T = 0.001, 0.125, 0.25, 0.50, 1.0$ GeV, from the top at the lowest temperature $T = 0.001$ GeV to the bottom at the highest temperature $T = 1.0$ GeV. (Upper panel) α_k as functions of k . (Lower panel) α_k as functions of \hat{k} . For a given temperature, there is a critical value \hat{k}_c separating the deep IR region (76) from the higher momentum region (75). The discontinuity of the derivative seen at \hat{k}_c comes from a crude approximation in which we have taken into account just the first linear term (i.e., $c_1 = c_2 = \dots = 0$) in the expansion (76), and can be avoided if we take into account higher-order terms as explained below (76). However, this is not essential to see qualitative behaviors of the solution of the flow equation.

pathology and to obtain the solution all the way down to the limit $k \rightarrow 0$, $a_{2,k}$ must grow relatively rapidly so that $|A_{2,k} + a_{2,k}| \ll 1$ towards the limit $k \rightarrow 0$.

An important observation of the flow equation (73) is that the explicit temperature dependence comes from the running coupling constant alone. At zero temperature, the running coupling constant is well parametrized by the fitting function [73]

$$\alpha_k = \frac{4\pi \times 0.709/N_c}{\ln[e + a_1(k^2)^{a_2} + b_1(k^2)^{b_2}]}, \quad (74)$$

where $a_1 = 5.292$, $a_2 = 2.324$, $b_1 = 0.034$, $b_2 = 3.169$ in units of GeV.

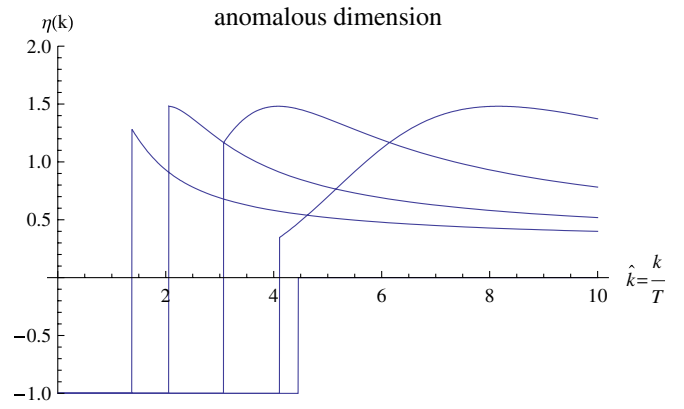


FIG. 7 (color online). The anomalous dimension η_k as functions of \hat{k} at $T = 0.001, 0.125, 0.25, 0.50, 1.0$ GeV. In each graph for a given temperature, there is a critical value \hat{k}_c of \hat{k} separating the deep IR region $\eta_k \approx -1$ from the higher momentum (intermediate and UV) region $\eta_k > 0$. The temperature is distinguished by \hat{k}_c ranging from the smallest value at the highest temperature $T = 1.0$ GeV to the largest value at the lowest temperature $T = 0.001$ GeV where $\eta_k \approx -1$ for $\hat{k} < \hat{k}_c$ and $\eta_k \approx 0$ for $\hat{k} > \hat{k}_c$. The discontinuity of the derivative seen at \hat{k}_c is due to the same reason as that explained in Fig. 6 and is not essential to see qualitative behaviors of the solution of the flow equation.

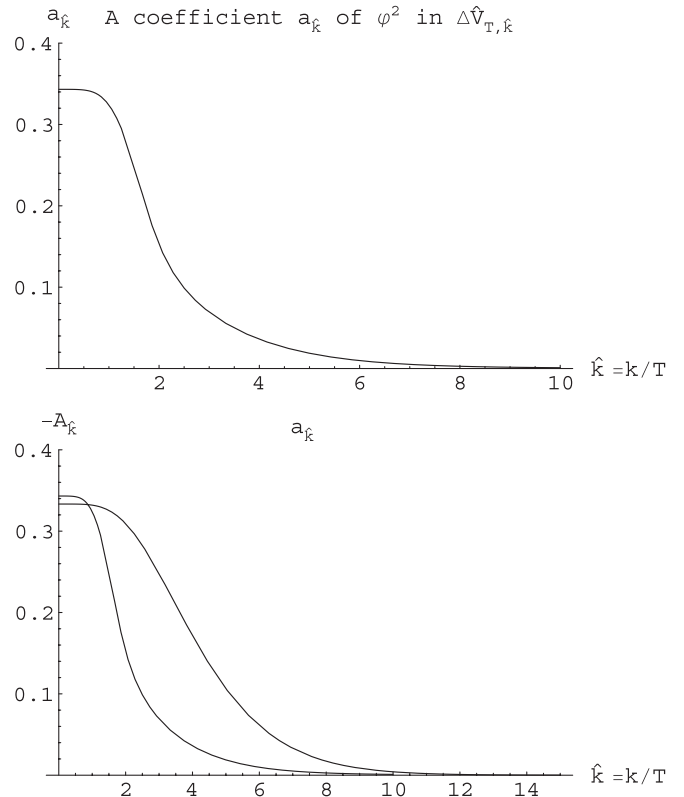


FIG. 8. $a_{2,k}$ vs $-A_{2,k}$ as functions of \hat{k} for $T < T_d$.

For the perturbative region $k \gg T$, i.e., $\hat{k} \gg 1$, we adopt this form: $k^2 = T^2 \hat{k}^2$,

$$\alpha_k = \frac{g_k^2}{4\pi} = \frac{4\pi \times 0.709/N_c}{\ln[e + a_1(T^2 \hat{k}^2)^{a_2} + b_1(T^2 \hat{k}^2)^{b_2}]} \quad (75)$$

For the nonperturbative region $k < 2\pi T$, i.e., $\hat{k} < O(1)$, we adopt the running coupling which is governed by an infrared fixed point [72]

$$\begin{aligned} \alpha_k &= \alpha_{3d}^* \frac{k}{T} + c_1 \left(\frac{k}{T}\right)^2 + c_2 \left(\frac{k}{T}\right)^3 + \dots \\ &= \alpha_{3d}^* \hat{k} + c_1 \hat{k}^2 + c_2 \hat{k}^3 + \dots, \end{aligned} \quad (76)$$

where coefficients c_1, c_2, \dots are determined such that the coupling at zero temperature (75) and its derivative with respect to k are connected continuously with this ansatz (76) at the scale set by the lowest nonvanishing bosonic Matsubara mode $\omega = 2\pi T$. However, the running coupling constant at small momenta (76) does not contribute to the explicit T dependence in the scaled flow equation, since it is written in terms of the scaled \hat{k} alone and hence denoted by a common curve going through the origin for any temperature T in the second figure of Fig. 6. Therefore, the running coupling at very small momentum region cannot be responsible for the confinement/deconfinement transition at finite temperature, if this observation is correct. As can be seen from the second figure of Fig. 6, the dominant contribution comes from the intermediate momentum region above $O(1)$ GeV. Thus, we can avoid the issue of the gauge-fixing artifact in the deep IR region due to Gribov copies in the zero-temperature case; see, e.g., [74,75] and reference therein.

We consider a solution of the reduced (or normalized) flow equation as a function of \hat{k} , rather than k , for a given temperature T . Then the difference between high- and low-temperature phases attributes to the behavior of the running coupling constant α_k as a function of $k = T\hat{k}$, which brings the explicit T dependence to the reduced flow equation. In the case of high-temperature $T \gg 1$, $k = T\hat{k}$ becomes large for a wide range of \hat{k} and the running coupling constant α_k remains relatively small. The resulting slow increase of $a_{2,k}$ keeps $a_{2,k}$ small such that $C_{2,k} = A_{2,k} + a_{2,k} < 0$ or $a_{2,k} < -A_{2,k}$ even at $k = 0$. This leads to the center-symmetry breaking at high temperature.

In the case of low temperature $T \ll 1$, $k = T\hat{k}$ becomes small for the same range of \hat{k} and the running coupling constant α_k gets into the intermediate region of $O(1)$ GeV rapidly and becomes larger as the temperature becomes smaller. At sufficiently low temperature, $a_{2,k}$ increases in decreasing \hat{k} so rapidly that $a_{2,k}$ eventually reaches to the point $A_{2,k} + a_{2,k} = 0$ or $a_{2,k} = -A_{2,k}$ at a certain value $\hat{k} = \hat{k}_0$. In other words, the graph of $a_{2,k}$ intersects with that of $-A_{2,k}$ at $\hat{k} = \hat{k}_0$. In the region $0 < k < k_0$ where $C_{2,k} = A_{2,k} + a_{2,k} > 0$ or $a_{2,k} > -A_{2,k}$, the flow equation reads

$$\partial_{\hat{k}} a_{2,k} \simeq -\frac{(1 + \frac{1}{5}\eta_k)}{6\pi^2} \frac{(A_{4,k} + a_{4,k})\hat{k}^4}{4\pi\alpha_k(A_{2,k} + a_{2,k})^2}, \quad (77)$$

the right-hand side gets small negative, and $a_{2,k}$ becomes flat near the IR limit; see Fig. 8. Finally, $a_{2,k}$ reaches the value realizing $C_{2,k} = A_{2,k} + a_{2,k} > 0$ or $a_{2,k} > -A_{2,k}$ at $k = 0$. This leads to the recovery of the center symmetry. The difference is clearly seen from the second figure of Fig. 6 where the running gauge coupling α_k is drawn as a function of \hat{k} for various temperatures.

In our treatment, the difference between the three-dimensional RG scale k_T and the four-dimensional one k is neglected by equating two scales $k_T = k$ just for simplifying the analysis, since it is enough for obtaining a qualitative understanding for the transition. This is not the case for obtaining quantitative results, see Appendix C of [33] for the precise treatment on this issue.

VII. QUARK PART AND GAUGED NONLOCAL NJL MODEL

We examine the quark self-interaction part $S_{\text{int}} = \int d^D x \int d^D y \frac{1}{2} \mathcal{J}^{\mu A}(x) g^2 (Q^{-1}[\mathcal{V}])_{\mu\nu}^{AB}(x, y) \mathcal{J}^{\nu B}(y)$. In estimating the effect of $Q^{-1}[\mathcal{V}]$, we take the same approximation as the above. Consequently, the inverse $(Q^{-1})_{\mu\nu}^{AB}[\mathcal{V}]$ is diagonal in the Lorentz indices:

$$\begin{aligned} (Q^{-1})_{\mu\nu}^{AB}[\mathcal{V}] &= g^{\mu\nu} (G^{-1})^{AB}[\mathcal{V}] \\ &= g^{\mu\nu} \begin{pmatrix} \frac{1}{2}[F_\varphi + F_{-\varphi}] & -\frac{1}{2i}[F_\varphi - F_{-\varphi}] & 0 \\ \frac{1}{2i}[F_\varphi - F_{-\varphi}] & \frac{1}{2}[F_\varphi + F_{-\varphi}] & 0 \\ 0 & 0 & F_0 \end{pmatrix}, \end{aligned} \quad (78)$$

where F_φ is defined by

$$\begin{aligned} F_\varphi(i\partial) &:= \frac{1}{(i\partial_\ell)^2 + (i\partial_0 + T\varphi)^2} \\ &= \frac{1}{(i\partial_\mu)^2 + (T\varphi)^2 + 2T\varphi i\partial_0}. \end{aligned} \quad (79)$$

In what follows, we consider only the diagonal parts of $(G^{-1})^{AB}[\mathcal{V}]$. This is achieved by the procedure

$$\frac{g^2}{2} (Q^{-1})_{\mu\nu}^{AB}(x, y) = g^{\mu\nu} \delta^{AB} \mathcal{G}(x - y), \quad (80)$$

which yields

$$\mathcal{G}(x - y) = \frac{g^2}{2} (Q^{-1})_{\mu\nu}^{AB}(x, y) \frac{g_{\mu\nu}}{D} \frac{\delta^{AB}}{N_c^2 - 1}. \quad (81)$$

Then the nonlocal interaction is obtained as

$$\mathcal{G}(x - y) = \frac{g^2 \text{tr}(G^{-1})}{2 N_c^2 - 1} = \frac{g^2}{2} \frac{F_\varphi + F_{-\varphi} + F_0}{3}. \quad (82)$$

This approximation is used just for simplifying the Fierz transformation performed below and hence it can be improved by taking into account the off-diagonal parts of G^{-1} if it is necessary to do so.

For $D = 4$, we use the Fierz identity [76] to rewrite the nonlocal current-current interaction as

$$\begin{aligned} S_{\text{int}} &= \int d^4x \int d^4y \mathcal{J}^{\mu A}(x) \mathcal{G}(x-y) \mathcal{J}^{\mu A}(y) \\ &= \int d^4x \int d^4y \mathcal{G}(x-y) \sum_{\alpha} c_{\alpha} (\bar{\psi}(x) \Gamma_{\alpha} \psi(y)) \\ &\quad \times (\bar{\psi}(y) \Gamma_{\alpha} \psi(x)) \\ &= \int d^4x \int d^4z \mathcal{G}(z) \sum_{\alpha} c_{\alpha} \{ \bar{\psi}(x+z/2) \Gamma_{\alpha} \psi(x-z/2) \} \\ &\quad \times \{ \bar{\psi}(x-z/2) \Gamma_{\alpha} \psi(x+z/2) \}, \end{aligned} \quad (83)$$

where the Γ_{α} are a set of Dirac spinor, color, and flavor matrices, resulting from the Fierz transform, with the property $\gamma_0 \Gamma_{\alpha}^{\dagger} \gamma_0 = \Gamma_{\alpha}$. Although the Fierz transformation induces mixings and recombinations among operators, the resulting theory must maintain the symmetries of the original QCD Lagrangian. A minimal subset of operators satisfying the global chiral symmetry $SU(2)_L \times SU(2)_R$ which governs low-energy QCD with two flavors is the color singlet of scalar-isoscalar and pseudoscalar-isovector operators. Thus, by restricting Γ_{α} hereafter to

$$\Gamma_{\alpha} := (\mathbf{1}, i\gamma_5 \vec{\tau}) \quad (84)$$

and ignoring other less relevant operators (vector and axial-vector terms in color singlet and color octet channels), we arrive at a nonlocal gauged NJL model

$$\begin{aligned} S_{\text{eff}}^{\text{gNJL}} &= \int d^4x \bar{\psi}(x) (i\gamma^{\mu} \mathcal{D}_{\mu}[\mathcal{V}] - \hat{m}_q + i\gamma^0 \mu_q) \psi(x) \\ &\quad + S_{\text{int}}, \\ S_{\text{int}} &= \int d^4x \int d^4z \mathcal{G}(z) [\bar{\psi}(x+z/2) \Gamma_{\alpha} \psi(x-z/2) \\ &\quad \times \bar{\psi}(x-z/2) \Gamma_{\alpha} \psi(x+z/2)]. \end{aligned} \quad (85)$$

This form is regarded as a gauged version of the nonlocal NJL model proposed in [22]. The function $\mathcal{G}(z)$ is replaced by a coupling constant G times a normalized distribution $\mathcal{C}(z)$

$$\mathcal{G}(z) := \frac{G}{2} \mathcal{C}(z), \quad \int d^4z \mathcal{C}(z) = 1. \quad (86)$$

The standard (local) gauged NJL model follows for the limiting case $\mathcal{C}(z) = \delta^4(z)$ with $\int d^4z \mathcal{C}(z) = 1$.

In contrast to [22], however, G and \mathcal{C} are determined in conjunction with the behavior of the Polyakov loop L or φ at temperature T : using the Fourier transform $\tilde{\mathcal{G}}(p)$ of \mathcal{G} , they are expressed as

$$\frac{G}{2} = \tilde{\mathcal{G}}(p=0), \quad \tilde{\mathcal{C}}(p) = \tilde{\mathcal{G}}(p) / \tilde{\mathcal{G}}(p=0), \quad (87)$$

where

$$\begin{aligned} \tilde{\mathcal{G}}(p) &= \frac{g^2}{2} \frac{\tilde{F}_{\varphi}(p) + \tilde{F}_{-\varphi}(p) + \tilde{F}_0(p)}{3}, \\ \tilde{F}_{\varphi}(p) &= \frac{1}{p^2 + (T\varphi)^2 + 2T\varphi p_0}. \end{aligned} \quad (88)$$

Note that $\tilde{F}_{\varphi}(p=0)$ and hence G diverge at $T=0$. This comes from an improper treatment of the $T=0$ part. To avoid this IR divergence at $T=0$, we add the $T=0$ contribution $M_0^2 \simeq M_{\chi}^2$ and replace $F_{\varphi}(i\partial)$ by

$$\begin{aligned} F_{\varphi}(i\partial) &= \frac{1}{(i\partial_{\ell})^2 + (i\partial_0 + T\varphi)^2 + M_0^2} \\ &= \frac{1}{(i\partial_{\mu})^2 + (T\varphi)^2 + 2T\varphi i\partial_0 + M_0^2}, \end{aligned} \quad (89)$$

and

$$\begin{aligned} \tilde{F}_{\varphi}(p) &= \frac{1}{p^2 + (p_0 + T\varphi)^2 + M_0^2} \\ &= \frac{1}{p^2 + (T\varphi)^2 + 2T\varphi p_0 + M_0^2}. \end{aligned} \quad (90)$$

In fact, such a contribution $\frac{1}{2} M_0^2$ comes in G^{AB} as an additional term $M_0^2 \delta^{AB}$ from the $O(\mathcal{X}^4)$ terms [note that $O(\mathcal{X}^3)$ terms are absent for $G = SU(2)$], as already mentioned above.

Another way to avoid this IR divergence is to introduce the regulator term which is needed to improve the one-loop perturbative result and obtain a nonperturbative one according to the Wilsonian renormalization group:

$$\begin{aligned} \Delta S_k &= \int d^Dx \frac{1}{2} \mathcal{X}_{\mu}^A(x) [\delta^{AB} g_{\mu\nu} R_k(i\partial)] \mathcal{X}_{\nu}^B(x) \\ &= \int \frac{d^Dp}{(2\pi)^D} \frac{1}{2} \tilde{\mathcal{X}}_{\mu}^A(-p) [\delta^{AB} g_{\mu\nu} \tilde{R}_k(p)] \tilde{\mathcal{X}}_{\nu}^B(p), \end{aligned} \quad (91)$$

where k is the RG scale and $\tilde{R}_k(p)$ is the Fourier transform of $R_k(i\partial)$. The regulator function R_k introduces a mass proportional to k^2 , which plays a similar role to M_0^2 in the above, as long as $k > 0$.

The NJL model [12] is well known as a low-energy effective theory of QCD to describe the dynamical breaking of chiral symmetry in QCD (at least in the confinement phase), see, e.g., [13,14]. The theory given above by $S_{\text{eff}}^{\text{QCD}} = S_{\text{eff}}^{\text{glue}} + S_{\text{eff}}^{\text{gNJL}}$ is able to describe chiral-symmetry breaking/restoration and quark confinement/deconfinement on an equal footing where the pure gluon part $S_{\text{eff}}^{\text{glue}}$ describes confinement/deconfinement transition signaled by the Polyakov-loop average. We can incorporate the information on confinement/deconfinement transition into the quark sector through the covariant derivative $\mathcal{D}[\mathcal{V}]$ and the nonlocal NJL interaction \mathcal{G} (G and \mathcal{C}), in sharp

contrast to the conventional PNJL model where the entanglement between chiral-symmetry breaking/restoration and confinement/deconfinement was incorporated through the covariant derivative $\mathcal{D}[\mathcal{V}]$ alone and the nonlocal NJL interaction \mathcal{G} is fixed to the zero-temperature case. In our theory, the nonlocal NJL interaction \mathcal{G} (G and \mathcal{C}) is automatically determined through the information of confinement/deconfinement dictated by the Polyakov loop L (nontrivial gluon background), while in the nonlocal PNJL model [22] the low-momentum (nonperturbative) behavior of \mathcal{C} was not controlled by first principles and was provided by the instanton model.

To study chiral dynamics, it is convenient to bosonize the gauged nonlocal NJL model as done in [22]. The nonlocal gauged NJL model (85) can be bosonised as follows: define

$$\Phi_\alpha(x) := (\sigma(x), \vec{\pi}(x)). \quad (92)$$

To eliminate the quadratic term in the nonlocal currents, we insert the unity

$$1 = \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp\left\{-\int d^4z \mathcal{C}(z) \int d^4x \frac{1}{2G} [\Phi_\alpha(x) + G\vec{\psi}(x+z/2)\Gamma_\alpha\psi(x-z/2)][\Phi_\alpha(x) + G\vec{\psi}(x+z/2)\Gamma_\alpha\psi(x-z/2)]^*\right\}, \quad (93)$$

where we have used $\int d^4z \mathcal{C}(z) = 1$. Then we have the gauged Yukawa model:

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_{\text{eff}}^{\text{gNJL}}} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp\{-S_{\text{eff}}^{\text{gY}}\}, \quad (94)$$

where with $x' := x + z/2$, $y' := x - z/2$,

$$S_{\text{eff}}^{\text{gY}} = \int d^4x' \int d^4y' \bar{\psi}(x') \left[\delta^4(x' - y') (-i\gamma^\mu \mathcal{D}_\mu[\mathcal{V}] + \hat{m}_q + i\gamma^4 \mu_q) + \frac{1}{2} \mathcal{C}(x' - y') \Gamma_\alpha \left[\Phi_\alpha \left(\frac{x' + y'}{2} \right) + \Phi_\alpha^* \left(\frac{x' + y'}{2} \right) \right] \right] \psi(y') + \int d^4x \frac{1}{2G} \Phi_\alpha(x) \Phi_\alpha^*(x), \quad (95)$$

or

$$S_{\text{eff}}^{\text{gY}} = \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \bar{\psi}(p) \left[(2\pi)^4 \delta^4(p - p') \times (-\gamma^\mu (p_\mu + g\tilde{V}_\mu(p)) + \hat{m}_q + i\gamma^4 \mu_q) + \frac{1}{2} \tilde{\mathcal{C}} \left(\frac{p + p'}{2} \right) \Gamma_\alpha [\Phi_\alpha(p - p') + \Phi_\alpha^*(p - p')] \right] \psi(p') + \int \frac{d^4p}{(2\pi)^4} \frac{1}{2G} \Phi_\alpha(p) \Phi_\alpha^*(p). \quad (96)$$

Finally, the bosonized theory of the gauged NJL model is obtained by way of the gauged Yukawa model by integrating out quark fields as

$$\begin{aligned} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_{\text{eff}}^{\text{gNJL}}} &= \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\{-S_{\text{eff}}^{\text{gY}}\} \\ &= \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp\{-S_{\text{eff}}^{\text{boson}}\}, \end{aligned} \quad (97)$$

where the bosonised action $S_{\text{eff}}^{\text{boson}}$ is

$$S_{\text{eff}}^{\text{boson}} = -\text{Tr} \ln \left\{ \delta^4(x' - y') (-i\gamma^\mu (\partial_\mu - ig\mathcal{V}_\mu) + i\gamma^4 \mu_q) + \hat{m}_q + \frac{1}{2} \mathcal{C}(x' - y') \Gamma_\alpha \left[\Phi_\alpha \left(\frac{x' + y'}{2} \right) + \Phi_\alpha^* \left(\frac{x' + y'}{2} \right) \right] \right\} + \int d^4x \frac{1}{2G} \Phi_\alpha(x) \Phi_\alpha^*(x), \quad (98)$$

or

$$S_{\text{eff}}^{\text{boson}} = -\text{Tr} \ln \left\{ (2\pi)^4 \delta^4(p - p') [-\gamma^\mu (p_\mu + g\tilde{V}_\mu) + i\gamma^4 \mu_q] + \hat{m}_q + \frac{1}{2} \tilde{\mathcal{C}} \left(\frac{p + p'}{2} \right) \Gamma_\alpha [\Phi_\alpha(p - p') + \Phi_\alpha^*(p - p')] \right\} + \int \frac{d^4p}{(2\pi)^4} \frac{1}{2G} \Phi_\alpha(p) \Phi_\alpha^*(p). \quad (99)$$

VIII. IMPLICATIONS OF THE POLYAKOV LOOP FOR CHIRAL-SYMMETRY BREAKING AT FINITE TEMPERATURE

The thermodynamics of QCD can be studied based on our effective theory derived in this paper in the similar way to the nonlocal PNJL model [22]. But this must be done by including the effect of gluon properly. In the PNJL model, the effect of the gluon was introduced by the standard minimal gauge coupling procedure, i.e., replacing the normal derivative ∂_μ by the covariant derivative $D_\mu[\mathcal{A}] := \partial_\mu - ig\mathcal{A}_\mu$. In our effective theory, the effect of the gluon is introduced through the NJL coupling constant G and the nonlocality function \mathcal{C} , in addition to the minimal coupling $D_\mu[\mathcal{V}]$. The nonlocal NJL interaction among quarks are mediated by gluons at finite temperature in QCD. Therefore, both G and \mathcal{C} characterizing nonlocal NJL interaction inevitably have temperature dependence, which could be different depending on whether quarks are in confinement or deconfinement phases.

At $T = 0$, QCD must be in the hadron phase where the chiral symmetry is spontaneously broken, which means that the NJL coupling constant $G(0)$ at zero temperature must be greater than the critical NJL coupling constant G_c

$$G(0) = g^2 \frac{1}{M_0^2} > G_c. \quad (100)$$

The nonlocality function or the form factor $\tilde{\mathcal{C}}(p)$ at $T = 0$ behaves as

$$\tilde{\mathcal{G}}(p) = \frac{g^2}{2} \frac{1}{p^2 + M_0^2}, \quad \tilde{\mathcal{G}}(0) = \frac{g^2}{2} \frac{1}{M_0^2}, \quad (101)$$

and

$$\tilde{\mathcal{C}}(p) = \frac{M_0^2}{p^2 + M_0^2}. \quad (102)$$

As an immediate outcome of our effective theory, this determines the temperature dependence of the coupling constant G of nonlocal NJL model. Using (87), we have

$$G(T) = \frac{1}{3} g^2 \left[\frac{2}{(T\varphi)^2 + M_0^2} + \frac{1}{M_0^2} \right], \quad (103)$$

which lead to the NJL coupling constant normalized at $T = 0$

$$G(T)/G(0) = \frac{1}{3} \left[\frac{2M_0^2}{(T\varphi)^2 + M_0^2} + 1 \right]. \quad (104)$$

In the presence of the dynamical quark $m_q < \infty$, the Polyakov loop is not an exact order parameter and does not show a sharp charge with discontinuous derivatives. Even in this case, we can introduce the pseudocritical temperature T_d^* as a temperature achieving the peak of the susceptibility. Below the deconfinement temperature T_d^* , i.e., $T < T_d^*$, therefore, $L \simeq 0$ or $\varphi \simeq \pi$, the NJL coupling constant G has the temperature dependence

$$G(T)/G(0) \simeq \frac{1}{3} \left[\frac{2M_0^2}{\pi^2 T^2 + M_0^2} + 1 \right] \quad (T < T_d^*). \quad (105)$$

This naive estimation gives a qualitative understanding for the existence of chiral phase transition. Since $G(T)$ is (monotonically) decreasing as the temperature T increases, it becomes smaller than the critical NJL coupling constant

$$G(0) = g^2 \frac{1}{M_0^2} > G_c, \quad T \uparrow \infty \Rightarrow G \downarrow 0. \quad (106)$$

Thus, the chiral transition temperature T_χ will be determined (if the chiral-symmetry restoration and confinement coexist or the chiral symmetry is restored in the confinement environment before deconfinement takes place, i.e., $T_\chi \leq T_d^*$) by solving

$$G(T_\chi) \equiv \frac{G(0)}{3} \left[\frac{2M_0^2}{T_\chi^2 \pi^2 + M_0^2} + 1 \right] = G_c. \quad (107)$$

Here we have assumed that the nonlocality function $\tilde{\mathcal{C}}(p)$ gives the dominant contribution at $p = 0$, namely, $\tilde{\mathcal{C}}(p) \leq \tilde{\mathcal{C}}(0) = \int d^4 z \mathcal{C}(z) = 1$ and that the occurrence of the chiral transition is determined by the NJL coupling constant alone.

At finite temperature T , the form factor reads

$$\begin{aligned} \mathcal{C}(x-y) &= T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \tilde{\mathcal{C}}(p_0 = \omega, \mathbf{p}) e^{ip \cdot (x-y)} \\ &= T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \frac{M_0^2}{3} \left[\frac{2}{p^2 + (\omega + T\varphi)^2 + M_0^2} \right. \\ &\quad \left. + \frac{1}{p^2 + \omega^2 + M_0^2} \right] e^{ip \cdot (x-y)} \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{M_0^2}{6\epsilon_p} \left[2 \frac{\sinh(\epsilon_p/T)}{\cosh(\epsilon_p/T) - \cos(\varphi)} \right. \\ &\quad \left. + \frac{\sinh(\epsilon_p/T)}{\cosh(\epsilon_p/T) - 1} \right] e^{ip \cdot (x-y)}, \end{aligned} \quad (108)$$

where we have defined $\epsilon_p := \sqrt{p^2 + M_0^2}$ and used

$$T \sum_{n \in \mathbb{Z}} \frac{1}{(\omega + C)^2 + \epsilon_p^2} = \frac{1}{2\epsilon_p} \frac{\sinh(\epsilon_p/T)}{\cosh(\epsilon_p/T) - \cos(C/T)}. \quad (109)$$

The form factor \mathcal{C} does not change so much around the deconfinement temperature $T \sim T_d^*$ (or $\varphi \sim \pi$). This is reasonable since the form factor is nearly equal to the \mathcal{X} correlator Q^{-1} , as already mentioned in Sec. II.

For more precise treatment, we must obtain the full effective potential $V_{\text{eff}}(\sigma, \varphi)$ as a function of two order parameters σ (or $\langle \bar{\psi} \psi \rangle$) and φ (or $\langle L \rangle$), and look for a set of values $(\sigma, \varphi) = (\sigma_0, \varphi_0)$ at which the minimum $V_{\text{eff}}(\sigma_0, \varphi_0)$ of $V_{\text{eff}}(\sigma, \varphi)$ is realized. Then φ must be replaced by φ_0 in the above consideration. For this goal, we must develop the RG treatment for the full theory. This issue will be studied in a subsequent paper.

IX. HOW TO UNDERSTAND THE ENTANGLEMENT BETWEEN CONFINEMENT AND CHIRAL-SYMMETRY BREAKING

To discuss the entanglement between confinement and chiral-symmetry breaking, we wish to obtain the total effective potential V^{QCD} of QCD written in terms of two order parameters, i.e., the Polyakov-loop average $\langle L \rangle$ and chiral condensate $\langle \bar{\psi} \psi \rangle$, so that its minima determine the vacuum for a given set of parameters m_q , T and μ_q when N_c and N_f are fixed. Here $m_q \uparrow \infty$ is the pure Yang-Mills limit and $m_q \downarrow 0$ is the chiral limit.

The effective potential for the quark part is obtained by integrating out quark degrees of freedom. The simplest form is obtained, e.g., from the bosonized model as

$$\begin{aligned} V^{\text{quark}} &= -\text{Tr} \ln \{ i \gamma^\mu \partial_\mu + m_q + \mathcal{C} \sigma - g \mathcal{A}_4 \gamma^4 + i \mu_q \gamma^4 \} \\ &\quad + \frac{1}{2G} \sigma^2. \end{aligned} \quad (110)$$

Then the RG scale k dependent effective potential V_k^{quark} for the quark part must be given as the solution of the flow

equation. In the same approximation as the above, it is written in terms of two order parameters σ and φ

$$V_k^{\text{quark}}(\sigma, \varphi)_{m_q, T, \mu_q} = -T \sum_{n \in \mathbb{Z}} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \ln [i\omega_n \gamma^0 - p_j \gamma^j + m_q + \mathcal{C}(p)\sigma - T\varphi T_3 \gamma^4 + i\mu_q \gamma^4 + R_k^{\text{quark}}] + \frac{1}{2G} \sigma^2, \quad (111)$$

where $T_3 = \sigma_3/2$ and R_k^{quark} is the regulator function for quarks. In the limit $k \downarrow 0$, indeed, V_k^{quark} (111) reduces to V^{quark} (110). The effective potential V_k^{quark} (111) depends on m_q , T , and μ_q when N_c and N_f are fixed. Because of the p_0 dependence of the ‘‘mass’’ function $M(p)$ which is an immediate consequence of the nonlocality of the present NJL model, it is difficult to obtain the closed analytical form by performing the summation over the Matsubara frequencies.

In our strategy, a full effective potential $V_{\text{eff},k}^{\text{QCD}}(\sigma, \varphi)$ of QCD is given by summing three parts:

$$V_{\text{eff},k}^{\text{QCD}}(\sigma, \varphi) = V_k^{\text{glue}}(\varphi) + V_k^{\text{quark}}(\sigma, \varphi) + \Delta V_k^{\text{QCD}}(\sigma, \varphi), \quad (112)$$

with the pure gluon part $V_k^{\text{glue}}(\varphi) = V_{T,k}$ (56);

$$V_k^{\text{glue}}(\varphi) = \text{Tr}[\ln[G^{AB} + \delta^{AB}R_k]], \\ = \text{Tr}[\ln[-\delta^{AB}\partial_\mu^2 + (\delta^{AB} - \delta^{A3}\delta^{B3})(T\varphi)^2 + 2\epsilon^{AB3}T\varphi\partial_0 + \delta^{AB}R_k]], \quad (113)$$

with the quark part (111),

$$V_k^{\text{quark}}(\sigma, \varphi) = \frac{1}{2G} \sigma^2 - \text{Tr} \ln \{ i\gamma^\mu \partial_\mu + m_q + \mathcal{C}\sigma - T\varphi T_3 \gamma^4 + i\mu_q \gamma^4 + R_k^{\text{quark}} \}; \quad (114)$$

and a nonperturbative part $\Delta V_k^{\text{QCD}}(\sigma, \varphi)$ induced in the RG evolution according to a flow equation. We assume that the total effective action of QCD obtained after integrating out the fields other than those relevant to chiral symmetry and confinement is the form

$$\Gamma_k = \int_0^{1/T} dx_4 \int d^3 x \left\{ \frac{1}{2} Z_0 [\partial_j \mathcal{V}_0(\mathbf{x})]^2 + \frac{1}{2} Z_\sigma [\partial_j \sigma(x)]^2 + V_{\text{eff},k}^{\text{QCD}}(\sigma, \varphi) \right\}, \quad (115)$$

and obeys the flow equation

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left\{ \left[\frac{\tilde{\delta}}{\delta \sigma^\dagger} \Gamma_k \frac{\tilde{\delta}}{\delta \sigma} + R_k \right]^{-1} \cdot \partial_t R_k \right\} + \frac{1}{2} \text{Tr} \left\{ \left[\frac{\tilde{\delta}}{\delta \mathcal{V}^\dagger} \Gamma_k \frac{\tilde{\delta}}{\delta \mathcal{V}} + R_k \right]^{-1} \cdot \partial_t R_k \right\}. \quad (116)$$

If the flow equation was solved, we would have obtained the effective potential of QCD, $V_{\text{eff},k}^{\text{QCD}}(\sigma, \varphi)$, which has the following power-series expansion with respect to two variables σ and $\tilde{\varphi}$ in the neighborhood of the transition point where $\sigma = 0 = L$ according to the Landau argument (as demonstrated in the pure glue case).

$$V_{\text{eff}}^{\text{QCD}}(\sigma, \tilde{\varphi}) = V^{\mathbb{S}}(\tilde{\varphi}) + V^{\mathbb{Q}}(\sigma) + V^{\mathbb{C}}(\sigma, \tilde{\varphi}), \\ V^{\mathbb{S}}(\tilde{\varphi}) = C_0 + C_1 \tilde{\varphi} + \frac{C_2}{2} \tilde{\varphi}^2 + \frac{C_3}{3} \tilde{\varphi}^3 + \frac{C_4}{4} \tilde{\varphi}^4 + O(\tilde{\varphi}^6), \\ V^{\mathbb{Q}}(\sigma) = \frac{E_2}{2} \sigma^2 + \frac{E_4}{4} \sigma^4 + O(\sigma^6), \\ V^{\mathbb{C}}(\sigma, \tilde{\varphi}) = F_1 \sigma^2 \tilde{\varphi} + \dots, \quad (117)$$

where $V^{\mathbb{S}}(\tilde{\varphi})$ denotes a part written in terms of $\tilde{\varphi}$ alone, and $V^{\mathbb{Q}}(\sigma)$ denotes a part written in terms of σ alone, while $V^{\mathbb{C}}(\sigma, \tilde{\varphi})$ denotes the cross term between σ and $\tilde{\varphi}$.

Once dynamical quarks are introduced, the exact center symmetry in pure Yang-Mills theory is no longer intact. Therefore, the QCD effective potential includes the explicitly center-symmetry breaking term. For $G = SU(2)$, the center symmetry $\tilde{\varphi} \rightarrow -\tilde{\varphi}$ is explicitly broken as $C_1 \neq 0$, $C_3 \neq 0$ in $V^{\mathbb{S}}(\varphi)$, and $F_1 \neq 0$ in $V^{\mathbb{C}}(\sigma, \varphi)$. The existence of the cross term is important to understand the entanglement between center symmetry and chiral symmetry, as pointed out by [19]. In fact, the one-loop calculation leads to $C_1 = 0.97434N_f > 0$ and $F_1 = -0.106103N_f < 0$ ($\mu_q = 0$ case) which appears to be a good indication for this purpose and serves as the initial condition in solving the flow equation.

In the paper by Schaefer, Pawłowski, and Wambach [26], a sort of backreaction from quarks has been introduced to improve the effective potential of the Polyakov loop, while the NJL coupling remains local. In contrast, this paper introduces a backreaction from gluons to improve the NJL interaction, leading to the nonlocal NJL coupling. However, this does not mean that two treatments are considered to be alternative. In the presence of dynamical quarks, the running coupling α is changed due to fermionic contributions. In [26], this effect has been taken into account as a modification of the expansion coefficient in the effective potential of the Polyakov loop, resulting in, e.g., the N_f flavor-dependent deconfinement temperature $T_d(N_f)$. Remembering that the input of our analysis is just a running coupling, a sort of backreaction from quarks considered in [26] is easily included into our framework by using the running coupling modified by quark contributions. Thus, the treatment in this paper is already able to take into account backreactions from quarks and gluons mentioned above.

This section is a sketch of our strategy of understanding the entanglement between center symmetry and chiral

symmetry. The detailed analysis will be given in a subsequent paper.³

X. CONCLUSION AND DISCUSSION

In this paper, we have presented a reformulation of QCD and suggested a framework for deriving a low-energy effective theory of QCD which enables one to study the deconfinement/confinement and chiral-symmetry restoration/breaking crossover transition simultaneously on an equal footing. A resulting low-energy effective theory based on this framework can be regarded as a modified (improved) version of the nonlocal PNJL model [22]. In our framework, the basic ingredients are a reformulation of QCD based on new variables and the flow equation of the Wetterich type for the Wilsonian renormalization group.

A lesson we learned in this study is that a perturbative (one-loop) result can be a good initial condition for solving the flow equation of the renormalization group to obtain the nonperturbative result. In gluodynamics, recently, it has been demonstrated [32,33] that the existence of confinement transition, i.e., recovery of the center symmetry signaled by the vanishing Polyakov-loop average can be shown by approaching the phase transition point from the high-temperature deconfinement phase in which the center symmetry is spontaneously broken. Indeed, the effective potential for the Polyakov loop obtained in the one-loop calculation which we call the Weiss potential leads to the nonvanishing Polyakov-loop average, i.e., spontaneous breaking of the center symmetry.

For the gluon sector, to understand the existence of confinement transition by approaching from the deconfinement side, we have given the Landau-Ginzburg description in the neighborhood of the (crossover) phase transition point by analyzing the flow equation of the functional renormalization group. The deconfinement/confinement phase transition is consistent with the second order transition for $G = SU(2)$, while the first-order transition is expected for $G = SU(3)$. The detailed study of the $SU(3)$ case will be given in a subsequent paper.

The input for solving the flow equation was just a running gauge coupling constant, in sharp contrast to the PNJL model including several parameters. From the viewpoint of a first-principle derivation, this is superior to phenomenological models with many input parameters.

³It is known that the appearance of a mixed term $\sigma^2 \tilde{\varphi}$ plays an essential role in the chiral-confinement entanglement. Such a term appears in the original PNJL model and leads to the 2 crossovers happening almost simultaneously. In the following paper posted to the archive after this paper was submitted for publication, it has been shown [77] that an effective Polyakov loop-dependent four-quark interaction derived by this paper yields stronger correlation between the chiral and deconfinement transitions, making $T_\chi \sim T_d$ more tight, than the usual PNJL model.

For the quark sector, it is possible to obtain the chiral-symmetry breaking/restoration transition from the first principle. However, we need more hard works, especially, to discuss the QCD phase diagram at finite density and the critical endpoint. A possibility in this direction from the first principle of QCD was demonstrated in one-flavor QCD based on the functional renormalization group [78]. It will be possible to treat chiral dynamics and confinement on an equal footing based on our framework along this line [79]. Still, however, we must overcome some technical issues to achieve the goal of understanding full phase structures of QCD. The detailed studies will be hopefully given in a subsequent paper.

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APPENDIX A: REFORMULATION OF QCD

We apply the decomposition (2) to the QCD Lagrangian. The quark part is decomposed according to (2) as

$$\begin{aligned} \mathcal{L}_q &:= \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu[\mathcal{A}] - \hat{m}_0 + i\mu\gamma^0)\psi \\ &= \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu[\mathcal{V}] - \hat{m}_0 + i\mu\gamma^0)\psi + g\bar{\psi}\gamma^\mu \mathcal{X}_\mu \psi, \end{aligned} \quad (\text{A1})$$

where the covariant derivative $\mathcal{D}_\mu[\mathcal{V}]$ is defined by

$$\mathcal{D}_\mu[\mathcal{V}] := \partial_\mu - ig\mathcal{V}_\mu. \quad (\text{A2})$$

The Yang-Mills part is treated as follows. For the general decomposition $\mathcal{A}_\mu(x) = \mathcal{V}_\mu(x) + \mathcal{X}_\mu(x)$, the field strength $\mathcal{F}_{\mu\nu}$ is decomposed as

$$\begin{aligned}
\mathcal{F}_{\mu\nu}[\mathcal{A}] &:= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu] \\
&= \mathcal{F}_{\mu\nu}[\mathcal{V}] + \partial_\mu \mathcal{X}_\nu - \partial_\nu \mathcal{X}_\mu - ig[\mathcal{V}_\mu, \mathcal{X}_\nu] \\
&\quad - ig[\mathcal{X}_\mu, \mathcal{V}_\nu] - ig[\mathcal{X}_\mu, \mathcal{X}_\nu] \\
&= \mathcal{F}_{\mu\nu}[\mathcal{V}] + D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu \\
&\quad - ig[\mathcal{X}_\mu, \mathcal{X}_\nu], \tag{A3}
\end{aligned}$$

where the covariant derivative $D_\mu[\mathcal{V}]$ in the background field \mathcal{V}_ν is defined by

$$D_\mu[\mathcal{V}] := \partial_\mu \mathbf{1} - ig[\mathcal{V}_\mu, \cdot], \tag{A4}$$

or, equivalently,

$$D_\mu[\mathcal{V}]^{AC} := \partial_\mu \delta^{AC} + gf^{ABC} \mathcal{V}_\mu^B. \tag{A5}$$

$$\begin{aligned}
\frac{1}{4}(D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu)^2 &= \frac{1}{2}(-\mathcal{X}_\mu \cdot D_\nu[\mathcal{V}]D^\nu[\mathcal{V}]\mathcal{X}^\mu + \mathcal{X}_\mu \cdot D_\nu[\mathcal{V}]D^\mu[\mathcal{V}]\mathcal{X}^\nu) \\
&= \frac{1}{2}\mathcal{X}^\mu \cdot \{-D_\rho[\mathcal{V}]D^\rho[\mathcal{V}]g_{\mu\nu} + D_\nu[\mathcal{V}]D_\mu[\mathcal{V}]\}\mathcal{X}^\nu \\
&= \frac{1}{2}\mathcal{X}^{\mu A}\{-(D_\rho[\mathcal{V}]D_\rho[\mathcal{V}])^{AB}g_{\mu\nu} - [D_\mu[\mathcal{V}], D_\nu[\mathcal{V}]]^{AB} + (D_\mu[\mathcal{V}]D_\nu[\mathcal{V}])^{AB}\}\mathcal{X}^{\nu B} \\
&= \frac{1}{2}\mathcal{X}^{\mu A}\{-(D_\rho[\mathcal{V}]D_\rho[\mathcal{V}])^{AB}g_{\mu\nu} + gf^{ABC}\mathcal{F}_{\mu\nu}^C[\mathcal{V}] + D_\mu[\mathcal{V}]^{AC}D_\nu[\mathcal{V}]^{CB}\}\mathcal{X}^{\nu B}, \tag{A8}
\end{aligned}$$

where we have used

$$\begin{aligned}
[D_\mu[\mathcal{V}], D_\nu[\mathcal{V}]]^{AB} &= [D_\mu[\mathcal{V}]^{AC}, D_\nu[\mathcal{V}]^{CB}] \\
&= -gf^{ABC}\mathcal{F}_{\mu\nu}^C[\mathcal{V}]. \tag{A9}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\mathcal{L}_{\text{YM}} &= -\frac{1}{4}\mathcal{F}_{\mu\nu}[\mathcal{V}]^2 - \frac{1}{2}\mathcal{F}^{\mu\nu}[\mathcal{V}] \cdot (D_\mu[\mathcal{V}]\mathcal{X}_\nu \\
&\quad - D_\nu[\mathcal{V}]\mathcal{X}_\mu) - \frac{1}{2}\mathcal{X}^{\mu A}W_{\mu\nu}^{AB}\mathcal{X}^{\nu B} \\
&\quad + \frac{1}{2}(D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu) \cdot ig[\mathcal{X}^\mu, \mathcal{X}^\nu] \\
&\quad - \frac{1}{4}(ig[\mathcal{X}_\mu, \mathcal{X}_\nu])^2, \tag{A10}
\end{aligned}$$

where we have defined

$$\begin{aligned}
W_{\mu\nu}^{AB} &:= -(D_\rho[\mathcal{V}]D^\rho[\mathcal{V}])^{AB}g_{\mu\nu} + 2gf^{ABC}\mathcal{F}_{\mu\nu}^C[\mathcal{V}] \\
&\quad + D_\mu[\mathcal{V}]^{AC}D_\nu[\mathcal{V}]^{CB}. \tag{A11}
\end{aligned}$$

In the usual background field method, the $\mathcal{O}(X)$ term is eliminated by requiring that the background field \mathcal{V} satisfies the equation of motion $D_\mu[\mathcal{V}]\mathcal{F}^{\mu\nu}[\mathcal{V}] = 0$

$$\begin{aligned}
&\frac{1}{2}\mathcal{F}^{\mu\nu}[\mathcal{V}] \cdot (D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu) \\
&= -\frac{1}{2}(D_\mu[\mathcal{V}]\mathcal{F}^{\mu\nu}[\mathcal{V}] \cdot \mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{F}^{\mu\nu}[\mathcal{V}] \cdot \mathcal{X}_\mu) \\
&= 0. \tag{A12}
\end{aligned}$$

In our framework, \mathcal{V} does not necessarily satisfy the equation of motion. Nevertheless, the $\mathcal{O}(X)$ term vanishes from the defining equations which specify the decomposition. For $G = SU(2)$, $D_\mu[\mathcal{V}]\mathbf{n} = 0$ and $\mathcal{X}_\mu \cdot \mathbf{n} = 0$ lead to

The Lagrangian density $\mathcal{L}_{\text{YM}} = -\frac{1}{4}\mathcal{F}_{\mu\nu}[\mathcal{A}] \cdot \mathcal{F}^{\mu\nu}[\mathcal{A}]$ of the Yang-Mills theory is decomposed as

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}\mathcal{F}_{\mu\nu}[\mathcal{A}]^2 \tag{A6}$$

$$\begin{aligned}
&= -\frac{1}{4}\mathcal{F}_{\mu\nu}[\mathcal{V}]^2 - \frac{1}{2}\mathcal{F}^{\mu\nu}[\mathcal{V}] \cdot (D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu) \\
&\quad - \frac{1}{4}(D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu)^2 \\
&\quad + \frac{1}{2}\mathcal{F}_{\mu\nu}[\mathcal{V}] \cdot ig[\mathcal{X}^\mu, \mathcal{X}^\nu] + \frac{1}{2}(D_\mu[\mathcal{V}]\mathcal{X}_\nu \\
&\quad - D_\nu[\mathcal{V}]\mathcal{X}_\mu) \cdot ig[\mathcal{X}^\mu, \mathcal{X}^\nu] - \frac{1}{4}(ig[\mathcal{X}_\mu, \mathcal{X}_\nu])^2. \tag{A7}
\end{aligned}$$

Here the third term on the right-hand side of the above equation is rewritten using integration by parts (or up to total derivatives) as

$$\begin{aligned}
&\mathcal{F}^{\mu\nu}[\mathcal{V}] \cdot (D_\mu[\mathcal{V}]\mathcal{X}_\nu) \\
&= G^{\mu\nu}\mathbf{n} \cdot (D_\mu[\mathcal{V}]\mathcal{X}_\nu) \\
&= G^{\mu\nu}[\partial_\mu(\mathcal{X}_\nu \cdot \mathbf{n}) - \mathcal{X}_\nu \cdot D_\mu[\mathcal{V}]\mathbf{n}] \\
&= 0. \tag{A13}
\end{aligned}$$

In order for the reformulated theory written in terms of new variables to be equivalent to the original QCD, we must impose the reduction condition [38]:

$$D_\mu[\mathcal{V}]\mathcal{X}^\mu = 0. \tag{A14}$$

This eliminates the last term of $W_{\mu\nu}^{AB}$ in (A11).

Moreover, the $\mathcal{O}(X^3)$ term is absent, i.e.,

$$\frac{1}{2}(D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu) \cdot ig[\mathcal{X}^\mu, \mathcal{X}^\nu] = 0, \tag{A15}$$

since $D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu$ is orthogonal to $[\mathcal{X}^\mu, \mathcal{X}^\nu]$; see [38,41,49].

Thus, the Yang-Mills Lagrangian density reads

$$\begin{aligned}
\mathcal{L}_{\text{YM}} &= -\frac{1}{4}\mathcal{F}_{\mu\nu}^A[\mathcal{V}]^2 - \frac{1}{2}\mathcal{X}^{\mu A}Q_{\mu\nu}^{AB}\mathcal{X}^{\nu B} \\
&\quad - \frac{1}{4}(ig[\mathcal{X}_\mu, \mathcal{X}_\nu])^2, \tag{A16}
\end{aligned}$$

where we have defined

$$Q_{\mu\nu}^{AB} := -(D_\rho[\mathcal{V}]D^\rho[\mathcal{V}])^{AB}g_{\mu\nu} + 2gf^{ABC}\mathcal{F}_{\mu\nu}^C[\mathcal{V}]. \tag{A17}$$

For $G = SU(2)$, the $\mathcal{O}(X^3)$ term is absent, because $\mathcal{F}_{\mu\nu}[\mathcal{V}]$ and $-ig[\mathcal{X}_\mu, \mathcal{X}_\nu]$ are parallel to \mathbf{n} (this is also the case for the sum $\mathcal{F}_{\mu\nu}[\mathcal{V}] - ig[\mathcal{X}_\mu, \mathcal{X}_\nu]$), while

$D_\mu[\mathcal{V}]\mathcal{X}_\nu - D_\nu[\mathcal{V}]\mathcal{X}_\mu$ is orthogonal to \mathbf{n} (which follows from the fact $\mathbf{n} \cdot \mathcal{X}_\mu = 0$). For $G = SU(2)$, therefore, we have

$$\mathcal{F}_{\mu\nu}^C[\mathcal{V}] = n^C G_{\mu\nu}[\mathcal{V}]. \quad (\text{A18})$$

Then the $SU(2)$ gluon part is rewritten into

$$\begin{aligned} \mathcal{L}_{\text{YM}} = & -\frac{1}{4}(G_{\mu\nu}[\mathcal{V}])^2 - \frac{1}{2}\mathcal{X}^{\mu A} Q_{\mu\nu}^{AB}[\mathcal{V}]\mathcal{X}^{\nu B} \\ & - \frac{1}{4}(ig[\mathcal{X}_\mu, \mathcal{X}_\nu])^2, \end{aligned} \quad (\text{A19})$$

where

$$\begin{aligned} Q_{\mu\nu}^{AB}[\mathcal{V}] = & -(D_\rho[\mathcal{V}]D^\rho[\mathcal{V}])^{AB}g_{\mu\nu} \\ & + 2g\epsilon^{ABC}n^C G_{\mu\nu}[\mathcal{V}]. \end{aligned} \quad (\text{A20})$$

APPENDIX B: COEFFICIENTS IN THE EFFECTIVE POTENTIAL

We expand $\hat{V}_{T,\hat{k}}$ defined by

$$\hat{V}_{T,\hat{k}} = \hat{V}_W + 4 \int_0^{\hat{k}_T} \frac{d\hat{p}\hat{p}^2}{(2\pi)^2} \{\ln(1 - 2e^{-\hat{k}_T} \cos\varphi + e^{-2\hat{k}_T}) - \ln(1 - 2e^{-\hat{p}} \cos\varphi + e^{-2\hat{p}})\}, \quad (\text{B1})$$

in a power series of $\tilde{\varphi}$ by using the expansion $-\cos\varphi = -\cos(\pi + \tilde{\varphi}) = \cos(\tilde{\varphi}) = 1 - \frac{1}{2}\tilde{\varphi}^2 + \frac{1}{24}\tilde{\varphi}^4 + O(\tilde{\varphi}^6)$ as follows:

$$\begin{aligned} \hat{V}_{T,\hat{k}} = & \hat{V}_W + \int_0^{\hat{k}_T} \frac{d\hat{p}\hat{p}^2}{\pi^2} \left\{ \ln \left[1 + 2e^{-\hat{k}_T} - e^{-\hat{k}_T} \tilde{\varphi}^2 + \frac{1}{12} e^{-\hat{k}_T} \tilde{\varphi}^4 + e^{-2\hat{k}_T} + O(\tilde{\varphi}^6) \right] \right. \\ & \left. - \ln \left[1 + 2e^{-\hat{p}} - e^{-\hat{p}} \tilde{\varphi}^2 + \frac{1}{12} e^{-\hat{p}} \tilde{\varphi}^4 + e^{-2\hat{p}} + O(\tilde{\varphi}^6) \right] \right\} \\ = & \hat{V}_W + \int_0^{\hat{k}_T} \frac{d\hat{p}\hat{p}^2}{\pi^2} \left\{ \ln \left[(1 + e^{-\hat{k}_T})^2 - e^{-\hat{k}_T} \tilde{\varphi}^2 + \frac{1}{12} e^{-\hat{k}_T} \tilde{\varphi}^4 + O(\tilde{\varphi}^6) \right] \right. \\ & \left. - \ln \left[(1 + e^{-\hat{p}})^2 - e^{-\hat{p}} \tilde{\varphi}^2 + \frac{1}{12} e^{-\hat{p}} \tilde{\varphi}^4 + O(\tilde{\varphi}^6) \right] \right\} \\ = & \hat{V}_W + \int_0^{\hat{k}_T} \frac{d\hat{p}\hat{p}^2}{\pi^2} \left\{ \ln(1 + e^{-\hat{k}_T})^2 + \ln \left[1 - \frac{e^{-\hat{k}_T}}{(1 + e^{-\hat{k}_T})^2} \tilde{\varphi}^2 + \frac{e^{-\hat{k}_T}}{12(1 + e^{-\hat{k}_T})^2} \tilde{\varphi}^4 + O(\tilde{\varphi}^6) \right] \right. \\ & \left. - \ln(1 + e^{-\hat{p}})^2 - \ln \left[1 - \frac{e^{-\hat{p}}}{(1 + e^{-\hat{p}})^2} \tilde{\varphi}^2 + \frac{e^{-\hat{p}}}{12(1 + e^{-\hat{p}})^2} \tilde{\varphi}^4 + O(\tilde{\varphi}^6) \right] \right\}. \end{aligned} \quad (\text{B2})$$

By using $\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$, therefore, $\hat{V}_{T,\hat{k}}$ has the polynomial expansion

$$\hat{V}_{T,\hat{k}} = A_{0,k} + \frac{A_{2,k}}{2} \tilde{\varphi}^2 + \frac{A_{4,k}}{4!} \tilde{\varphi}^4 + O(\tilde{\varphi}^6), \quad (\text{B3})$$

where the coefficient is given by the integral form

$$\begin{aligned} \frac{A_{2,k}}{2} = & -\frac{1}{6} + \int_0^{\hat{k}_T} \frac{d\hat{p}\hat{p}^2}{\pi^2} \left[\frac{e^{-\hat{p}}}{(1 + e^{-\hat{p}})^2} - \frac{e^{-\hat{k}_T}}{(1 + e^{-\hat{k}_T})^2} \right], \\ \frac{A_{4,k}}{4!} = & \frac{1}{12\pi^2} - \int_0^{\hat{k}_T} \frac{d\hat{p}\hat{p}^2}{\pi^2} \left[\frac{-6e^{-2\hat{p}} + e^{-\hat{p}}(1 + e^{-\hat{p}})^2}{12(1 + e^{-\hat{p}})^4} - \frac{-6e^{-2\hat{k}_T} + e^{-\hat{k}_T}(1 + e^{-\hat{k}_T})^2}{12(1 + e^{-\hat{k}_T})^4} \right], \\ A_{0,k} = & \int_0^{\hat{k}_T} \frac{d\hat{p}\hat{p}^2}{\pi^2} \{\ln(1 + e^{-\hat{k}_T})^2 - \ln(1 + e^{-\hat{p}})^2\}. \end{aligned} \quad (\text{B4})$$

The integration can be performed analytically and the coefficient has the closed form

$$\begin{aligned} \frac{A_{2,k}}{2} = & -\frac{1}{6} + \frac{1}{\pi^2} \left[-\frac{e^s s^3}{3(1 + e^s)^2} + \frac{e^s s^2}{1 + e^s} - 2 \log(1 + e^s) s - 2 \text{Li}_2(-e^s) - \frac{\pi^2}{6} \right] \Big|_{s=\hat{k}_T}, \\ \frac{A_{4,k}}{4!} = & \frac{1}{12\pi^2} + \frac{e^{2s}(-2s^3 + (s^2 + 6) \cosh(s)s + 6s + 3(s^2 - 2) \sinh(s) - 3 \sinh(2s))}{18(1 + e^s)^4 \pi^2} \Big|_{s=\hat{k}_T}, \end{aligned} \quad (\text{B5})$$

where $\text{Li}_n(z) = \text{PolyLog}[n, z]$ is the polylogarithm function, and, in particular, the dilogarithm satisfies $\text{Li}_2(z) = \int_z^0 \frac{\log(1-t)}{t} dt$ which is known as the Spence integral.

Note that the function $\frac{e^{-x}}{(1+e^{-x})^2}$ is monotonically decreasing in x and hence the second term in $A_{2,k}$ is positive (non-negative). The coefficient $A_{2,k}$ is negative and monotonically increasing in k and approaches zero for $k \rightarrow \infty$

$$\frac{A_{2,k}}{2} = -\frac{1}{6}, \quad -\frac{1}{6} \leq \frac{A_{2,k}}{2} < 0 \quad \text{for } k \in [0, \infty), \quad (\text{B6})$$

or

$$-\frac{1}{3} \leq \frac{\partial^2}{\partial \tilde{\varphi}^2} \hat{V}_{T,\hat{k}}|_{\tilde{\varphi}=0} < 0 \quad \text{for } k \in [0, \infty). \quad (\text{B7})$$

This is because

$$\int_0^{\hat{k}_T} d\hat{p} \hat{p}^2 \frac{e^{-\hat{p}}}{(1+e^{-\hat{p}})^2} \rightarrow \int_0^\infty d\hat{p} \hat{p}^2 \frac{e^{-\hat{p}}}{(1+e^{-\hat{p}})^2} = \frac{1}{6} \pi^2, \quad (\text{B8})$$

$$\int_0^{\hat{k}_T} d\hat{p} \hat{p}^2 \frac{e^{-\hat{k}_T}}{(1+e^{-\hat{k}_T})^2} = \frac{1}{3} \hat{k}_T^3 \frac{e^{-\hat{k}_T}}{(1+e^{-\hat{k}_T})^2} \rightarrow 0. \quad (\text{B9})$$

The coefficient $A_{4,k}$ is positive and approaches 0 for $k \rightarrow \infty$, although $A_{4,k}$ is not monotonically decreasing in k .

$$\frac{A_{4,0}}{4!} = \frac{1}{12\pi^2}, \quad \frac{A_{4,k}}{4!} > 0 \quad \text{for } k \in [0, \infty). \quad (\text{B10})$$

This is because

$$\int_0^{\hat{k}_T} d\hat{p} \hat{p}^2 \left[\frac{-6e^{-2\hat{p}} + e^{-\hat{p}}(1+e^{-\hat{p}})^2}{12(1+e^{-\hat{p}})^4} \right] \rightarrow \frac{1}{12}, \quad (\text{B11})$$

$$\int_0^{\hat{k}_T} d\hat{p} \hat{p}^2 \left[\frac{-6e^{-2\hat{k}_T} + e^{-\hat{k}_T}(1+e^{-\hat{k}_T})^2}{12(1+e^{-\hat{k}_T})^4} \right] = \frac{1}{3} \hat{k}_T^3 \left[\frac{-6e^{-2\hat{k}_T} + e^{-\hat{k}_T}(1+e^{-\hat{k}_T})^2}{12(1+e^{-\hat{k}_T})^4} \right] \rightarrow 0. \quad (\text{B12})$$

APPENDIX C: FLOW EQUATION FOR THE COEFFICIENT

Suppose that $\Delta \hat{V}_{\hat{k}}$ is of the form

$$\Delta \hat{V}_{\hat{k}} = a_{0,k} + a_{1,k} \tilde{\varphi} + \frac{a_{2,k}}{2} \tilde{\varphi}^2 + \frac{a_{3,k}}{3!} \tilde{\varphi}^3 + \frac{a_{4,k}}{4!} \tilde{\varphi}^4 + O(\tilde{\varphi}^6). \quad (\text{C1})$$

The left-hand side of the flow equation reads

$$\partial_{\hat{k}} \Delta \hat{V}_{\hat{k}} = \partial_{\hat{k}} a_{0,k} + \partial_{\hat{k}} a_{1,k} \tilde{\varphi} + \partial_{\hat{k}} \frac{a_{2,k}}{2} \tilde{\varphi}^2 + \partial_{\hat{k}} \frac{a_{3,k}}{3!} \tilde{\varphi}^3 + \partial_{\hat{k}} \frac{a_{4,k}}{4!} \tilde{\varphi}^4 + O(\tilde{\varphi}^6). \quad (\text{C2})$$

The flow equation $\partial_{\hat{k}} a_{n,k}$ for the coefficient of $\tilde{\varphi}^n$ is extracted by differentiating both sides of the flow equation n times and by putting $\tilde{\varphi} = 0$. The left-hand side is

$$\partial_{\hat{k}} a_{n,k} = \frac{\partial^n}{\partial \tilde{\varphi}^n} \partial_{\hat{k}} \Delta \hat{V}_{\hat{k}}|_{\tilde{\varphi}=0}. \quad (\text{C3})$$

Define

$$f(\varphi) := \frac{4\pi\alpha_k}{\hat{k}^2} (\hat{V}_{T,\hat{k}} + \Delta \hat{V}_{\hat{k}}). \quad (\text{C4})$$

The right-hand sides of the flow equation $\partial_{\hat{k}} a_{n,k}$ are calculated from

$$\frac{\partial}{\partial \tilde{\varphi}} \left[\frac{1}{1 + \partial_{\tilde{\varphi}}^2 f(\varphi)} \right] = \frac{-\partial_{\tilde{\varphi}}^3 f(\varphi)}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^2}, \quad (\text{C5})$$

$$\frac{\partial^2}{\partial \tilde{\varphi}^2} \left[\frac{1}{1 + \partial_{\tilde{\varphi}}^2 f(\varphi)} \right] = \frac{-\partial_{\tilde{\varphi}}^4 f(\varphi)}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^2} - 2 \frac{-[\partial_{\tilde{\varphi}}^3 f(\varphi)]^2}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^3}, \quad (\text{C6})$$

$$\begin{aligned} & \frac{\partial^3}{\partial \tilde{\varphi}^3} \left[\frac{1}{1 + \partial_{\tilde{\varphi}}^2 f(\varphi)} \right] \\ &= \frac{-\partial_{\tilde{\varphi}}^5 f(\varphi)}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^2} - 2 \frac{-3\partial_{\tilde{\varphi}}^4 f(\varphi) \partial_{\tilde{\varphi}}^3 f(\varphi)}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^3} \\ &+ 6 \frac{-[\partial_{\tilde{\varphi}}^3 f(\varphi)]^3}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^4}, \end{aligned} \quad (\text{C7})$$

$$\begin{aligned} & \frac{\partial^4}{\partial \tilde{\varphi}^4} \left[\frac{1}{1 + \partial_{\tilde{\varphi}}^2 f(\varphi)} \right] \\ &= \frac{-\partial_{\tilde{\varphi}}^6 f(\varphi)}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^2} - 2 \frac{-4\partial_{\tilde{\varphi}}^5 f(\varphi) \partial_{\tilde{\varphi}}^3 f(\varphi) - 3[\partial_{\tilde{\varphi}}^4 f(\varphi)]^2}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^3} \\ &+ 6 \frac{-6\partial_{\tilde{\varphi}}^4 f(\varphi) [\partial_{\tilde{\varphi}}^3 f(\varphi)]^2}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^4} - 24 \frac{-[\partial_{\tilde{\varphi}}^3 f(\varphi)]^4}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^5}, \dots \end{aligned} \quad (\text{C8})$$

If f is an even polynomial in $\tilde{\varphi}$, then the flow equation is simplified

$$\partial_{\hat{k}} a_{1,k} \simeq \frac{\partial}{\partial \tilde{\varphi}} \left[\frac{1}{1 + \partial_{\tilde{\varphi}}^2 f(\varphi)} \right] \Big|_{\tilde{\varphi}=0} = 0, \quad (\text{C9})$$

$$\partial_{\hat{k}} a_{2,k} \simeq \frac{\partial^2}{\partial \tilde{\varphi}^2} \left[\frac{1}{1 + \partial_{\tilde{\varphi}}^2 f(\varphi)} \right] \Big|_{\tilde{\varphi}=0} = \frac{-\partial_{\tilde{\varphi}}^4 f(\varphi)}{[1 + \partial_{\tilde{\varphi}}^2 f(\varphi)]^2} \Big|_{\tilde{\varphi}=0}, \quad (\text{C10})$$

$$\partial_{\hat{k}} a_{3,k} \simeq \frac{\partial^3}{\partial \tilde{\varphi}^3} \left[\frac{1}{1 + \partial_{\varphi}^2 f(\varphi)} \right] \Big|_{\tilde{\varphi}=0} = 0, \quad (\text{C11})$$

$$\begin{aligned} \partial_{\hat{k}} a_{4,k} &\simeq \frac{\partial^4}{\partial \tilde{\varphi}^4} \left[\frac{1}{1 + \partial_{\varphi}^2 f(\varphi)} \right] \Big|_{\tilde{\varphi}=0} \\ &= \frac{-\partial_{\varphi}^6 f(\varphi)}{[1 + \partial_{\varphi}^2 f(\varphi)]^2} \Big|_{\tilde{\varphi}=0} + 6 \frac{[\partial_{\varphi}^4 f(\varphi)]^2}{[1 + \partial_{\varphi}^2 f(\varphi)]^3} \Big|_{\tilde{\varphi}=0}, \dots \end{aligned} \quad (\text{C12})$$

Therefore, with an initial condition, $a_{1,k} = 0 = a_{3,k}$ at $k = \Lambda$, the flow equations in the above

$$\partial_{\hat{k}} a_{1,k} = 0, \quad \partial_{\hat{k}} a_{3,k} = 0, \quad (\text{C13})$$

guarantee the solution

$$a_{1,k} \equiv 0, \quad a_{3,k} \equiv 0 \quad (0 \leq k \leq \Lambda). \quad (\text{C14})$$

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