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Gravity as the square of gauge theory

Zvi Bern,[1] Tristan Dennen,[1] Yu-tin Huang,[1] and Michael Kiermaier[2]

[1]*Department of Physics and Astronomy, UCLA, Los Angeles, California 90095-1547, USA*

[2]*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA*

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We explore consequences of the recently discovered duality between color and kinematics, which states that kinematic numerators in a diagrammatic expansion of gauge-theory amplitudes can be arranged to satisfy Jacobi-like identities in one-to-one correspondence to the associated color factors. Using on-shell recursion relations, we give a field-theory proof showing that the duality implies that diagrammatic numerators in gravity are just the product of two corresponding gauge-theory numerators, as previously conjectured. These squaring relations express gravity amplitudes in terms of gauge-theory ingredients, and are a recasting of the Kawai, Lewellen, and Tye relations. Assuming that numerators of loop amplitudes can be arranged to satisfy the duality, our tree-level proof immediately carries over to loop level via the unitarity method. We then present a Yang-Mills Lagrangian whose diagrams through five points manifestly satisfy the duality between color and kinematics. The existence of such Lagrangians suggests that the duality also extends to loop amplitudes, as confirmed at two and three loops in a concurrent paper. By "squaring" the novel Yang-Mills Lagrangian we immediately obtain its gravity counterpart. We outline the general structure of these Lagrangians for higher points. We also write down various new representations of gauge-theory and gravity amplitudes that follow from the duality between color and kinematics.

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I. INTRODUCTION

A key lesson from studies of scattering amplitudes is that weakly coupled gauge and gravity theories have a far simpler and richer structure than is evident from their usual Lagrangians. A striking example of this is Witten's remarkable conjecture that scattering amplitudes in twistor space are supported on curves of a degree controlled by their helicity and loop order [1]. At weak coupling another remarkable structure visible in on-shell tree amplitudes is the Kawai-Lewellen-Tye (KLT) relations, which express gravity tree-level amplitudes as sums of products of gauge-theory amplitudes [2,3]. These relations were originally formulated in string theory, but hold just as well in field theory. In fact, in many cases, they hold even when no string theory lives above the field theory [4].

The KLT relations have recently been recast into a much simpler form in terms of numerators of diagrams with only three-point vertices. In the new representation the diagrammatic numerators in gravity are simply a product of two corresponding gauge-theory numerators [5]. Underlying these numerator "squaring relations" is a newly conjectured duality between kinematic numerators of gauge theory and their associated color factors, by Carrasco, Johansson, and one of this paper's authors (BCJ). The BCJ duality states that gauge-theory amplitudes can be nontrivially rearranged into a form where diagrammatic numerators satisfy a set of identities in one-to-one correspondence to the Jacobi identities obeyed by color factors. The duality appears to hold in large classes of theories including pure Yang-Mills theory and $\mathcal{N} = 4$ super-Yang-Mills (SYM). At four points the duality is automatically satisfied, as noted 30 years ago [6] to explain certain zeros in cross sections. BCJ also conjectured that the numerators of gravity diagrams are simply the product of two corresponding gauge-theory numerators that satisfy the duality. These squaring relations were verified in Ref. [5] at tree level up to eight points. Interestingly, the duality also leads to a set of nontrivial relations between gauge-theory amplitudes [5], which are now well understood in string theory [7]. The numerator duality relations have also been understood from the vantage point of string theory [8–10]. In particular, the heterotic string offers important insight into these relations, because of the parallel treatment of color and kinematics [9].

In this paper we describe two complementary approaches to developing a field-theory understanding of the duality between color and kinematics, and its relation to gravity as two copies of gauge theory. In the first approach we use Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion relations to prove that the squaring relations are satisfied if the numerators of gauge-theory diagrams satisfy the BCJ duality. Our proof is inductive, starting with three points where it is simple to verify the double-copy property for candidate gravity theories. For higher points, we apply the BCFW recursion relations to gauge-theory amplitudes whose numerators are arranged to satisfy the duality. The BCFW recursion relations, however, in general do not respect the duality. This requires us to apply a "generalized gauge transformation" to rearrange terms in the recursion relation in a way that restores the duality, which is a key ingredient in our proof. These generalized gauge transformations correspond to the most general rearrangements of amplitude numerators that do not alter their values. (Such transformations need not correspond to gauge transformations in the traditional sense.) To apply this to gravity we make use of the fact that BCFW recursion relations for color-dressed gauge-theory amplitudes [11] are closely related to the gravity ones [12]. By also applying a generalized gauge transformation to the BCFW recursion relation in gravity, we show that the squaring relations indeed reproduce the gravity amplitude correctly.

The generalized gauge invariance contains an enormous freedom in rearranging amplitudes, and for some rearrangements the squaring relations between gravity and gauge theory hold. Such generalizations of the squaring relations at five points were discussed in Refs. [9,10]. Here we present an all-n generalization of the squaring relations given in an asymmetric form, in which only one of the two sets of gauge-theory numerators is required to satisfy the BCJ duality.[1]

In our second approach to understand the color-kinematics duality, we use a more traditional Lagrangian viewpoint. A natural question is: what Lagrangian generates diagrams that automatically satisfy the BCJ duality? We shall describe such a Lagrangian here, and present its explicit form up to five points, leaving the question of the more complicated explicit higher-point forms to the future. We have also worked out the six-point Lagrangian and outline its structure, and make comments about the all-orders form of the Lagrangian. We find that a covariant Lagrangian whose diagrams satisfy the duality is necessarily nonlocal. We can make this Lagrangian local by introducing auxiliary fields. Remarkably we find that, at least through six points, the Lagrangian differs from ordinary Feynman gauge simply by the addition of an appropriate zero, namely, terms that vanish by the color Jacobi identity. Although the additional terms vanish when summed, they appear in diagrams in just the right way so that the BCJ duality is satisfied. Based on the structures we find, it seems likely that any covariant Lagrangian where diagrams with an arbitrary number of external legs satisfy the duality must have an infinite number of interactions.

In Ref. [14], the problem was posed of how to construct a Lagrangian that reflects the double-copy property of gravity. That reference carried out some initial steps, showing that one can factorize the graviton indices into ‘‘left’’ and ‘‘right’’ classes consistent with the factorization observed in the KLT relations. (See also Ref. [15].) Unfortunately, beyond three points the relationship of the constructed gravity Lagrangian to gauge theory was rather obscure. As it turns out, a key ingredient was missing: the duality between color and kinematics, which was discovered much later [5]. Using the modified local version of the gauge-theory Lagrangian whose Feynman diagrams respect the BCJ duality, we construct a Lagrangian for gravity valid through five points, as a double copy of the gauge-theory one. The likely appearance of an infinite number of interactions in the modified gauge-theory Lagrangian is perhaps natural, because we expect any covariant gravity Lagrangian to also have an infinite number of terms.

The unitarity method [16] immediately implies that gravity loop amplitudes must have the double-copy property, if the corresponding gauge-theory loop amplitudes can be put in a form that satisfies the BCJ duality, as does indeed appear to be the case [13]. The squaring relations then apply to gravity numerators for *any* value of loop momenta, i.e. with no cut conditions applied. This is to be contrasted with the KLT relations, which are valid only at tree level, and can be applied at loop level only on unitarity cuts that decompose loop amplitudes into tree amplitudes [17]. The KLT relations take a different functional form for every cut of a given amplitude, depending on the precise tree-amplitude factors involved in the cut. The squaring relations, on the other hand, take a simple universal form for any choice of loop momenta.

We also present a simple application of the BCJ duality. Since the BCJ duality states that diagrammatic numerators have the same algebraic structure as color factors, we can immediately make use of different known color representations of amplitudes to write dual formulas where color and kinematic numerators are swapped. In particular, Del Duca, Dixon, and Maltoni [18] have given a color decomposition of tree amplitudes using adjoint-representation color matrices. They derived this color decomposition using the color Jacobi identity and Kleiss-Kuijf relations [19]. By swapping color and numerator factors in their derivation, we immediately obtain novel forms of both gauge-theory and gravity tree amplitudes.

This paper is organized as follows. In Sec. II we review BCJ duality. We discuss invariances of the amplitudes in Sec. III, and also present a new asymmetric form of the squaring relations. We give our proof that the kinematic diagrammatic numerators for gravity are double copies of the gauge-theory ones in Sec. IV. Then, in Sec. V, we turn to the question of constructing a gauge-theory Lagrangian that generates Feynman diagrams that respect the BCJ duality. A few simple implications of BCJ duality are given in Sec. VI.

II. REVIEW OF BCJ DUALITY

A. General considerations

Consider a gauge-theory amplitude, which we write in a diagrammatic form,

[1] H. Johansson independently realized that only one set of numerators needs to satisfy the BCJ duality to obtain gravity from the numerator squaring relation; see Ref. [13] for a nontrivial loop-level application.

$$\frac{1}{g^{n-2}}\mathcal{A}_n^{\rm tree}(1,2,3,\ldots,n) = \sum_{{\rm diags.}\,i}\frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{2.1}$$

where the sum runs over all diagrams i with only three-point vertices, the c_i are color factors, the n_i are kinematic numerators, and the s_{α_i} are the inverse propagators associated with the channels α_i of the diagram i. Any gauge-theory amplitude can be put into this form by replacing contact terms with numerator factors canceling propagators, i.e. s_α/s_α and assigning the contribution to the proper diagram according to the color factor. The value of the color coefficient c_i of each term is obtained from the diagram i by dressing each three-point vertex with a structure constant $\tilde{f}^{abc}$, where

$$\tilde{f}^{abc} \equiv i\sqrt{2}f^{abc} = {\rm Tr}([T^a, T^b]T^c), \tag{2.2}$$

and dressing each internal line with δ^{ab}.

A key property of the $\tilde{f}^{abc}$ is that they satisfy the Jacobi identity. Consider, for example, the color factors of the three diagrams illustrated in Fig. 1. They take the schematic form,

$$c_s \equiv \ldots \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}\ldots, \qquad c_t \equiv \ldots \tilde{f}^{a_1a_4b}\tilde{f}^{ba_2a_3}\ldots,$$
$$c_u \equiv \ldots \tilde{f}^{a_1a_3b}\tilde{f}^{ba_4a_2}\ldots, \tag{2.3}$$

where the "…"s signify factors common to all three diagrams. The color factors then, of course, satisfy the Jacobi identity

$$c_s + c_t + c_u = 0. \tag{2.4}$$

Here we have chosen a sign convention[2] that differs from Ref. [5].

The BCJ conjecture states that numerators n_i can always be found that satisfy Jacobi relations in one-to-one correspondence with the color Jacobi identities,

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0, \tag{2.5}$$

where i, j, and k label diagrams whose color factors are related by a Jacobi identity. (In general the relative signs between the color factors in all Jacobi identities cannot be taken to be globally positive, but according to the BCJ conjecture the relative signs always match between the color and kinematic identities.) In addition, BCJ duality also requires that the n_i satisfy the same self-antisymmetry relations as the c_i. That is, if a color factor is antisymmetric under an interchange of two legs, the corresponding numerator satisfies the same antisymmetry relations,

$$c_i \to -c_i \Rightarrow n_i \to -n_i. \tag{2.6}$$

We note that when the color-ordered partial amplitudes are expressed in terms of numerators satisfying these self-antisymmetry relations, they automatically satisfy the Kleiss-Kuijf relations [18,19] between color-ordered partial amplitudes [5]. Here we will also assume that local numerators exist which satisfy the BCJ duality. For pure Yang-Mills amplitudes through six points, we have confirmed the existence of such numerators by explicitly constructing them.

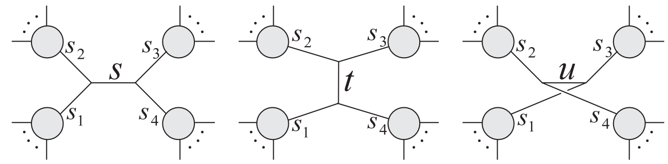

FIG. 1. A Jacobi relation between color factors of diagrams. According to the BCJ duality the diagrammatic numerators of amplitudes can be arranged so that they satisfy relations in one-to-one correspondence to the color Jacobi identities.

B. Five-point example and generalized Jacobi-like structures

Consider the five-point case as a simple example, discussed already in some detail from various viewpoints in Refs. [5,8–10]. At 5 points there are 15 numerators and 9 independent duality relations, leaving 6 numerators. Of these remaining numerators, 4 can be chosen arbitrarily due to a "generalized gauge invariance." By choosing the remaining two n_i to correctly give two of the partial amplitudes, nontrivial relations between color-ordered amplitudes can be derived from the condition that the remaining partial amplitudes are also reproduced correctly. For example,

$$s_{35}A_5^{\rm tree}(1,2,4,3,5) - (s_{13}+s_{23})A_5^{\rm tree}(1,2,3,4,5) - s_{13}A_5^{\rm tree}(1,3,2,4,5) = 0. \tag{2.7}$$

This relation has generalizations for an arbitrary number of external legs [5], which have been derived using string theory [7].

As discussed in Refs. [9,10], Eq. (2.7) is equivalent to a relation that exhibits a Jacobi-like structure,

$$\frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_{11} + n_{13}}{s_{24}} - \frac{n_3 - n_1 + n_{12}}{s_{12}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} = 0, \tag{2.8}$$

where only the sum over terms is required to vanish. In this equation, the Jacobi-like structure involves additional minus signs because we follow the sign conventions given in Ref. [5] for the expansion of the five-point amplitude,

[2]In any given Jacobi relation the relative signs are arbitrary since they always can be moved between color factors and kinematic numerators.

$$A_5^{\rm tree}(1,2,3,4,5) \equiv \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}},$$
$$A_5^{\rm tree}(1,2,4,3,5) \equiv \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}},$$
$$A_5^{\rm tree}(1,3,2,4,5) = \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{23}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{23}} - \frac{n_{11}}{s_{51}s_{24}}. \tag{2.9}$$

As explained in Ref. [9,10], relations of the form (2.8) are the natural gauge-invariant numerator identities that emerge from string theory. Because of the generalized gauge invariance, these relations are less stringent than the BCJ duality. Indeed, the individual terms in Eq. (2.8) are not required to vanish, but only their sum. We note that the heterotic string offers some important insight into the BCJ duality (2.5): in the heterotic string both color and kinematics arise from world-sheet fields, making the duality more natural [9]. Identities of the form (2.8), though interesting, will not play a role in the analysis below. In the remainder of this paper we will only be concerned with numerators n_i that satisfy the more stringent BCJ-duality requirements of Eq. (2.5).

C. Gravity squaring relations

Another conjecture in Ref. [5] is that gravity tree amplitudes can be constructed directly from the n_i through "squaring relations." Consider two gauge-theory amplitudes,

$$\frac{1}{g^{n-2}}\mathcal{A}_n^{\rm tree}(1,2,3,\ldots,n) = \sum_{{\rm diags.}\,i}\frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}},$$
$$\frac{1}{\tilde g^{n-2}}\tilde{\mathcal{A}}_n^{\rm tree}(1,2,3,\ldots,n) = \sum_{{\rm diags.}\,i}\frac{\tilde n_i \tilde c_i}{\prod_{\alpha_i} s_{\alpha_i}}. \tag{2.10}$$

These two amplitudes do not have to be from the same theory, and can have differing gauge groups and particle contents. In Ref. [5] the requirement that *both* the n_i and the $\tilde n_i$ satisfy the BCJ duality was imposed, i.e. they satisfy all duality conditions $n_i + n_j + n_k = 0$ and $\tilde n_i + \tilde n_j + \tilde n_k = 0$. The conjectured squaring relations state that gravity amplitudes are given simply by

$$\frac{-i}{(\kappa/2)^{n-2}}\mathcal{M}_n^{\rm tree}(1,2,3,\ldots,n) = \sum_{{\rm diags.}\,i}\frac{n_i \tilde n_i}{\prod_j s_{\alpha_i}}, \tag{2.11}$$

where the sum runs over the same set of diagrams as in Eq. (2.10). The states appearing in the gravity theory are just direct products of gauge-theory states, and their interactions are dictated by the product of the gauge-theory momentum-space three-point vertices. The squaring relations (2.11) were explicitly checked through eight points and have recently been understood from the KLT relations in heterotic string theory [9].

Using standard factorization arguments it is simple to see why one would expect the BCJ duality to imply that gravity numerators are a double copy of gauge-theory numerators. Let us assume that the numerators of all n-point gauge-theory amplitudes (2.1) satisfy the BCJ duality (2.5). Let us also assume that we have already proven that the squaring relations (2.11) hold for amplitudes with fewer legs. Consider an ansatz for the n-point graviton amplitude given in terms of diagrams by the double-copy formula (2.11). We now step through all possible factorization channels using real momenta. By general field-theory considerations we know that in each channel the diagrams break up into products of lower-point diagrams. The sum over diagrams on each side of the factorization pole forms a lower-point amplitude. Since each numerator factor of the n-point expression satisfies the duality condition, we expect the lower-point tree diagrams on each side of the factorized propagator to inherit this property when we choose special kinematics to factorize a diagram. Thus on each side of the pole we have a correct set of double-copy numerators for the lower-point gravity amplitudes. Stepping through all factorization channels we see that we have correct diagram-by-diagram factorizations in all channels. This provides a strong indication that the double-copy property follows from BCJ duality. In Sec. IV, we will make this conclusion rigorous using a BCFW construction.

III. INVARIANCES OF AMPLITUDES AND GENERALIZED SQUARING RELATIONS

In this section we discuss the invariances of gauge-theory and gravity amplitudes. This leads to a new, more general squaring relation for gravity, in which the numerators of only one of the gauge-theory factors are required to satisfy the BCJ duality. (See also Ref. [13].) As already noted, there is a substantial freedom in choosing the numerators, which we will generically call generalized gauge invariance, even though much of the freedom cannot be attributed to conventional gauge invariance. Our proof of the squaring relations will rely on an understanding of the most general form of this freedom at n points.

A. Generalized gauge invariance

Consider a shift of the $\tilde n_i$ in Eq. (2.10),

$$\tilde n_i \to \tilde n_i + \Delta_i. \tag{3.1}$$

The key constraint that the Δ_i must satisfy is that they do not alter the value of the amplitude, immediately leading to,

$$\sum_{\text{diags.}\, i} \frac{\Delta_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} = 0. \tag{3.2}$$

Any set of Δ_i that satisfies this constraint can be viewed as a valid generalized gauge transformation since it leaves the amplitude invariant. Ordinary gauge transformations, of course, satisfy this property. We may take Eq. (3.2) as the fundamental constraint satisfied by any generalized gauge transformation.

A key observation is that it is only the algebraic properties of the c_i, and not their explicit values, that enter into the cancellations in Eq. (3.2). This is so because the equation holds for any gauge group. Thus any object that shares the algebraic properties of the c_i will satisfy a similar constraint. Since the numerators n_i of the BCJ proposal satisfy exactly the same algebraic properties as the c_i, we immediately have

$$\sum_{\text{diags.}\, i} \frac{\Delta_i n_i}{\prod_{\alpha_i} s_{\alpha_i}} = 0, \tag{3.3}$$

as the key statement of generalized gauge invariance. This holds for any Δ_i that satisfies the constraint (3.2). In particular, note that we do *not* need to require the Δ_i to satisfy any Jacobi-like relations.

The freedom in making these shifts leads to an enormous freedom in writing different representations of either gauge-theory or gravity amplitudes. In the gravity case, besides shifts of either the n_i or the $\tilde{n}_i$, we can also shift the n_i and $\tilde{n}_i$ simultaneously as long as the interference terms vanish as well.

B. A direct derivation of the identity

It is instructive to directly demonstrate Eq. (3.3) in a way that goes beyond the explanation above. If we take the Δ_i to be local, then they move contributions between diagrams by canceling propagators in such a way that they can be absorbed into other diagrams. We can thus decompose each Δ_i as

$$\Delta_i = \sum_{\alpha_i} \Delta_{i,\alpha_i} s_{\alpha_i}, \tag{3.4}$$

where the α_i label the different propagators in diagram i. For simplicity, here we take the Δ_i to be local and linear in inverse propagators, i.e. to contain no terms that are quadratic or higher order in the inverse propagators s_{α_i}. In this case, the decomposition (3.4) is unique because the inverse propagators of any diagram i are independent under momentum conservation.

Consider three diagrams labeled by i, j, and k whose color factors are related by the Jacobi identity. These three diagrams share all propagators except for one, as illustrated in Fig. 1. For definiteness, let us denote the distinct inverse propagators of diagrams i, j, k by s, t, and u, respectively. Note that $s + t + u \neq 0$ (except for four-point amplitudes). Instead, $s + t + u$ is the sum of the invariant "masses" of the four legs that enter the two vertices connected to the s propagator in diagram i (which is the same as the four legs entering the vertices of propagator t in j, etc.). If one of these legs is external, its mass vanishes. Otherwise, this leg is another internal propagator shared by the diagrams i, j, k and its mass simply the associated variable s_α. Denoting the invariant masses of the four neighboring legs by s_1, s_2, s_3, and s_4, we have

$$s + t + u = s_1 + s_2 + s_3 + s_4. \tag{3.5}$$

Any color-ordered amplitude must contain either none or two of the diagrams i, j, k. For definiteness, consider the color-ordered amplitude that contains the diagrams i and j. With the sign conventions (2.5), n_i and n_j must enter this color-ordered amplitude with opposite sign. The contributions of the generalized gauge transformation (3.1) to this color-ordered amplitude is given by

$$\frac{\Delta_{i,s} s}{\prod_{\alpha_i} s_{\alpha_i}} - \frac{\Delta_{j,t} t}{\prod_{\alpha_j} s_{\alpha_j}} + \cdots = \frac{\Delta_{i,s} - \Delta_{j,t}}{\prod'_{\alpha_i} s_{\alpha_i}} + \cdots, \tag{3.6}$$

where $\prod'_{\alpha_i} s_{\alpha_i}$ represents the product of inverse propagators of diagram i except for s, i.e. $\prod'_{\alpha_i} s_{\alpha_i} = s^{-1} \prod_{\alpha_i} s_{\alpha_i}$. This contribution must cancel by itself, because all other contributions, represented by the "$\cdots$" in Eq. (3.6), have a different propagator structure and are therefore independent within this color-ordered amplitude. This independence is true because the diagram k, which contributes to a different ordering, is absent. [If we also had a contribution from diagram k, we could use Eq. (3.5) to relate contributions that have distinct propagator structures.] We conclude that $\Delta_{i,s} = \Delta_{j,t}$. Repeating this analysis for the other color-ordered amplitudes containing two of the diagrams i, j, k, we obtain the constraints

$$\Delta_{i,s} = \Delta_{j,t}, \qquad \Delta_{j,t} = \Delta_{k,u}, \\ \Delta_{i,s} = \Delta_{k,u} \Rightarrow \Delta_{i,s} = \Delta_{j,t} = \Delta_{k,u} \equiv \delta. \tag{3.7}$$

Now we have assembled all the ingredients to prove (3.3). With the decomposition (3.4) for the Δ_i, (3.3) reads

$$\sum_{\text{diags.}\, i} \frac{n_i (\sum_{\alpha_i} \Delta_{i,\alpha_i} s_{\alpha_i})}{\prod_{\alpha_i} s_{\alpha_i}} = 0. \tag{3.8}$$

Let us organize all terms in the decomposition (3.8) according to their propagator structure. For definiteness, we isolate the terms with the inverse propagator structure $\prod'_{\alpha_i} s_{\alpha_i}$. This gives

$$\frac{n_i \Delta_{i,s} + n_j \Delta_{j,t} + n_k \Delta_{k,u}}{\prod'_{\alpha_i} s_{\alpha_i}} = \delta \times \frac{n_i + n_j + n_k}{\prod'_{\alpha_i} s_{\alpha_i}} = 0, \tag{3.9}$$

since we have taken the n_i to satisfy the BCJ duality. We can repeat the same analysis for all other propagator structures appearing in Eq. (3.8), and each of them vanishes separately. This then explicitly exhibits the cancellation (3.3) for local Δ_i with linear contact terms. For nonlocal Δ_i, or Δ_i with quadratic or higher contact terms, the cancellations are similar but more involved.

C. Generalized squaring relations

We now apply the identity (3.3) to find a generalization of the BCJ squaring relations. The latter express the gravity amplitude as

$$\frac{-i}{(\kappa/2)^{n-2}}\mathcal{M}_n = \sum_{\text{diags.}\, i}\frac{n_i\tilde{n}_i}{\prod_{\alpha_i}s_{\alpha_i}}, \qquad (3.10)$$

where both n_i and $\tilde{n}_i$ are in the BCJ representation. Consider now a set of gauge-theory numerators $\tilde{n}_i'$ that do not satisfy the duality relations. Defining $\tilde{\Delta}_i = \tilde{n}_i' - \tilde{n}_i$, we find

$$\frac{-i}{(\kappa/2)^{n-2}}\mathcal{M}_n = \sum_{\text{diags.}\, i}\frac{n_i\tilde{n}_i}{\prod_{\alpha_i}s_{\alpha_i}} = \sum_{\text{diags.}\, i}\left[\frac{n_i\tilde{n}_i'}{\prod_{\alpha_i}s_{\alpha_i}} - \frac{n_i\tilde{\Delta}_i}{\prod_{\alpha_i}s_{\alpha_i}}\right]. \qquad (3.11)$$

It follows from the identity (3.3) that the second term vanishes. We thus conclude that

$$\frac{-i}{(\kappa/2)^{n-2}}\mathcal{M}_n = \sum_{\text{diags.}\, i}\frac{n_i\tilde{n}_i'}{\prod_{\alpha_i}s_{\alpha_i}}, \qquad (3.12)$$

where the n_i satisfy the duality but the $\tilde{n}_i'$ do not need to. Interestingly, such asymmetric constructions should work just as well at loop level [13].

Note that we cannot also relax the BCJ duality condition on the n_i in (3.12). Indeed, performing an arbitrary generalized gauge transformation Δ_i on the n_i would create cross terms of the form

$$\sum_{\text{diags.}\, i}\frac{\Delta_i\tilde{n}_i'}{\prod_{\alpha_i}s_{\alpha_i}} \neq 0, \qquad (3.13)$$

which generically do not vanish because neither Δ_i nor $\tilde{n}_i'$ satisfy the duality relations.

IV. SQUARING RELATIONS BETWEEN GAUGE AND GRAVITY THEORIES

We now derive the squaring relations (2.11) between gravity and gauge theories. Our derivation requires two gauge theories whose amplitudes have diagrammatic expansions with numerators that satisfy the BCJ duality. We show that the corresponding gravity numerators are then simply the product of these gauge-theory numerators. Our proof relies on the existence of on-shell recursion relations for both gravity and gauge theory based on the same shifted momenta. As we will explain, this is, for example, satisfied for the pure Yang-Mills/gravity pair in any dimension, and for the $\mathcal{N} = 4$ SYM/$\mathcal{N} = 8$ supergravity pair in $D = 4$.

A. Derivation of squaring relations for tree amplitudes

First, we consider the case of Einstein gravity obtained from two copies of pure Yang-Mills theories. The direct product of two Yang-Mills theories with $(D-2)$ states each (not counting color) gives $(D-2)^2$ states corresponding to a theory with a graviton, an antisymmetric tensor and dilaton. At tree level, however, we can restrict ourselves to the pure-graviton sector since the other states do not enter as intermediate states.

We will assume that one can always obtain local Yang-Mills numerators that satisfy the BCJ duality. We will prove inductively, using on-shell recursion relations with the lower-point amplitudes in the BCJ representation, that the n-point gravity numerator is the square of the n-point Yang-Mills numerator in the BCJ representation.

For three points, the squaring relations are trivial: there is only one "diagram" with no propagators, and the relation simply states [20]

$$\frac{-i}{\kappa/2}\mathcal{M}_3 = (A_3)^2, \qquad (4.1)$$

where A_3 is the color-ordered Yang-Mills three-point amplitude.

For larger numbers of external legs, we proceed inductively. To carry out our derivation of the squaring relations we make use of on-shell recursion relations. These are derived using complex deformations of the external momenta of the amplitude,

$$p_a \to \hat{p}_a(z) = p_a + zq_a \qquad a = 1, \ldots, n,$$
$$\hat{p}_a^2(z) = 0, \qquad \sum_{a=1}^{n} q_a = 0. \qquad (4.2)$$

Note that both momentum conservation and the on-shell conditions are preserved. To have valid recursion relations we demand that both the gravity and the gauge-theory tree amplitude vanish as we take the deformation parameter to infinity[3]:

$$\hat{\mathcal{M}}_n(z) \to 0, \qquad \hat{\mathcal{A}}_n(z) \to 0, \qquad \hat{\tilde{\mathcal{A}}}_n(z) \to 0 \quad \text{as } z \to \infty. \qquad (4.3)$$

The details of this complex shift (such as the number of shifted lines or the particular choice of q_a) will not play a role in our analysis, but we note that a large variety of shifts that satisfy (4.3) is known [11,21]. The simplest of these

[3]Here, both gauge-theory factors are pure Yang-Mills amplitudes and thus $\tilde{\mathcal{A}}_n = \mathcal{A}_n$. However, keeping the later generalization to other gravity/gauge-theory pairs in mind, we do not make use of this equality in the following discussion.

are BCFW two-line shifts. At least one BCFW shift exists for any choice of two external lines a and b, such that both the gauge-theory and the gravity amplitude vanish at large z [22]. Such shifts are also known to work in $D \geq 4$ dimensions. In our analysis we initially pick one arbitrary (but fixed) such shift.

We also pick an arbitrary local choice of gauge-theory numerators n_i that satisfies the BCJ duality (2.5). The assumption that such a choice exists at all is crucial for the following derivation. As the n_i are local, their complex deformations $\hat{n}_i(z)$ under the shift are polynomial in z; in particular $\hat{n}_i(z)$ has no poles in z. (To ensure this property, one has to choose the polarization vectors such that they do not contain poles in z.)

As a first step, let us analyze the recursion relation for the gauge-theory amplitude $\mathcal{A}_n$ that arises from the complex shift. The amplitude contains poles at values $z = z_\alpha$ where an internal propagator $1/s_\alpha$ goes on shell, i.e. $\hat{s}_\alpha(z_\alpha) = 0$. We obtain an expression for $\mathcal{A}_n$ as a sum over residues,

$$\mathcal{A}_n = \sum_\alpha \frac{\hat{\mathcal{A}}_n^\alpha}{s_\alpha}, \quad \text{with} \quad \hat{\mathcal{A}}_n^\alpha = i\hat{\mathcal{A}}_L(z_\alpha)\hat{\mathcal{A}}_R(z_\alpha). \quad (4.4)$$

This is illustrated in Fig. 2(a). The residue $\hat{\mathcal{A}}_n^\alpha$ at z_α with $\hat{s}_\alpha(z_\alpha) = 0$ factorizes into the product of a left and right subamplitude, and it does not depend on the representation of the amplitude. We can thus analyze each term in the sum over α separately, without ambiguity. Note that in Eq. (4.4) and from now on, whenever there is a product of left/right factors, an implicit sum over the polarizations of the on-shell intermediate state is assumed. Plugging in the left and right subamplitudes in the BCJ representation, we obtain

$$\frac{1}{g^{n-2}}\hat{\mathcal{A}}_n^\alpha = \sum_{\alpha\text{-diags.}\, i} \frac{i\hat{n}_{L,i}^\alpha \hat{n}_{R,i}^\alpha c_i}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)}. \quad (4.5)$$

Here, the sum goes only over diagrams i that contain the channel α, and again the prime on the product indicates that the propagator corresponding to that channel is not included, $\prod'_{\alpha_i} s_{\alpha_i} = \prod_{\alpha_i \neq \alpha} s_{\alpha_i}$. The color factor c_i arises from the color factors of the left and right subamplitudes after summing over the states of the intermediate gluon. The diagrammatic representation of the residue $\hat{\mathcal{A}}_n^\alpha$ is illustrated in Fig. 2(b).

On the other hand, we can directly express $\hat{\mathcal{A}}_n^\alpha$ in terms of the original representation (2.1) of $\mathcal{A}_n$. To this end, we evaluate (2.1) at shifted momenta and read off the poles in the deformation parameter z from the right hand side. We obtain

$$\frac{1}{g^{n-2}}\hat{\mathcal{A}}_n^\alpha = \sum_{\alpha\text{-diags.}\, i} \frac{\hat{n}_i(z_\alpha) c_i}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)}. \quad (4.6)$$

Note that the shifted numerators $\hat{n}_i(z)$ satisfy the BCJ duality for any z because the n_i satisfy the BCJ duality for any choice of on-shell external momenta, including shifted ones. Comparing Eq. (4.5) to Eq. (4.6), we see that $i\hat{n}_{L,i}^\alpha \hat{n}_{R,i}^\alpha$ and $\hat{n}_i(z_\alpha)$ are related by some generalized gauge transformation:

$$\hat{n}_i(z_\alpha) = i\hat{n}_{L,i}^\alpha \hat{n}_{R,i}^\alpha + \Delta_i^\alpha, \quad (4.7)$$

where the Δ_i^α satisfy

$$\sum_{\alpha\text{-diags.}\, i} \frac{\Delta_i^\alpha c_i}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)} = 0. \quad (4.8)$$

It may seem unusual that we evaluate both sides in Eq. (4.7) at shifted momenta. Unlike in usual applications of BCFW recursion relations, however, the unshifted amplitude is already an input in our construction and we are not interested in computing it. Instead, we are using (4.7) to relate the shifted numerators $\hat{n}_i(z_\alpha)$ to the diagrammatic numerators of BCFW subamplitudes. Note that the Δ_i^α are only unambiguously defined at $z = z_\alpha$, and should therefore not be thought of as a function of z. Also note that this is not a single generalized gauge transformation, but a distinct one for each choice of α, and we can analyze it separately for each α. It will be important in the following that the Δ_i^α satisfy all duality constraints that relate diagrams containing the internal line α, i.e.

$$\Delta_i^\alpha + \Delta_j^\alpha + \Delta_k^\alpha = 0. \quad (4.9)$$

To see this, note that $\Delta_i^\alpha = \hat{n}_i - i\hat{n}_{L,i}\hat{n}_{R,i}$, and as the $\hat{n}_i$ satisfy all duality relations it is sufficient to examine the duality properties of $i\hat{n}_{L,i}\hat{n}_{R,i}$. The diagrams i, j, k in

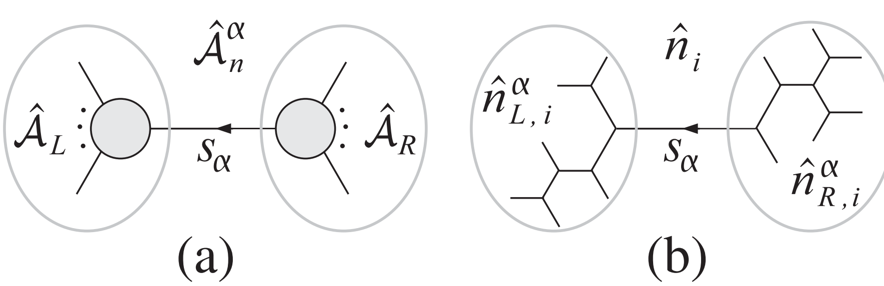

FIG. 2. (a) In an on-shell recursion relation, a given residue $\hat{\mathcal{A}}_n^\alpha$ is determined by diagrams sharing the same propagator labeled by s_α. (b) We can obtain a diagrammatic expansion of the recursion relation either from the numerators $\hat{n}_i$ of the shifted full amplitude $\hat{\mathcal{A}}_n$, or from the numerators $\hat{n}_{L,i}^\alpha$, $\hat{n}_{R,i}^\alpha$ of the subamplitudes.

Eq. (4.9) share all but one propagator, and thus they either share the entire left or the entire right subdiagram of the factorized amplitude. For definiteness, let us consider the case where they share the entire right subdiagram, and thus $\hat{n}_{R,i} = \hat{n}_{R,j} = \hat{n}_{R,k}$. Then the duality relation (4.9) immediately follows from the corresponding duality relation for the numerators in the left subdiagram, $\hat{n}_{L,i} + \hat{n}_{L,j} + \hat{n}_{L,k} = 0$. This is illustrated in Fig. 3.

Note that we needed to introduce the Δ_i^α, which only satisfy a partial set of duality relations, because the recursion relation (4.5) by itself does not yield a BCJ-compatible representation of the amplitude. In fact, from Eq. (4.5), one can immediately read off an implied numerator representation n_i' of $\mathcal{A}_n$ given by

$$n_i' = \sum_{\alpha_i} i \hat{n}_{L,i}^{\alpha_i} \hat{n}_{R,i}^{\alpha_i} \prod_{\beta_i \neq \alpha_i} \frac{s_{\beta_i}}{\hat{s}_{\beta_i}(z_{\alpha_i})}. \tag{4.10}$$

Generically, these numerators n_i' do not satisfy any duality relations.

We now turn to gravity. Applying the recursion relation for gravity to the amplitude $\mathcal{M}_n$, we obtain

$$\mathcal{M}_n = \sum_\alpha \frac{i}{s_\alpha} \hat{\mathcal{M}}_L(z_\alpha) \hat{\mathcal{M}}_R(z_\alpha). \tag{4.11}$$

Using our inductive assumption that the squaring relations are valid for lower-point amplitudes we can plug in the squaring relations (2.11) for the subamplitudes $\hat{\mathcal{M}}_L$, $\hat{\mathcal{M}}_R$ and obtain

$$\frac{1}{(\kappa/2)^{n-2}} \mathcal{M}_n = \sum_\alpha \frac{i}{s_\alpha} \sum_{\alpha\text{-diags.}\,i} \frac{[i\hat{n}_{L,i}^\alpha \hat{n}_{R,i}^\alpha][i\hat{\tilde{n}}_{L,i}^\alpha \hat{\tilde{n}}_{R,i}^\alpha]}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)}. \tag{4.12}$$

We can now use the gauge-theory relation (4.7) to rewrite the gravity amplitude as

$$\begin{aligned} \frac{1}{(\kappa/2)^{n-2}} \mathcal{M}_n &= \sum_\alpha \frac{i}{s_\alpha} \sum_{\alpha\text{-diags.}\,i} \frac{[\hat{n}_i(z_\alpha) - \Delta_i^\alpha][\hat{\tilde{n}}_i(z_\alpha) - \tilde{\Delta}_i^\alpha]}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)} \\ &= \sum_\alpha \frac{i}{s_\alpha} \sum_{\alpha\text{-diags.}\,i} \Bigg[\frac{\hat{n}_i(z_\alpha)\hat{\tilde{n}}_i(z_\alpha)}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)} \\ &\quad - \frac{\Delta_i^\alpha \hat{\tilde{n}}_i(z_\alpha) + \tilde{\Delta}_i^\alpha \hat{n}_i(z_\alpha)}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)} + \frac{\Delta_i^\alpha \tilde{\Delta}_i^\alpha}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)} \Bigg]. \end{aligned} \tag{4.13}$$

The cross terms involving the numerators $\tilde{\Delta}_i^\alpha \hat{n}_i(z_\alpha)$, $\Delta_i^\alpha \hat{\tilde{n}}_i(z_\alpha)$ vanish due to the identity (3.3), because the n_i satisfy the BCJ duality. We will now argue that the last term also vanishes:

$$\sum_{\alpha\text{-diags.}\,i} \frac{\Delta_i^\alpha \tilde{\Delta}_i^\alpha}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)} = 0. \tag{4.14}$$

To see this, we proceed analogously to the derivation of the identity (3.3), treating the factor Δ_i^α as the generalized gauge transformation, and the other factor $\tilde{\Delta}_i^\alpha$ as the Jacobi-satisfying coefficient. We have shown above that the Δ_i^α, $\tilde{\Delta}_i^\alpha$ satisfy the duality relations within the class of diagrams that contain the line α [see Eq. (4.9)]. One may worry that this is not sufficient to guarantee (4.14), because there is also one duality relation that relates the diagram i to two diagrams in which line α is replaced by its t- and u-channel analogue. These diagrams do not contain the line α and thus do not appear in Eq. (4.14). To see that this complication is harmless, we expand Δ_i^α in its distinct contact term contributions, just like we expanded Δ_i in Eq. (3.4):

$$\Delta_i^\alpha = \sum_{\alpha_i \neq \alpha} \Delta_{i,\alpha_i}^\alpha \hat{s}_{\alpha_i}(z_\alpha). \tag{4.15}$$

Note that there is no contact term ambiguity in Δ_i^α associated with s_α because $\hat{s}_\alpha(z_\alpha) = 0$. In the derivation of (3.3), the duality relation between three diagrams i, j, k was important to cancel the contact term ambiguities associated with the propagators s, t, u [see Eq. (3.9)]. In our case there is no such ambiguity associated with the propagator s_α, so we only need the duality relations (4.9) to argue that (4.14) holds. We conclude that

$$\frac{-i}{(\kappa/2)^{n-2}} \mathcal{M}_n = \sum_\alpha \frac{1}{s_\alpha} \sum_{\alpha\text{-diags.}\,i} \frac{\hat{n}_i(z_\alpha)\hat{\tilde{n}}_i(z_\alpha)}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)}. \tag{4.16}$$

We now define

$$\frac{-i}{(\kappa/2)^{n-2}} \mathcal{M}_n' \equiv \sum_{\text{diags.}\,i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{4.17}$$

where, as previously stated, the n_i are the n-point Yang-Mills numerator in the BCJ representation. Since the n_i are local, we see from (4.16) that $\hat{\mathcal{M}}_n'(z)$ and $\hat{\mathcal{M}}_n(z)$ have precisely the same pole structure. Indeed, the falloff (4.3) of $\hat{\mathcal{M}}_n(z)$ at large z, together with (4.16), imply that

$$\frac{-i}{(\kappa/2)^{n-2}} \hat{\mathcal{M}}_n(z) = \sum_\alpha \frac{1}{s_\alpha(z)} \sum_{\alpha\text{-diags.}\,i} \frac{\hat{n}_i(z_\alpha)\hat{\tilde{n}}_i(z_\alpha)}{\prod'_{\alpha_i} \hat{s}_{\alpha_i}(z_\alpha)}. \tag{4.18}$$

This form makes manifest that $\hat{\mathcal{M}}_n(z)$ and $\hat{\mathcal{M}}_n'(z)$ have coinciding residues for all finite-z poles. However, in principle the two functions could still differ by a function $\mathcal{P}$ of momenta and polarization vectors that is polynomial in z under the complex shift,

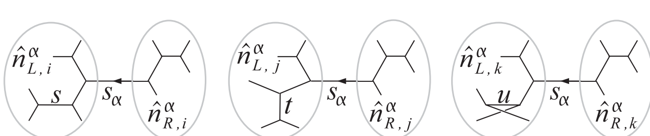

FIG. 3. The product $i\hat{n}_{L,i}\hat{n}_{R,i}$ satisfies the duality relations satisfied by its factors $\hat{n}_{L,i}$ and $\hat{n}_{R,i}$.

$$\mathcal{P} = \mathcal{M}'_n - \mathcal{M}_n, \qquad \hat{\mathcal{P}}(z) = \text{polynomial in } z. \tag{4.19}$$

The on-shell gravity amplitude $\mathcal{M}_n$ is of course invariant under gravity on-shell gauge transformations, i.e. under shifts of the polarization tensors that are proportional to the corresponding external graviton momentum. This gauge invariance actually also holds for $\mathcal{M}'_n$. To see this, note that we can break up the gravity gauge transformation into two Yang-Mills on-shell gauge transformations acting separately on the factors n_i and $\tilde{n}_i$ in the numerators of $\mathcal{M}'_n$. We note that such a true on-shell gauge transformation leaves the duality property of the n_i intact.[4] Invariance of $\mathcal{M}'_n$ then immediately follows from the identity (3.3). We conclude that $\mathcal{P}$ is gauge-invariant.

The shift analysis above is not yet sufficient to argue that $\mathcal{P}$ is also local. While it cannot have poles in z, it could in principle have propagators in its denominator that are invariant under the particular shift that we chose. Note, however, that we could repeat the analysis above for any other shift under which both the gravity and the gauge-theory amplitude vanish. We conclude that $\mathcal{P}$ must be polynomial in z under *any* such shift. As explained above, many such shifts are available; in particular, there is a valid BCFW shift for any choice of two external lines. As no propagator can be invariant under all of these shifts, we conclude that $\mathcal{P}$ must be local.

From dimensional analysis, we also know that $\mathcal{P}$ must be quadratic in momenta. Note that this is only true because we are considering a gravity theory that is not modified by higher-dimension operators. For example, if we had allowed for α' corrections to gravity, the expression $\mathcal{P}$ could contain contributions that are higher order in momenta.

$\mathcal{P}$ is thus a gauge-invariant, local expression quadratic in momenta. No such expression exists because the matrix elements of any contractions $D^m R^n$ of the Riemann tensor with covariant derivatives contain at least $2n + m$ powers of momenta.[5] We conclude that

$$\mathcal{P} = 0, \tag{4.20}$$

and therefore

$$\frac{-i}{(\kappa/2)^{n-2}}\mathcal{M}_n = \frac{-i}{(\kappa/2)^{n-2}}\mathcal{M}'_n = \sum_{\text{diags.}\, i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{4.21}$$

where both n_i and $\tilde{n}_i$ are in the BCJ representation. As discussed in Sec. III, it then immediately follows from the identity (3.3) that the squaring relations also hold if we only impose the duality relations on one of the two copies of gauge-theory numerators. This concludes our derivation of the squaring relations for pure Einstein gravity in arbitrary dimensions $D \geq 4$.

B. A five-point example

We now illustrate some crucial steps of our general derivation with the simplest possible nontrivial example, the $D = 4$ five-point maximally helicity-violating (MHV) amplitude. The simplicity of this example allows us to give compact explicit expressions for the numerators n_i and the required generalized gauge transformations Δ_i^α. We use these to display some of the key relations that we derived above on general grounds, in particular, Eqs. (4.7), (4.9), (4.13), and (4.14).

At five points, a basis of color-ordered amplitudes under the Kleiss-Kuijf relations [19] is given by

$$\begin{aligned}
A_5(1,2,3,4,5) &= \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}},\\
A_5(1,4,3,2,5) &= \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}},\\
A_5(1,3,4,2,5) &= \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}},\\
A_5(1,2,4,3,5) &= \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}},\\
A_5(1,4,2,3,5) &= \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}},\\
A_5(1,3,2,4,5) &= \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}},
\end{aligned} \tag{4.22}$$

[4] To see this, recall that the n_i are functions of momenta p_a^μ and polarization vectors ϵ_a^μ. A shift of the polarization vectors $\delta\epsilon_a^\mu \propto p_a^\mu$ treats all n_i on equal footing, and thus can never spoil a duality relation $n_i + n_j + n_k = 0$.

[5] Another way to reach the same conclusion for the special case of $D = 4$ dimensions is to note that a local, gauge-invariant expression must be expressible as a polynomial in the angle and square brackets of the spinor-helicity formalism. No expression quadratic in angle and square brackets can have the correct little-group scaling property [1] of an n-point amplitude.

where we follow the notation of Ref. [5], including the signs in the duality relations. Specifying the negative-helicity lines to be 1 and 5, we can compute the first two color-ordered amplitudes above directly from the Parke-Taylor [23] formula:

$$A_5(1^-2^+3^+4^+5^-) = i\frac{\langle 15\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle},$$
$$A_5(1^-4^+3^+2^+5^-) = i\frac{\langle 15\rangle^4}{\langle 14\rangle\langle 43\rangle\langle 32\rangle\langle 25\rangle\langle 51\rangle}. \tag{4.23}$$

Let us furthermore specify the BCFW shift $[1,5\rangle$ for the following analysis:

$$|1] \to |\hat{1}] = |1] - z|5], \qquad |5\rangle \to |\hat{5}\rangle = |5\rangle + z|1\rangle. \tag{4.24}$$

We could start with a general local expression for the n_i to ensure that they are polynomial in z under the shift. In fact, there is a 225-parameter family of such local n_i that satisfy the duality relations and reproduce all Kleiss-Kuijf relations [19] correctly. For our purposes, however, it is more convenient to construct a simple choice of n_i by hand. To reproduce the first amplitude $A_5(1^-2^+3^+4^+5^-)$ correctly, we pick

$$n_1 = -i\frac{\langle 15\rangle^3}{\langle 23\rangle\langle 34\rangle}\times[12][45],$$
$$n_2 = n_3 = n_4 = n_5 = 0. \tag{4.25}$$

It is obvious that n_1 is polynomial under the BCFW shift ([11]), because this shift leaves the angle brackets $\langle 23\rangle$ and $\langle 34\rangle$ invariant. The duality relations $n_4 - n_2 + n_7 = 0$ and $n_3 - n_5 + n_8 = 0$ immediately imply that n_7 and n_8 vanish. To reproduce the second amplitude $A_5(1^-4^+3^+2^+5^-)$ correctly, we thus need to set

$$n_6 = -i\frac{\langle 15\rangle^3}{\langle 23\rangle\langle 34\rangle}\times[14][25], \qquad n_7 = n_8 = 0. \tag{4.26}$$

n_6 is also manifestly polynomial under the specified BCFW shift. All other numerators $n_9, \ldots, n_{15}$ are now determined through the duality relations and are of course also polynomial under the shift. In summary, we obtain the following choice of numerators:

$$n_1 = n_{12} = n_{15} = -i\frac{\langle 15\rangle^3}{\langle 23\rangle\langle 34\rangle}\times[12][45],$$
$$n_6 = n_9 = n_{14} = -i\frac{\langle 15\rangle^3}{\langle 23\rangle\langle 34\rangle}\times[14][25],$$
$$n_{10} = -n_{13} = i\frac{\langle 15\rangle^3}{\langle 23\rangle\langle 34\rangle}\times([12][45]-[14][25]),$$
$$n_2 = n_3 = n_4 = n_5 = n_7 = n_8 = n_{11} = 0. \tag{4.27}$$

Let us now consider the contribution of the factorization channel $s_\alpha = s_{45}$ to the BCFW recursion relation. All shifted expressions must thus be evaluated at $z = z_\alpha \equiv -[45]/[14]$. The amplitude factorizes into a right three-point anti-MHV subamplitude[6]

$$\hat{A}_R(4^+, \hat{5}^-, -\hat{P}^+) = \hat{n}_R = -i\frac{[4,-\hat{P}]^4}{[45][5,-\hat{P}][-\hat{P},4]}, \tag{4.28}$$

and a left four-point subamplitude. The latter is MHV and takes the form

$$\hat{A}_L(\hat{1}^-, 2^+, 3^+, \hat{P}^-) = \frac{\hat{n}_{L,s}}{\hat{s}_{12}} - \frac{\hat{n}_{L,t}}{\hat{s}_{23}}. \tag{4.29}$$

We can pick an arbitrary (local or nonlocal) representation of $\hat{n}_{L,s}$, $\hat{n}_{L,t}$. One choice is given by

$$\hat{n}_{L,s} = i\frac{\langle 12\rangle\langle 1\hat{P}\rangle[23]^2}{2\langle 23\rangle[3\hat{P}]}, \qquad \hat{n}_{L,t} = -i\frac{\langle 1\hat{P}\rangle[23]^3}{2[\hat{1}2][3\hat{P}]},$$
$$\hat{n}_{L,u} = -\hat{n}_{L,s} - \hat{n}_{L,t}. \tag{4.30}$$

With this choice the s- and t-channel contributions to $\hat{A}_L$ happen to coincide, but any other choice of $n_{L,s}$, $\hat{n}_{L,t}$ that reproduces $\hat{A}_L$ correctly would merely alter the generalized gauge transformations needed to match to the n_i satisfying the BCJ-duality. Combining left and right subamplitudes, the color factors of $\hat{n}_{L,s}\hat{n}_R$, $\hat{n}_{L,t}\hat{n}_R$, and $\hat{n}_{L,u}\hat{n}_R$, are c_1, c_4, and c_{15}, respectively. Their corresponding numerators satisfy the duality relation $n_1 - n_4 - n_{15} = 0$. We thus define

$$\begin{aligned}
\Delta_1^\alpha &= \hat{n}_1 - i\hat{n}_{L,s}\hat{n}_R \\
&= -i\frac{\langle 1\hat{5}\rangle^3[\hat{1}2][45]}{\langle 23\rangle\langle 34\rangle} - i\frac{\langle 12\rangle\langle 1\hat{P}\rangle[4\hat{P}]^3[23]^2}{2\langle 23\rangle[3\hat{P}][45][5\hat{P}]} \\
&= i\frac{\langle 1\hat{5}\rangle^3[45]}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle}\times\hat{s}_{12}, \\
\Delta_4^\alpha &= \hat{n}_4 + i\hat{n}_{L,t}\hat{n}_R = -i\frac{\langle 1\hat{P}\rangle[\hat{2}3]^3[4\hat{P}]^3}{2[\hat{1}2][3\hat{P}][45][5\hat{P}]} \\
&= -i\frac{\langle 1\hat{5}\rangle^3[45]}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle}\times\hat{s}_{23}, \\
\Delta_{15}^\alpha &= \hat{n}_{15} + i\hat{n}_{L,u}\hat{n}_R = -i\frac{\langle 1\hat{5}\rangle^3[45]}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle}\times\hat{s}_{13}.
\end{aligned} \tag{4.31}$$

Note that these Δ_i^α indeed satisfy the duality relation on the pole: $\Delta_1^\alpha - \Delta_4^\alpha - \Delta_{15}^\alpha = 0$.

The crucial step in our derivation of the squaring relations was the cancellation of the $\Delta_i^\alpha\hat{n}_i$ and $(\Delta_i^\alpha)^2$ pieces in Eq. (4.13). It is now straightforward to verify this cancellation directly in our current example:

[6] We adopt the convention that all external momenta are incoming, and the internal momentum P is incoming in the left subamplitude, and outgoing in the right subamplitude. We use the spinor conventions $|-P\rangle = i|P\rangle$, $|-P] = i|P]$, $s_{ab} = -[ab]\langle ab\rangle$.

$$\frac{\Delta_1^\alpha \hat{n}_1}{\hat{s}_{12}} + \frac{\Delta_4^\alpha \hat{n}_4}{\hat{s}_{23}} + \frac{\Delta_{15}^\alpha \hat{n}_{15}}{\hat{s}_{13}} = i\frac{\langle 1\hat{5}\rangle^3[45]}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle} \times (\hat{n}_1 - \hat{n}_4 - \hat{n}_{15}) = 0,$$
$$\frac{(\Delta_1^\alpha)^2}{\hat{s}_{12}} + \frac{(\Delta_1^\alpha)^2}{\hat{s}_{23}} + \frac{(\Delta_{15}^\alpha)^2}{\hat{s}_{13}} = -\left(\frac{\langle 1\hat{5}\rangle^3[45]}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle}\right)^2 \times (\hat{s}_{12} + \hat{s}_{23} + \hat{s}_{13}) = 0, \quad (4.32)$$

where we used the pole condition $\hat{s}_{123} = \hat{s}_{45} = 0$.

C. Generalization to other gravity/gauge-theory pairs

The derivation of the squaring relations in the previous section specifically pertained to pure gravity and pure Yang-Mills theory. However, only a few steps in the derivation depended on this specific choice of theories. For a more general gravity/gauge-theory pair, the above derivation goes through if the following three conditions are satisfied:

(1) Every amplitude in the gauge theory can be expressed using local numerators that satisfy the BCJ duality.
(2) There exist "valid" complex shifts of the external momenta, i.e. shifts such that both gauge-theory and gravity tree amplitudes vanish at large z. Such shifts give rise to on-shell recursion relations.
(3) Each propagator of every gravity amplitude must develop a pole under at least one of these valid complex shifts. This property was crucial for our conclusion above that $\mathcal{P}$ defined in Eq. (4.19) vanishes identically.

An interesting candidate gravity/gauge-theory pair are the $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity theories in four dimensions. Just as for pure Yang-Mills theory, it remains to be shown that amplitudes in $\mathcal{N} = 4$ SYM can be expressed using local numerators satisfying the BCJ duality. Although we expect the duality to work in supersymmetric theories [5,8,24], naively, the conditions (2) and (3) above seem hard to satisfy; while each $\mathcal{N} = 4$ SYM amplitude with $n > 4$ external legs admits at least one valid BCFW shift [25,26] and a variety of valid holomorphic shifts [26–28], the same does not hold for the amplitudes of $\mathcal{N} = 8$ supergravity [29]. For certain amplitudes, we seem to have no valid BCFW shifts available at all, let alone sufficiently many to conclude that $\mathcal{P}$ vanishes.

Fortunately, there is a simple fix to this problem: We promote the numerators n_i to on-shell superfields $\mathfrak{n}_i$ and the amplitudes $\mathcal{A}_n$, $\mathcal{M}_n$ to superamplitudes $\mathfrak{A}_n$, $\mathfrak{M}_n$, which depend on Grassmann parameters $\eta_{a,A}$ (where a and A denote the particle index and the $SU(\mathcal{N})$ index, respectively). The superamplitudes $\mathfrak{A}_n$ and $\mathfrak{M}_n$ are η-polynomials that encode all n-point amplitudes of SYM and supergravity as their coefficients.

At the MHV level, we can circumvent a new derivation of the squaring relations altogether. The tree-level pure-gluon amplitudes of SYM are identical to the ones of pure Yang-Mills theory. The pure-graviton amplitudes in supergravity and pure gravity also coincide. The squaring relations then immediately apply, in particular, to the gluon/graviton MHV amplitude pair $\mathcal{A}_n^{--+\cdots+}$, $\mathcal{M}_n^{--+\cdots+}$. Choosing duality-satisfying numerators $n_i^{--+\cdots+}$ for the gluon amplitude, we define "supernumerators"

$$\mathfrak{n}_i = \frac{\delta^{(8)}(\bar{Q}_A)}{\langle 12\rangle^8} \times n_i^{--+\cdots+}, \qquad \bar{Q}_A = \sum_a |a\rangle \eta_{a,A}. \quad (4.33)$$

These supernumerators satisfy all duality relations, because

$$\mathfrak{n}_i + \mathfrak{n}_j + \mathfrak{n}_k \propto n_i^{--+\cdots+} + n_j^{--+\cdots+} + n_k^{--+\cdots+} = 0. \quad (4.34)$$

The $\mathfrak{n}_i$, though nonlocal, also manifestly satisfy the squaring relations:

$$\frac{-i}{\kappa^{n-2}}\mathfrak{M}_n = \frac{-i\delta^{(16)}(\bar{Q}_A)}{\kappa^{n-2}\langle 12\rangle^{16}}\mathcal{M}_n^{--+\cdots+} = \frac{\delta^{(16)}(\bar{Q}_A)}{\langle 12\rangle^{16}} \times \sum_i \frac{(n_i^{--+\cdots+})^2}{\prod_{\alpha_i} s_{\alpha_i}} = \sum_i \frac{\mathfrak{n}_i \tilde{\mathfrak{n}}_i}{\prod_{\alpha_i} s_{\alpha_i}}. \quad (4.35)$$

It then follows from the identity (3.3) that all BCJ numerators must satisfy the squaring relations at the MHV level.

Beyond the MHV level we make use of Refs. [30,31], where it was shown that the superamplitudes $\mathfrak{A}_n$ and $\mathfrak{M}_n$ vanish under a *super*-BCFW shift of any two lines a and b:

$$|a] \to |\hat{a}] = |a] - z|b], \qquad |b\rangle \to |\hat{b}\rangle = |b\rangle + z|a\rangle,$$
$$\eta_{a,A} \to \hat{\eta}_{a,A} = \eta_{a,A} - z\eta_{b,A}. \quad (4.36)$$

We thus have a large number of valid super-BCFW shifts available for the superamplitudes $\mathfrak{A}_n$ and $\mathfrak{M}_n$, and conditions (2) and (3) are easily satisfied for this gravity/gauge-theory pair. Instead of performing sums over intermediate polarizations in the derivation of Sec. IVA [for example in the product $\hat{n}_{L,i}\hat{n}_{R,i}$ of Eq. (4.5)], we perform integrals over the Grassmann parameters $\eta_{P,A}$ associated with the internal line:

$$\hat{n}^\alpha_{L,i}\hat{n}^\alpha_{R,i} \to \int d^4\eta_{P,A}\hat{\mathfrak{n}}^\alpha_{L,i}\hat{\mathfrak{n}}^\alpha_{R,i}. \quad (4.37)$$

The remaining analysis carries through without modification, establishing the squaring relations for $\mathcal{N} = 4$ SYM/$\mathcal{N} = 8$ supergravity.

A similar analysis can be repeated for other gravity/gauge-theory pairs by systematically verifying the conditions (1)–(3) above. Whether the KLT relations are valid for a particular gravity/gauge-theory pair is usually addressed using the $\alpha' \to 0$ limit of string theory amplitudes. Our three conditions above for the squaring relations, on the other hand, give purely field-theoretic criteria for the validity of "gravity = (gauge theory) × (gauge theory)".

D. Extension to loops

We note that our tree-level derivation of the squaring relations (2.11) from the BCJ duality (2.5) immediately extends to loops via the unitarity method [16]. In the unitarity method, no shifts of momenta are required and there is no issue with large-z behavior, if the cuts are evaluated in D dimensions, ensuring cut constructability [32]. Assuming that gauge-theory loop amplitudes satisfy the duality, a gravity ansatz in terms of diagrams built by taking double copies of numerators will have all the correct cuts in all channels, since the numerators of all tree diagrams appearing in the cuts are double copies. This immediately implies that the gravity amplitude so constructed is correct.

Given two gauge theories whose L-loop numerators n_i, $\tilde{n}_i$ can be arranged to satisfy the BCJ duality, and whose tree amplitudes are related through the squaring relations to a corresponding gravity theory, we can immediately write down the gravity L-loop amplitude [13]:

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n+2L-2}} \mathcal{M}_n^{L\text{-loop}} = \sum_{\text{diags.}\, i} \int \prod_{a=1}^{L} \frac{d^D l_a}{(2\pi)^D} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}, \quad (4.38)$$

where the numerators $n_i \tilde{n}_i$ and propagators $1/s_{\alpha_i}$ depend on external and loop momenta, and the sum runs over all L-loop diagrams with only three-point vertices. We note that this squaring relation holds at loop level for arbitrary loop momenta, while the traditional KLT relations only apply to unitarity cuts that factorize the loop amplitude into a product of tree amplitudes.

In the next section we construct Lagrangians whose diagrams reflect the BCJ duality, suggesting that the gauge-theory duality does indeed extend to loop level. Interestingly, not only has the extension of the duality to loop level been explicitly demonstrated in a pure Yang-Mills two-loop example and an $\mathcal{N} = 4$ super-Yang-Mills three-loop example, but the double-copy property of the corresponding gravity loop amplitudes has also been confirmed [13].

V. A LAGRANGIAN GENERATING DIAGRAMS WITH BCJ DUALITY

We now turn to the question of finding a Lagrangian which generates amplitudes with numerators that manifestly satisfy the BCJ duality. If a local Lagrangian of this type could be found, it would enable us to construct a corresponding gravity Lagrangian whose squaring relations with Yang-Mills theory are manifest. We show that such a construction is indeed possible, and we present the explicit form of a Yang-Mills Lagrangian which generates diagrams that respect the BCJ duality up to five points. We use it to construct the corresponding Lagrangian for gravity. We also outline the structure of Lagrangians that preserve the duality in higher-point diagrams.

A. General strategy of the construction

A Yang-Mills Lagrangian with manifest BCJ duality can only differ from the conventional Yang-Mills Lagrangian by terms that do not affect the amplitudes. The amplitudes are unaffected, for example, by adding total derivative terms or by carrying out field redefinitions. In fact, the MHV Lagrangian [33] for the Cachazo-Svrček-Witten [34] expansion is an example where identities or structures of tree-level amplitudes can be derived through a field redefinition of the original Lagrangian. Such a construction has the additional complication that a Jacobian can appear at loop level. Surprisingly, we find that not only does a Lagrangian with manifest BCJ duality exist, it differs from the conventional Lagrangian by terms whose sum is identically zero by the color Jacobi identity! Although the sum over added terms vanishes, they cause the necessary rearrangements so that the BCJ duality holds. Another curious property is that the additional terms are necessarily nonlocal, at least if we want a covariant Lagrangian without auxiliary fields.

For example, consider five-gluon tree amplitudes. To obtain diagrams that satisfy the BCJ duality one is required to add terms to the Lagrangian of the form

$$\mathcal{L}'_5 \sim \mathrm{Tr}[A^\nu, A^\rho] \frac{1}{\Box} ([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho]), \quad (5.1)$$

along with other contractions. If we expand the commutators, the added terms immediately vanish by the color Jacobi identity. If, however, the commutators are reexpressed in terms of group-theory structure constants, they generate terms that get distributed across different diagrams and color factors. We find similar results up to six points, suggesting that it is a general feature for any number of points.

Since the BCJ duality relates the structure of kinematic numerators and color factors of diagrams with only three-point vertices, the desired Lagrangian should contain only three-point interactions. To achieve this we introduce auxiliary fields. The auxiliary fields not only reduce the interactions down to only three points, they also convert the newly introduced nonlocal terms into local interactions. This procedure introduces a large set of auxiliary fields into the Lagrangian. This is not surprising since we want a double copy of this Lagrangian to describe gravity. The ordinary gravity Lagrangian contains an infinite set of contact terms; if we were to write it in terms of three-point interactions we would need to introduce a new set of auxiliary fields for each new contact term in the expansion.

Since in our approach the gravity Lagrangian is essentially the square of the Yang-Mills Lagrangian, it is natural to expect that the desired Lagrangian contains a large (perhaps infinite) number of auxiliary fields. We now begin our construction of a Yang-Mills Lagrangian with manifest BCJ duality.

B. The Yang-Mills Lagrangian through five points

We write the Yang-Mills Lagrangian as

$$\mathcal{L}_{\mathrm{YM}} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \cdots \tag{5.2}$$

where $\mathcal{L}$ is the conventional Yang-Mills Lagrangian and $\mathcal{L}'_n$, $n > 4$ are the additional terms required so that the BCJ duality is satisfied. At four points, the BCJ duality is trivially satisfied in any gauge [5,6], so $\mathcal{L}$ by itself will generate diagrams whose numerators satisfy Eq. (2.5). For simplicity we choose Feynman gauge for $\mathcal{L}$,[7] though similar conclusions hold for other gauges. All contact terms are uniquely assigned to the three-vertex diagram carrying the corresponding color factor. The $\mathcal{L}'_n$ are required to leave scattering amplitudes invariant, and they must rearrange the numerators of diagrams in a way so that the BCJ duality is satisfied. It turns out that the set of terms with the desired properties is not unique. Indeed, "self-BCJ" terms that satisfy the BCJ duality by themselves can also be added. This ambiguity is due to the residual "generalized gauge invariance" that remains after solving the duality identities [5,9].

By imposing the constraint that the generated five-point diagrams satisfy the BCJ duality (2.5), we find the following Lagrangian:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}A^a_\mu \Box A^{a\mu} - g f^{a_1a_2a_3}\partial_\mu A^{a_1}_\nu A^{a_2\mu}A^{a_3\nu} \\ &\quad - \frac{1}{4}g^2 f^{a_1a_2b}f^{ba_3a_4}A^{a_1}_\mu A^{a_2}_\nu A^{a_3\mu}A^{a_4\nu}\, \mathcal{L}'_5 \\ &= -\frac{1}{2}g^3 f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5}(\partial_{[\mu}A^{a_1}_{\nu]}A^{a_2}_\rho A^{a_3\mu} \\ &\quad + \partial_{[\mu}A^{a_2}_{\nu]}A^{a_3}_\rho A^{a_1\mu} + \partial_{[\mu}A^{a_3}_{\nu]}A^{a_1}_\rho A^{a_2\mu})\frac{1}{\Box}(A^{a_4\nu}A^{a_5\rho}).\end{aligned} \tag{5.3}$$

The numerators n_i are derived from this action by first computing the contribution from the three-point vertices, which gives a set of three-vertex diagrams with unique numerators. Then the contributions from the four- and five-point interaction terms are assigned to the various diagrams with only three-point vertices according to their color factors. Since these terms will contain fewer propagators than those obtained by using only three-point vertices, their contributions to the numerators contain inverse propagators. Finally, we combine all diagrams with the same color factor, however they arose in the procedure above, into a single diagram. Its kinematic coefficient is the desired numerator that satisfies the BCJ duality. In this light, the purpose of $\mathcal{L}'_5$ is to restore the BCJ duality (2.5) violated by the interaction terms of $\mathcal{L}$.

Although $\mathcal{L}'_5$ is not explicitly local, as we mentioned, we can make it local by the introduction of auxiliary fields. It turns out that without auxiliary fields there is no solution for a local Lagrangian in any covariant gauge that generates numerators satisfying the BCJ duality. The nonlocality explains the difficulty of stumbling onto this Lagrangian without knowing its desired property ahead of time.

As previously mentioned, $\mathcal{L}'_5$ is identically zero by the color Jacobi identity. To see this we can relabel color indices to obtain

$$\begin{aligned}\mathcal{L}'_5 &= -\frac{1}{2}g^3(f^{a_1a_2b}f^{ba_3c} + f^{a_2a_3b}f^{ba_1c} \\ &\quad + f^{a_3a_1b}f^{ba_2c})f^{ca_4a_5}\partial_{[\mu}A^{a_1}_{\nu]}A^{a_2}_\rho A^{a_3\mu}\frac{1}{\Box}(A^{a_4\nu}A^{a_5\rho}).\end{aligned} \tag{5.4}$$

As apparent in (5.4), the canceling terms have different color factors and thus appear in different channels. For the individual diagrams these terms are nonvanishing. Furthermore, they alter the numerators of the individual diagrams such that the BCJ duality (2.5) is satisfied.

It is interesting to note that there is one other term that can be added to Yang-Mills at five points which preserves the relation (2.5):

$$\begin{aligned}\mathcal{D}_5 &= \frac{-\beta}{2}g^3 f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5}(\partial_{(\mu}A^{a_1}_{\nu)}A^{a_2}_\rho A^{a_3\mu} \\ &\quad + \partial_{(\mu}A^{a_2}_{\nu)}A^{a_3}_\rho A^{a_1\mu} + \partial_{(\mu}A^{a_3}_{\nu)}A^{a_1}_\rho A^{a_2\mu})\frac{1}{\Box} \\ &\quad \times (A^{a_4\nu}A^{a_5\rho}),\end{aligned} \tag{5.5}$$

where β is an arbitrary parameter. $\mathcal{D}_5$ also vanishes identically by the color Jacobi identity. Since $\mathcal{D}_5$ does not serve to correct lower-point contributions to make the BCJ duality relations hold through five points, we do not need to include it. It does however show that there are families of Lagrangians with the desired properties.

C. Toward gravity

Now that we have a Lagrangian that gives the desired numerators n_i for gauge theory (2.10), we use it to construct the tree-level gravity Lagrangian by demanding that it gives diagrams whose numerators are a double copy of the gauge-theory numerators, as in Eq. (2.11). However, as explained above, we need to first bring the Yang-Mills Lagrangian into a cubic form to achieve this. We can do so by introducing an auxiliary field $B^a_{\mu\nu}$:

[7] We are considering only tree level at this point. Therefore we ignore ghost terms.

$$\mathcal{L}_{\rm YM} = \frac{1}{2}A^{a\mu}\Box A^a_\mu + B^{a\mu\nu}B^a_{\mu\nu} - gf^{abc}(\partial_\mu A^a_\nu + B^a_{\mu\nu})A^{b\mu}A^{c\nu}. \tag{5.6}$$

This is equivalent to the ordinary Yang-Mills Lagrangian as we can immediately verify by integrating out $B^a_{\mu\nu}$, i.e. by substituting the equation of motion of $B^a_{\mu\nu}$,

$$B^a_{\mu\nu} = \frac{g}{2}f^{abc}(A^b_\mu A^c_\nu). \tag{5.7}$$

Since the BCJ duality is trivially satisfied through four points, naively one would take the square of this action to obtain a tree-level action for gravity valid through four points. However, since the squaring is with respect to the numerators n_i and not numerator over propagator, n_i/s_α, we need the auxiliary fields to generate the numerators with the inverse propagators directly, instead of multiplying and dividing by inverse propagators afterward. This implies that the auxiliary fields must become dynamical (to generate the required propagator) and that their interactions must contain additional derivatives to produce the inverse propagator necessary to cancel the propagator. At four points this leads to the Lagrangian

$$\mathcal{L}_{\rm YM} = \frac{1}{2}A^{a\mu}\Box A^a_\mu - B^{a\mu\nu\rho}\Box B^a_{\mu\nu\rho} - gf^{abc}(\partial_\mu A^a_\nu + \partial^\rho B^a_{\rho\mu\nu})A^{b\mu}A^{c\nu}, \tag{5.8}$$

where the equation of motion for the auxiliary field $B^a_{\mu\nu\rho}$ becomes

$$\Box B^a_{\mu\nu\rho} = \frac{g}{2}f^{abc}\partial_\mu(A^b_\nu A^c_\rho). \tag{5.9}$$

We are now ready to construct a gravity action that gives the correct four-point amplitude. We begin in momentum space, where the identification

$$A_\mu(k)\tilde{A}_\nu(k) \rightarrow h_{\mu\nu}(k) \tag{5.10}$$

can be trivially implemented. We first demonstrate how the gravity action can be derived from the four-point Yang-Mills Lagrangian. We write the Yang-Mills Lagrangian in momentum space. Since gravity does not have any color indices, we encode the information of the structure constants in the antisymmetrization and cyclicity of the interaction terms. We drop the coupling constant for now; it can easily be restored in the final gravity action. We arrive at

$$\begin{aligned}\mathcal{S}_{\rm YM} \sim \frac{1}{2}\int & d^4k_1 d^4k_2 \delta^4(k_1+k_2)k_2^2[A^\mu(k_1)A_\mu(k_2) \\ & - 2B^{\mu\nu\rho}(k_1)B_{\mu\nu\rho}(k_2)] + \int d^4k_1 d^4k_2 d^4k_3 \\ & \times \delta^4(k_1+k_2+k_3)P_6\{[k_{1\mu}A_\nu(k_1) \\ & + k_1^\rho B_{\rho\mu\nu}(k_1)]A^\mu(k_2)A^\nu(k_3)\},\end{aligned} \tag{5.11}$$

where P_6 indicates a sum over all permutations of $\{k_1, k_2, k_3\}$ with the antisymmetrization signs included. From here, we can read off a gravity action valid through four points:

$$\mathcal{S}_{\rm grav} = \mathcal{S}_{\rm kin} + \mathcal{S}_{\rm int}, \tag{5.12}$$

with

$$\begin{aligned}\mathcal{S}_{\rm kin} \sim \frac{1}{4}\int & d^4k_1 d^4k_2 \delta^4(k_1+k_2)k_2^2 \\ & \times [A^\mu(k_1)A_\mu(k_2) - 2B^{\mu\nu\rho}(k_1)B_{\mu\nu\rho}(k_2)] \\ & \times [\tilde{A}^\sigma(k_1)\tilde{A}_\sigma(k_2) - 2\tilde{B}^{\sigma\tau\lambda}(k_1)\tilde{B}_{\sigma\tau\lambda}(k_2)], \\ \mathcal{S}_{\rm int} \sim \int & d^4k_1 d^4k_2 d^4k_3 \delta^4(k_1+k_2+k_3)P_6 \\ & \times \{[k_{1\mu}A_\nu(k_1) + k_1^\rho B_{\rho\mu\nu}(k_1)]A^\mu(k_2)A^\nu(k_3)\}P_6 \\ & \times \{[k_{1\lambda}\tilde{A}_\sigma(k_1) + k_1^\tau \tilde{B}_{\tau\lambda\sigma}(k_1)]\tilde{A}^\lambda(k_2)\tilde{A}^\sigma(k_3)\}.\end{aligned} \tag{5.13}$$

In extracting the Feynman rules from this action the left and right fields each contract independently. For example, for the propagators we have

$$\begin{aligned}\langle A_\mu(k_1)\tilde{A}_\rho(k_1)A_\nu(k_2)\tilde{A}_\sigma(k_2)\rangle &= \frac{i\eta_{\mu\nu}\eta_{\rho\sigma}}{k_1^2}\delta^4(k_1+k_2), \\ \langle A_\mu(k_1)\tilde{B}_{\rho\sigma\tau}(k_1)A_\nu(k_2)\tilde{B}_{\eta\kappa\lambda}(k_2)\rangle &= -\frac{i\eta_{\mu\nu}\eta_{\rho\eta}\eta_{\sigma\kappa}\eta_{\tau\lambda}}{2k_1^2} \times \delta^4(k_1+k_2).\end{aligned} \tag{5.14}$$

By construction, this action will give the correct three- and four-graviton tree-level amplitudes.

We note that one can construct the coordinate-space action by combining the left-right fields as

$$\begin{aligned} A^\mu\tilde{A}^\nu &\rightarrow h^{\mu\nu}, & A^\mu\tilde{B}^{\nu\rho\sigma} &\rightarrow g^{\mu\nu\rho\sigma}, \\ B^{\mu\rho\sigma}\tilde{A}^\nu &\rightarrow \tilde{g}^{\mu\rho\sigma\nu}, & B^{\mu\rho\sigma}\tilde{B}^{\nu\tau\lambda} &\rightarrow f^{\mu\rho\sigma\nu\tau\lambda},\end{aligned} \tag{5.15}$$

where $h^{\mu\nu}$ is the physical field, which includes the graviton, antisymmetric tensor, and dilaton. The kinetic terms in x space take the form

$$\begin{aligned}\mathcal{S}_{\rm kin} = -\frac{1}{2}\int d^Dx[& h^{\mu\nu}\Box h_{\mu\nu} - 2g^{\mu\nu\rho\sigma}\Box g_{\mu\nu\rho\sigma} \\ & - 2\tilde{g}^{\mu\nu\rho\sigma}\Box\tilde{g}_{\mu\nu\rho\sigma} + 4f^{\mu\nu\rho\sigma\tau\lambda}\Box f_{\mu\nu\rho\sigma\tau\lambda}].\end{aligned} \tag{5.16}$$

The interaction terms can similarly be constructed, but we do not display them here as there are 144 of them.

To move on to five points, we need to introduce a new set of auxiliary fields to rewrite the nonlocal terms in a local and cubic form. We simply give the result:

$$\begin{aligned}\mathcal{L}'_5 \to{}& Y^{a\mu\nu}\Box X^a_{\mu\nu} + D^{a\mu\nu\rho}_{(3)}\Box C^a_{(3)\mu\nu\rho} \\ &+ D^{a\mu\nu\rho\sigma}_{(4)}\Box C^a_{(4)\mu\nu\rho\sigma} + g f^{abc}(Y^{a\mu\nu}A^b_\mu A^c_\nu \\ &+ \partial_\mu D^{a\mu\nu\rho}_{(3)} A^b_\nu A^c_\rho - \frac{1}{2}\partial_\mu D^{a\mu\nu\rho\sigma}_{(4)}\partial_{[\nu}A^b_{\rho]}A^c_\sigma) \\ &+ g f^{abc} X^{a\mu\nu}\left(\frac{1}{2}\partial_\rho C^{b\rho\sigma}_{(3)\mu}\partial_{[\sigma}A^c_{\nu]} + \partial_\rho C^{b\rho\sigma}_{(4)\nu[\mu}A^c_{\sigma]}\right).\end{aligned} \tag{5.17}$$

Note that these new auxiliary fields do not couple to $B^{\mu\nu\rho}$. It is now straightforward to transform (5.17) to momentum space and, through the squaring process, obtain a gravity Lagrangian valid through five points.

D. Beyond five points

As we increase the number of legs we find new violations of manifest BCJ duality, so we need to add further terms. We have constructed a six-point correction to the interactions so that the Lagrangian generates numerators with manifest BCJ duality.

The general structure of $\mathcal{L}'_6$ is similar to that of $\mathcal{L}'_5$; after relabeling color indices, we can arrange $\mathcal{L}'_6$ to vanish manifestly via two Jacobi identities:

$$\begin{aligned}0 &= (f^{a_1a_2b}f^{ba_3c} + f^{a_2a_3b}f^{ba_1c} + f^{a_3a_1b}f^{ba_2c})f^{cda_6}f^{da_4a_5}, \\ 0 &= f^{a_1a_2b}(f^{ba_3c}f^{cda_6} + f^{bdc}f^{ca_6a_3} + f^{ba_6c}f^{ca_3d})f^{da_4a_5}.\end{aligned} \tag{5.18}$$

The first of these two color factors is contracted with 59 different terms having a schematic form[8]

$$\frac{1}{\Box}(A^{a_1}A^{a_2}A^{a_3})\frac{1}{\Box}(A^{a_4}A^{a_5})A^{a_6}, \tag{5.19}$$

where the parentheses indicate which fields the $\frac{1}{\Box}$ acts on. The second color factor contracts with 49 terms of the form

$$\frac{1}{\Box}(A^{a_1}A^{a_2})A^{a_3}\frac{1}{\Box}(A^{a_4}A^{a_5})A^{a_6}. \tag{5.20}$$

In each term, there are an additional two partial derivatives in the numerator acting on the gauge fields. The large number of terms arises from the many different ways to contract the eight Lorentz indices. We have found that the coefficients of these 108 terms depend on 30 distinct free parameters, in addition to the β that showed up at five points (5.5). Thus, there is a 30-parameter family of self-BCJ six-point interactions.

We anticipate that this structure continues to higher orders, with the addition of new vanishing combinations of terms. We have seen no indication that the Lagrangian will terminate; for each extra leg that we add to an amplitude, we will likely need to add more terms to the action to ensure that the diagrams satisfy the BCJ duality. A key outstanding problem is to find a pattern or symmetry that would enable us to write down the all-order BCJ-corrected action without having to analyze each n-point level individually.

If the construction of a Lagrangian to all orders succeeds, it would be a fully off-shell realization of the BCJ duality at the classical level. It would be interesting to then study nonperturbative phenomena such as instantons using this Lagrangian to see whether BCJ duality and the squaring relations can elucidate physics beyond the regime of scattering amplitudes. Our off-shell construction suggests that BCJ duality may also work at loop level. Of course, one would need to account for the ghost structure and, more importantly, demonstrate that the loop amplitudes so constructed do indeed have the desired duality properties manifest.

VI. A FEW SIMPLE IMPLICATIONS

In this short section, we point out that the BCJ duality immediately leads to some novel forms of gauge and gravity amplitudes. Del Duca, Dixon, and Maltoni [18] presented an alternative color decomposition from the usual one,

$$\mathcal{A}^{\mathrm{tree}}_n(1,2,\ldots,n) = g^{n-2}\sum_{\sigma\in S_{n-2}} c_{1,\sigma_2,\ldots,\sigma_{n-1},n} \times A^{\mathrm{tree}}_n(1,\sigma_2,\ldots,\sigma_{n-1},n), \tag{6.1}$$

where $\mathcal{A}^{\mathrm{tree}}_n$ is the full color-dressed n-gluon amplitude, and the A^{tree}_n are the usual color-ordered partial gauge-theory amplitudes. The sum runs over all permutations of $n-2$ legs. The color factors are

$$c_{1,\sigma_2,\ldots,\sigma_{n-1},n} \equiv \tilde{f}^{a_1a_{\sigma_2}x_1}\tilde{f}^{x_1a_{\sigma_3}x_2}\cdots\tilde{f}^{x_{n-3}a_{\sigma_{n-1}}a_n}. \tag{6.2}$$

Diagrammatically, this color factor is associated with Fig. 4. This form is derived starting from Eq. (2.1) and using color Jacobi identities along with the Kleiss-Kuijf relations, which are equivalent to the self-antisymmetry of the diagrammatic numerator factors.

A simple observation is that when diagram numerators are chosen to satisfy the BCJ duality, they have precisely the same algebraic structure as color factors. Thus, we can immediately write a dual formula decomposing the full amplitude into numerators instead of color factors:

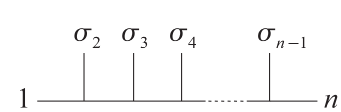

FIG. 4. A graphical representation of the color basis $c_{1,\sigma_2,\ldots,\sigma_{n-1},n}$ introduced in Ref. [18]. Each vertex represents a structure constant $\tilde{f}^{abc}$, while each bond indicates contracted indices between the $\tilde{f}^{abc}$. This is also precisely the diagram associated with the kinematic numerator $n_{1,\sigma_2,\ldots,\sigma_{n-1},n}$.

[8] Momentum conservation can alter these counts but we give them as an indication of the number of terms involved.

$$\mathcal{A}_n^{\text{tree}}(1, 2, \ldots, n) = g^{n-2} \sum_{\sigma \in S_{n-2}} n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \times A_n^{\text{scalar}}(1, \sigma_2, \ldots, \sigma_{n-1}, n), \quad (6.3)$$

where A_n^{scalar} is a dual partial scalar amplitude with ordered legs obtained by replacing the gauge-theory numerator factors with group-theory color factors. The numerator factors $n_{1,\sigma_2,\ldots,\sigma_{n-1},n}$ are the numerators of the diagrams displayed in Fig. 4. Note that all other numerators can be expressed as linear combinations of the $n_{1,\sigma_2,\ldots,\sigma_{n-1},n}$ through the duality relations, and the form (6.3) for the gauge-theory amplitude makes this property manifest, after expanding A_n^{scalar} in terms of diagrams. This form is related to an unusual color decomposition of gauge-theory amplitudes which follows from applying KLT relations to the low-energy limit of heterotic strings [4].

Similarly, this immediately gives us a new representation for graviton amplitudes in terms of gauge-theory amplitudes,

$$\mathcal{M}_n^{\text{tree}}(1, 2, \ldots, n) = i\kappa^{n-2} \sum_{\sigma \in S_{n-2}} n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \times A_n^{\text{tree}}(1, \sigma_2, \ldots, \sigma_{n-1}, n), \quad (6.4)$$

where A_n^{tree} is the usual gauge-theory color-ordered amplitude.

VII. CONCLUSIONS

In this paper we investigated consequences of a curious duality between color and kinematic numerators of gauge-theory diagrams [5]. In particular, using BCFW recursion relations, we proved that the duality implies that numerators of gravity amplitudes are just a product of two gauge-theory numerators, as conjectured in Ref. [5]. The inductive proof of these "squaring relations" makes use of a generalized gauge invariance of gauge-theory amplitudes, which we use to rearrange the gravity BCFW recursion relations. We also explained how the proof extends to other theories including $\mathcal{N} = 8$ supergravity as two copies of $\mathcal{N} = 4$ super-Yang-Mills theory. We showed that the squaring relations even hold in a generalized, asymmetric form, in which only one set of gauge-theory numerators is required to satisfy the BCJ duality. If we assume the duality works at loop level as well, the unitarity method straightforwardly allows us to conclude that the squaring relations hold at loop level [13].

In a complementary approach we described the construction of a Lagrangian whose Feynman diagrams obey the duality. Remarkably, through at least six points, and presumably for any number of points, the Lagrangian differs from the usual Feynman-gauge Lagrangian by terms that vanish identically by the color Jacobi identity. The extra terms, however, have the effect of shuffling terms between diagrams to make the duality hold. These extra higher-point terms are necessarily nonlocal, but with the use of auxiliary fields we can make the Lagrangian local at least through six points.

For the case of $\mathcal{N} = 4$ super-Yang-Mills theory we took some initial steps to recast the BCJ duality into a form where both the duality and supersymmetry are manifest. It would be interesting to study this further, and to connect this to the recently uncovered [35] Grassmannian forms of tree-level scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory.

Another interesting problem would be to see if we can construct a complete Lagrangian respecting the duality valid to all orders. The key open problem for doing so is to find a pattern in the additional terms that generalizes to higher points. Although we have constructed a Lagrangian valid through six points (not presented here), it contains 108 terms and 30 parameters. Clearly, one should first resolve its seeming complexity before attempting to construct an all-order form. An intriguing question is whether the form of the Lagrangian can be fixed by imposing symmetry requirements (prior to applying color Jacobi identities which make the additional terms vanish). If an all-order form of the off-shell gauge-theory Lagrangian and the corresponding double-copy gravity Lagrangian could indeed be constructed, then it would be natural to try to find a mapping between their classical solutions. Such Lagrangians would thus lend themselves to addressing nonperturbative implications of the BCJ duality.

Finally, we note that our partial construction of Lagrangians that generate diagrams respecting the gauge-theory duality between color and kinematics and the gravity double-copy property provides new evidence that these properties may extend to loop level as well. Indeed, this does appear to be the case, as demonstrated in a concurrent paper [13]. We hope that a combined effort of on-shell methods, Lagrangian approaches, and string theory will shed further light on the origins, scope, and implications of the BCJ duality.

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