

Digitizing the Neveu-Schwarz model on the lightcone worldsheet

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The purpose of this article is to extend the light-cone worldsheet lattice description of string theory to include the Neveu-Schwarz model. We model each component of the fermionic worldsheet field by a critical Ising model. We show that a simple choice of boundary conditions for the Ising variables leads to the half-integer modes required by the model. We identify the G -parity operation within the Ising model and formulate the procedure for projecting onto the even G -parity sector. We construct the lattice version of the three open string vertex, with the necessary operator insertion at the interaction point. We sketch a formalism for summing planar open string multiloop amplitudes, and we discuss prospects for numerically summing them. If successful, the methods described here could provide an alternative to lattice gauge theory for computations in large N QCD.

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I. INTRODUCTION

Non-Abelian gauge theory in 4 space-time dimensions can be regarded as the zero slope limit ($\alpha' \rightarrow 0$) of open string theory [1] in a higher dimension (e.g. $D = 10$), provided that the open strings are required to end on D3-branes [2]. For the conformally invariant $\mathcal{N} = 4$ supersymmetric gauge theory, the AdS/CFT correspondence asserts that, after reinterpreting the planar open string multiloop diagrams as closed string trees, the $\alpha' \rightarrow 0$ limit remains a string theory on the curved background $\text{AdS}^5 \times \text{S}^5$ [3]. In this sense $\mathcal{N} = 4$ is not just the limit of string theory: it is a (closed) string theory. Moreover, the conformal invariance of $\mathcal{N} = 4$ means that the gauge coupling is a true parameter, and this closed string theory can be analyzed semiclassically in the limit of large 't Hooft coupling Ng^2 .

For QCD, however, asymptotic freedom precludes such a parametric semiclassical closed string limit. Even though 't Hooft's large N limit [4] of QCD should still have an interpretation in terms of some kind of closed string background, the absence of a semiclassical approximation to find and analyze that background casts doubt on the practical value of a closed string interpretation in actual calculations. We entertain here the possibility that, in this circumstance, it may be more profitable to forego the equivalence of QCD to some, as yet to be discovered, closed string theory and instead to exploit the known connection of large N QCD to open string theory, keeping $\alpha' > 0$ as a regulator until the end of the analysis. The hope is that some nonperturbative aspects of QCD, such as quark confinement and the meson spectrum in "t Hooft's" large N limit, will be more tractable treated as an open string theory with $\alpha' > 0$ than as a quantum field theory, its $\alpha' \rightarrow 0$ limit.

Several years ago my collaborators and I showed how to represent each gauge theory planar diagram as a light-cone open string worldsheet path integral [5]. This work established a direct formal connection of large N QCD to an open string theory with $\alpha' = 0$. The problem with keeping $\alpha' = 0$, however, is that the UV divergences of quantum field theory are not properly regulated, so that various counterterms must be bought in, order by order in the loop expansion, to cancel UV induced artifacts that violate Lorentz invariance [6]. Keeping $\alpha' > 0$ certainly mitigates these problems, although it remains to be seen whether it completely removes them. In any case the formalism for planar graph summation [7] is on firmer foundation, and is actually somewhat easier to implement, with $\alpha' > 0$.

The large N limit of gauge theory amounts to summing all the planar Feynman diagrams of perturbation theory. These diagrams are the $\alpha' \rightarrow 0$ limit of the planar open string multiloop diagrams. Mandelstam [8] has given a remarkably simple and intuitive representation of the latter diagrams as path integrals over the worldsheets of the light-cone quantized string [9]. Each loop in a given multiloop diagram is represented as an internal worldsheet boundary whose beginning describes a breaking string and whose end describes two strings joining. For each fixed configuration of these internal boundaries, the path integral is a Gaussian integral over the transverse string coordinates and the measure is precisely the natural lattice measure for the quadratic action. Summing over the number, location, and lengths of these internal boundaries with the same measure accomplishes the sum over planar open string diagrams [7].

While there is still some optimism that the large N limit of the maximal supersymmetric ($\mathcal{N} = 4$) gauge theory might be exactly solvable [3], it is doubtful that this will be possible for the pure gauge theory underlying QCD. However, it might well turn out that the representation of the sum of planar diagrams by a light-cone worldsheet

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lattice system, as described in the previous paragraph, can be studied on a computer, perhaps using Monte Carlo algorithms. Such a numerical attack on QCD via its connection to open string theory could have strengths and weaknesses complementary to those of standard lattice gauge theory simulations.

The basic framework for this approach to summing open string planar loop diagrams was set up by Giles and me (GT) over three decades ago [7] in the context of bosonic open string theory in 26-dimensional space-time. The zero slope limit of these diagrams (modulo complications from the presence of the open string tachyon) would be 26- (not 4-) dimensional gauge theory. Recently, I explained how to incorporate D3-branes into the GT formalism [10] in order to arrange zero slope limits that were four dimensional gauge theories, albeit coupled to 22 massless scalars corresponding to vibrations in the extra dimensions. There are mechanisms to decouple such massless scalars (e.g. see [11]), but the difficulties of the open bosonic string tachyon would remain.

The open string Neveu-Schwarz model [12,13], restricted to even G -parity open string states (NS +), has no open string tachyon [14,15], so the multiloop planar diagrams of the $D = 4$ version of NS+ would have a large N 4 dimensional gauge theory as a clean zero slope limit [16]. But since the standard light-cone quantization works only in the critical dimension [17–19], it is simpler to use the ten-dimensional version of the model, employing D3-branes to yield a four dimensional gauge theory. The 6 massless scalars can then be suppressed either by orbifold projections or by using non-Abelian D3-branes [11].

The purpose of this article is to adapt the GT worldsheet lattice formalism to the Neveu-Schwarz model. This requires providing a viable lattice definition of the fermionic worldsheet field $H^\mu(\sigma, \tau)$ of that model. Each (transverse) component of H is a two-dimensional Majorana spinor field. As always, putting fermion fields on a lattice involves difficulties. Besides the inevitable fermion doubling, for which there are a variety of remedies (Wilson fermions, staggered fermions, domain wall fermions), there is the difficulty that Grassmann path integrands do not have a probabilistic interpretation, a prerequisite for Monte Carlo methods. This last difficulty could be dealt with by integrating out the fermions, but that would sacrifice locality, rendering numerical simulations very costly. Fortunately, for the two-dimensional worldsheet, there is another option which we pursue here. This is based on the well-known fact that the physics of the critical two-dimensional Ising model is that of a free Majorana Fermi field. The partition func-

tion of the Ising model is a sum over Ising spins $s_{ij} = \pm 1$ with a positive definite Boltzmann factor, the exponent of which is local in the spins. By replacing each component of H with an Ising spin system, the light-cone lattice formalism for summing the planar diagrams of the Neveu-Schwarz model can be analyzed with Monte-Carlo methods.

In this article we present and study a version of the Ising model which achieves this purpose. In Sec. II we recall the basic features of Onsager's solution of the model which employs a transfer matrix representation. We review the diagonalization of the bulk transfer matrix in terms of anticommuting spin matrices. In Sec. III we turn to the issue of boundary conditions on an open strip, and show that a simple condition on the original Ising spins at the boundary produces the 1/2 integer modes required of the Neveu-Schwarz field H . Section IV deals with the explicit construction of the eigenoperators of the transfer matrix. An important symmetry of the Neveu-Schwarz string is the so-called G parity. In Sec. V we identify the symmetry of the Ising model that becomes G parity in the continuum limit. This is important for imposing the even G -parity restriction in lattice simulations. Unfortunately, imposing this restriction, in the most straightforward way, allows minus signs and/or nonlocality to creep back into the path summand, again posing potential difficulties for Monte-Carlo methods. In Sec. VI, we discuss the 3 open string vertex, which requires an operator insertion at the interaction point. It is argued that, provided the even G -parity restriction is maintained, these insertion factors can be taken into the exponent and interpreted as a modification of the worldsheet action. In Sec. VII we give a brief discussion of the resulting representation of the sum over planar diagrams. Concluding discussion is in a final Sec. VIII.

II. BULK PROPERTIES OF THE ISING MODEL

We consider the two dimensional Ising model on an $M \times N$ lattice specified by the partition function

$$Z = \sum_{\sigma_i^j = \pm 1} e^{\sum_{ij} (J\sigma_i^j\sigma_{i+1}^j + J'\sigma_i^j\sigma_i^{j+1})/2}, \quad (1)$$

where $1 \leq i \leq M$ and $1 \leq j \leq N$. Onsager's transfer matrix representation of Z is

$$Z = (e^{J'} - e^{-J'})^{MN/2} \langle f | \mathcal{T}(M)^N | i \rangle, \quad (2)$$

$$(e^{J'} - e^{-J'})^{M/2} \mathcal{T}(M) = \prod_k (e^{J'/2} + \sigma_k^x e^{-J'/2}) \exp\left\{ \frac{J}{2} \sum_{i=1}^{M-1} \sigma_i^z \sigma_{i+1}^z \right\}, \quad (3)$$

$$\mathcal{T}(M) = \exp\left\{ \frac{\xi}{2} \sum_{i=1}^M \sigma_i^x \right\} \exp\left\{ \frac{J}{2} \sum_{i=1}^{M-1} \sigma_i^z \sigma_{i+1}^z \right\}.$$

$$\tanh \frac{\xi}{2} = e^{-J'}, \quad (4)$$

where the states $|i\rangle, |f\rangle$ belong to an M -fold tensor product of 2-spinors, and they are determined by the boundary conditions at $j = 1, N$, respectively. Here the $\sigma_i^{x,y,z}$ are M independent sets of 2×2 Pauli spin matrices:

$$\{\sigma_i^a, \sigma_j^b\} = 2\delta_{ab}, \quad [\sigma_i^a, \sigma_j^b] = 0, \quad \text{for } i \neq j. \quad (5)$$

An important property of the eigenvalue spectrum of the transfer matrix $\mathcal{T}(M)$ is an easy consequence of the representation (3): it is geometrically symmetric about 1. To see this note that if \mathcal{T} has the eigenvalue T on the state $|T\rangle$, then it has the eigenvalue T^{-1} on the state $e^{\xi \sum_k \sigma_k^{y,z}} \prod_{k=\text{odd}} \sigma_k^x \prod_l \sigma_l^z |T\rangle$. As shown by Onsager, the Ising model has a critical point when $\xi = J$. For the isotropic case $J' = J$, this occurs when $\sinh J = 1$, or $J = \ln(1 + \sqrt{2})$.

As usual, it is most convenient to replace the Pauli matrix dynamical variables with variables that anticommute for different i using the Jordan-Wigner trick

$$S_i^{y,z} \equiv \frac{\sigma_i^{y,z}}{\sqrt{2}} \prod_{k=1}^{i-1} \sigma_k^x, \quad (6)$$

$$\{S_i^a, S_k^b\} = \delta_{ab} \delta_{ik}, \quad (7)$$

whereupon the Onsager transfer matrix becomes

$$\mathcal{T}(M) = e^{-i\xi \sum_{k=1}^M S_k^y S_k^z} e^{-iJ \sum_{k=1}^{M-1} S_{k+1}^z S_k^y}. \quad (8)$$

This choice for \mathcal{T} imposes a particular set of boundary conditions at $k = 1, M$ appropriate for the Ising spins living on an open strip. If they lived on a cylinder, periodic or antiperiodic boundary conditions would be appropriate. We shall show that the boundary conditions chosen here lead to a half-integer moded fermion field in the continuum limit, which is what is needed to describe the Neveu-Schwarz model [12,13].

It is straightforward to calculate the action of \mathcal{T} by conjugation on the S variables:

$$\begin{aligned} \mathcal{T} S_k^z \mathcal{T}^{-1} &= c_J c_\xi S_k^z - i c_J s_\xi S_k^y + i s_J c_\xi S_{k-1}^y - s_J s_\xi S_{k-1}^z, \\ &1 < k \leq M, \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{T} S_k^y \mathcal{T}^{-1} &= c_J c_\xi S_k^y + i c_J s_\xi S_k^z - i s_J c_\xi S_{k+1}^z - s_J s_\xi S_{k+1}^y, \\ &1 \leq k < M, \end{aligned} \quad (10)$$

$$\mathcal{T} S_1^z \mathcal{T}^{-1} = c_\xi S_1^z - i s_\xi S_1^y, \quad (11)$$

$$\mathcal{T} S_M^y \mathcal{T}^{-1} = c_\xi S_M^y + i s_\xi S_M^z, \quad (12)$$

where we have introduced the shorthand notation $c_J \equiv \cosh J$, $s_J \equiv \sinh J$, $t_J \equiv \tanh J$, and similarly for $J \rightarrow \xi$.

We see that \mathcal{T} acts linearly on the S 's, which means that the problem of finding eigenoperators for \mathcal{T} is one of

linear algebra. The form of the right sides of (9) and (10) suggests that an expansion of the S 's in eigenoperators will involve plane wave k dependence:

$$S_k^z = \sum_\lambda A_\lambda e^{i\lambda k}, \quad S_k^y = \sum_\lambda B_\lambda e^{i\lambda k}, \quad (13)$$

$$\mathcal{T}(A_\lambda, B_\lambda) \mathcal{T}^{-1} = t_\lambda(A_\lambda, B_\lambda).$$

We first consider the implications of the bulk recursion relations, focusing on a particular mode λ . Suppressing the λ subscripts on A, B, t , we find

$$\begin{aligned} tA &= c_J c_\xi A - i c_J s_\xi B + i s_J c_\xi B e^{-i\lambda} - s_J s_\xi A e^{-i\lambda}, \\ tB &= c_J c_\xi B + i c_J s_\xi A - i s_J c_\xi A e^{+i\lambda} - s_J s_\xi B e^{+i\lambda}. \end{aligned}$$

The consistency of these two equations requires

$$|t - c_J c_\xi + e^{-i\lambda} s_J s_\xi|^2 = |c_J s_\xi - e^{-i\lambda} s_J c_\xi|^2, \quad (14)$$

which simplifies to

$$t^2 - 2t(c_J c_\xi - s_J s_\xi \cos \lambda) + 1 = 0. \quad (15)$$

Since this equation is even under $\lambda \rightarrow -\lambda$ each of the two solutions,

$$t_\pm = c_J c_\xi - s_J s_\xi \cos \lambda \mp \sqrt{(c_J c_\xi - s_J s_\xi \cos \lambda)^2 - 1}, \quad (16)$$

belong to both left ($\lambda < 0$) and right ($\lambda > 0$) moving plane waves. This degeneracy is important in realizing definite boundary conditions. Also from the equation, it immediately follows that $t_+ t_- = 1$.

The eigenoperators A_a , with eigenvalue t_a , connect eigenstates of \mathcal{T} with different eigenvalues. Because the state space is of finite dimension, there is a maximum eigenvalue of \mathcal{T} , T_{\max} , and a minimum eigenvalue: $T_{\min} = T_{\max}^{-1}$ by the symmetry of the eigenvalue spectrum. Let $|G\rangle$ be the state with the maximum eigenvalue. Then, we must have

$$A_a |G\rangle = 0, \quad \text{whenever } t_a > 1, \quad (17)$$

because if $t_a > 1$, then the eigenvalue of the state on the left would be $t_a T_{\max} > T_{\max}$. Assuming completeness of the eigenoperators, i.e. that monomials of eigenoperators acting on $|G\rangle$ span the whole state space, then the state $\prod_{t_a < 1} A_a |G\rangle$ is the eigenstate with the minimum eigenvalue $T_{\min} = T_{\max} \prod_{t_a < 1} t_a$. It follows that $T_{\max}^2 = \prod_{t_a > 1} t_a$:

$$T_{\max} = \sqrt{\prod_{t_a > 1} t_a}. \quad (18)$$

Identifying $\ln T_{\max} = -E_G$ the ground state energy, we can write this relation as

$$E_G = -\frac{1}{2} \sum_{t_a > 1} \ln t_a = -\frac{1}{2} \sum_{t_a > 1} \omega_a, \quad (19)$$

where we have defined the energy created by an eigenop-

erator as $\omega = -\text{Int}$. Then, since the eigenvalues t come in reciprocal pairs, the ω 's come in opposite sign pairs $\omega_- = -\omega_+$, and we can write $\omega_{\pm} = \pm\omega$, with the convention that $\omega > 0$.

In order that infinitely long waves ($\lambda \rightarrow 0$) cost zero energy, we see from (16) that the Ising system must be critical, that is $\xi = J$. In this critical case we can simplify

$$t_{\pm} = 1 + (1 - \cos\lambda)s_J^2 \mp \sqrt{2(1 - \cos\lambda)s_J^2 + (1 - \cos\lambda)^2 s_J^4}, \quad (20)$$

$$= 1 + 2s_J^2 \sin^2 \frac{\lambda}{2} \mp 2 \left| s_J \sin \frac{\lambda}{2} \right| \sqrt{1 + s_J^2 \sin^2 \frac{\lambda}{2}}, \quad (21)$$

$$\lambda \xrightarrow{\sim} 0 \quad 1 \mp |\lambda s_J|. \quad (22)$$

Once t is determined by consistency, we can then solve for B in terms of A :

$$B = \frac{t - c_J c_{\xi} + e^{-i\lambda} s_J s_{\xi}}{i(e^{-i\lambda} s_J c_{\xi} - c_J s_{\xi})} A \equiv RA. \quad (23)$$

Notice that, according to the consistency condition, the modulus of the numerator of R is equal to the modulus of the denominator, so R is a pure phase. Putting in the explicit forms for t_{\pm} , we get

$$R_{\pm} = \frac{-is_J s_{\xi} \sin\lambda \mp \sqrt{(c_J c_{\xi} - s_J s_{\xi} \cos\lambda)^2 - 1}}{i(e^{-i\lambda} s_J c_{\xi} - c_J s_{\xi})}. \quad (24)$$

Specializing to the critical case, the form simplifies

$$R_{\pm} = e^{i\lambda/2} \frac{-is_J \cos(\lambda/2) \mp \text{sgn}(\lambda) \sqrt{1 + s_J^2 \sin^2(\lambda/2)}}{c_J}, \quad (25)$$

$$\xi = J.$$

When we apply boundary conditions to an open chain in the next section, it will be necessary to consider linear combinations of left and right moving plane waves. Then we will take A, B as the coefficients of $e^{i\lambda k}$ and A', B' as the coefficients of $e^{-i\lambda k}$ with $\lambda > 0$ in both cases. Then Eq. (24) stands and

$$B' = R'A', \quad (26)$$

$$R'_{\pm} = \frac{+is_J s_{\xi} \sin\lambda \mp \sqrt{(c_J c_{\xi} - s_J s_{\xi} \cos\lambda)^2 - 1}}{i(e^{+i\lambda} s_J c_{\xi} - c_J s_{\xi})} = -R_{\pm}^*. \quad (27)$$

For the critical case, these equations become

$$R_{\pm} = e^{i\lambda/2} \frac{-is_J \cos(\lambda/2) \mp \sqrt{1 + s_J^2 \sin^2(\lambda/2)}}{c_J}, \quad (28)$$

$$\xi = J,$$

$$R'_{\pm} = e^{-i\lambda/2} \frac{-is_J \cos(\lambda/2) \pm \sqrt{1 + s_J^2 \sin^2(\lambda/2)}}{c_J} = -R_{\pm}^*, \quad (29)$$

$$\xi = J.$$

III. BOUNDARY CONDITIONS

Now, we turn to the issue of boundary conditions. The simplest open chain model, given by the transfer matrix of (8), will suffice for our purposes. The action of \mathcal{T} on the first (S_1^a) and last (S_M^a) spins (11) and (12) must be made to agree with the bulk recursive formulas (9) and (10) which were satisfied with our plane wave ansatz. This can be arranged by suitable definitions of S_0^a and S_{M+1}^a , which do not appear in \mathcal{T} . The bulk recursions read for $k = 1, M$:

$$\mathcal{T} S_1^z \mathcal{T}^{-1} = c_J c_{\xi} S_1^z - ic_J s_{\xi} S_1^y + is_J c_{\xi} S_0^y - s_J s_{\xi} S_0^z \mathcal{T} S_M^y,$$

$$\mathcal{T}^{-1} = c_J c_{\xi} S_M^y + ic_J s_{\xi} S_M^z - is_J c_{\xi} S_{M+1}^z$$

$$- s_J s_{\xi} S_{M+1}^y.$$

Agreement will occur if we impose

$$(c_J - 1)(c_{\xi} S_1^z - is_{\xi} S_1^y) + s_J(ic_{\xi} S_0^y - s_{\xi} S_0^z) = 0, \quad (30)$$

$$(c_J - 1)(c_{\xi} S_M^y + is_{\xi} S_M^z) + s_J(-ic_{\xi} S_{M+1}^z - s_{\xi} S_{M+1}^y) = 0. \quad (31)$$

Then plugging the plane wave ansatz

$$S_k^z = A e^{i\lambda k} + A' e^{-i\lambda k}, \quad S_k^y = B e^{i\lambda k} + B' e^{-i\lambda k} \quad (32)$$

into these equations and rearranging leads to two different expressions for α defined by $A' = \alpha A$:

$$\alpha = -\frac{e^{i\lambda}(c_J - 1)(c_{\xi} - iR s_{\xi}) + s_J(iR c_{\xi} - s_{\xi})}{e^{-i\lambda}(c_J - 1)(c_{\xi} - iR' s_{\xi}) + s_J(iR' c_{\xi} - s_{\xi})}, \quad (33)$$

$$\alpha = -e^{2iM\lambda} \frac{(c_J - 1)(R c_{\xi} + is_{\xi}) + e^{i\lambda} s_J(-ic_{\xi} - R s_{\xi})}{(c_J - 1)(R' c_{\xi} + is_{\xi}) - e^{-i\lambda} s_J(ic_{\xi} + R' s_{\xi})}. \quad (34)$$

Remembering that R and $R' = -R^*$ have already been determined in (24) and (27), we see that these two equations determine α and a quantization condition on λ from the consistency of the two equations. We are particularly interested in the critical case $\xi = J$, for which the eigenvalue condition simplifies to

$$e^{2iM\lambda} = \frac{(e^{i\lambda} - 1)c_J - e^{i\lambda} - 1}{(e^{i\lambda} - 1)c_J + e^{i\lambda} + 1} = -\frac{1 - ic_J \tan(\lambda/2)}{1 + ic_J \tan(\lambda/2)}$$

$$\equiv e^{i\theta(\lambda)}. \quad (35)$$

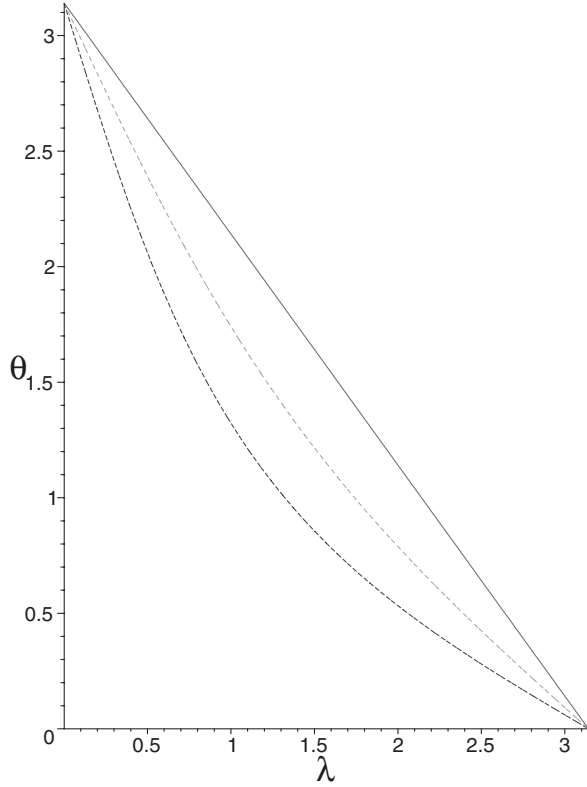


FIG. 1. $\theta(\lambda)$ for the critical case $\xi = J$, and $J = 0$ (highest curve), $J = 1$ (middle curve), and $J = 1.5$ (lowest curve).

We plot $\theta(\lambda)$ in the critical case for three different values of $J = \xi$ in Fig. 1. The graphical solution of (35) is shown in Fig. 2, for $J = 1.5$ and $M = 5$. This graph makes it clear that there are precisely M nontrivial eigenoperator solutions for any value of c_J . The apparently $(M + 1)$ th solution $\lambda = \pi$ is spurious. This is because, in this case, the left and right moving k dependence is identical ($e^{ik\pi} = e^{-ik\pi}$) and the limit $\lambda \rightarrow \pi$ implies $C' \rightarrow -C$ and $D' \rightarrow -D$. Thus the corresponding eigenoperators, proportional to $C + C'$ and $D + D'$, vanish. For comparison, we show $\theta(\lambda)$ and the solution of the eigenvalue equation for several noncritical values $J < \xi$ in Fig. 3.

Applying an eigenoperator to an eigenstate of \mathcal{T} with eigenvalue T_0 produces another eigenstate of \mathcal{T} with eigenvalue tT_0 . We can say that the eigenoperator has increased the energy of the state by an amount $\omega_{\pm} = -\ln t_{\pm} = \pm\omega$. We show $\omega(\lambda)$ for a critical case $\xi = J$ in Fig. 4 and for a noncritical case in Fig. 5. The fact that the eigenvalue spectrum is nondegenerate guarantees that the set of eigenoperators is complete.

The critical case allows a nontrivial continuum limit $M \rightarrow \infty$: In that limit there are finite energy excitations which have $\lambda = O(1/M)$. In this limit the eigenvalue condition simplifies even further to

$$e^{i(2M+c_J)\lambda} = -1 + O(\lambda^2), \quad \text{for } M \rightarrow \infty, \quad \lambda M \text{ fixed.} \quad (36)$$

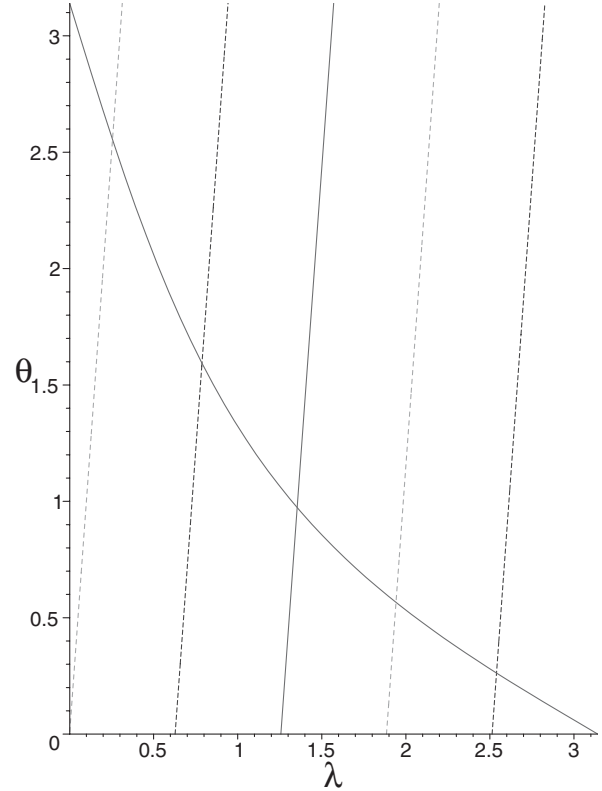


FIG. 2. Solution of (35) for $J = 1.5$. The straight lines are $2M\lambda - 2n\pi$ for $n = 0, 1, \dots, M - 1$ for the case $M = 5$. Note that $\lambda = \pi$ corresponding to $n = M$, though a solution of (35), is spurious as discussed in the text.

or $\lambda = (n + 1/2)\pi/M$, which is the mode quantization of the Neveu-Schwarz worldsheet field.

IV. CONSTRUCTION OF EIGENOPERATORS

In this section we express the eigenoperators in terms of the S 's. We write the eigenoperator as

$$A = \sum_{k=1}^M (S_k^z U_k + S_k^y V_k), \quad \mathcal{T} A \mathcal{T}^{-1} = tA. \quad (37)$$

Then we easily derive the recursion relations satisfied by the U, V :

$$tU_k = c_J c_{\xi} U_k - s_J s_{\xi} U_{k+1} + i c_J s_{\xi} V_k - i s_J c_{\xi} V_{k-1}, \quad k = 2, \dots, M - 1, \quad (38)$$

$$tV_k = c_J c_{\xi} V_k - s_J s_{\xi} V_{k-1} - i c_J s_{\xi} U_k + i s_J c_{\xi} U_{k+1}, \quad k = 2, \dots, M - 1, \quad (39)$$

$$tU_1 = c_{\xi} U_1 - s_J s_{\xi} U_2 + i c_J s_{\xi} V_1, \quad (40)$$

$$tV_1 = -i s_{\xi} U_1 + c_J c_{\xi} V_1 + i s_J c_{\xi} U_2, \quad (41)$$

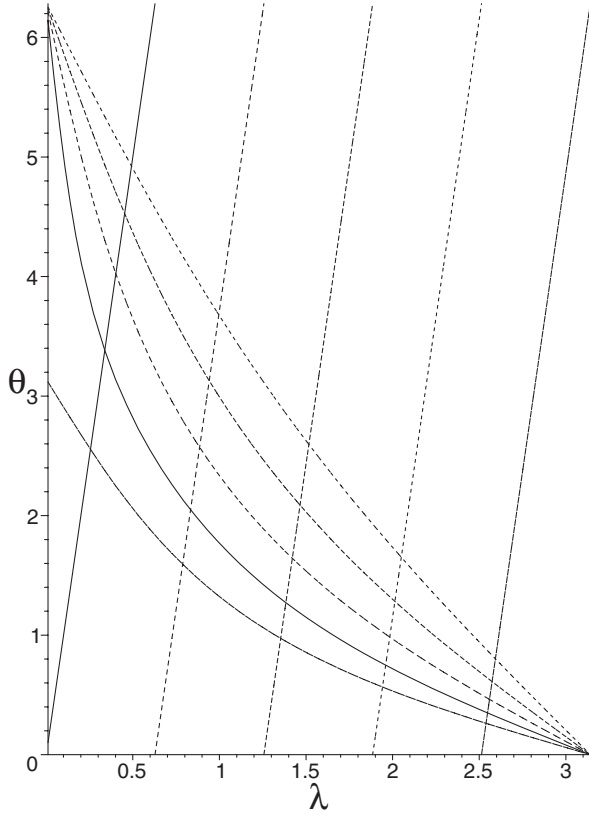


FIG. 3. $\theta(\lambda)$ for some noncritical cases: $\xi = 1.5$ and $J = 0.3, 0.6, 0.9, 1.2$ from the highest to second lowest curves. The lowest curve is the critical case $J = 1.5$, included for comparison. The straight lines are $2M\lambda - 2n\pi$ for $n = 0, 1, \dots, M-1$ for the case $M = 5$. Their intersections with the $\theta(\lambda)$ curves show how the solutions of the eigenvalue equation for the noncritical cases approach the solution for the critical case as $J \rightarrow \xi$.

$$tU_M = c_J c_\xi U_M - i s_J c_\xi V_{M-1} + i s_\xi V_M, \quad (42)$$

$$tV_M = c_\xi V_M - i c_J s_\xi U_M - s_J s_\xi V_{M-1}. \quad (43)$$

The last four special cases will be included in the first two generic equations if we put

$$i s_J V_0 = (c_J - 1) U_1, \quad -i s_J U_{M+1} = (c_J - 1) V_M, \quad (44)$$

which specify the boundary conditions. As before, these relations can be solved with the plane wave ansatz:

$$U_k = C e^{ik\lambda} + C' e^{-ik\lambda}, \quad V_k = D e^{ik\lambda} + D' e^{-ik\lambda}, \quad (45)$$

leading to

$$\frac{D}{C} = -i \frac{t - c_J c_\xi + e^{i\lambda} s_J s_\xi}{c_J s_\xi - e^{-i\lambda} s_J c_\xi}, \quad (46)$$

$$\frac{D'}{C'} = -\frac{D^*}{C^*}, \quad (47)$$

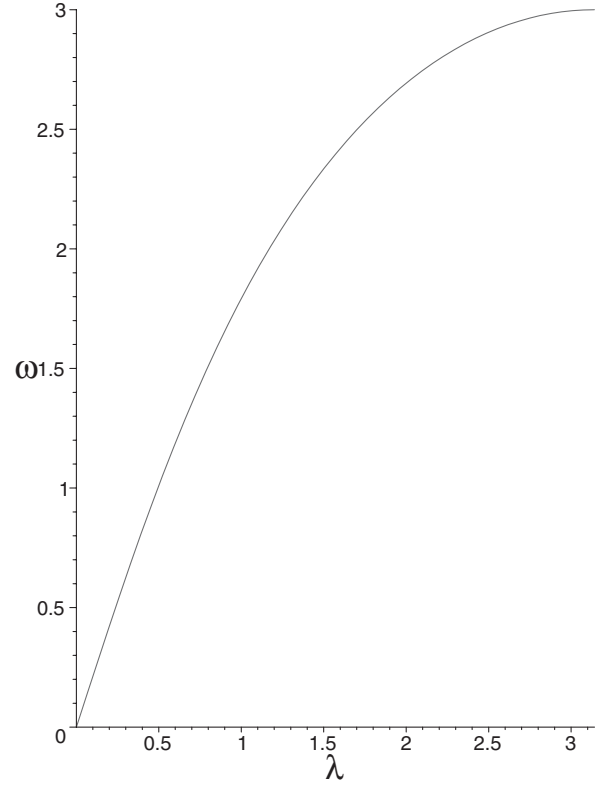


FIG. 4. The excitation energy $\omega(\lambda) = -\ln t_+(\lambda)$ for the critical case $\xi = J = 1.5$.

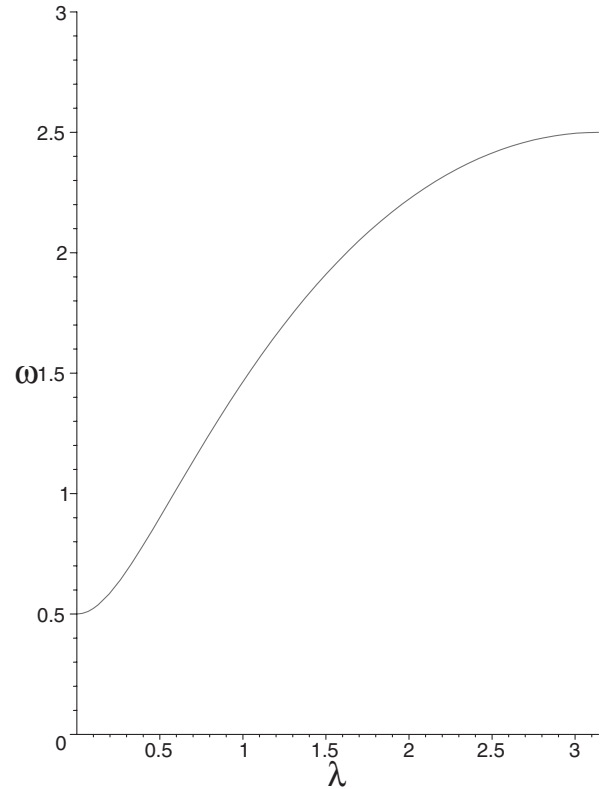


FIG. 5. The excitation energy $\omega(\lambda) = -\ln t_+(\lambda)$ for the non-critical case $J = 1, \xi = 1.5$.

together with the consistency condition

$$t^2 - 2t(c_J c_\xi - s_J s_\xi \cos \lambda) + 1 = 0, \quad (48)$$

which is identical to (15). Note that D/C is similar to B/A , but not identical to it. This is because the matrix we are diagonalizing is not Hermitian. Of course, since the characteristic equation is the same for both eigenvalue problems, the eigenvalue spectrum will be identical. The boundary conditions read

$$is_J(D + D') = (c_J - 1)(Ce^{i\lambda} + C'e^{-i\lambda}), \quad (49)$$

$$\begin{aligned} & -is_J(Ce^{i(M+1)\lambda} + C'e^{-i(M+1)\lambda}) \\ & = (c_J - 1)(De^{iM\lambda} + D'e^{-iM\lambda}), \end{aligned} \quad (50)$$

which lead to

$$\frac{C'}{C} = \frac{is_J(D/C) - (c_J - 1)e^{i\lambda}}{is_J(D^*/C^*) + (c_J - 1)e^{-i\lambda}}, \quad (51)$$

and to the consistency condition

$$e^{2iM\lambda} = \frac{(is_J(D/C) - (c_J - 1)e^{i\lambda})(-is_Je^{-i\lambda} + (c_J - 1)(D^*/C^*))}{(is_J(D^*/C^*) + (c_J - 1)e^{-i\lambda})((c_J - 1)(D/C) + is_Je^{i\lambda})}, \quad (52)$$

$$= \frac{is_J(1 + |D/C|^2) + 2\operatorname{Re}(e^{-i\lambda}D/C) + 2ic_J \operatorname{Im}(e^{-i\lambda}D/C)}{is_J(1 + |D/C|^2) - 2\operatorname{Re}(e^{-i\lambda}D/C) + 2ic_J \operatorname{Im}(e^{-i\lambda}D/C)}, \quad (53)$$

$$= \frac{is_J + \operatorname{Re}(e^{-i\lambda}D/C) + ic_J \operatorname{Im}(e^{-i\lambda}D/C)}{is_J - \operatorname{Re}(e^{-i\lambda}D/C) + ic_J \operatorname{Im}(e^{-i\lambda}D/C)}. \quad (54)$$

The last line follows because one can show that C/D is a pure phase. Plugging the solution for t in the equation for D/C and a little manipulation shows that

$$e^{-i\lambda} \frac{D}{C} = -i \frac{\pm \sqrt{(c_J c_\xi - s_J s_\xi \cos \lambda)^2 - 1} + is_J s_\xi \sin \lambda}{c_J s_\xi \cos \lambda - s_J c_\xi + ic_J s_\xi \sin \lambda}, \quad (55)$$

$$\rightarrow - \frac{\pm \sqrt{1 + s_J^2 \sin^2(\lambda/2)} + is_J \cos(\lambda/2)}{c_J(i \sin(\lambda/2) + \cos(\lambda/2))}, \quad (56)$$

where the last line shows the simplifying critical limit $\xi \rightarrow J$. In this limit we also evaluate

$$\operatorname{Re} \left[e^{-i\lambda} \frac{D}{C} \right] = -\frac{1}{c_J} \cos \frac{\lambda}{2} \left(\pm \sqrt{1 + s_J^2 \sin^2 \frac{\lambda}{2}} + s_J \sin \frac{\lambda}{2} \right), \quad (57)$$

$$s_J + c_J \operatorname{Im} \left[e^{-i\lambda} \frac{D}{C} \right] = \sin \frac{\lambda}{2} \left(\pm \sqrt{1 + s_J^2 \sin^2 \frac{\lambda}{2}} + s_J \sin \frac{\lambda}{2} \right). \quad (58)$$

Then the quantization condition on λ simplifies dramatically to

$$\begin{aligned} e^{2iM\lambda} &= - \frac{\cos(\lambda/2) - ic_J \sin(\lambda/2)}{\cos(\lambda/2) + ic_J \sin(\lambda/2)} \\ &= - \frac{1 - ic_J \tan(\lambda/2)}{1 + ic_J \tan(\lambda/2)}, \end{aligned} \quad (59)$$

which is, of course, identical to (35).

V. G PARITY ON THE LATTICE

In the Neveu-Schwarz model the G parity is a crucial symmetry concept. The fermionic raising and lowering operators $b_r = b_{-r}^\dagger$ are moded with respect to $R = \sum_{r>0} r b_{-r} b_r$ by half odd integers. G parity is defined by $G b_r G = -b_r$, and $G^2 = 1$. Defining fermion number as $N_F = \sum_{r>0} b_{-r} b_r$, G is essentially $(-)^{N_F}$. Because of the circumstance that the b 's carry half odd mode number, we could also write $(-)^{N_F} = (-)^{2R}$ which leads to technical simplifications in analyzing the model. The importance of the symmetry is that, since the interactions conserve G parity, it is consistent to restrict the open string spectrum to be even under G parity. Since the lowest odd G -parity state is a tachyon, and the lowest even G -parity state is a massless vector, the restriction to even G parity removes the tachyon, leaving a massless sector that can describe a gauge boson. With this convention for even and odd, $G = -(-)^{N_F} = -(-)^{2R}$.

There is no lattice analog of N_F that commutes with the transfer matrix of the Ising model. However, there is a candidate for $(-)^{N_F}$, namely $G = \eta \prod_k \sigma_k^x$, with $\eta^2 = 1$. This operator, which anticommutes with $S_k^{y,z}$, commutes with the transfer matrix. The Neveu-Schwarz model in D space-time dimensions requires $D - 2$ transverse components H and therefore $D - 2$ Ising systems, describable by $D - 2$ sets of Pauli matrices $\sigma_{iA}^{x,y,z}$, $A = 1, \dots, D - 2$. Then the total G -parity operator will be $\eta \prod_{kA} \sigma_{kA}^x$. It remains to fix the value of $\eta = \pm 1$. According to the conventions of the Neveu-Schwarz model, we would like it to have the value -1 on the ground state of the system when it is critical. Since the ground state at general J has only been determined implicitly by rather complicated

equations, it would seem a daunting task to evaluate the action of G on it. However, because G takes on only the values ± 1 , its value cannot change under the variation of a continuous parameter. Thus, we can exploit simplifications in the state that occur when $J \rightarrow 0$ at fixed ξ and M .

As we can see from Fig. 3, the eigenvalue problem varies continuously as J decreases continuously from $J = \xi$ to $J = 0$. In the limit $J \rightarrow 0$ the eigenvalue equation smoothly approaches

$$e^{2iM\lambda} = e^{-2i\lambda}, \quad \Rightarrow \lambda = \frac{n\pi}{M+1}, \quad n = 1, \dots, M. \quad (60)$$

Furthermore, the eigenvalues $t_{\pm}(\lambda) \rightarrow e^{\mp\xi}$, independent of lambda, and the eigenoperators approach

$$A_n^{\pm} \rightarrow 2iC \sum_{k=1}^M (S_k^z \pm iS_k^y) \sin \frac{n\pi k}{M+1}. \quad (61)$$

The A_n^- increase the eigenvalue of a state by the factor e^{ξ} . Therefore, in the limit the ground state of the system must be annihilated by all the A_n^- . Since the mode functions $\sin(n\pi k/(M+1))$ are complete, this limiting state must satisfy

$$(S_k^z - iS_k^y)|G_0\rangle = 0, \quad \text{for all } k = 1, \dots, M. \quad (62)$$

But $\sigma_k^x = i\sigma_k^z\sigma_k^y = 2iS_k^zS_k^y$, so it follows that

$$\sigma_k^x|G_0\rangle = 2iS_k^z(-iS_k^z)|G_0\rangle = |G_0\rangle, \quad \text{for all } k, \quad (63)$$

$$\prod_k \sigma_k^x|G_0\rangle = +|G_0\rangle. \quad (64)$$

By continuity, we conclude that the ground state at all $J < \xi$ will have this same eigenvalue. This conclusion depends on the equations determining the ground state not changing discontinuously at some point. For example, if the eigenvalue of an eigenoperator changed from $t > 1$ to $t < 1$ (from $\omega < 0$ to $\omega > 0$) as J was varied, it would no longer annihilate the ground state and vice versa. But the character of the eigenvalue solutions does not allow this to happen, as is evident from Fig. 3. In particular, note that the curve in Fig. 5 stays well away from $\omega = 0$ throughout. We conclude that $\eta = -1$, so the correct G -parity operator to use in describing the Neveu-Schwarz model is $G = -\prod_{kA} \sigma_{kA}^x$.

Finally, we work out the meaning of G -parity in the language of the original Ising partition function (1). To do this, focussing on a single Ising system, we consider the product of the appropriate factors of G with the transfer matrix (3):

$$\prod_k \sigma_k^x \mathcal{T} = (e^{J'} - e^{-J'})^{-M/2} \prod_k (e^{-J'/2} + \sigma_k^x e^{J'/2}) \times \exp\left\{\frac{J}{2} \sum_k \sigma_k^z \sigma_{k+1}^z\right\}, \quad (65)$$

compared to

$$\mathcal{T} = (e^{J'} - e^{-J'})^{-M/2} \prod_k (e^{J'/2} + \sigma_k^x e^{-J'/2}) \times \exp\left\{\frac{J}{2} \sum_k \sigma_k^z \sigma_{k+1}^z\right\}. \quad (66)$$

Thus, inserting a G somewhere in the matrix element (2) has the effect on (1) of multiplying Z by an overall factor (-1) and reversing the sign of *one* of the terms in the sum over j , $J' \sum_j (\sum_{kA} \sigma_{kA}^j \sigma_{kA}^{j+1})/2$. Technically, this is accomplished by inserting a factor $-e^{-J' \sum_{kA} \sigma_{kA}^l \sigma_{kA}^{l+1}}$, for a particular time slice l , in the summand of (1). To project onto even G -parity states, simply insert the factor $(1 - e^{-J' \sum_{kA} \sigma_{kA}^l \sigma_{kA}^{l+1}})/2$. Or perhaps a better way to say it is to make the replacement

$$e^{J' \sum_{kA} \sigma_{kA}^l \sigma_{kA}^{l+1}/2} \rightarrow \sinh\left\{\frac{J'}{2} \sum_{kA} \sigma_{kA}^l \sigma_{kA}^{l+1}\right\} \quad (67)$$

on at least one time slice of each open string propagator. This sinh factor is non local and also can be negative. The nonlocality would add to the computational cost of a Monte Carlo simulation. This is because the probabilistic criterion for retaining an update of the spin on a site k , l requires knowledge of all the other sites on the time slice l , as well as those on neighboring time slices. However, the one-dimensionality of this nonlocality may help keep the cost manageable. The strict probabilistic interpretation of the summand of Z , which is the theoretical basis for Monte Carlo simulations, is marred by the fact that the sinh is negative for some spin configurations. However, the spin configurations where the argument of the sinh is negative are strongly suppressed by the exponential factors on the many other time slices. One can then hope that those configurations will cause a minimal degradation of the convergence of the simulation.

VI. OPEN STRING VERTEX

A striking aspect of string theory is that interactions among strings are inherent in the nature of free string. This is because a single string can make a transition to two strings by simply breaking at a point. Using light-cone quantization of the free string [9] Mandelstam's interacting string formalism [8] provides the most concrete realization of this concept. As illustrated in Fig. 6, Mandelstam's three string vertex is simply a worldsheet path integral with the

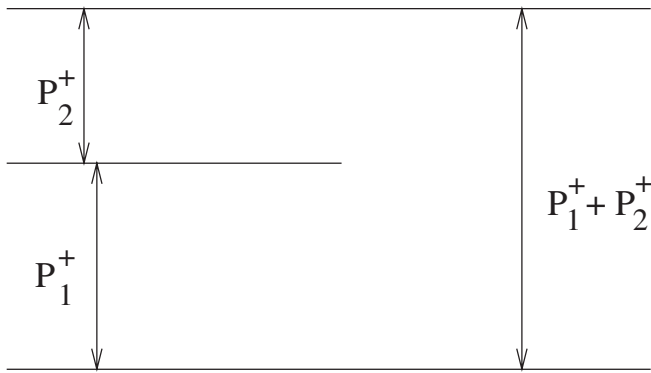


FIG. 6. Light-cone parameter domain for a three open string vertex. Light-cone time τ is the horizontal axis and σ is the vertical axis.

free string action, but for which the worldsheet fields live on a two dimensional domain corresponding to two strings joining ends to become a single string (or the time reversed process). Discretizing this domain on a worldsheet grid, Giles and I [7] clarified the nature of the singularity induced by the string joining/splitting process for the bosonic string in D space-time dimensions. Write $T = aN$, $P_1^+ + P_2^+ \equiv P^+ = MaT_0$, so the diagram has dimensions $P^+ \times T$, and the associated lattice is $M \times N$. We found that, in the continuum limit $M, N \rightarrow \infty$, the 360° corner at the interaction point induced a behavior $M^{-(D-2)/16} \times \text{Finite}$. For the critical dimension, $D = 26$, this scaling behavior accounts precisely for the $(P_1^+ P_2^+ P^+)^{-1/2} = (MaT_0)^{-3/2} (P_1^+ P_2^+ / P^+)^{-1/2}$ factor required by Poincaré invariance.

For the Neveu-Schwarz model, a fermionic worldsheet field, $H^\mu(z) = \sum_r b_r^\mu z^{-r}$ with r ranging over half odd integers, is introduced. Of course, only the transverse components H^k , $k = 1, \dots, D-2$ play a role on the light-cone worldsheet. The main thrust of this article is to represent these fields by $D-2$ independent critical Ising models on the worldsheet lattice. The contribution of the Ising degrees of freedom to the singular behavior at the interaction point can be inferred by realizing that a Majorana fermion is, roughly speaking, a half boson. More precisely, two Majorana fermion worldsheet fields can be interpreted, through bosonization, as a single bosonic worldsheet field. Thus, the singular factor, including Ising and coordinate variables, should be $M^{-3(D-2)/32}$. If this were the end of the story, the Lorentz invariant critical dimension would be $D = 16 + 2 = 18$, not the Neveu-Schwarz critical dimension $D = 10$.

The point is that the Neveu-Schwarz vertex is not simply the overlap represented by the diagram of Fig. 6, but there is also an operator insertion at the joining point [19]. In the continuum limit, the insertion is just the density of the superconformal generators, $\mathbf{H} \cdot \dot{\mathbf{x}}$. Mandelstam showed that if this insertion is placed a distance ϵ from the inter-

action point, then the amplitude $\sim \epsilon^{-3/4}$ as $\epsilon \rightarrow 0$. More generally, if the operator insertion had conformal weight J , the singular behavior would be $\epsilon^{-J/2}$.

We now translate these conclusions to a lattice calculation. Then, there is no need to introduce ϵ : one simply inserts the operator a lattice step or two away from the interaction site. Next, with lattice normalization the insertion operator itself would correspond to a^J times the continuum expression. For example, instead of $\dot{\mathbf{x}}$, one would insert $\mathbf{x}_i^{j+1} - \mathbf{x}_i^j \rightarrow a\dot{\mathbf{x}}$. Likewise, instead of \mathbf{H} which has delta function normalization, one would insert $S_i^j \rightarrow a^{1/2}\mathbf{H}$, which has Kronecker delta normalization. Thus, putting $\epsilon = a$ the continuum analysis of Mandelstam would lead to the behavior $a^J \times a^{-J/2} = a^{J/2}$. Finally, in the lattice setup there is no reference to a , only to the number of lattice sites. Thus, this estimate translates to the behavior $1/M^{J/2}$ or $1/M^{3/4}$ for the case $J = 3/2$ of interest here. Putting this together with the result from the overlap leads to the overall scaling behavior

$$V_{\text{NS}} \sim \left(\frac{1}{M}\right)^{3(D-2)/32+3/4}, \quad (68)$$

which is seen to give the correct Lorentz invariance power $3/2$ for $D = 8 + 2 = 10$ in accord with the known properties of the model. This result has been obtained by translating the analysis done in the continuum theory into expectations for the results of a lattice calculation. It would be very nice to also see it from a direct lattice calculation, but we do not attempt that here.

Now let us look more closely at the operator insertion from the Ising model point of view. We first need to decide which Ising model will describe the $D-2 \equiv d$ Fermi fields $S_{kA}^{y,z}$, $A = 1, \dots, d$. The most straightforward choice would be simply d decoupled Ising models as defined by (2), with each $S_{kA}^{y,z}$ built from the Pauli matrices following (6). Then to make the S_{kA} for different A anticommute, one could define individual G -parity operators $G_A = \prod_{k=1}^M \sigma_{kA}^x$, and include a factor of $\prod_{B=1}^{A-1} G_B$ in the definition of S_{kA} :

$$S_{kA}^{y,z} = \sigma_{kA}^{y,z} \prod_{B=1}^{A-1} G_B \prod_{l=1}^{k-1} \sigma_{lA}^x, \quad \text{JW I.} \quad (69)$$

The trouble with this choice is that the insertion operator $\dot{\mathbf{x}}_A S_A$ will have a residual nonlocality when expressed in terms of the original Ising variables.

There is a better choice, which can be described as follows. Extend the index of $S_k^{y,z}$ to the range $k = 1, \dots, Md$. Then identify the first d of these with S_{1A} , the next d with S_{2A} , etc. Then relate the anticommuting $S_k^{y,z}$ to the Ising $\sigma_k^{y,z}$ by the standard Jordan-Wigner transform (6):

$$S_k^{y,z} = \sigma_k^{y,z} \prod_{l=1}^{k-1} \sigma_l^x, \quad k = 1, \dots, Md, \quad \text{JW II. (70)}$$

The advantage of this version of the Jordan-Wigner trick is that the nonlocality of the insertion operator is subsumed in a factor which is proportional to the G -parity operator for one of the strings entering the vertex. When all strings entering the vertex are restricted to even G parity, the nonlocality in the Ising variables disappears! Thus the insertion will be local in *both* the Pauli matrix and fermionic representations. This is very welcome, because it is the Pauli matrix form that will be more amenable to numerical analysis. On the unrestricted state space, it is only the fermionic representation that is local.

There is a price for this choice however. Since we demand that the transfer matrix expressed in terms of the fermionic spin variables be unchanged, the new Jordan-Wigner transform leads to a more complicated Ising

model. To see how we plug the new Jordan-Wigner transform into the transfer matrix

$$\begin{aligned} \mathcal{T}(M) &= e^{-i\xi \sum_{A=1}^d \sum_{k=1}^M S_{kA}^y S_{kA}^z} e^{-iJ \sum_{A=1}^d \sum_{k=1}^{M-1} S_{k+1,A}^z S_{kA}^y} \\ &= e^{-i\xi \sum_{k=1}^{Md} S_k^y S_k^z} e^{-iJ \sum_{k=1}^{(M-1)d} S_{k+d}^z S_k^y}. \end{aligned} \quad (71)$$

The terms in the exponent of the first factor behave exactly as before:

$$\begin{aligned} -i \sum_{k=1}^{Md} S_k^y S_k^z &= -\frac{i}{2} \sum_{k=1}^{Md} \sigma_k^y \sigma_k^z = \frac{1}{2} \sum_{k=1}^{Md} \sigma_k^x \\ (e^{J'} - e^{-J'})^{Md/2} e^{-i\xi \sum_{k=1}^{Md} S_k^y S_k^z} &= \prod_{k=1}^{Md} (e^{J'} + \sigma_k^x e^{-J'}). \end{aligned} \quad (72)$$

However, the exponent of the second factor changes substantially:

$$\begin{aligned} -iJ \sum_{k=1}^{(M-1)d} S_{k+d}^z S_k^y &= \frac{J}{2} \sum_{k=1}^{(M-1)d} \sigma_{k+d}^z \sigma_k^y \prod_{l=k+1}^{k+D-3} \sigma_l^x \\ (e^{J'} - e^{-J'})^{Md/2} \mathcal{T}(M) &= \prod_{k=1}^{Md} (e^{J'/2} + \sigma_k^x e^{-J'/2}) \exp\left\{ \frac{J}{2} \sum_{k=1}^{(M-1)d} \sigma_{k+d}^z \sigma_k^y \prod_{l=k+1}^{k+d-1} \sigma_l^x \right\}. \end{aligned} \quad (73)$$

Only for $d = 1$ does this reduce to the usual Ising model. The form of the Ising model corresponding to this transfer matrix is worked out in the Appendix.

VII. SUMMING PLANAR LOOPS

Once the three open string vertex has been established as in the previous section, the complete perturbation series is determined [8,19]. A generic planar multiloop diagram in the series is the worldsheet path integral using the free open string worldsheet action, but with the worldsheet variables living on a domain such as depicted in Fig. 7.

In order to digitize the sum over planar diagrams, we fix the overall dimensions of the domain to $P^+ \times T$, and then set up a worldsheet grid of dimensions $M \times N$, with $T = aN$ and $P^+ = aT_0 M$ [7]. The discretized worldsheet action is constructed so that the internal horizontal lines shown in Fig. 7 represent open string boundaries. For the Neveu-Schwarz model, the worldsheet variables are the $d = D - 2$ transverse coordinates $\mathbf{x}(\sigma, \tau) \rightarrow \mathbf{x}_i^j$ and the d fermionic fields $\sqrt{a} \mathbf{H}(\sigma, \tau) \rightarrow \mathbf{S}_i^j$ where the \mathbf{S}_i^j are taken to be the Jordan-Wigner transformed Pauli spin matrices of d independent Ising models. For each end of a horizontal line, which depicts the breaking or joining of open strings, there is a factor of the open string coupling g and also the operator insertion $\mathcal{S}_i^j \cdot (\mathbf{x}_i^{j+1} - \mathbf{x}_i^j)$ as explained in the previous section. The precise location of this insertion is somewhat flexible, as long as it is within one or two lattice steps from the end of the horizontal line. For definiteness,

we will always place it on the longest string participating in the vertex, with i the spatial location of the horizontal line, and $j - 1$ or $j + 2$ the time of the end of the horizontal line, with the choice determined so that the insertion lies completely on the longest string participating in the vertex.

Next, we turn to G -parity restrictions. The vertex obtained in the previous section, and adopted in this section,

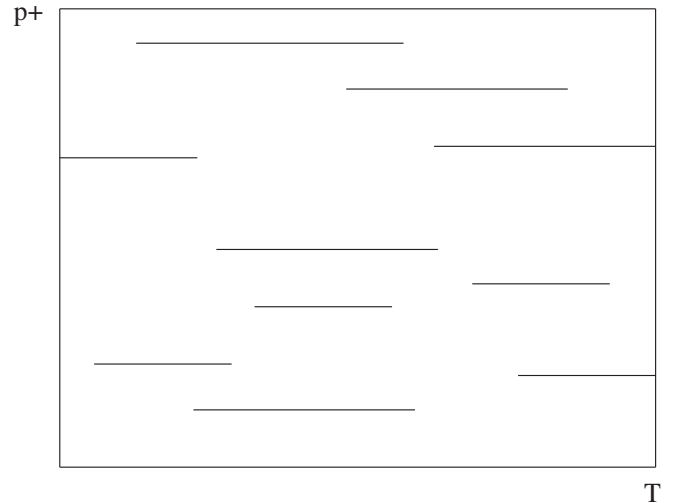


FIG. 7. A planar multiloop light-cone interacting string diagram. The horizontal lines form the boundaries of the propagator worldsheets for intermediate open strings. This diagram has 7 loops and 5 external strings.

conserves G parity: it connects 3 even G -parity open strings with each other or 2 odd G -parity strings to an even one. We would like to restrict the open string states to even G parity only. When a diagram involves one or more loops, it is not sufficient to restrict the external states to even G parity, because a pair of odd G -parity states can be produced from an even G -parity state. Thus each internal propagator in a multiloop diagram such as depicted in Fig. 7 must include a projector $(1 + G)/2$ onto the even G -parity sector.

As we have discussed in the previous section, the presence of even G -parity projectors throughout the diagram introduces nonlocality into the worldsheet dynamics. The projectors also produce negative contributions to the path integrand. These could potentially lead to inefficiencies and inaccuracies in the applications of Monte Carlo algorithms to this system. However, there are beneficial aspects of the presence of the projectors. One, already mentioned, is that restriction to the even G -parity states renders the nonlocality in the relation between the S 's and the σ 's harmless. Thus, the operator insertions, necessary to describe the Neveu-Schwarz model, will have a local representation in terms of the original Ising variables. Even so, the insertions are awkward because they are indefinite in sign and have no obvious interpretation as terms in the worldsheet action.

It would be nice if these factors could be taken into the exponent where they would become a modification of the action.¹ If we do that and then expand the exponential, the effect would be to replace the desired insertion with a sum of all possible powers of the insertion, including a term with no insertion at all. The higher powers are innocuous because they will either renormalize the zeroth and first powers or introduce operators of higher conformal weight which would be suppressed relative to the zeroth and first powers in the continuum limit. It is the zeroth term that poses the difficulty. In the continuum limit it would dominate over the desired linear term. On the other hand, it does not couple 3 even G -parity open strings together. Thus, if the even G -parity restriction is enforced throughout, this troublesome term will not contribute. So by including the projectors, we enable the interpretation of the operator insertions as modifications of the action.

VIII. CONCLUDING REMARKS

In this article we have extended the GT lattice light-cone string formalism to include the Neveu-Schwarz model, representing the Fermi fields via Ising spin systems. That representation has the virtue of avoiding the minus signs

¹When H is represented as a Grassmann variable this procedure would have the bizarre effect of adding Grassmann odd terms to the action. However, in the Ising model representation, it would simply mean adding terms linear in the spin to the action—no weirder than an external magnetic field.

inherent in a description of fermions as Grassmann fields. If there were no need to include operator insertions at the interaction points and no need to make the even G -parity restriction, the formalism would be ready for immediate analysis via Monte Carlo simulations.

Alas, that would give neither the Neveu-Schwarz (NS) model nor its even G -parity projected descendent (NS +). Indeed, the simulations would be dominated by tachyon effects, most likely yielding the same sort of almost trivial outcome as the bosonic string [20], namely, the sum of planar open string diagrams would produce the propagator of a free closed string. What makes the NS+ model dynamically interesting is precisely what makes its Monte Carlo analysis problematic: the operator insertions in the NS model are of indefinite sign and the projectors necessary to describe the NS+ model are nonlocal. Intriguingly, for the NS+ model, we argued that the restriction to even G -parity provides a resolution to the operator insertion difficulty. In its present status, all of the sign and nonlocality problems of the formalism reside entirely in the projection procedure. We think it is some progress to be able to attribute all of the nonlocality and negativity to such a well-defined source, in which they may turn well out to be relatively benign. In any case, we suspect that a better formulation will be found to circumvent any difficulties that remain.

Turning to more mundane matters, we recall that our main motivation for keeping $\alpha' > 0$ was to mitigate the need for uncontrolled counterterms to maintain Lorentz invariance in the loop expansion. While this should effectively deal with the usual field theoretic UV divergences, it does not by itself take care of the potential need for the worldsheet contact counterterms studied, for example, in [21]. This issue clearly needs further study. However, it is comforting to know that the fundamentally stringy regularization identified in [22] seems to be at least partially realized by the GT light-cone lattice. This mechanism works by regarding each open string loop on the worldsheet as an emission or absorption of a closed string in the vacuum: giving that closed string a nonzero momentum regulates the worldsheet divergence. By studying the one loop open string self energy, we have found that discretizing $P^+ = Ma$, which is the discretization of GT light-cone lattice, has a similar regulating effect as $p \neq 0$. For this example, it guarantees that the open string gluon remains massless. If this continues to happen in more complicated multiloop processes, we will be in business.

ACKNOWLEDGMENTS

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APPENDIX: ALTERNATE DESCRIPTION OF d ISING SPIN SYSTEMS

Here we work out the Ising model corresponding to the transfer matrix (73). The way to do this is to evaluate the transfer matrix elements between basis states which are

eigenstates of the σ_k^z . Then \mathcal{T}^N can be expressed as sums over these eigenvalues. However, it is more convenient to factor \mathcal{T} into $d = D - 2$ factors, and work out the matrix elements of each factor. So write

$$(e^{J'} - e^{-J'})^{Md/2} \mathcal{T}(M) = \prod_{m=1}^d \left[\prod_{k=1}^M (e^{J'/2} + \sigma_{m+(k-1)d}^x e^{-J'/2}) \prod_{k=1}^{M-1} \exp \left\{ \frac{J}{2} \sigma_{m+kd}^z \sigma_{m+(k-1)d}^z \prod_{l=m+1+(k-1)d}^{m-1+kd} \sigma_l^x \right\} \right], \quad (\text{A1})$$

and we evaluate the matrix elements of each T_m , where

$$\begin{aligned} T_m &= \prod_{k=1}^M (e^{J'/2} + \sigma_{m+(k-1)d}^x e^{-J'/2}) \prod_{k=1}^{M-1} \exp \left\{ \frac{J}{2} \sigma_{m+kd}^z \sigma_{m+(k-1)d}^z \prod_{l=m+1+(k-1)d}^{m-1+kd} \sigma_l^x \right\} \\ &= \prod_{k=1}^M (e^{J'/2} + \sigma_{m+(k-1)d}^x e^{-J'/2}) \prod_{k=1}^{M-1} \left\{ \cosh \frac{J}{2} + \sigma_{m+kd}^z \sigma_{m+(k-1)d}^z \prod_{l=m+1+(k-1)d}^{m-1+kd} \sigma_l^x \sinh \frac{J}{2} \right\}. \end{aligned} \quad (\text{A2})$$

Notice that in the last expression no σ^x appears more than linearly. Thus we can use the identities

$$\langle \sigma^l | \sigma^x | \sigma \rangle = \frac{1 - \sigma^l \sigma}{2}, \quad \langle \sigma^l | I | \sigma \rangle = \frac{1 + \sigma^l \sigma}{2}, \quad (\text{A3})$$

where $|\sigma\rangle$ is an eigenstate of σ^z with eigenvalue σ . Then a few minutes thought leads to

$$\begin{aligned} \langle \{\sigma^l\} | T_m | \{\sigma\} \rangle &= \exp \left\{ \frac{J'}{2} \sum_{k=1}^M \sigma_{m+(k-1)d}^l \sigma_{m+(k-1)d} \right\} \prod_{k=1}^{M-1} \left[\prod_{l=m+1+(k-1)d}^{m-1+kd} \frac{1 + \sigma_l^l \sigma_l}{2} \cosh \frac{J}{2} \right. \\ &\quad \left. + \sigma_{m+kd} \sigma_{m+(k-1)d} \prod_{l=m+1+(k-1)d}^{m-1+kd} \frac{1 - \sigma_l^l \sigma_l}{2} \sinh \frac{J}{2} \right]. \end{aligned} \quad (\text{A4})$$

The matrix element of the full transfer matrix is then

$$\langle \{\sigma^l\} | \prod_{m=1}^d T_m | \{\sigma\} \rangle = \sum_{\{\{\sigma_i\}\}} \langle \{\sigma^l\} | T_1 | \{\sigma_1\} \rangle \langle \{\sigma_1\} | T_2 | \{\sigma_2\} \rangle \cdots \langle \{\sigma_{D-3}\} | T_d | \{\sigma\} \rangle. \quad (\text{A5})$$

In effect we can think of each actual time slice as d coincident time slices, so that the computational scale of this new version of the Ising model would correspond to an $M \times dN$ lattice as compared to an $M \times N$ grid of the simple Ising model. And of course as complicated as this formulation seems, it has identical physics to d decoupled Ising models. The only reason for suffering these extra complications is that in the application to the interacting NS+ model, the insertion operators will be local in the new representation, whereas they would be nonlocal in the original formulation.

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