Detailed discussions and calculations of quantum Regge calculus of Einstein-Cartan theory

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This article presents detailed discussions and calculations of the recent paper "Quantum Regge calculus of Einstein-Cartan theory" in [9]. The Euclidean space-time is discretized by a four-dimensional simplicial complex. We adopt basic tetrad and spin-connection fields to describe the simplicial complex. By introducing diffeomorphism and local Lorentz invariant holonomy fields, we construct a regularized Einstein-Cartan theory for studying the quantum dynamics of the simplicial complex and fermion fields. This regularized Einstein-Cartan action is shown to properly approach to its continuum counterpart in the continuum limit. Based on the local Lorentz invariance, we derive the dynamical equations satisfied by invariant holonomy fields. In the mean-field approximation, we show that the averaged size of 4-simplex, the element of the simplicial complex, is larger than the Planck length. This formulation provides a theoretical framework for analytical calculations and numerical simulations to study the quantum Einstein-Cartan theory.

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I. INTRODUCTION

Since the Regge calculus [1,2] was proposed for the discretization of gravity theory in 1961, many progresses have been made in the approach of quantum Regge calculus [3,4] and its variant dynamical triangulations [5]. In particular, the renormalization-group treatment is applied to discuss any possible scale dependence of gravity [3]. Inspired by the success of lattice regularization of non-Abelian gauge theories, the gauge-theoretic formulation [6] of quantum gravity using connection variables on a flat hypercubic lattice of the space-time was studied in the Lagrangian formalism. The canonical quantization approaches to the Regge calculus in Hamiltonian formulation are studied in Ref. [7]. A locally finite model for gravity has been recently proposed [8]. All these studies are very important steps to understand the Einstein general relativity for gravitational fields in the framework of *quantum field theory*. In the brief paper [9] based on the scenario of quantum Regge calculus, we present a diffeomorphism and local Lorentz invariant (i.e., *local* gauge-invariant) regularization and quantization of Euclidean Einstein-Cartan (EC) theory. Detailed calculations and discussions are presented in this article.

The four-dimensional Euclidean space-time is discretized by a simplicial complex, analogously to the formulation of the Regge calculus. In the framework of the Einstein-Cartan theory, we adopt basic gravitational variables, i.e., a pair of tetrad and spin-connection fields to describe the simplicial complex. Introducing diffeomorphism and local Lorentz invariant (i.e., *local* gauge-invariant) holonomy fields in terms of tetrad and spin-connection fields along loops, we propose an invariantly regularized EC theory for the dynamics of simplicial complex, which couples to fermion spinor fields. We show that in the continuum limit when the wavelengths of tetrad and spin-connection fields are much larger than the Planck length, this regularized EC action properly approaches to the continuum EC action. The quantum dynamics of the simplicial complex is described by the Euclidean partition function that is a Feynman path-integral overall quantum tetrad, spin connection, and fermion fields with the weight of regularized EC action. Based on *local* gauge invariance, we derive the dynamical equations satisfied by invariant holonomy fields of tetrad, spin-connection, and fermion fields. In the mean-field approximation, we show the averaged size of 4-simplex (and its 3-simplex and 2-simplex), elements of the simplicial complex, has to be larger than the Planck length. This formulation provides a theoretical framework for analytical calculations, in particular, numerical simulations to study the Einstein-Cartan theory as a *quantum field theory*.

This article is organized as follows: In Sec. II, we give a brief review of the continuum EC theory. In Sec. III, we discuss the regularized EC theory based on (1) the description of simplicial complex by tetrad and spin-connection fields; (2) parallel transport equations in simplicial complex; (3) invariant holonomy fields and regularized EC action and their continuum limit; (4) the Euclidean partition function. In Secs. IV and V, we study chiral gauge symmetric bilinear and quadralinear-fermion actions, and derive dynamical equations for holonomy fields. In Sec. VI, we adopt the method of the mean-field approximation to show the averaged size of the 4-simplex has to be larger than the Planck length. In the last section, we give some concluding remarks, and detailed calculations are arranged in Appendices A, B, C, D, and E.

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II. CONTINUUM EINSTEIN-CARTAN THEORY

The basic gravitational variables in the Einstein-Cartan theory constitute a pair of tetrad and spin-connection fields $[e_\mu{}^a(x), \omega^{ab}_\mu(x)]$, whose Dirac-matrix values

$$e_\mu(x) = e_\mu{}^a(x)\gamma_a \quad \text{and} \quad \omega_\mu(x) = \omega^{ab}_\mu(x)\sigma_{ab}. \tag{1}$$

The fields $e_\mu{}^a(x)$ and $\omega^{ab}_\mu(x)$ are 1-form real fields on the four-dimensional Euclidean space-time $\mathcal{R}^4$, taking values, respectively, in the local Lorentz vector space $V_{\mathcal{L}}$ and in the Lie algebra $so(4)$ of the Lorentz group $SO(4)$ of the linear transformations of $V_{\mathcal{L}}$ preserving $\delta^{ab} = (+, +, +, +)$. In this local Lorentz vector space $V_{\mathcal{L}}$, fermions are spinor fields $\psi(x)$, Dirac γ matrices obey

$$\{\gamma_a, \gamma_b\} = -2\delta_{ab}, \tag{2}$$

$\gamma_a^\dagger = -\gamma_a$ and $\gamma_a^2 = -1$ $(a = 0, 1, 2, 3)$; the Hermitian γ_5 matrix

$$\gamma_5 = \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_0\gamma_1\gamma_2\gamma_3, \tag{3}$$

$\gamma_5^\dagger = \gamma_5$ and $\gamma_5^2 = 1$; the Hermitian spinor matrix,

$$\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]. \tag{4}$$

Totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{abcd}e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d$. The space-time metric of four-dimensional Euclidean manifold $\mathcal{R}^4$ is

$$g_{\mu\nu}(x) = e_\mu{}^a(x)e_\nu{}^b(x)\delta_{ab} = -\tfrac{1}{2}\{e_\mu, e_\nu\}. \tag{5}$$

And the Lorentz scalar components of the metric tensor are then simply

$$\delta_{ab} = g_{\mu\nu}e^\mu{}_a e^\nu{}_b, \tag{6}$$

where the inverse of the tetrad fields $e^\mu{}_a e_\nu{}^a = \delta^\mu{}_\nu$ and $e_\mu{}^b e^\mu{}_a = \delta^b{}_a$.

Two gauge invariances due to the equivalence principle have to be respected: (1) the diffeomorphism invariance under the general coordinate transformation $x \to x'(x)$; (2) the *local* gauge invariance under the local Lorentz coordinate transformation $\xi(x) \to \xi'(x)$, i.e.,

$$\xi'^a(x) = [\Lambda(x)]^a_b \xi^b(x). \tag{7}$$

Under the local Lorentz coordinate transformation (7), the finite *local* gauge transformation is

$$\begin{aligned} \mathcal{V}(\xi) &= \exp i[\theta^{ab}(\xi)\sigma_{ab}] \in SO(4), \\ \mathcal{V}(\xi)\gamma_a\mathcal{V}^\dagger(\xi) &= [\Lambda^{-1}(x)]^b_a\gamma_b, \end{aligned} \tag{8}$$

where $\theta^{ab}(\xi)$ is the antisymmetric tensor and an arbitrary function of $\xi = \xi(x)$. The Dirac-matrix valued fields e_μ, ω_μ and fermion spinor field ψ are transformed as follows:

$$e_\mu(\xi) \to e'_\mu(\xi) = \mathcal{V}(\xi)e_\mu(\xi)\mathcal{V}^\dagger(\xi); \tag{9}$$

$$\begin{aligned} \omega_\mu(\xi) &\to \omega'_\mu(\xi) \\ &= \mathcal{V}(\xi)\omega_\mu(\xi)\mathcal{V}^\dagger(\xi) + \mathcal{V}(\xi)\partial_\mu\mathcal{V}^\dagger(\xi), \end{aligned} \tag{10}$$

$$\psi(\xi) \to \psi'(\xi) = \mathcal{V}(\xi)\psi(\xi); \tag{11}$$

$$\mathcal{D}'_\mu = \mathcal{V}(\xi)\mathcal{D}_\mu\mathcal{V}^\dagger(\xi), \tag{12}$$

where the derivative $\partial_\mu = e^a_\mu(\partial/\partial\xi^a)$, the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - ig\omega_\mu(\xi), \tag{13}$$

and g is the gauge coupling. Corresponding to the finite *local* gauge transformations (9)–(11), infinitesimal *local* gauge transformations for fields e_μ, ω_μ and ψ are

$$\delta e_\mu(\xi) = \theta^{ab}(\xi)d_{ab,c}e^c_\mu(\xi); \tag{14}$$

$$\delta\omega_\mu(\xi) = 2\gamma_5\epsilon_{abcd}\omega^{ab}_\mu\theta^{cd}(\xi) - i\sigma_{ab}\partial_\mu\theta^{ab}(\xi); \tag{15}$$

$$\delta\psi(\xi) = i\theta^{ab}(\xi)\sigma_{ab}\psi(\xi), \tag{16}$$

where

$$d_{ab,c} = i[\sigma_{ab}, \gamma_c] = 2(\delta_{bc}\gamma_a - \delta_{ac}\gamma_b), \tag{17}$$

and we use the commutator relation

$$\{\sigma^{\alpha\beta}, \sigma^{\delta\gamma}\} = -2i\gamma^5\epsilon^{\alpha\beta\delta\gamma}, \tag{18}$$

to obtain Eq. (15).

In an $SU(2)$ gauge theory, gauge field $A_a(\xi_E)$ can be viewed as a connection $\int A_a(\xi_E)d\xi^a_E$ on the global flat manifold. On a locally flat manifold, the spin connection $\omega_\mu dx^\mu = \omega_a(\xi)d\xi^a$, where $\omega_a(\xi) = \omega_\mu e^\mu{}_a$, one can identify that the spin-connection field $\omega_\mu(x)$ or $\omega_a(\xi)$ is the gravity analog of gauge field and its *local* curvature is given by

$$R^{ab} = d\omega^{ab} - g\omega^{ae} \wedge \omega^b{}_e, \tag{19}$$

and the Dirac-matrix valued curvature $R_{\mu\nu} = R^{ab}_{\mu\nu}\sigma_{ab}$. Under the gauge transformation (9) and (10),

$$R'^{ab} = \mathcal{V}(\xi)R^{ab}(\xi)\mathcal{V}^\dagger(\xi). \tag{20}$$

The diffeomorphism invariance under the general coordinate transformation $x \to x'(x)$ is preserved by all derivatives and d-form fields on $\mathcal{R}^4$ made to be coordinate scalars with the help of tetrad fields $e_\mu{}^a = \partial\xi^a/\partial x^\mu$ (see Ref. [10]). The diffeomorphism and *local* gauge-invariant EC action for gravity coupling to fermions is given by the Palatini action S_P and host modification S_H for the gravitational field,

$$S_{\rm EC}(e, \omega) = S_P(e, \omega) + S_H(e, \omega) + S_F(e, \omega, \psi), \tag{21}$$

$$S_P(e, \omega) = \frac{1}{4\kappa}\int d^4x \det(e)\epsilon_{abcd}e^a \wedge e^b \wedge R^{cd}, \tag{22}$$

$$S_H(e, \omega) = \frac{1}{2\kappa\tilde{\gamma}} \int d^4x \det(e) e_a \wedge e_b \wedge R^{ab}, \tag{23}$$

and fermion action S_F (see Refs. [11,12]),

$$S_F(e, \omega, \psi) = \frac{1}{2} \int d^4x \det(e)[\bar{\psi} e^\mu \mathcal{D}_\mu \psi + \text{H.c.}], \tag{24}$$

where $\kappa \equiv 8\pi G$, the Newton constant $G = 1/m^2_{\text{Planck}}$, $\det(e)$ is the Jacobi of mapping $x \to \xi(x)$ and the integration $\int d^4x \equiv \int_{\mathcal{R}^4} d^4x$. The complex Ashtekar connection [13] with reality condition and the real Barbero connection [14] are linked by a canonical transformation of the connection with a finite complex Immirzi parameter $\tilde{\gamma} \neq 0$ [15], which is crucial for *loop quantum gravity* [16].

Classical equations of motion can be obtained by the stationarity of the EC action (21) under variations (9)–(11),

$$\delta S_{\text{EC}}(e, \omega, \psi) = \frac{\delta S_{\text{EC}}}{\delta e_\mu} \delta e_\mu + \frac{\delta S_{\text{EC}}}{\delta \psi(x)} \delta \psi(x) + \frac{\delta S_{EC}}{\delta \omega_\mu} \delta \omega_\mu = 0. \tag{25}$$

From Eqs. (14)–(16), we find that Eq. (25) can be expressed in terms of independent bases γ_5, γ_μ, and σ_{ab} of the Dirac matrices. Therefore, for arbitrary function $\theta_{ab}(\xi)$, Eq. (25) leads to the following three equalities:

$$\frac{\delta S_{\text{EC}}}{\delta \psi} = 0; \qquad \frac{\delta S_{\text{EC}}}{\delta e_\mu} = 0; \qquad \frac{\delta S_{\text{EC}}}{\delta \omega_\mu} = 0. \tag{26}$$

The first and second equations, respectively, lead to the Dirac equation,

$$e^\mu \mathcal{D}_\mu \psi(x) = 0, \tag{27}$$

and the Einstein equation

$$\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}[\omega(e)] = \kappa \bar{\psi}(x)(e \wedge \mathcal{D}) \psi(x), \tag{28}$$

where the energy-momentum tensor is

$$\bar{\psi}(e \wedge \mathcal{D})\psi \equiv \tfrac{1}{2} \bar{\psi}[e_\mu \mathcal{D}_\nu - \mathcal{D}_\mu e_\nu]\psi. \tag{29}$$

The gauge invariance of the EC action (21) under the gauge transformation (15) leads to the third constraint equation $\delta S_{\text{EC}}/\delta \omega_\mu = 0$ of Eq. (26), which is the Cartan structure equation,

$$de^a - g\omega^{ab} \wedge e_b - T^a = 0, \tag{30}$$

where the nonvanishing torsion field,

$$T^a = \kappa g e_b \wedge e_c J^{ab,c}, \tag{31}$$

relating to the fermion spin current

$$J^{ab,c} = i\bar{\psi}\{\sigma^{ab}, \gamma^c\}\psi = \epsilon^{abcd} \bar{\psi} \gamma_d \gamma^5 \psi, \tag{32}$$

$$\{\sigma^{ab}, \gamma^c\} = i\epsilon^{abcd} \gamma^5 \gamma_d. \tag{33}$$

The fermion spin current (32) contributes only to the pseudotrace axial vector of torsion tensor, which is one of irreducible parts of torsion tensor [17]. The solution to Eq. (30) is

$$\omega^{ab}_\mu = \omega^{ab}_\mu(e) + \tilde{\omega}^{ab}_\mu, \qquad \tilde{\omega}^{ab}_\mu = \kappa g e^c_\mu J^{ab}{}_c, \tag{34}$$

where the connection $\omega^{ab}_\mu(e)$ obeys Eq. (30) for torsion-free case $T^a = 0$,

$$de^a - g\omega^{ab}(e) \wedge e_b = 0. \tag{35}$$

Replacing the spin-connection field ω^{ab}_μ in the Einstein-Cartan action (22) and (24), by Eq. (34),

$$S_P[e, \omega] \to S_P[e, \omega(e)] + \kappa g^2 \int d^4x \det(e)(\bar{\psi}\gamma^d\gamma^5\psi) \times (\bar{\psi}\gamma_d\gamma^5\psi); \tag{36}$$

$$S_F[e, \omega, \psi, \bar{\psi}] \to S_F[e, \omega(e), \psi, \bar{\psi}] + 2\kappa g^2 \int d^4x \det(e) \times (\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi), \tag{37}$$

one obtains the well-known Einstein-Cartan theory: the standard tetrad action of torsion-free gravity coupling to fermions with four-fermion interactions,

$$S_{\text{EC}}[e, \omega(e), \psi, \bar{\psi}] = S_P[e, \omega(e)] + S_F[e, \omega(e), \psi, \bar{\psi}] + 3\kappa g^2 \int d^4x \det(e)(\bar{\psi}\gamma^d\gamma^5\psi) \times (\bar{\psi}\gamma_d\gamma^5\psi). \tag{38}$$

Note that the four-fermion interaction actually is the coupling of two fermion spin currents (32). Taking into account the host action (23), one obtains

$$S_{\text{EC}}[e, \omega(e), \psi] = S_P[e, \omega(e)] + S_H[e, \omega(e)] + S_F[e, \omega(e), \psi] + S_{4F}(e, \psi), \tag{39}$$

$$S_{4F}(e, \psi) = 3\zeta\kappa g^2 \int d^4x \det(e)(\bar{\psi}\gamma^d\gamma^5\psi)(\bar{\psi}\gamma_d\gamma^5\psi), \tag{40}$$

where $\zeta = \tilde{\gamma}^2/(\tilde{\gamma}^2 + 1)$ [18]. Using the commutator relations (18) and $[\sigma_{ab}, \gamma_5] = 0$, one can show that $(\bar{\psi}\gamma_d\gamma^5\psi)$ is a pseudovector and (40) is invariant under the gauge transformation (11).

As we can see from Eqs. (24) to (39), the bilinear term (24) of massless fermion fields coupled to the spin-connection field (13) is bound to yield a nonvanishing torsion field T^a (30), which is local and static (see, for example, Refs. [12,19]). As a result, the spin-connection ω_μ is no longer torsion-free and acquires a torsion-related spin connection $\tilde{\omega}^{ab}_\mu$ (34), in addition to the torsion-free spin connection $\omega^{ab}_\mu(e)$. The torsion-related spin connection $\tilde{\omega}^{ab}_\mu$ is related to the fermion spin current (32). The quadratic term of the spin-connection field ω in the curvature (19) and the coupling between the spin-connection field ω and fermion spin current in Eqs. (13) and (24) lead to the quadralinear terms of fermion fields in Eqs. (36) and (37). Another way to see this is to treat the static torsion-related spin connection $\tilde{\omega}^{ab}_\mu$ (34) as a static auxiliary field, which has its quadratic term and linear coupling to the spin-current of fermion fields. Performing the Gaussian integral of the static auxiliary field, we exactly obtain the quadralinear term (40), in addition to the torsion-free EC action.

The action (21) and classical Eqs. (27)–(30) can be separated into left- and right-handed parts [20], with respect to the local $SU_L(2)$ and $SU_R(2)$ symmetries of the Lorentz group $SO(4) = SU_L(2) \otimes SU_R(2)$. This can be shown by writing Dirac fermions $\psi = \psi_L + \psi_R$, where Weyl fermions $\psi_{L,R} \equiv P_{L,R}\psi$, $P_{L,R} = (1 \mp \gamma_5)/2$; and Dirac-matrix valued tetrad field $e^\mu = e^\mu_L + e^\mu_R$, $e^\mu_{L,R} \equiv P_{L,R}e^\mu$, as well as Dirac-matrix valued spin-connection fields $\omega_\mu = \omega^\mu_L + \omega^\mu_R$, $\omega^\mu_{L,R} \equiv P_{L,R}\omega^\mu$.

III. THE REGULARIZED EINSTEIN-CARTAN THEORY

A. Simplicial complex

The four-dimensional Euclidean manifold $\mathcal{R}^4$ is discretized as an ensemble of $\mathcal{N}_0$ space-time points (vertexes) "$x \in \mathcal{R}^4$" and $\mathcal{N}_1$ links (edges) "$l_\mu(x)$" connecting two neighboring vertexes. This ensemble forms a simplicial manifold $\mathcal{M}$ embedded into the $\mathcal{R}^4$. The way to construct a simplicial manifold depends also on the assumed topology of the manifold, which gives geometric constrains on the numbers of subsimplices ($\mathcal{N}_0, \mathcal{N}_1, \ldots$, see Ref. [5]). In this article, analogously to the simplicial manifold adopted by the Regge calculus we consider the simplicial manifold $\mathcal{M}$ as a simplicial complex, whose elementary building block is a 4-simplex (pentachoron). The 4-simplex has five vertexes—0-simplex (a space-time point "x"), five "faces"—3-simplex (a tetrahedron), and each 3-simplex has four faces—2-simplex [a triangle $h(x)$], and each 2-simplex has three faces—1-simplex [an edge or a link "$l_\mu(x)$"]. Different configurations of the simplicial complex correspond to variations of relative vertex-positions $\{x\}$, edges "$\{l_\mu(x)\}$" and "deficit angles" associating to 2-simplices $h(x)$. These configurations will be described by the configurations of dynamical tetrad fields $e_\mu(x)$ and spin-connection fields $\omega_\mu(x)$ assigned to 1-simplexes (edges) of the simplicial complex in this article. We are not clear now how to relate configurations of fields $e_\mu(x)$ and $\omega_\mu(x)$ to topological constrained configurations of the simplicial complex in dynamical triangulations.

1. Edges: 1-simplexes

The edge (1-simplex) denoted by (x, μ), connecting two neighboring vertexes labeled by x and $x + a_\mu$, can be represented as a four-vector field $l_\mu(x)$, defined at the vertex "x" by its forward direction μ pointing from x to $x + a_\mu$ and its length

$$a_\mu(x) \equiv |l_\mu(x)| \neq 0, \tag{41}$$

which is the distance between two vertexes x and $x + a_\mu$. The fundamental tetrad field $e_\mu(x)$ is assigned to each edge (1-simplex) of the simplicial complex to describe the edge location "x," direction "μ" and length $a_\mu(x)$. We use the tetrad field $e_\mu(x)$, defined at the vertex x, to characterize the edge (1-simplex) $l_\mu(x)$

$$l_\mu(x) \equiv a e_\mu(x), \tag{42}$$

where the Planck length $a \equiv (8\pi G)^{1/2} = \kappa^{1/2}$, and

$$|l_\mu(x)| \equiv \frac{a}{2}\{|\mathrm{tr}[e_\mu(x) \cdot e_\mu(x)]|\}^{1/2}. \tag{43}$$

By definition, either $l_\mu(x)$ or $e_\mu(x)$ is a Dirac-matrix valued four-vector field, defined at the vertex "x."

2. Triangles: 2-simplexes

We consider an orienting 2-simplex (triangle) (see Fig. 1). This 2-simplex (triangle) has three edges connecting three neighboring vertexes that are labeled by x, $x + a_\mu$ and $x + a_\nu$. This triangle (2-simplex) has two orientations: (i) the anti-clocklike $h(x)$ $[x \overset{\mu}{\mapsto} x + a_\mu \overset{\rho}{\mapsto} x + a_\nu \overset{\nu}{\mapsto} x]$ and (ii) the clocklike $h^\dagger(x)$ $[x \overset{-\nu}{\mapsto} x + a_\nu \overset{-\rho}{\mapsto} x + a_\mu \overset{-\mu}{\mapsto} x]$.

Along the triangle path of the anti-clocklike 2-simplex $h(x)$ $[x \overset{\mu}{\mapsto} x + a_\mu \overset{\rho}{\mapsto} x + a_\nu \overset{\nu}{\mapsto} x]$, three edges and their forward directions are represented by: (1) $l_\mu(x)$ and μ pointing from x to $x + a_\mu$; (2) $l_\rho(x + a_\mu)$ and ρ pointing from $x + a_\mu$ to $x + a_\nu$; (3) $l_\nu(x + a_\nu)$ and ν pointing from $x + a_\nu$ to x. The lengths of three edges are, respectively, represented by edge spacings a_μ, a_ρ and a_ν [see Eqs. (41) and (43)]. We use the tetrad fields

$$e_\mu(x), \qquad e_\rho(x + a_\mu), \qquad e_\nu(x + a_\nu), \tag{44}$$

defined at x, $x + a_\mu$ and $x + a_\nu$, to, respectively, characterize locations, forward directions and lengths of three edges: (42) and

$$\begin{aligned} l_\rho(x + a_\mu) &= a e_\rho(x + a_\mu), \\ l_\nu(x + a_\nu) &= a e_\nu(x + a_\nu), \end{aligned} \tag{45}$$

of the anti-clocklike 2-simplex $h(x)$ [see Fig. 1 and Eqs. (42) and (43)].

B. Parallel transports and curvature

The fundamental spin-connection fields $\{\omega_\mu(x)\}$ are assigned to 1-simplices (edges) of the simplicial complex, i.e., each edge (x, μ) we associate with it $\omega_\mu(x)$. The torsion-free Cartan Eq. (35) is actually an equation for infinitesimal parallel transports of tetrad fields $e^a_\nu(x)$. Applying this equation to the 2-simplex $h(x)$, as shown in Fig. 1, we show that $e^a_\nu(x)$ $[e^a_\mu(x)]$ undergoes its parallel transport to $\bar{e}^a_\nu(x + a_\mu)$ $[\bar{e}^a_\mu(x + a_\nu)]$ along the μ (ν) direction for an edge spacing $a_\mu(x)$ $[a_\nu(x)]$, following the discretized Cartan equations:

$$\bar{e}^{\,a}_\nu(x + a_\mu) - e^a_\nu(x) - a_\mu g \omega^{ab}_\mu(x) \wedge e_{\nu b}(x) = 0, \tag{46}$$

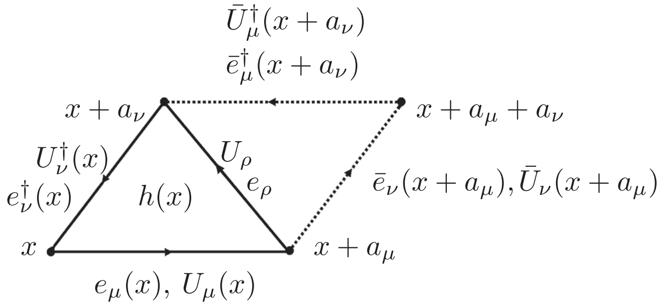

FIG. 1. We sketch a 2-simplex (triangle) $h(x)$ formed by three edges $l_\mu(x) = ae_\mu(x)$, $l_\rho(x + a_\mu) = ae_\rho(x + a_\mu)$ and $l_\nu(x + a_\mu) = ae_\nu(x + a_\nu)$ $[a = 1]$ connecting three vertexes x, $x + a_\mu$ and $x + a_\nu$. Assuming three edge spacings a_μ, a_ν and a_ρ (41) are so small that the geometry of the interior of each 4-simplex and its subsimplex (3- and 2-simplex) is approximately flat, we assign a local Lorentz frame to each 4-simplex. On the local Lorentz manifold $\xi^a(x)$ at a space-time point "x", we sketch a closed parallelogram $\mathcal{C}_P(x)$ lying in the 2-simplex $h(x)$. Its two edges $e_\mu(x)$ and $e^\dagger_\nu(x)$ are two edges of the 2-simplex $h(x)$, and other two edges (dashed lines) $\bar{e}^\dagger_\mu(x + a_\nu)$ and $\bar{e}_\nu(x + a_\mu)$ are parallel transports of $e^\dagger_\mu(x)$ and $e^\dagger_\nu(x)$ along ν and μ directions, respectively [see Eqs. (46), (47), (62), and (63)]. Each 2-simplex in the simplicial complex has a closed parallelogram lying in it. Group-valued gauge fields $U_\mu(x)$ and $U^\dagger_\nu(x)$ are, respectively, associated to edges $e_\mu(x)$ and $e^\dagger_\nu(x)$ of the 2-simplex $h(x)$, as indicated. The fields $e_\rho \equiv e_\rho(x + a_\mu)$ and $U_\rho \equiv U_\rho(x + a_\mu)$ are associated to the third edge $(x + a_\mu, \rho)$ of the 2-simplex $h(x)$. The group fields $\bar{U}_\nu(x + a_\mu)$ and $\bar{U}^\dagger_\mu(x + a_\nu)$ indicate the parallel transports of $U^\dagger_\nu(x)$ and $U_\mu(x)$ [see Eqs. (48), (49), (82), and (83)] for the zero curvature case. Note that the point $(x + a_\mu + a_\nu)$ is not a vertex of the simplicial complex, points: $(x - a_\mu)$, $(x - a_\nu)$, $(x + a_\mu + a_\mu)$, $(x + a_\mu - a_\rho)$, and $(x + a_\nu + a_\rho)$, which are not shown in the sketch, are not vertexes of the simplicial complex as well. Parallel transports $\bar{e}_\nu(x + a_\mu)$ and $\bar{e}^\dagger_\mu(x + a_\nu)$, as well as $\bar{U}_\nu(x + a_\mu)$ and $\bar{U}^\dagger_\mu(x + a_\nu)$ are not associated to any edge of the simplicial complex. Throughout this article, the notations $\bar{e}$ and $\bar{U}$ indicates parallel transports that are not associated to any edge of the simplicial complex.

$$\bar{e}^{\,a}_{\mu}(x + a_\nu) - e^a_\mu(x) - a_\nu g\omega^{ab}_\nu(x) \wedge e_{\mu b}(x) = 0. \qquad (47)$$

The parallel transports $\bar{e}^a_\nu(x + a_\mu)$ and $\bar{e}^a_\mu(x + a_\nu)$ are neither independent fields, nor assigned to any edges of the simplicial complex. They are related to $e^\dagger_\nu(x)[e_\mu(x)]$ and $\omega_\mu(x)[\omega_\nu(x)]$ fields assigned to the edges $(x, -\nu)$ and (x, μ) of the 2-simplex $h(x)$ by the Cartan Eq. (46) and (47). Because of torsion-free, $e_\mu(x)$, $e^\dagger_\nu(x)$ and their parallel transports $\bar{e}^\dagger_\mu(x + a_\nu)$, $\bar{e}_\nu(x + a_\mu)$ form a *closed* parallelogram $\mathcal{C}_P(x)$ (Fig. 1). Otherwise this would means the curved space-time could not be approximated locally by a flat space-time [21]. Note that the point $(x + a_\mu + a_\nu)$ at the *closed* parallelogram $\mathcal{C}_P(x)$ (Fig. 1) is not any vertex of the simplicial complex.

For the zero curvature case $R^{ab}_{\nu\mu}(x) = 0$, the curvature Eq. (19) can be discretized as

$$\bar{\omega}^{\,ab}_{\nu}(x + a_\mu) - \omega^{ab}_\nu(x) - a_\mu g\omega^{ae}_\mu(x) \wedge \omega^b_{e\nu}(x) = 0, \qquad (48)$$

$$\bar{\omega}^{\,ab}_{\mu}(x + a_\nu) - \omega^{ab}_\mu(x) - a_\nu g\omega^{ae}_\nu(x) \wedge \omega^b_{e\mu}(x) = 0, \qquad (49)$$

where $\bar{\omega}^{ab}_\nu(x + a_\mu)$ and $\bar{\omega}^{ab}_\mu(x + a_\nu)$ are, respectively, parallel transports of $\omega^{ab}_\nu(x)$ and $\omega^{ab}_\mu(x)$ in the μ and ν directions. Analogously to the parallel transports $\bar{e}^a_\nu(x + a_\mu)$ and $\bar{e}^a_\mu(x + a_\nu)$ given by Eqs. (46) and (47), parallel transports $\bar{\omega}^{ab}_\nu(x + a_\mu)$ and $\bar{\omega}^{ab}_\mu(x + a_\nu)$ are neither independent fields, nor assigned to any edge of the simplicial complex. They are related to $\omega_\mu(x)$ and $\omega_\nu(x)$ fields assigned to the edges (x, μ) and $(x + a_\nu, \nu)$ of the 2-simplex $h(x)$ by the parallel transport Eqs. (48) and (49). The fields $\omega_\mu(x)$, $\omega_\nu(x)$ and their parallel transports $\bar{\omega}_\mu(x + a_\nu)$, $\bar{\omega}_\nu(x + a_\mu)$ also form a *closed* parallelogram, analogously to the one $\mathcal{C}_P(x)$ formed by the tetrad fields $e_\mu(x)$, $e_\nu(x)$ and their parallel transports $\bar{e}_\mu(x + a_\nu)$, $\bar{e}_\nu(x + a_\mu)$ (see Fig. 1).

Whereas, for the nonzero curvature case $R^{ab}_{\nu\mu}(x) \neq 0$, the curvature Eq. (19) can be discretized as

$$\omega^{ab}_\nu(x + a_\mu) - \omega^{ab}_\nu(x) - a_\mu g\omega^{ae}_\mu(x) \wedge \omega^b_{e\nu}(x) = a_\mu R^{ab}_{\mu\nu}(x), \qquad (50)$$

$$\omega^{ab}_\mu(x + a_\nu) - \omega^{ab}_\mu(x) - a_\nu g\omega^{ae}_\nu(x) \wedge \omega^b_{e\mu}(x) = a_\nu R^{ab}_{\nu\mu}(x), \qquad (51)$$

which define fields $\omega^{ab}_\nu(x + a_\mu)$ and $\omega^{ab}_\mu(x + a_\nu)$ in terms of fields $\omega^{ab}_\nu(x)$, $\omega^{ab}_\mu(x)$ and curvature $R^{ab}_{\nu\mu}(x)$. These fields $\omega^{ab}_\nu(x + a_\mu)$ and $\omega^{ab}_\mu(x + a_\nu)$ are neither independent fields, nor assigned to any edge of the simplicial complex. They are related not only to $\omega^{ab}_\mu(x)$ and $\omega^{ab}_\nu(x)$ fields assigned to the edges (x, μ) and $(x + a_\nu, \nu)$ of the 2-simplex $h(x)$, but also to the curvature $R^{ab}_{\mu\nu}$ (50) and $R^{ab}_{\nu\mu}$ (51).

These fields $\omega^{ab}_\nu(x + a_\mu)$ and $\omega^{ab}_\mu(x + a_\nu)$ are no longer parallel transports $\bar{\omega}^{ab}_\nu(x + a_\mu)$ and $\bar{\omega}^{ab}_\mu(x + a_\nu)$ defined by Eqs. (48) and (49). The difference between $\omega^{ab}_\nu(x + a_\mu)$ and $\bar{\omega}^{ab}_\nu(x + a_\mu)$ [or between $\omega^{ab}_\mu(x + a_\nu)$ and $\bar{\omega}^{ab}_\mu(x + a_\nu)$] is the curvature $a_\mu R^{ab}_{\mu\nu}(x)$ $[a_\nu R^{ab}_{\nu\mu}(x)]$,

$$\omega^{ab}_\nu(x + a_\mu) - \bar{\omega}^{ab}_\nu(x + a_\mu) = a_\mu R^{ab}_{\mu\nu}(x), \qquad (52)$$

$$\omega^{ab}_\mu(x + a_\nu) - \bar{\omega}^{ab}_\mu(x + a_\nu) = a_\nu R^{ab}_{\nu\mu}(x). \qquad (53)$$

The fields $\omega_\mu(x)$, $\omega_\nu(x)$ and fields $\omega_\mu(x + a_\nu)$, $\omega_\nu(x + a_\mu)$ do not form a *closed* parallelogram, due to the nonzero curvature $R^{ab}_{\nu\mu}(x) \neq 0$.

C. Group-valued fields

Instead of a $\omega_\mu(x)$ field, we assign a group-valued field $U_\mu(x)$ to each edge (1-simplex) of the simplicial complex.

On the edge (x, μ) connecting two vertexes x and $x + a_\mu$ in the forward direction μ, we place an $SO(4)$ group-valued spin-connection fields,

$$U_\mu(x) = e^{iga\omega_\mu(x)}, \tag{54}$$

whereas the same edge $(x + a_\mu, -\mu)$ in the backward direction $-\mu$, we associate with it

$$U_{-\mu}(x + a_\mu) \equiv U^\dagger_\mu(x) = U^{-1}_\mu(x), \tag{55}$$

analogously to the definition of link fields in lattice gauge theories. On the three edges in forward directions (x, μ), $(x + a_\mu, \rho)$ and $(x + a_\nu, \nu)$ of the anti-clocklike 2-simplex $h(x)$ ($\mu \neq \nu \neq \rho$ see Fig. 1), we define $SO(4)$ group-valued spin-connection fields,

$$U_\mu(x) = e^{iga\omega_\mu(x)}, \tag{56}$$

$$U_\rho(x + a_\mu) = e^{iga\omega_\rho(x+a_\mu)}, \tag{57}$$

$$U_\nu(x + a_\nu) = e^{iga\omega_\nu(x+a_\nu)}, \tag{58}$$

which take values of the fundamental representation of the compact group $SO(4)$. On the three edges in backward directions $(x, -\nu)$, $(x + a_\nu, -\rho)$ and $(x + a_\mu, -\mu)$ of the clocklike 2-simplex $h^\dagger(x)$ (see Fig. 1), we define $SO(4)$ group-valued spin-connection fields,

$$U_{-\nu}(x) = U^\dagger_\nu(x + a_\nu) = e^{-iga\omega_\nu(x+a_\nu)}, \tag{59}$$

$$U_{-\rho}(x + a_\nu) = U^\dagger_\rho(x + a_\mu) = e^{-iga\omega_\rho(x+a_\mu)}, \tag{60}$$

$$U_{-\mu}(x + a_\mu) = U^\dagger_\mu(x) = e^{-iga\omega_\mu(x)}. \tag{61}$$

These uniquely define group-valued spin-connection fields on the anti-clocklike and clocklike 2-simplex.

1. Unitary operators for parallel transports of $e_\mu(x)$ fields

Actually, these group-valued fields (56)–(61) can be viewed as unitary operators for finite parallel transportations. The parallel transportation (Cartan) Eqs. (46) and (47) can be generalized to ($\mu \neq \nu$)

$$\bar{e}_\nu(x + a_\mu) = U^\dagger_\mu(x)e_\nu(x)U_\mu(x), \tag{62}$$

$$\bar{e}_\mu(x + a_\nu) = U^\dagger_\nu(x)e_\mu(x)U_\nu(x), \tag{63}$$

and using Eq. (55) these equations can be equivalently rewritten as

$$e_\nu(x) = U^\dagger_{-\mu}(x + a_\mu)\bar{e}_\nu(x + a_\mu)U_{-\mu}(x + a_\mu), \tag{64}$$

$$e_\mu(x) = U^\dagger_{-\nu}(x + a_\nu)\bar{e}_\mu(x + a_\nu)U_{-\nu}(x + a_\nu). \tag{65}$$

While for ($\mu = \nu$), we similarly have the following parallel transportation equations:

$$\begin{aligned} \bar{e}_\mu(x + a_\mu) &= U^\dagger_\mu(x)e_\mu(x)U_\mu(x), \\ e_\mu(x) &= U^\dagger_{-\mu}(x + a_\mu)\bar{e}_\mu(x + a_\mu)U_{-\mu}(x + a_\mu), \end{aligned} \tag{66}$$

indicating that $e_\mu(x)$ is parallel transported to $\bar{e}_\mu(x + a_\mu)$ in the μ forward direction, and $\bar{e}_\mu(x + a_\mu)$ is parallel transported to $e_\mu(x)$ in the $-\mu$ backward direction. Similar discussions can be made for parallel transports with the unitary operator $U_\rho(x + a_\mu)$.

2. Unitary operators for parallel transports of $e^\dagger_\mu(x)$ fields

In the simplicial complex, each edge (1-simplex) connecting two vertexes has only one direction. One can identify each edge by its starting vertex and direction pointing to its ending vertex. On the basis of the tetrad field $e_\mu(x)$ (42) defined at the vertex "x" for the edge (x, μ) starting from the vertex "x" in the forward direction (μ) to the vertex "$x + a_\mu$" below, using the unitary operator $U_\mu(x)$ for parallel transports, we will uniquely introduce the "conjugated" field $e^\dagger_\mu(x)$ defined at the vertex "x" to describe the same edge $(x + a_\mu, -\mu)$ but in the backward direction $-\mu$ starting from the vertex "$x + a_\mu$" to the vertex "x." Analogously to Eq. (42), this edge starting from the vertex "$x + a_\mu$" in the backward direction $(-\mu)$ can be formally represented by

$$l_{-\mu}(x + a_\mu) \equiv ae_{-\mu}(x + a_\mu). \tag{67}$$

By the parallel transport, we define the field $e_{-\mu}(x + a_\mu)$ as

$$e_{-\mu}(x + a_\mu) \equiv U^\dagger_\mu(x)e^\dagger_\mu(x)U_\mu(x) = e^\dagger_\mu(x + a_\mu) \tag{68}$$

in terms of the unitary operator $U_\mu(x)$ and conjugated tetrad fields $e^\dagger_\mu(x)$ defined at the vertex "x." From the definition in Eq. (68), we rewrite

$$e^\dagger_\mu(x) \equiv U_\mu(x)e_{-\mu}(x + a_\mu)U^\dagger_\mu(x) = \bar{e}_{-\mu}(x). \tag{69}$$

The second equalities in Eqs. (68) and (69) are given by the definition of parallel transports by unitary operators [see Eq. (62)]. Equation (68) means that we can associate the conjugated field

$$e^\dagger_\mu(x) = U_\mu(x)e^\dagger_\mu(x + a_\mu)U^\dagger_\mu(x), \tag{70}$$

with the same edge $(x + a_\mu, -\mu)$ but in backward direction $-\mu$ and write

$$l^\dagger_\mu(x) \equiv ae^\dagger_\mu(x). \tag{71}$$

As a result, the edge (x, μ) $[(x + a_\mu, -\mu)]$ in the forward (backward) direction is uniquely described by the field $e_\mu(x)$ $[e^\dagger_\mu(x)]$ defined at the vertex x. Note that the conjugated field $e^\dagger_\mu(x)$ is given by the parallel transport (70) from $x + a_\mu$ to x in the direction $(-\mu)$. In addition, Eqs. (68) and (69) indicate that conjugated fields mean the inverse of field's direction ($\mu \to -\mu$).

This prescription shows that the edge (x, μ) is completely described by the fields $e_\mu(x)$ and $e^\dagger_\mu(x)$, latter is a function of fields $e_\mu(x)$ and $U_\mu(x)$, as required by the principle of local gauge symmetries and the gauge field $U_\mu(x)$ corresponds a parallel transport between x and $x + a_\mu$. In consequence, any edge (1-simplex) of the simplicial complex is uniquely identified by its location and direction (z, σ), and described by the fields $e_\sigma(z)$ and $U_\sigma(z)$.

Using the properties $(\gamma_a)^\dagger = -\gamma_a$ [see Eq. (2)] and the definition of tetrad field $e_\mu(x) = e_\mu{}^a(x)\gamma_a$, where the index μ is fixed, we have

$$e^\dagger_\mu(x) = [e_\mu{}^a(x)\gamma_a]^\dagger = (\gamma_a)^\dagger[e_\mu{}^a(x)]^\dagger, = -e_\mu(x), \quad (72)$$

where because of the index μ being fixed, the real tetrad-field component $e_\mu{}^a(x) \equiv \partial\xi^a/\partial x^\mu$ can be viewed as a one-row matrix $(e_\mu{}^0, e_\mu{}^1, e_\mu{}^2, e_\mu{}^3)$ and $[e_\mu{}^a(x)]^\dagger$ a one-column matrix $(e_\mu{}^0, e_\mu{}^1, e_\mu{}^2, e_\mu{}^3)^\dagger$. Analogously to Eq. (43), the length of the edge (71) in backward direction $-\mu$,

$$|l^\dagger_\mu(x)| = \frac{a}{2}\left\{|\mathrm{tr}[e^\dagger_\mu(x)\cdot e^\dagger_\mu(x)]|\right\}^{1/2} = |l_\mu(x)|, \quad (73)$$

which is the same as the length of the edge in the forward direction μ.

We turn to the discussion of other two backward-direction edges $(x + a_\nu, -\nu)$ and $(x + a_\mu, -\rho)$ of the clocklike 2-simplex $h^\dagger(x)$ (see Fig. 1). Analogously to Eqs. (68) and (69), we have in the $(-\nu)$ direction,

$$\begin{aligned} e_{-\nu}(x) &\equiv U_\nu(x)e^\dagger_\nu(x + a_\nu)U^\dagger_\nu(x) = e^\dagger_\nu(x), \\ e^\dagger_\nu(x + a_\nu) &\equiv U^\dagger_\nu(x)e_{-\nu}(x)U_\nu(x) = \bar{e}_{-\nu}(x + a_\nu), \end{aligned} \quad (74)$$

and in the $(-\rho)$ direction

$$\begin{aligned} e_{-\rho}(x + a_\nu) &\equiv U^\dagger_\rho(x + a_\mu)e^\dagger_\rho(x + a_\mu)U_\rho(x + a_\mu) \\ &= e^\dagger_\rho(x + a_\nu). \\ e^\dagger_\rho(x + a_\mu) &\equiv U_\rho(x + a_\mu)e_{-\rho}(x + a_\nu)U^\dagger_\rho(x + a_\mu) \\ &= \bar{e}_{-\rho}(x + a_\mu), \end{aligned} \quad (75)$$

As a result, the edge $(x + a_\nu, \nu)$ $[(x + a_\nu, -\nu)]$ in the forward (backward) direction is uniquely described by the field $e_\nu(x + a_\nu)$ $[e^\dagger_\nu(x + a_\nu)]$ defined at the vertex $x + a_\nu$

$$e^\dagger_\nu(x + a_\nu) = U^\dagger_\nu(x)e^\dagger_\nu(x)U_\nu(x), \quad (76)$$

see Eq. (74). Note that the conjugated field $e^\dagger_\nu(x + a_\nu)$ is given by the parallel transport (76) from x to $x + a_\nu$ in the direction (ν). We can write

$$l^\dagger_\nu(x + a_\nu) \equiv a e^\dagger_\nu(x + a_\nu). \quad (77)$$

Similarly, the edge $(x + a_\mu, \rho)$ $[(x + a_\mu, -\rho)]$ in the forward (backward) direction is uniquely described by the field $e_\rho(x + a_\mu)$ $[e^\dagger_\rho(x + a_\mu)]$ defined at the vertex $x + a_\mu$

$$e^\dagger_\rho(x + a_\mu) = U_\rho(x + a_\mu)e^\dagger_\rho(x + a_\nu)U^\dagger_\rho(x + a_\mu), \quad (78)$$

see Eq. (75). Note that the conjugated field $e^\dagger_\rho(x + a_\mu)$ is given by the parallel transport (78) from $x + a_\nu$ to $x + a_\mu$ in the direction $(-\rho)$. We can write

$$l^\dagger_\rho(x + a_\mu) \equiv a e^\dagger_\rho(x + a_\mu). \quad (79)$$

This prescription shows that the edge $(x + a_\nu, \nu)$ is completely described by the fields $e_\nu(x + a_\nu)$ and $U_\nu(x + a_\nu)$, and the edge $(x + a_\mu, \rho)$ by the fields $e_\rho(x + a_\mu)$ and $U_\rho(x + a_\mu)$. The field $U_\nu(x + a_\nu)$ $[U_\rho(x + a_\mu)]$ corresponds a parallel transport between x and $x + a_\mu$ ($x + a_\mu$ and $x + a_\nu$).

Along the triangle path of the clocklike 2-simplex $h^\dagger(x)$ $[x \overset{-\nu}{\mapsto} x + a_\nu \overset{-\rho}{\mapsto} x + a_\mu \overset{-\mu}{\mapsto} x]$ (see Fig. 1), these three edges and their backward directions are formally represented by (1) $l_{-\mu}(x + a_\mu)$ and $-\mu$ pointing from $x + a_\mu$ to x; (2) $l_{-\nu}(x)$ and $-\nu$ pointing from x to $x + a_\nu$; (3) $l_{-\rho}(x + a_\nu)$ and $-\rho$ pointing from $x + a_\nu$ to $x + a_\mu$. Based on Eqs. (68), (74), (75), (70), (76), and (78), we use the conjugated tetrad fields

$$e^\dagger_\mu(x), \qquad e^\dagger_\nu(x + a_\nu), \qquad e^\dagger_\rho(x + a_\mu), \quad (80)$$

which are, respectively, defined at vertexes x, $x + a_\nu$, $x + a_\mu$, to characterize both backward directions and lengths of three edges (71), (77), and (79) of the clocklike 2-simplex $h^\dagger(x)$.

In the simplicial complex, each edge (1-simplex), described by tetrad field $e_\mu(x)$, is uniquely identified by its location and direction (x, μ), and each triangle (2-simplex) $h(x)$ has a definite orientation, as indicated in Fig. 1, either anti-clocklike or clocklike. Thus each triangle, for example, the one presented in Fig. 1 is completely described by the tetrad fields $e_\mu(x)$, $e_\nu(x + a_\nu)$, $e_\rho(x + a_\mu)$, and unitary operators $U_\mu(x)$, $U_\nu(x + a_\nu)$, $U_\rho(x + a_\mu)$.

3. Unitary operators and curvature

In the zero curvature case, the group-valued fields for parallel transports $\bar{\omega}_\mu(x + a_\nu)$ and $\bar{\omega}_\nu(x + a_\mu)$, defined by parallel transport Eqs. (48) and (49), are given by

$$\begin{aligned} \bar{U}_\mu(x + a_\nu) &= e^{iga\bar{\omega}_\mu(x+a_\nu)}, \\ \bar{U}_\nu(x + a_\mu) &= e^{iga\bar{\omega}_\nu(x+a_\mu)}. \end{aligned} \quad (81)$$

Similarly to Eqs. (62) and (63), the parallel transport Eqs. (48) and (49) can be generalized to

$$\bar{U}_\nu(x + a_\mu) = U^\dagger_\mu(x)U_\nu(x)U_\mu(x), \quad (82)$$

$$\bar{U}_\mu(x + a_\nu) = U^\dagger_\nu(x)U_\mu(x)U_\nu(x). \quad (83)$$

The parallel transport fields $\bar{U}_\nu(x + a_\mu)$ and $\bar{U}_\mu(x + a_\nu)$ together with $U_\mu(x)$ and $U_\nu(x)$ form a *closed* parallelogram, see Fig. 1. This closed parallelogram is not the same as the parallelogram $\mathcal{C}_P(x)$ formed by e and $\bar{e}$ fields.

In the nonzero curvature case, corresponding to the fields $\omega_\mu(x+a_\nu)$ and $\omega_\nu(x+a_\mu)$ defined by Eqs. (50) and (51), the group-valued fields can be similarly given by

$$U_\mu(x+a_\nu) = e^{iga\omega_\mu(x+a_\nu)}, \quad U_\nu(x+a_\mu) = e^{iga\omega_\nu(x+a_\mu)}, \tag{84}$$

whose values obviously depend on the curvature $R_{\mu\nu}(x)$. The same as the fields $\omega_\mu(x+a_\nu)$ and $\omega_\nu(x+a_\mu)$, these group-valued fields $U_\nu(x+a_\mu)$ and $U_\mu(x+a_\nu)$ are neither independent fields, nor assigned to any edge of the simplicial complex. They are related to $U_\mu(x)$ and $U_\nu(x)$ fields assigned to the edges (x,μ) and (x,ν) of the 2-simplex $h(x)$ by

$$U_\nu(x+a_\mu) \equiv U_\mu^\dagger(x)U_\nu(x)U_\mu(x), \tag{85}$$

$$U_\mu(x+a_\nu) \equiv U_\nu^\dagger(x)U_\mu(x)U_\nu(x), \tag{86}$$

which are generalized from Eqs. (50) and (51). The fields $U_\nu(x+a_\mu)$ and $U_\mu(x+a_\nu)$ defined in Eqs. (85) and (86) encode the information of a nontrivial curvature. They do not form a *closed* parallelogram together with $U_\mu(x)$ and $U_\nu(x)$, at the point $(x+a_\mu+a_\nu)$ (see Fig. 1).

In order to see the nontrivial curvature information encoded in the fields $U_\nu(x+a_\mu)$ and $U_\mu(x+a_\nu)$ defined by Eqs. (84)–(86), based on Eqs. (85) and (86), we introduce quantities

$$U_{\mu\nu}(x) \equiv U_\nu(x)U_\mu(x) = U_\mu(x)U_\nu(x+a_\mu), \tag{87}$$

$$U_{\nu\mu}(x) \equiv U_\mu(x)U_\nu(x) = U_\nu(x)U_\mu(x+a_\nu), \tag{88}$$

and calculate their expressions in the naive continuum limit. In the *naive continuum limit*: $ag\omega_\mu \ll 1$ (small coupling g or weak ω_μ field), indicating that the wavelengths of weak and slow-varying fields $\omega_\mu(x)$ are much larger than the edge spacing a_μ, we obtain (see Appendix A)

$$U_{\mu\nu}(x) = \exp\Big\{iga[\omega_\mu(x)+\omega_\nu(x)] + iga^2\partial_\mu\omega_\nu(x) - \tfrac{1}{2}(ga)^2[\omega_\mu(x),\omega_\nu(x)] + \mathcal{O}(a^3)\Big\}, \tag{89}$$

where $\mathcal{O}(a^3)$ indicates high-order powers of $ag\omega_\mu$. It is shown that the quantity $U_{\mu\nu}(x)$ [Eq. (89)] is related to the curvature $R_{\mu\nu}(x)$ in Appendix A. For the sake of simplicity in the following calculations to show the naive continuum limit, the quantities introduced by (87) and (88), and their expressions in the naive continuum limit (89) are useful.

D. Triangle constrain and area

Three tetrad fields $e_\mu(x)$, $e_\rho(x+a_\mu)$ and $e_\nu(x+a_\nu)$ [see Eq. (44)] are three edges of the anti-clocklike 2-simplex $h(x)$, satisfying the triangle constraint

$$e_\rho(x+a_\mu) = e_{-\nu}(x) - e_\mu(x) = e_\nu^\dagger(x) - e_\mu(x). \tag{90}$$

Equivalently, three tetrad fields $e_\mu^\dagger(x)$, $e_\nu^\dagger(x+a_\nu)$ and $e_\rho^\dagger(x+a_\mu)$ [see Eqs. (70), (76), and (78) or (80)] of the clocklike 2-simplex $h^\dagger(x)$, satisfying the triangle constraint

$$e_{-\rho}(x+a_\nu) = e_\mu(x) - e_{-\nu}(x) = e_\mu(x) - e_\nu^\dagger(x), \tag{91}$$

where $e_{-\rho}(x+a_\nu) = e_\rho^\dagger(x+a_\nu)$ [see Eq. (75)]. Also, Eq. (74) is used for $e_{-\nu}(x) = e_\nu^\dagger(x)$ in the second equality of Eqs. (90) and (91). Two of three edges are independent for a given anti-clocklike (clocklike) 2-simplex $h(x)$ [$h^\dagger(x)$].

However, in Eqs. (90) and (91), vector fields defined at different vertexes are related without being parallel transported to the same vertex, thus these relationships are not proper and does not properly transform under local gauge transformations. This is an exactly essential point of local gauge symmetries, that gauge fields U for parallel transports are needed to relate variations of gauge freedom at different coordinate points. Using the parallel transport by the unitary operator $U_\mu(x)$, we rewrite the triangle constraint (90) for the anti-clocklike 2-simplex $h(x)$ as

$$U_\mu(x)e_\rho(x+a_\mu)U_\mu^\dagger(x) = e_\nu^\dagger(x) - e_\mu(x), \tag{92}$$

where in the left-handed side, $e_\rho(x+a_\mu)$ is parallel transported from the vertex $x+a_\mu$ to the vertex x to be related to $e_\nu^\dagger(x)$ and $e_\mu(x)$ at the same vertex x in the right-handed side. Using $\bar{e}_\rho(x) = U_\mu(x)e_\rho(x+a_\mu)U_\mu^\dagger(x)$, we rewrite Eq. (92) as

$$e_\nu(x) + e_\mu(x) + \bar{e}_\rho(x) = 0. \tag{93}$$

Using the parallel transport by the unitary operator $U_\nu(x)$, we rewrite the triangle constraint (91) for the clocklike 2-simplex $h^\dagger(x)$ as

$$U_\nu(x)e_\rho^\dagger(x+a_\nu)U_\nu^\dagger(x) = e_\mu(x) - e_\nu^\dagger(x), \tag{94}$$

where in the left-handed side $e_\rho^\dagger(x+a_\nu)$ is parallel transported from the vertex $x+a_\nu$ to the vertex x to be related to $e_\nu^\dagger(x)$ and $e_\mu(x)$ at the same vertex x in the right-handed side. Equation (94) is identical to Eq. (92) or Eq. (93), if we consider $\bar{e}_\rho^\dagger(x) = U_\nu(x)e_\rho^\dagger(x+a_\nu)U_\nu^\dagger(x)$ and $\bar{e}_\rho^\dagger(x) = -\bar{e}_\rho(x)$. The proper parallel transports by unitary operators can shift the triangle constrain to other vertexes, for example, $x+a_\mu$ and $x+a_\nu$.

We are now in the position of discussing the area of the 2-simplex $h(x)$. We define the fundamental area operator of the anti-clocklike 2-simplex $h(x)$ (see Fig. 1)

$$S^h_{\mu\nu}(x) \equiv a^2 e_\mu(x) \wedge e_{-\nu}(x) \tag{95}$$

at the vertex x. In addition, we can also define the following area operators:

$$S^h_{\rho\mu}(x+a_\mu) \equiv a^2 e_\rho(x+a_\mu) \wedge e_{-\mu}(x+a_\mu) \tag{96}$$

at the vertex $x+a_\mu$, and

$$S^h_{\nu\rho}(x+a_\nu) \equiv a^2 e_\nu(x+a_\nu) \wedge e_{-\rho}(x+a_\nu) \tag{97}$$

at the vertex $x + a_\nu$. Using Eqs. (68), (74), and (75), we rewrite the area operators (95)–(97) of the anti-clocklike 2-simplex $h(x)$ as

$$S^h_{\mu\nu}(x) \equiv a^2 e_\mu(x) \wedge e^\dagger_\nu(x), \tag{98}$$

$$S^h_{\rho\mu}(x + a_\mu) \equiv a^2 e_\rho(x + a_\mu) \wedge e^\dagger_\mu(x + a_\mu), \tag{99}$$

$$S^h_{\nu\rho}(x + a_\nu) \equiv a^2 e_\nu(x + a_\nu) \wedge e^\dagger_\rho(x + a_\nu). \tag{100}$$

In the following, we show that area operators (98)–(100), defined at three vertexes x, $x + a_\mu$, and $x + a_\nu$ are universal up to parallel transports by unitary operators. Using Eqs. (68) and (92), we obtain

$$\begin{aligned} S^h_{\rho\mu}(x + a_\mu) &= a^2 U^\dagger_\mu(x)[e^\dagger_\nu(x) - e_\mu(x)]U_\mu(x) \\ &\quad \wedge U^\dagger_\mu(x) e^\dagger_\mu(x) U_\mu(x), \\ &= a^2 U^\dagger_\mu(x)[e^\dagger_\nu(x) \wedge e^\dagger_\mu(x)]U_\mu(x), \\ &= U^\dagger_\mu(x) S^h_{\mu\nu}(x) U_\mu(x). \end{aligned} \tag{101}$$

Analogously, using Eqs. (74) and (94), we obtain

$$\begin{aligned} S^h_{\nu\rho}(x + a_\nu) &= a^2 U^\dagger_\nu(x) e_\nu(x) U_\nu(x) \\ &\quad \wedge U^\dagger_\nu(x)[e_\mu(x) - e^\dagger_\nu(x)]U_\nu(x) \\ &= a^2 U^\dagger_\nu(x) e_\nu(x) \wedge e_\mu(x) U_\nu(x) \\ &= U^\dagger_\nu(x) S^h_{\mu\nu}(x) U_\nu(x). \end{aligned} \tag{102}$$

In Eqs. (101) and (102), we use $e^\dagger_\mu(x) = -e_\mu(x)$, $e_\mu(x) \wedge e_\mu(x) = e^\dagger_\mu(x) \wedge e^\dagger_\mu(x) = e^\dagger_\mu(x) \wedge e_\mu(x) = 0$ and the same for $(\mu \to \nu)$. This shows that the area operators (98)–(100) defined at three vertexes of the 2-simplex $h(x)$ are universal up to parallel transports.

Therefore, Eq. (95) or (98) defines the area operator of the 2-simplex $h(x)$

$$\begin{aligned} S^h_{\mu\nu}(x) &\equiv \frac{a^2}{2}[e_\mu(x) e^\dagger_\nu(x) - e^\dagger_\nu(x) e_\mu(x)] \\ &= a^2 \frac{i}{2} \sigma_{ab}[e^a_\mu(x) e^b_\nu(x) - e^a_\nu(x) e^b_\mu(x)], \end{aligned} \tag{103}$$

up to parallel transports. As consequence, the area of the 2-simplex $h(x)$ is uniquely determined by

$$S_h(x) \equiv |S^h_{\mu\nu}(x)|, \quad S^2_h(x) \equiv \tfrac{1}{8}\mathrm{tr}[S^h_{\mu\nu}(x) \cdot S^{h\dagger}_{\mu\nu}(x)]. \tag{104}$$

Its uniqueness [independence of the vertexes x, $x + a_\mu$ and $x + a_\nu$ of the 2-simplex $h(x)$], i.e.,

$$S_h(x) \equiv |S^h_{\mu\nu}(x)| = |S^h_{\rho\mu}(x + a_\mu)| = |S^h_{\nu\rho}(x + a_\nu)|, \tag{105}$$

can be shown by using Eqs. (101) and (102).

In the same way as Eqs. (95)–(97), we define the area operators of the clocklike 2-simplex $h^\dagger(x)$:

$$\begin{aligned} S^h_{\nu\mu}(x) &\equiv a^2 e_{-\nu}(x) \wedge e_\mu(x) = -S^h_{\mu\nu}(x) = S^{h\dagger}_{\mu\nu}(x), \\ S^h_{\mu\rho}(x + a_\mu) &\equiv a^2 e_{-\mu}(x + a_\mu) \wedge e_\rho(x + a_\mu) \\ &= -S^h_{\rho\mu}(x + a_\mu) = S^{h\dagger}_{\rho\mu}(x + a_\mu), \\ S^h_{\rho\nu}(x + a_\nu) &\equiv a^2 e_{-\rho}(x + a_\nu) \wedge e_\nu(x + a_\nu) \\ &= -S^h_{\nu\rho}(x + a_\nu) = S^{h\dagger}_{\nu\rho}(x + a_\nu), \end{aligned} \tag{106}$$

whose directions are opposite to the counterparts of anti-clocklike 2-simplex $h(x)$. However, the area of the clocklike 2-simplex $h^\dagger(x)$ is equal to the area (104).

Based on the definition of 2-simplex $h(x)$ area (104), we can define a volume element around the vertex "x"

$$dV(x) = \sum_{h(x)} dV_h(x), \qquad dV_h(x) \equiv S^2_h(x), \tag{107}$$

where $dV_h(x)$ indicates the volume element contributed from a 2-simplex $h(x)$, and $\sum_{h(x)}$ indicates the sum over all 2-simplices $h(x)$ that share the same vertex x. This definition of volume element (107) indicates that a 2-simplex $h(x)$ contributes the volume element S^2_h at its three vertexes x, $x + a_\mu$ and $x + a_\nu$.

Before ending this section, we note that using the parallel transports (68), (74), and (75), one can obtain parallel transports of area operators (95)–(97) of triangles (2-simplexes),

$$\begin{aligned} \bar{S}_{\mu\nu}(x + a_\mu) &= U^\dagger_\mu(x) S^h_{\mu\nu}(x) U_\mu(x), \\ \bar{S}_{\mu\nu}(x + a_\nu) &= U^\dagger_\nu(x) S^h_{\mu\nu}(x) U_\nu(x), \cdots, \end{aligned} \tag{108}$$

which are consistent with the definitions of unitary operators $U_\mu(x)$ and $U_\nu(x)$ for parallel transports (62) and (63) of edges (1-simplexes). The notation "$\bar{S}_{\mu\nu}$" instead of $S^h_{\mu\nu}$ in the left-handed side of Eqs. (108) indicates that the parallel transport "$\bar{S}_{\mu\nu}$" is not associated to any triangle of the simplicial complex.

E. Local gauge transformations

In accordance with Eq. (10), the bilocal gauge transformations of three U fields (56)–(58) of the anti-clocklike 2-simplex $h(x)$ are,

$$\begin{aligned} U_\mu(x) &\to \mathcal{V}(x) U_\mu(x) \mathcal{V}^\dagger(x + a_\mu), \\ U_\nu(x + a_\nu) &\to \mathcal{V}(x + a_\nu) U_\nu(x + a_\nu) \mathcal{V}^\dagger(x), \\ U_\rho(x + a_\mu) &\to \mathcal{V}(x + a_\mu) U_\rho(x + a_\mu) \mathcal{V}^\dagger(x + a_\nu), \end{aligned} \tag{109}$$

and their inverses (59)–(61) of the clocklike 2-simplex $h^\dagger(x)$ transform as

$$\begin{aligned} U^\dagger_\mu(x) &\to \mathcal{V}(x + a_\mu) U^\dagger_\mu(x) \mathcal{V}^\dagger(x), \\ U^\dagger_\nu(x + a_\nu) &\to \mathcal{V}(x) U^\dagger_\nu(x + a_\nu) \mathcal{V}^\dagger(x + a_\nu), \\ U^\dagger_\rho(x + a_\mu) &\to \mathcal{V}(x + a_\nu) U^\dagger_\rho(x + a_\mu) \mathcal{V}^\dagger(x + a_\mu). \end{aligned} \tag{110}$$

In accordance with Eq. (9), the tetrad fields $e_\mu(x)$, $e_\nu(x+a_\nu)$ and $e_\rho(x+a_\mu)$ for the anti-clocklike 2-simplex $h(x)$ transform under local gauge transformations

$$\begin{aligned} e_\mu(x) &\to e'_\mu(x) = \mathcal{V}(x) e_\mu(x) \mathcal{V}^\dagger(x), \\ e_\nu(x+a_\nu) &\to e'_\nu(x+a_\nu) \\ &= \mathcal{V}(x+a_\nu) e_\nu(x+a_\nu) \mathcal{V}^\dagger(x+a_\nu), \\ e_\rho(x+a_\mu) &\to e'_\rho(x+a_\mu) \\ &= \mathcal{V}(x+a_\mu) e_\rho(x+a_\mu) \mathcal{V}^\dagger(x+a_\mu), \end{aligned} \tag{111}$$

respectively at the vertexes x, $x+a_\nu$, and $x+a_\mu$ where they are defined. Using above local gauge transformations (109)–(111), we obtain the following local gauge transformations of the conjugated fields $e^\dagger_\mu(x)$, $e^\dagger_\nu(x+a_\nu)$ and $e^\dagger_\rho(x+a_\mu)$ defined by Eqs. (68), (74), and (75) for the clocklike 2-simplex $h^\dagger(x)$,

$$\begin{aligned} e^\dagger_\mu(x) &\to e^{\dagger'}_\mu(x) = \mathcal{V}(x) e^\dagger_\mu(x) \mathcal{V}^\dagger(x), \\ e^\dagger_\nu(x+a_\nu) &\to e^{\dagger'}_\nu(x+a_\nu) \\ &= \mathcal{V}(x+a_\nu) e^\dagger_\nu(x+a_\nu) \mathcal{V}^\dagger(x+a_\nu), \\ e^\dagger_\rho(x+a_\mu) &\to e^{\dagger'}_\rho(x+a_\mu) \\ &= \mathcal{V}(x+a_\mu) e^\dagger_\rho(x+a_\mu) \mathcal{V}^\dagger(x+a_\mu). \end{aligned} \tag{112}$$

These local gauge transformations (112) of the conjugated fields at the vertexes x, $x+a_\nu$ and $x+a_\mu$ are in the same manner as that given by Eqs. (111). This means that each edge (1-simplex) $l_\mu(x)$ of the simplicial complex is uniquely described by tetrad fields $e_\mu(x)$ and $e^\dagger_\mu(x)$, that are defined at the vertex x, and covariantly transformed under local gauge transformation.

It is worthwhile to mention that the transformations (112) are just conjugated transformations (111), and consistent with the following local gauge transformations:

$$\begin{aligned} e_{-\mu}(x+a_\mu) &\to e'_{-\mu}(x+a_\mu) \\ &= \mathcal{V}(x+a_\mu) e_{-\mu}(x+a_\mu) \mathcal{V}^\dagger(x+a_\mu), \\ e_{-\nu}(x) &\to e'_{-\nu}(x) = \mathcal{V}(x) e_{-\nu}(x) \mathcal{V}^\dagger(x), \\ e_{-\rho}(x+a_\nu) &\to e'_{-\rho}(x+a_\nu) \\ &= \mathcal{V}(x+a_\nu) e_{-\rho}(x+a_\nu) \mathcal{V}^\dagger(x+a_\nu), \end{aligned} \tag{113}$$

which follow the transformation rules of Eq. (111).

It is shown that the tetrad fields (44) and their conjugated fields (80) given by Eqs. (70), (76), and (78), as well as the triangle constraints (92) and (94), are gauge covariant, and properly transformed under local gauge transformations (109)–(112). The length (43) or (73) of edges (1-simplexes) is unique and invariant under local gauge transformations (109)–(112).

Under local gauge transformations (109)–(112), the fundamental area operators (98)–(100) of the anti-clocklike 2-simplex $h(x)$ are gauge covariant and transform

$$\begin{aligned} S^h_{\mu\nu}(x) &\to S^{h'}_{\mu\nu}(x) = \mathcal{V}(x) S^h_{\mu\nu}(x) \mathcal{V}^\dagger(x), \\ S^h_{\nu\rho}(x+a_\nu) &\to S^{h'}_{\nu\rho}(x+a_\nu) \\ &= \mathcal{V}(x+a_\nu) S^h_{\nu\rho}(x+a_\nu) \mathcal{V}^\dagger(x+a_\nu), \\ S^h_{\rho\mu}(x+a_\mu) &\to S^{h'}_{\rho\mu}(x+a_\mu) \\ &= \mathcal{V}(x+a_\mu) S^h_{\rho\mu}(x+a_\mu) \mathcal{V}^\dagger(x+a_\mu), \end{aligned} \tag{114}$$

which are consistent with Eqs. (101), (102), (109), and (110), and their counterparts [see Eq. (106)] of the clocklike 2-simplex $h^\dagger(x)$ transform in the same manner. The parallel transports (108) of area operators transform consistently with Eqs. (109), (110), and (114). However, the area (104) of the 2-simplex $h(x)$ is unique and invariant under local gauge transformations.

It is worthwhile to mention that under local gauge transformation (109)–(111), parallel transport fields (62) and (63) transform locally

$$\begin{aligned} \bar{e}_\mu(x+a_\nu) &\to \bar{e}'_\mu(x+a_\nu) \\ &= \mathcal{V}(x+a_\nu) \bar{e}_\mu(x+a_\nu) \mathcal{V}^\dagger(x+a_\nu), \\ \bar{e}_\nu(x+a_\mu) &\to \bar{e}'_\nu(x+a_\mu) \\ &= \mathcal{V}(x+a_\mu) \bar{e}_\nu(x+a_\mu) \mathcal{V}^\dagger(x+a_\mu), \end{aligned} \tag{115}$$

in accordance with local gauge transformations (111) for tetrad fields. Therefore, the *closed* parallelogram $\mathcal{C}_P(x)$ (see Fig. 1), formed by $e_\mu(x)$, $e_\nu(x)$ and their parallel transports $\bar{e}_\mu(x+a_\nu)$, $\bar{e}_\nu(x+a_\mu)$, is invariant under local gauge transformation. This is consistent with the torsion-free condition for the existence of local Lorentz frames at each points of a curved space-time.

The prescription of using tetrad fields $e_\sigma(z)$ and gauge fields $U_\sigma(z)$ for parallel transports to describe edges (1-simplexes) and triangles (2-simplexes) of the simplicial complex fully respects the principle of local gauge symmetries. Therefore, this prescription is independent of a particular vertex z, oriented edge $l_\sigma(z)$ and triangle $h(z)$, because of the gauge invariance. The formulation of defining tetrad fields $e_\sigma(z)$ at one of edge endpoints "z" and direction "σ," and each triangle has a definite orientation is gauge invariant.

However, the gauge transformation properties of fields $U_\nu(x+a_\mu)$ and $U_\mu(x+a_\nu)$ defined by Eqs. (85) and (86), as well as $U_{\mu\nu}(x)$ and $U_{\nu\mu}(x)$ introduced by Eqs. (87) and (88), are very complicate under the bilocal gauge transformations (109) and (110). This implies that we could not use these fields to construct a gauge-invariant object. We need to study the object of three U fields, $U_\mu(x)$, $U_\rho(x+a_\mu)$ and $U_\nu(x+a_\nu)$ along a closed triangle path of each 2-simplex $h(x)$ (see Fig. 1), which will be discussed in the next section.

F. Regularized EC action

To illustrate how to construct a gauge-invariantly regularized EC theory describing dynamical configurations of the simplicial complex, we consider anti-clocklike 2-simplex (triangle) $h(x)$ and clocklike 2-simplex (triangle) $h^\dagger(x)$ (see Figs. 1 and 2).

For simplifying notations, we henceforth do not explicitly write negative signs $-\mu$, $-\nu$, $-\rho$ to indicate the backward directions of edges. In terms of the tetrad fields $e_\mu(x)$ and $e_\nu(x)$ of the 2-simplex $h(x)$ (see Fig. 1), we introduce the following vertex fields $v_{\mu\nu}(x)$:

$$v_{\mu\nu}(x) \equiv \gamma_5 e_{\mu\nu}(x), \tag{116}$$

$$\begin{aligned} e_{\mu\nu}(x) &\equiv \sigma_{ab}[e^a(x)\wedge e^b(x)]_{\mu\nu} \\ &\equiv \frac{1}{2}\sigma_{ab}[e^a_\mu(x)e^b_\nu(x) - e^a_\nu(x)e^b_\mu(x)] \\ &= \frac{i}{2}[e_\mu(x)e_\nu(x) - e_\nu(x)e_\mu(x)], \end{aligned} \tag{117}$$

which have properties: $v_{\mu\nu}(x) = -v_{\nu\mu}(x)$, $\mathrm{tr}[v_{\mu\nu}(x)] = 0$ and $v^\dagger_{\mu\nu}(x) = v_{\nu\mu}(x)$ (see Appendix B). Under the local gauge transformation (9) and (111), the vertex fields (116) and (117) transform locally at a vertex x,

$$v_{\mu\nu}(x) \to \mathcal{V}(x)v_{\mu\nu}(x)\mathcal{V}^\dagger(x), \tag{118}$$

which is transformed in the same manner as area operators (114). In addition to the vertex field $e_{\mu\nu}(x)$ (117) at the vertex (x), we can define in the same way the vertex fields $e_{\rho\mu}(x + a_\mu)$ at the vertex $(x + a_\mu)$, and $e_{\nu\rho}(x + a_\nu)$ at the vertex $(x + a_\nu)$ of the anti-clocklike 2-simplex $h(x)$ (see Fig. 1). Actually, the vertex fields $e_{\mu\nu}(x)$ (117), $e_{\rho\mu}(x + a_\mu)$ and $e_{\nu\rho}(x + a_\nu)$ are related to the fundamental area operators $S^h_{\mu\nu}(x)$ (98), $S^h_{\rho\mu}(x + a_\mu)$ (99) and $S^h_{\nu\rho}(x + a_\nu)$ (100), e.g.,

$$S^h_{\mu\nu}(x) = ia^2 e_{\mu\nu}(x). \tag{119}$$

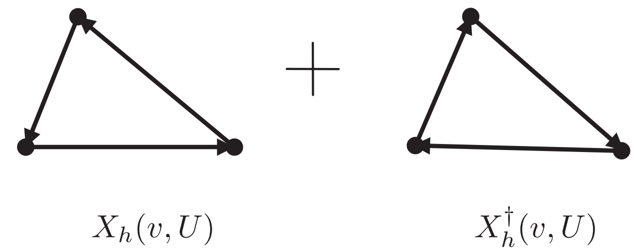

FIG. 2. The smallest holonomy field along a closed triangle path of the 2-simplex $h(x)$: the anti-clocklike orientation $X_h(v, U)$ [left]; the clocklike orientation $X^\dagger_h(v, U)$ [right].

As discussions for three area operators in Eqs. (95)–(103), only one of three vertex fields $e_{\mu\nu}(x)$, $e_{\rho\mu}(x + a_\mu)$ and $e_{\nu\rho}(x + a_\nu)$ is independent for the anti-clocklike 2-simplex $h(x)$. As for an clocklike 2-simplex $h^\dagger(x)$, vertex fields can be obtained by using the relations $e^\dagger_{\mu\nu}(x) = e_{\nu\mu}(x)$ and $e_{\mu\nu}(x) = -e_{\nu\mu}(x)$.

Using the tetrad fields $e_\mu(x)$ and vertex fields $v_{\mu\nu}(x)$ to construct coordinate and Lorentz scalars to preserve the diffeomorphism and *local* gauge invariance, we define a smallest holonomy field along the closed triangle path of the 2-simplex $h(x)$ (see Fig. 1):

$$\begin{aligned} X_h(v, U) = \mathrm{tr}[&v_{\nu\mu}(x)U_\mu(x)v_{\mu\rho}(x + a_\mu) \\ &\times U_\rho(x + a_\mu)v_{\rho\nu}(x + a_\nu)U_\nu(x + a_\nu)], \end{aligned} \tag{120}$$

whose orientation is anti-clocklike, as shown the left graphic in Fig. 2. Considering the clocklike orientation, as shown the right graphic in Fig. 2, we have

$$\begin{aligned} X^{\text{clocklike}}_h(v, U) &= \mathrm{tr}[v_{\mu\nu}(x)U_\nu(x)v_{\nu\rho}(x + a_\nu)U_\rho(x \\ &\quad + a_\nu)v_{\rho\mu}(x + a_\mu)U_\mu(x + a_\mu)] \\ &= X_h(v, U)|_{\mu\leftrightarrow\nu}. \end{aligned} \tag{121}$$

On the other hand,

$$\begin{aligned} X^\dagger_h(v, U) &= \mathrm{tr}[U^\dagger_\nu(x + a_\nu)v^\dagger_{\rho\nu}(x + a_\nu)U^\dagger_\rho(x + a_\mu)v^\dagger_{\mu\rho}(x + a_\mu)U^\dagger_\mu(x)v^\dagger_{\nu\mu}(x)] \\ &= \mathrm{tr}[U_\nu(x)v_{\nu\rho}(x + a_\nu)U_\rho(x + a_\nu)v_{\rho\mu}(x + a_\mu)U_\mu(x + a_\mu)v_{\mu\nu}(x)] \\ &= \mathrm{tr}[v_{\mu\nu}(x)U_\nu(x)v_{\nu\rho}(x + a_\nu)U_\rho(x + a_\nu)v_{\rho\mu}(x + a_\mu)U_\mu(x + a_\mu)] = X^{\text{clocklike}}_h(v, U) \end{aligned} \tag{122}$$

where in the second line of equation, we use the properties $U^\dagger_\nu(x + a_\nu) = U_\nu(x)$, $U^\dagger_\rho(x + a_\mu) = U_\rho(x + a_\nu)$, $U^\dagger_\mu(x) = U_\mu(x + a_\mu)$ and $v^\dagger_{\mu\nu}(x) = v_{\nu\mu}(x)$. Therefore, we have

$$X_h(v, U) + \text{H.c.} = X_h(v, U) + X^{\text{clocklike}}_h(v, U). \tag{123}$$

Equations (121)–(123) are invariant under gauge transformations (109), (110), and (118).

Using Eqs. (120)–(123), we are ready to construct the diffeomorphism and *local* gauge-invariant regularized EC action. First we consider the case $v_{\mu\nu}(x) = e_{\mu\nu}(x)\gamma_5$:

$$\mathcal{A}_P(e, U) = \frac{1}{8g^2}\sum_{h\in\mathcal{M}}\{X_h(v, U) + \text{H.c.}\}, \tag{124}$$

where $\sum_{h\in\mathcal{M}}$ is the sum over all 2-simplices h of the simplicial complex. In the naive continuum limit: $ag\omega_\mu \ll 1$, Eq. (124) becomes (see Appendix B)

$$\mathcal{A}_P(e, U_\mu) = \frac{1}{a^2}\sum_{h\in\mathcal{M}} S^2_h(x)\epsilon_{cdab}e^c\wedge e^d\wedge R^{ab} + \mathcal{O}(a^4), \tag{125}$$

where the 2-simplex $h(x)$ contributed volume element $S_h^2(x)$ is given in Eq. (104) or Eq. (B17). Based the volume element $dV(x)$ (107) around the vertex "x"

$$\sum_{h\in\mathcal{M}} S_h^2(x) = \frac{1}{3}\sum_x dV(x) \tag{126}$$

where $\sum_x$ stands for a sum overall vertexes (0-simplices) of the simplicial complex, and the factor $1/3$ is due to each 2-simplex contributing its area to its three vertexes. The interior of the 4-simplex is approximately flat, leading to

$$\sum_x dV(x) \Rightarrow \int d^4\xi(x) = \int d^4x \det[e(x)]. \tag{127}$$

As a result, Eq. (125) approaches to $S_P(e, \omega)$ (22) with an effective Newton constant

$$G_{\rm eff} = \tfrac{3}{4}gG, \tag{128}$$

and $\kappa_{\rm eff} \equiv 8\pi G_{\rm eff}$. The second we consider the case $v_{\mu\nu}(x) = e_{\mu\nu}(x)$:

$$\mathcal{A}_H(e, U_\mu) = \frac{1}{8g^2\gamma}\sum_{h\in\mathcal{M}}[X_h(v, U) + {\rm H.c.}], \tag{129}$$

where the real parameter $\gamma = i\tilde{\gamma}$ [see Eq. (23)]. Analogously, in the naive continuum limit: $ag\omega_\mu \ll 1$, Eq. (129) approaches to $S_H(e, \omega)$ (23) [see Appendix B],

$$\mathcal{A}_H(e, U_\mu) = \frac{1}{2\kappa_{\rm eff}\tilde{\gamma}}\int d^4x \det[e(x)] e_a \wedge e_b \wedge R^{ab} + \mathcal{O}(a^4), \tag{130}$$

with the effective Newton constant $\kappa_{\rm eff} \equiv 8\pi G_{\rm eff}$ (128). The diffeomorphism and *local* gauge-invariant regularized EC action is then given by

$$\mathcal{A}_{\rm EC} = \mathcal{A}_P + \mathcal{A}_H. \tag{131}$$

In addition, we can generalize the link field $U_\mu(x)$ to be all irreducible representations j of the gauge group $SO(4)$. The regularized EC action (131) should be a sum over all irreducible representations j,

$$\mathcal{A}_{\rm EC} = \sum_j \frac{4}{d_j}[\mathcal{A}_P^j(e_\mu, U_\mu) + \mathcal{A}_H^j(e_\mu, U_\mu)], \tag{132}$$

where d_j is the dimensions of the irreducible representations j and $d_j = 4$ for the fundamental representation, which is the dimension of the Dirac spinor space.

G. Invariant holonomy fields along a large loop

We consider the following diffeomorphism and *local* gauge-invariant holonomy fields along a loop $\mathcal{C}$ on the Euclidean manifold $\mathcal{R}^4$

$$X_\mathcal{C}(v, \omega) = \mathcal{P}_C \,{\rm tr}\exp\left\{ig\oint_\mathcal{C} v_{\mu\nu}(x)\omega^\mu(x)dx^\nu\right\}, \tag{133}$$

where $\mathcal{P}_C$ is the path-ordering and "tr" denotes the trace over spinor space. We attempt to regularize these holonomy fields (133) on the simplicial complex $\mathcal{M}$. Suppose that an orientating loop $\mathcal{C}$ passes space-time points (vertexes) $x_1, x_2, x_3, \cdots, x_N = x_1$ and edges connecting between neighboring points in the simplicial complex $\mathcal{M}$ (see the diagram in the left-hand side of graphic equation, Fig. 3). At each point x_i two tetrad fields $e_\mu(x_i)$ and $e_{\mu'}(x_i)$ ($\mu \neq \mu'$), respectively, orientating path incoming to ($i - 1 \to i$) and outgoing from ($i \to i + 1$) the point x_i, we have the vertex field $v_{\mu\mu'}(x_i)$ defined by Eqs. (116) and (117). Link fields $U_\mu(x_i)$ are defined on edges lying in the loop $\mathcal{C}$. Recalling the relationship $U_{-\mu}(x_{i+1}) = U_\mu^\dagger(x_i)$ [see Eqs. (59)–(61)], we can write the regularization of the holonomy fields (133) as

$$\begin{aligned} X_\mathcal{C}(v, U) = \mathcal{P}_C\,{\rm tr}[&v_{\mu\mu'}(x_1)U_{\mu'}(x_1)v_{\mu'\nu}(x_2)U_\nu(x_2) \\ &\cdots v_{\rho\rho'}(x_i)U_{\rho'}(x_i)v_{\rho'\sigma}(x_{i+1}) \\ &\cdots v_{\lambda\mu}(x_{N-1})U_\mu^\dagger(x_{N-1})], \end{aligned} \tag{134}$$

which preserve diffeomorphism and *local* gauge invariances. The holonomy fields $X_\mathcal{C}(e, U)$ are functionals of fields (v, U) and loop $\mathcal{C}$. Consistently with the holonomy fields $X_\mathcal{C}(e, U)$ [Eq. (134)], the holonomy field $X_h(e, U)$

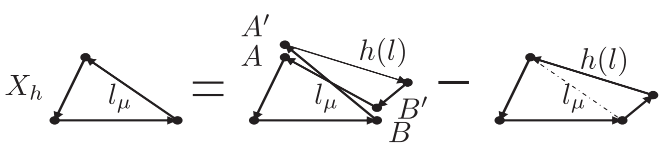

FIG. 3. We sketch a graphic representation of the dynamical Eq. (165) for the general holonomy field $X_\mathcal{C}$ (134). The diagram in the left-hand side of the graphic equation indicates the first term in Eq. (165). The first and second diagrams in the right-hand side of the graphic equation, respectively, indicate the third and second terms in Eq. (165). We indicate the edge l_μ, where the *local* gauge transformation is made. In the right-hand side of graphic equation, the summation over all 2-simplices $h(l)$ associated to this edge l_μ is made.

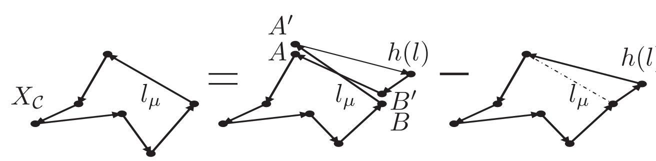

FIG. 4. We sketch a graphic representation of the dynamical Eq. (165) for the smallest holonomy field $X_h(v, U)$ (120). The diagram in the left-hand side of the graphic equation indicates the first term in Eq. (165). The first and second diagrams in the right-hand side of the graphic equation, respectively, indicate the third and second terms in Eq. (165). Note that A and A' are the same vertex, so are B and B'. We indicate the edge l_μ, where the *local* gauge transformation is made. In the right-hand side of the graphic equation, the summation over all 2-simplices $h(l)$ associated to this edge l_μ is made.

[Eq. (120)] is the one with the smallest loop, i.e., the closed path of the 2-simplex (triangle) $h(x)$, see Fig. 1.

H. Euclidean partition function

The partition function $Z_{\rm EC}$ and effective action $\mathcal{A}^{\rm eff}_{\rm EC}$ are given by

$$Z_{\rm EC} = \exp - \mathcal{A}^{\rm eff}_{\rm EC} = \int \mathcal{D}e\mathcal{D}U \exp - \mathcal{A}_{\rm EC}, \qquad (135)$$

with the diffeomorphism and *local* gauge-invariant measure

$$\int \mathcal{D}e\mathcal{D}U \equiv \prod_{l_\mu(x)\in\mathcal{M}} \int_{l_\mu(x)} de_\mu(x) dU_\mu(x) \delta(\Delta), \qquad (136)$$

where $\prod_{l_\mu(x)\in\mathcal{M}}$ indicates the product of overall edges (1-simplices) of the four-dimensional simplicial complex $\mathcal{M}$. As already mentioned, the configuration $\{l_\mu(x) \in \mathcal{M}\}$ is formulated such that each edge $l_\mu(x) = ae_\mu(x)$ is defined by giving its coordinate (vertex) $x \in \mathcal{M}$ in one of the endpoint coordinates x and $x + a_\mu$, and giving its forward direction μ pointing from x to $x + a_\mu$. This endpoint coordinate x and forward direction μ have to be uniquely chosen for each edge $l_\mu(x) \in \mathcal{M}$. Beside, on such defined edge $l_\mu(x)$, we place an independent gauge field $U_\mu(x)$ corresponding a parallel transport between x and $x + a_\mu$. The gauge-invariant properties, discussed in Sec. III E, guarantee that the change of a formulation does not lead to the change in the measure of the configuration $\{l_\mu(x) \in \mathcal{M}\}$. In addition, the triangle constraint (92) and (93) must be imposed in the measure (136), symbolically indicated as $\delta(\Delta)$, a δ functional of Eq. (92) or Eq. (93).

In the single edge measure [see Eq. (136)]

$$\int_{l_\mu(x)} de_\mu(x) dU_\mu(x), \qquad (137)$$

$dU_\mu(x)$ is the invariant Haar measure of the compact gauge group $SO(4)$ or $SU_L(2) \otimes SU_R(2)$, and $de_\mu(x)$ is the measure of the Dirac-matrix valued field $e_\mu(x) = \sum_a e^a_\mu(x)\gamma_a$, determined by the functional measure $de^a_\mu(x)$ of the bosonic field $e^a_\mu(x)$. The single edge measure has to be the measure over fields only $e_\mu(x)$ and $U_\mu(x)$ of the edge in the forward direction μ, because $e^\dagger_\mu(x)$ and $U^\dagger_\mu(x)$ of the edge in the backward direction $-\mu$ are related to the fields $e_\mu(x)$ and $U_\mu(x)$ by Eqs. (55), (68), (70), and (72) so that the single edge measure (137) is actually over all degrees of fields assigned on the edge.

It should be mentioned that the measure (136) is just a lattice form of the standard DeWitt functional measure [22] over the continuum degrees, with the integral of the spin-connection field $\omega_\mu(x)$ replaced by the Haar integral over the $U_\mu(x)$'s, analytical integration or numerical simulations runs overall configuration space of continuum degrees and no gauge fixing is needed. In addition, it should be noted that the measure (136) does not contain parallel transport fields $\bar{e}$ and $\bar{U}$, for examples $\bar{e}_\nu(x + a_\mu)$ and $\bar{e}_\mu(x + a_\nu)$ (see Fig. 1) given by the Cartan Eqs. (46) and (47), since parallel transport fields are not associated to any edges of the four-dimensional simplicial complex. This means that the torsion-free Cartan equation has been taken into account.

In this path-integral quantization formalism, the partition function (135) presents all dynamical configurations of the simplicial complex, described by the configurations of dynamical fields $e_\mu(x)$ and $U_\mu(x)$ in the weight of $\exp - \mathcal{A}_{\rm EC}$. The effective action $\mathcal{A}^{\rm eff}_{\rm EC}$ (135) contains all one-particle irreducible (1PI) functions (operators), i.e., all truncated n-point Green-functions. The vacuum expectation values (vevs) of diffeomorphism and *local* gauge-invariant quantities, for instance holonomy fields (134), are given by

$$\langle X_C(v, U)\rangle = \frac{1}{Z_{\rm EC}} \int \mathcal{D}e\mathcal{D}U [X_C(v, U)] \exp - \mathcal{A}_{\rm EC}. \quad (138)$$

In the action (124) and (129), $X_h(v, U)$ [Eq. (120)] contains the quadratic term of $e_\mu(x)$ field associated to each edge of 2-simplex $h(x)$, the partition function $Z_{\rm EC}$ (135) and vev (138) are not divergent for large fluctuating e_μ fields, provided the action $\mathcal{A}_{\rm EC}$ is positive definite, see discussions below. On the other hand, all edge lengths do not vanish $[|e_\mu(x)| \neq 0$, see Eqs. (41) and (42)], and all simplicial triangle inequalities and their higher dimensional analogs should be imposed [2,3]. Integrating spin-connection fields U_μ over the Haar measure of compact gauge groups is similar to that in the Wilson-lattice QCD, the difference is that the $X_h(v, U)$ (120) contains three U fields in a 2-simplex h, while the Wilson action contains four U fields in a plaquette. Equation (138) can be calculated by numerical Monte Carlo simulations. We are trying do some numerical Monte Carlo simulations, it will take time so that the results will be published in a separate paper.

Before ending this section, we make some discussions on the convergences of the partition function (135) and vevs (138). Suppose that we first integrate Eqs. (135) and (138) over the compact Haar measure of the $SO(4)$ gauge group, roughly speaking, the result gives, in addition to a polynomial of tetrad fields e, a combination of both decreasing exponents $\exp[-\mathcal{A}^{(+)}(e)]$ and increasing exponents $\exp[-\mathcal{A}^{(-)}(e)]$ as functions of increasing tetrad fields e. From the regularized action (120), one can find that $\mathcal{A}^{(\pm)}(e)$ depend on 2-simplex area operators S_h (104) and are the sum over all 2-simplexes. $\mathcal{A}^{(\pm)}(e)$ are either some extremal values of the action $\mathcal{A}_{\rm EC}$ (131) with respect to group-valued U fields, or those values taken at the boundary points of the compact $SO(4)$ gauge group. Clearly, for the case of decreasing exponents $\exp[-\mathcal{A}^{(+)}(e)]$, integrations Eqs. (135) and (138) over tetrad fields e are convergent. This is certainly the case for perturbative weak U fields, i.e., $U \sim 1$. While for the case of increasing exponents $\exp[-\mathcal{A}^{(-)}(e)]$, integrations Eqs. (135) and (138) over tetrad fields e are divergent.

To avoid these possible divergences, it is necessary to add into the regularized action $\mathcal{A}_{\rm EC}$ (131) an additional term of another dimensionality: either a curvature squared R^2 term: $X_h^2(v, U)$ + H.c. with a new coupling parameter; or a bare cosmological term: $\mathcal{A}_\Lambda(e)$. We consider here an additional bare cosmological term $\mathcal{A}_\Lambda$ to the regularized action $\mathcal{A}_{\rm EC}$ (131): $\mathcal{A}_{\rm EC} \to \mathcal{A}_{\rm EC} + \mathcal{A}_\Lambda$,

$$\mathcal{A}_\Lambda(e) = \frac{\lambda}{4\cdot(4!)^2}\epsilon^{\mu\nu\rho\sigma}\sum_x \mathrm{tr}[\gamma_5 e_\mu(x)e_\nu(x)e_\rho(x)e_\sigma(x)] + \mathrm{H.c.} = \lambda\sum_x \det[e^a_\mu(x)] + \mathrm{H.c.} \tag{139}$$

where the cosmological parameter $\lambda \equiv \Lambda a^2$ and Λ is the bare cosmological constant. The bare cosmological term $\mathcal{A}_\Lambda(e)$ is a four-dimensional volume term (sum over all vertexes x), which is independent of configurations of group-valued U fields. The exponent $\exp[-\mathcal{A}_\Lambda(e)]$ decreases with strong tetrad fields e, large volume configurations. Bare parameters g, γ and λ play an important role for convergences of the partition function (135) and vacuum expectation values (138). It needs further studies to find the region of bare parameters g, γ and λ for the convergences, and the scaling invariant region $(g_c, \gamma_c, \lambda_c)$ for the physically sensible continuum limit, see the discussions in the last Sec. VII.

I. *Local* gauge symmetry

Analogously to Eq. (25), the *local* gauge invariance of the partition function (135), i.e., $\delta Z_{\rm EC} = 0$ under the gauge transformation (109) and (118), leads to (no summation over index μ)

$$\left\langle \frac{\delta\mathcal{A}_{\rm EC}}{\delta e_\mu}\delta e_\mu + \frac{\delta\mathcal{A}_{\rm EC}}{\delta\omega_\mu}\delta\omega_\mu + \mathrm{H.c.}\right\rangle = 0. \tag{140}$$

Based on δe_μ and $\delta\omega_\mu$ (14) and (15) for an arbitrary function $\theta^{ab}(x)$ and the independent bases of Dirac matrices γ_5, γ_μ and σ_{ab}, we obtain the "averaged" Cartan Eq. (35) for the torsion-free case,

$$\left\langle U_\mu\frac{\delta\mathcal{A}_{\rm EC}}{\delta U_\mu} - U^\dagger_\mu\frac{\delta\mathcal{A}_{\rm EC}}{\delta U^\dagger_\mu}\right\rangle = 0, \tag{141}$$

where we use

$$\frac{\delta\mathcal{A}_{\rm EC}}{\delta\omega_\mu} = iag\left\{U_\mu\frac{\delta\mathcal{A}_{EC}}{\delta U_\mu} - U^\dagger_\mu\frac{\delta\mathcal{A}_{\rm EC}}{\delta U^\dagger_\mu}\right\}, \tag{142}$$

for the group-valued field $U_\mu(x) = \exp[iag\omega_\mu(x)]$ (56). The averaged torsion-free Cartan Eq. (141) actually shows the impossibility of spontaneous breaking of the *local* gauge symmetry. This should not be surprised, since the torsion-free (30) is a necessary condition to have a *local* Lorentz frame, therefore a *local* gauge invariance, as required by the equivalence principle.

IV. INCLUDING FERMION FIELDS

A. Bilinear and quadralinear-fermion actions

Introducing dimensionless fermion field $\psi'(x) \equiv a^{3/2}\psi(x)$ (drop "prime" henceforth) and using the relations $\gamma^0(\gamma_a)^\dagger\gamma^0 = \gamma_a$, $\gamma^0(\sigma_{ab})^\dagger\gamma^0 = \sigma_{ab}$ and

$$\gamma^0 e^\dagger_\mu\gamma^0 = e_\mu; \qquad \gamma^0 U^\dagger_\mu\gamma^0 = U^\dagger_\mu, \tag{143}$$

we consider the following regularized kinetic action of fermion fields,

$$\mathcal{A}_F(e, U, \psi) = \frac{1}{2}\sum_{x,\mu}[\bar\psi(x)e^\mu(x)U_\mu(x)\psi(x+a_\mu) - \bar\psi(x+a_\mu)U^\dagger_\mu(x)e^\mu(x)\psi(x)], \tag{144}$$

where fermion fields $\psi(x)$ and $\psi(x+a_\mu)$ are defined at two neighboring points (vertexes) of the edge $(x, x+a_\mu)$, (see Fig. 1), where fields $U_\mu(x)$ and $e_\mu(x)$ are added to preserve *local* gauge and diffeomorphism invariances, and $\sum_{x,\mu}$ is the sum over all edges (1-simplexes) of the simplicial complex.

Using Eq. (142) and performing a variation of the regularized fermion action (144) with respect to the spin-connection field $\omega_\mu(x)$, i.e., $\delta\mathcal{A}_F(e, U, \psi)/\delta\omega_\mu$, we obtain the nonvanishing torsion field $T^a = \kappa g e_b \wedge e_c \mathcal{J}^{ab,c}$, where the regularized fermion spin current is

$$\mathcal{J}^{ab,c} = \epsilon^{abcd}\bar\psi(x)\gamma_d\gamma^5 U_\mu(x)\psi(x+a_\mu), \qquad \mu\ \text{fixed}, \tag{145}$$

[see Eq. (32)]. Instead of solving regularized Cartan equation and finding an effective theory, as what is done in the continuum case (25)–(32), we assume that the $U_\mu(x)$ in Eqs. (144) and (145) is the group-valued spin-connection field $\omega_\mu(e)$ for the torsion-free case (35), i.e., $U_\mu(x) = \exp[iag\omega_\mu(e)]$. Thus, the regularization of the effective EC theory (39) and (40) is given by Eqs. (131) and (144) and the regularized four-fermion interaction

$$\mathcal{A}_{4F}(U, \psi) = 3\zeta g^2\sum_{x,\mu}[\bar\psi(x)\gamma^d\gamma^5 U^\mu(x)\psi(x+a_\mu)] \times [\bar\psi(x+a_\mu)U^\dagger_\mu(x)\gamma_d\gamma^5\psi(x)], \tag{146}$$

where $\zeta = \tilde\gamma^2/(\tilde\gamma^2+1) = \gamma^2/(\gamma^2+1)$ [see Eq. (40)]. In the naive continuum limit $ag\omega_\mu \ll 1$, the regularized fermion action $\mathcal{A}_F(e, U, \psi)$ (144) approaches to the continuum fermion action $S_F(e, \omega_\mu, \psi)$ (24), and Eqs. (145) and (146), respectively approach to their continuum counterparts $J^{ab,c}$ (32) and S_{4F} (40). The diffeomorphism and *local* gauge-invariant regularized EC action is then given by

$$\mathcal{A}_{EC} = \mathcal{A}_P + \mathcal{A}_H + \mathcal{A}_F + \mathcal{A}_{4F}. \tag{147}$$

The partition function Z_{EC} and effective action $\mathcal{A}^{eff}_{EC}$ are

$$Z_{EC} = \exp - \mathcal{A}^{eff}_{EC} = \int \mathcal{D}e\mathcal{D}U\mathcal{D}\psi \exp - \mathcal{A}_{EC}, \tag{148}$$

with the diffeomorphism and *local* gauge-invariant measure

$$\int \mathcal{D}e\mathcal{D}U\mathcal{D}\psi \equiv \prod_{l_\mu(x)\in\mathcal{M}} \int_{l_\mu(x)} de_\mu(x) dU_\mu(x)\delta(\Delta) \cdot \prod_{x\in\mathcal{M}} \int d\psi(x) d\bar{\psi}(x), \tag{149}$$

where $d\psi(x)d\bar{\psi}(x)$ is the measure of Grassmann anticommuting fields. Analogously to Eq. (132), Eqs. (147)–(149) can be straightforwardly generalized to include all irreducible representations j of the gauge group $SO(4)$ that couple to corresponding spinor states of fermion fields.

B. Holonomy fields with fermions

We consider the following diffeomorphism and *local* gauge-invariant quantities

$$X_{\mathcal{L}}(e, \omega, \psi) = \bar{\psi}(x_1)\mathcal{P}\exp\left\{ig\int_{\mathcal{L}} v_{\mu\nu}(x)\omega^\mu(x)dx^\nu\right\}\psi(x_N), \tag{150}$$

where $\mathcal{L}$ stands for an orientating ($\mathcal{P}$) path connecting two vertexes x_1 and x_N ($x_1 \neq x_N$) on the simplicial complex $\mathcal{M}$. In Eq. (150), $X_{\mathcal{L}}(e, \omega, \psi)$ represents the evolution of the spin of fermion fields from the vertex x_N to the vertex x_1 under the gravitational field influence. Analogously to discussions in Sec. III G for the holonomy fields (133), we regularize these quantities (150) on the simplicial complex as follows:

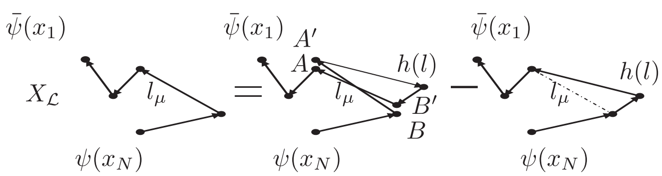

FIG. 5. We sketch a graphic representation of the dynamical Eq. (165) for the field $X_{\mathcal{L}}$ (151). The diagram in the left-hand side of the graphic equation indicates the first term in Eq. (165). The first and second diagrams in the right-hand side of the graphic equation, respectively, indicate the third and second terms in Eq. (165). Note that A and A' are the same vertex, so are B and B'. We indicate the edge l_μ, where the *local* gauge transformation is made. We also indicate the fermion field $\psi(x_N)$ at staring point x_N and the fermion field $\bar{\psi}(x_1)$ at ending point x_1 of the path $\mathcal{L}$. In the right-hand side of the graphic equation, the summation over all 2-simplices $h(l)$ associated to this edge l_μ is made.

$$X_{\mathcal{L}}(e, U, \psi) = \bar{\psi}(x_1)\mathcal{P}[U_{\mu'}(x_1)v_{\mu'\nu}(x_2)U_\nu(x_2) \cdots v_{\rho\rho'}(x_i)U_{\rho'}(x_i)v_{\rho'\sigma}(x_{i+1}) \cdots v_{\lambda\mu}(x_N)U^\dagger_\mu(x_N)]\psi(x_N), \tag{151}$$

which preserves diffeomorphism and *local* gauge invariances. The graphic representation of $X_{\mathcal{L}}(e, U, \psi)$ can be found in Fig. 5 (see the diagram in the left-hand side of graphic equation).

C. Chiral gauge symmetries

Analogously to the discussions in the continuum EC theory (see the end of Sec. II), the regularized EC action (147) can be separated into left- and right-handed parts. Fermion fields ψ are decomposed into their left- and right-handed Weyl fields: $\psi = \psi_L + \psi_R$ and $\psi_{L,R} \equiv P_{L,R}\psi$, where the chiral projector $P_{L,R} = (1 \mp \gamma_5)/2$ and the commutators $[\sigma^{ab}, P_{L,R}] = 0$ and $[\gamma^a\gamma^b, P_{L,R}] = 0$. The 4×4 Dirac spinor space is split into two independent left- and right-handed 2×2 Weyl spinor spaces. In the chiral representation of matrices γ^a and σ^{ab}

$$\gamma^0 = i\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{152}$$

$$\sigma^{ij} = \begin{pmatrix} \Sigma^{ij} & 0 \\ 0 & \Sigma^{ij} \end{pmatrix}, \qquad \sigma^{0i} = i\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}; \tag{153}$$

where $\Sigma^{ij} = \epsilon^{ij}_k\sigma^k$ and $\sigma^i (i = 1, 2, 3)$ are the Pauli matrices, we define $\gamma^a_{L,R} \equiv P_{L,R}\gamma^a$:

$$P_L\gamma^0 = i\begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix}, \qquad P_L\gamma^i = \begin{pmatrix} 0 & 0 \\ -\sigma^i & 0 \end{pmatrix}, \qquad P_R\gamma^0 = i\begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix}, \qquad P_R\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ 0 & 0 \end{pmatrix}; \tag{154}$$

and $\sigma^{ab}_{L,R} \equiv P_{L,R}\sigma^{ab}$:

$$P_L\sigma^{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma^{ij} \end{pmatrix}, \qquad P_L\sigma^{0i} = i\begin{pmatrix} 0 & 0 \\ 0 & -\sigma^i \end{pmatrix}; \qquad P_R\sigma^{ij} = \begin{pmatrix} \Sigma^{ij} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R\sigma^{0i} = i\begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}. \tag{155}$$

Using Eq. (154), we separate tetrad fields e^μ into their left- and right-handed fields: $e^\mu = e^\mu_L + e^\mu_R$, $e^\mu_{L,R} \equiv P_{L,R}e^\mu$. Using Eq. (155), we separate spin-connection fields ω^μ and vertex fields $v_{\mu\nu}$ into their left- and right-handed fields: $\omega^\mu = \omega^\mu_L + \omega^\mu_R$, $\omega^\mu_{L,R} \equiv P_{L,R}\omega^\mu$; and $v_{\mu\nu} = v^L_{\mu\nu} + v^R_{\mu\nu}$, $v^{L,R}_{\mu\nu} \equiv P_{L,R}v_{\mu\nu}$. This splits the Lie algebra of the group $SO(4)$ into two independent Lie algebra of sub groups $SU_L(2) \otimes SU_R(2)$. Therefore, the four-dimensional rotational group $SO(4)$ is split into two commuting and independent groups $SU_L(2) \otimes SU_R(2)$. The link fields

$U_\mu(x) = U^R_\mu(x) \oplus U^L_\mu(x)$, where $U^R_\mu(x) \in SU_R(2)$ and $U^L_\mu(x) \in SU_L(2)$ respectively.

The regularized EC theory (147)–(149) possesses exact chiral gauge symmetries, as consequences, the holonomy fields (120), (134), and (151) can be split into the left- and right-handed parts:

$$X_h(e, U) = X^L_h(e^L, U^L) + X^R_h(e^R, U^R); \tag{156}$$

$$X_{\mathcal{C}}(e, U) = X^L_{\mathcal{C}}(e^L, U^L) + X^R_{\mathcal{C}}(e^R, U^R); \tag{157}$$

$$X_{\mathcal{L}}(e, U, \psi) = X^L_{\mathcal{L}}(e^L, U^L, \psi_L) + X^R_{\mathcal{L}}(e^R, U^R, \psi_R), \tag{158}$$

where notations in the right-handed side of equations, for instance, $X^L_{\mathcal{L}}(e^L, U^L, \psi_L)$ indicates the same function $X_{\mathcal{L}}(e, U, \psi)$ (151) with replacements $e \to e^L$, $U \to U^L$ and $\psi \to \psi_L$. The fermion action (144) and four-fermion interaction (146) are also separated into the left- and right-handed parts:

$$\mathcal{A}_F(e, U, \psi) = \mathcal{A}^L_F(e^L, U^L, \psi_L) + \mathcal{A}^R_F(e^R, U^R, \psi_R); \tag{159}$$

$$\mathcal{A}_{4F}(U, \psi) = \mathcal{A}^L_{4F}(U^L, \psi_L) + \mathcal{A}^R_{4F}(U^R, \psi_R). \tag{160}$$

The chiral gauge symmetries of the regularized EC theory (147)–(149) are crucial for formulating the parity-violating (chiral) gauge symmetries $SU_L(2) \otimes U_Y(1)$, e.g., the standard model for particle physics, onto the simplicial complex described by the dynamical tetrad fields $e_\mu(x)$ and group-valued spin-connection fields $U_\mu(x)$. We only discuss the case of Weyl fermions (massless Dirac fermions), and the discussions on the case of Majorana fermions are the same, thus not presented in this article.

V. DYNAMICAL EQUATIONS FOR HOLONOMY FIELDS

Under a *local* gauge transformation (9)–(11), equivalently (9), (11), and (109), the *local* gauge invariance of holonomy fields $\langle X\rangle$ [Eq. (138)], i.e., $\delta\langle X\rangle = 0$, leads to the dynamical equations for the holonomy fields X_h (120), $X_{\mathcal{C}}$ (134) and $X_{\mathcal{L}}$ (151),

$$\left\langle \frac{\delta X}{\delta e_\mu}\delta e_\mu - X\frac{\delta \mathcal{A}_{\rm EC}}{\delta e_\mu}\delta e_\mu\right\rangle + \left\langle \frac{\delta X}{\delta\psi}\delta\psi - X\frac{\delta \mathcal{A}_{\rm EC}}{\delta\psi}\delta\psi\right\rangle + iag\left\langle X\delta\omega_\mu\right\rangle - \left\langle X\frac{\delta\mathcal{A}_{\rm EC}}{\delta\omega_\mu}\delta\omega_\mu\right\rangle + {\rm H.c.} = 0, \tag{161}$$

where the index μ is fixed, and for the variation $\delta X/\delta\omega_\mu$ we use Eq. (142) and the relationship

$$\sum_{ab} U^{ab}_\mu \frac{\delta X}{\delta U^{ab}_\mu} = X; \quad \text{or} \quad \sum_{ab} U^{ab\dagger}_\mu \frac{\delta X}{\delta U^{ab\dagger}_\mu} = X. \tag{162}$$

Analogously to the analysis in Sec. III I, we obtain the dynamical equations for the holonomy fields $X = X_h$, $X_{\mathcal{C}}$ and $X_{\mathcal{L}}$

$$\left\langle \frac{\delta X}{\delta e_\mu}\delta e_\mu - X\frac{\delta \mathcal{A}_{\rm EC}}{\delta e_\mu}\delta e_\mu\right\rangle + {\rm H.c.} = 0, \tag{163}$$

$$\left\langle \frac{\delta X}{\delta\psi}\delta\psi - X\frac{\delta \mathcal{A}_{\rm EC}}{\delta\psi}\delta\psi\right\rangle + {\rm H.c.} = 0, \tag{164}$$

and

$$\langle X\rangle + \left\langle X\left(U^\dagger_\mu\frac{\delta\mathcal{A}_{\rm EC}}{\delta U^\dagger_\mu}\right)\right\rangle - \left\langle X\left(U_\mu\frac{\delta\mathcal{A}_{\rm EC}}{\delta U_\mu}\right)\right\rangle = 0. \tag{165}$$

Equation (165) has the same form as the Dyson-Schwinger equation for the Wilson loops in lattice gauge theories. In Figs. 3–5, we show the graphic representations of the dynamical Eqs. (165) for the holonomy fields and $X_{\mathcal{C}}$ (134) and X_h (120) as well as $X_{\mathcal{L}}$ (151).

VI. MEAN-FIELD APPROXIMATION

A. Mean-field approach

In this section, we try to approximately calculate the partition function (135), the vacuum expectation values of the 2-simplex area (104) and the volume element (107) by using the approach of the mean-field approximation. In the regularized action $X_h(v, U)$ (120) associating to the 2-simplex $h(x)$ (Fig. 1), we replace the vertex fields $v_{\mu\rho}(x + a_\mu)$ and $v_{\rho\nu}(x + a_\nu)$ by assuming a nonvanishing mean-field value $M_h > 0$,

$$(M^2_h)\delta^{\alpha\beta} \equiv [\langle v_{\mu\rho}v_{\rho\nu}\rangle]^{\alpha\beta}, \tag{166}$$

where α, β are Dirac spinor indexes. The definition of mean-field value (166) does not depend on whether $v_{\mu\rho}$ and $v_{\rho\nu}$ contain the matrix γ_5 or not, due to $\gamma_5^2 = 1$ and $[\gamma_5, \sigma_{ab}] = 0$. The mean-field value M_h is independent of any specific vertex, edge and 2-simplex of the simplicial complex. Based on the definitions of the 2-simplex area (104) and the volume element (107), the mean-field values for the 2-simplex area and the volume element are given by

$$\langle S_h(x)\rangle = a^2 M_h, \qquad \langle dV(x)\rangle = a^4 N_h M^2_h, \tag{167}$$

where N_h is the mean value of the number of 2-simplices $h(x)$ that share the same vertex. Note that in this preliminary calculations in the mean-field approximation, we do not take into account the cosmological term (139), since the path integrals are convergent (see below) for positive mean-field value $M_h > 0$.

Based on the mean-field value (166), the smallest holonomy field $X_h(v, U)$ (120) is approximated by its mean-field counterpart

$$\bar X_h(v, U) = {\rm tr}[v_{\nu\mu}(x)U_\mu(x)U_\rho(x + a_\mu)U_\nu(x + a_\nu)]M^2_h, \tag{168}$$

$$\bar X^\dagger_h(v, U) = {\rm tr}[v_{\mu\nu}(x)U_\nu(x)U_\rho(x + a_\nu)U_\mu(x + a_\mu)]M^2_h, \tag{169}$$

where using Eqs. (121) and (122) for $\mu \neq \nu$ we obtain $\bar{X}_h^\dagger(v, U)$. Note that two of three vertex fields $v(x)$ in the $X_h(v, U)$ (120), i.e., $v_{\mu\rho}(x + a_\mu)$ and $v_{\rho\nu}(x + a_\nu)$ are replaced by their mean-field values M_h, and the 2-simplex $h(x)$ shown in Fig. 1 can also be identified by three different indexes $\mu \neq \nu \neq \rho$ (no summation over these indexes). Equations (168) and (169) depend on U_ρ, and the fields (e_μ, U_μ) and (e_ν, U_ν) associated to two edges (x, μ) and (x, ν) of the 2-simplex (triangle) $h(x)$ (see Fig. 1). Using Eqs. (168) and (169), we define the *local* mean-field action $\bar{\mathcal{A}}_h$ for the 2-simplex $h(x)$

$$\begin{aligned}\bar{\mathcal{A}}_h &= \frac{1}{8g^2}[\bar{X}_h(v, U) + \bar{X}_h^\dagger(v, U)]_{v_{\mu\nu}=\gamma_5 e_{\mu\nu}} + \frac{1}{8g^2\gamma} \\ &\quad \times [\bar{X}_h(v, U) + \bar{X}_h^\dagger(v, U)]_{v_{\mu\nu}=e_{\mu\nu}} \\ &= \mathrm{tr}[e_\nu(x)\Gamma^h_{\nu\mu}(x)e_\mu(x) - e_\mu(x)\Gamma^h_{\nu\mu}(x)e_\nu(x)],\end{aligned} \tag{170}$$

where

$$\begin{aligned}\Gamma^h_{\nu\mu}(x) &= \frac{1}{8g^2}\left(\gamma_5 - \frac{1}{\gamma}\right)H_{\nu\mu}(x) \\ &= \frac{1}{8g^2}\left(\frac{i}{2}\right)M_h^2\left(\gamma_5 - \frac{1}{\gamma}\right)[U_\nu(x)U_\rho(x + a_\nu)U_\mu^\dagger(x)] \\ &\quad + \mathrm{H.c.}\end{aligned} \tag{171}$$

The detailed derivation is given in Appendix D. In this mean-field approximation, all 2-simplices $\{h(x)\}$ in the simplicial complex $\mathcal{M}$ have the same *local* action (170), namely, the single 2-simplex mean-field action $\bar{\mathcal{A}}_h$ (170) and operator $\Gamma^h_{\nu\mu}$ (171) are independent of the vertex "x". With the *local* mean-field action (170), we define the *local* mean-field partition function

$$\bar{Z}_h = \int_h \mathcal{D}U\mathcal{D}e \exp - \bar{\mathcal{A}}_h, \tag{172}$$

where the *local* mean-field measure is defined by

$$\int_h \mathcal{D}U\mathcal{D}e \equiv \int_h dU_\mu dU_\nu dU_\rho de_\mu de_\nu, \tag{173}$$

for each 2-simplex h. Thus, the regularized EC action $\mathcal{A}_{\mathrm{EC}}$ (131) is approximated by its mean-field counterpart,

$$\bar{\mathcal{A}}_{\mathrm{EC}} = \sum_{h\in\mathcal{M}} \bar{\mathcal{A}}_h, \tag{174}$$

which is the sum of the mean-field actions $\bar{\mathcal{A}}_h$ over all 2-simplices h. With the mean-field approximated action (174), we define the mean-field approximated partition function

$$\bar{Z}_{\mathrm{EC}} = \prod_{h\in\mathcal{M}} \int_h \mathcal{D}U\mathcal{D}e \exp - \bar{\mathcal{A}}_{\mathrm{EC}} = \prod_{h\in\mathcal{M}} \bar{Z}_h, \tag{175}$$

which is the mean-field counterpart of the partition function (135).

Using the mean-field EC action $\bar{\mathcal{A}}_{\mathrm{EC}}$ (170) and partition function $\bar{Z}_{\mathrm{EC}}$ (175), we have the following identity

$$Z_{\mathrm{EC}} \equiv \bar{Z}_{\mathrm{EC}}\langle e^{-(\mathcal{A}_{\mathrm{EC}} - \bar{\mathcal{A}}_{\mathrm{EC}})}\rangle_\circ, \tag{176}$$

where $\langle\cdots\rangle_\circ$ is the vacuum expectation value with respect to the mean-field partition function $\bar{Z}_{\mathrm{EC}}$ (175). Using the convexity inequality [23]

$$\langle e^{-(\mathcal{A}_{\mathrm{EC}} - \bar{\mathcal{A}}_{\mathrm{EC}})}\rangle_\circ \geq e^{-\langle\mathcal{A}_{\mathrm{EC}} - \bar{\mathcal{A}}_{\mathrm{EC}}\rangle_\circ}, \tag{177}$$

one can derive the following inequality

$$-\ln Z_{\mathrm{EC}} \leq -\ln\bar{Z}_{\mathrm{EC}} + \langle\mathcal{A}_{\mathrm{EC}} - \bar{\mathcal{A}}_{\mathrm{EC}}\rangle_\circ, \tag{178}$$

where $-\ln Z_{\mathrm{EC}}$ and $-\ln\bar{Z}_{\mathrm{EC}}$ are proportional to the free energies. We define the right-handed side of the inequality (178) as an approximate free energy (or approximate effective action)

$$\mathcal{F}^{\mathrm{app}}_{\mathrm{EC}}(M_h, g, \gamma) \equiv -\ln\bar{Z}_{\mathrm{EC}} + \langle\mathcal{A}_{\mathrm{EC}} - \bar{\mathcal{A}}_{\mathrm{EC}}\rangle_\circ. \tag{179}$$

The validity of the mean-field approximation approach bases on the inequality (178) that gives a low bound of the approximate free energy $\mathcal{F}^{\mathrm{app}}_{\mathrm{EC}}(M_h, g, \gamma)$. We determine the mean-field value $M_h^*(g, \gamma)$ of the *local* mean-field action (170), which minimizes the approximate free energy (179) and thus optimizes the low bound in Eq. (178), by satisfying the condition

$$\left[\frac{\delta}{\delta M_h}\mathcal{F}^{\mathrm{app}}_{\mathrm{EC}}(M_h, g, \gamma)\right]_{M_h=M_h^*} = 0. \tag{180}$$

Using the mean-field value $M_h^*(g, \gamma)$ and corresponding minimum of the approximate free energy $\mathcal{F}^{\mathrm{app}}_{\mathrm{EC}}[M_h^*(g, \gamma), g, \gamma]$ (179), we can gain some insights into the value of the 2-simplex area (166) and (167), and the critical points of the second-order phase transition, in terms of the gauge coupling g and Immirzi parameter γ. In addition, we can use the mean-field action (170) with the value M_h^* to calculate mean-field vacuum expectation values $\langle\cdots\rangle_\circ$ to approximate true vacuum expectation values $\langle\cdots\rangle$ that we discussed in Secs. III H, III I, and V.

B. Analytical calculations

We can analytically calculate the mean-field partition function (175). First we integrate over quantized tetrad $e_\mu(x)$ and $e_\nu(x)$ fields, which is quadratic in Eq. (170) (see Appendix E). Using the formula (E2), we have

$$\prod_{h\in\mathcal{M}} \int de_\mu de_\nu \exp - \bar{\mathcal{A}}_{\mathrm{EC}} = \prod_{h\in\mathcal{M}} \overset{-1}{\det}[I - \Gamma^h] \tag{181}$$

and the Cayley-Hamilton formula for a determinant [24]

$$\overset{-1}{\det}[I - \Gamma^h] = \exp[-\mathrm{tr}\ln(I - \Gamma^h)] = 1 + \sum_a \Gamma^h_{aa} + \frac{1}{2}\sum_{a,b}(\Gamma^h_{aa}\Gamma^h_{bb} + \Gamma^h_{ab}\Gamma^h_{ba}) + \cdots + \frac{1}{n!}\sum_{a_1\cdots a_n}\sum_P \Gamma^h_{a_1 a_{P_1}}\cdots\Gamma^h_{a_n a_{P_n}} \quad (182)$$

where P indicates permutations of $(1, \cdots, n)$ and Eq. (182) is a sum of traces of symmetrized tensor products. The expression (182) stops at the n-th order for a finite n-dimensional matrix Γ^h in the space of the gauge group.

Second we integrate over group-valued spin-connection $U_\rho(x + a_\mu)$, $U_\mu(x)$ and $U_\nu(x)$ fields defined at edges $(x + a_\mu, \rho)$, (x, μ) and (x, ν) of the 2-simplex $h(x)$ by using the properties of the invariant Haar measure:

$$\int dU_\mu(x) = 1, \quad (183)$$

$$\int dU_\mu(x)U_\mu(x) = 0, \quad (184)$$

$$\int dU_\mu(x)U^{ab}_\mu(x)U^{\dagger cd}_\sigma(x') = \frac{1}{d_j}\delta_{\mu\sigma}\delta^{ac}\delta^{bd}\delta(x - x'), \quad (185)$$

where $d_j = n_{j_L}n_{j_R}$ $(n_{j_L,j_R} = 2j_{L,R} + 1;\ j_{L,R} = 1/2, 3/2, \cdots)$ is the dimensions of irreducible representations $j = (j_L, j_R)$ of the gauge group $SU_L(2) \otimes SU_R(2)$, $j_R = j_L = 1/2$ and $d_j = 4$ for the fundamental representation. In Appendix E, we give more detailed calculations to obtain the mean-field partition function (175),

$$\bar{Z}_{\mathrm{EC}} = \prod_{h\in\mathcal{M}}\left[1 + \frac{\gamma^2 + 1}{64g^4\gamma^2 d_j^3}M_h^4\right], \quad (186)$$

where $\prod_{h\in\mathcal{M}}$ is the product of all 2-simplices h of the simplicial complex $\mathcal{M}$. The mean-field entropy is given by

$$\bar{\mathcal{S}} = \ln\bar{Z}_{\mathrm{EC}} = \sum_{h\in\mathcal{M}}\ln\left[1 + \frac{\gamma^2 + 1}{64g^4\gamma^2 d_j^3}M_h^4\right] = \mathcal{N}\ln\left[1 + \frac{\gamma^2 + 1}{64g^4\gamma^2 d_j^3}M_h^4\right], \quad (187)$$

where $\mathcal{N} = \sum_{h\in\mathcal{M}}$ is the total number of 2-simplexes, and the mean-field free energy

$$\bar{\mathcal{F}} = -\frac{1}{\beta}\ln\bar{Z}_{\mathrm{EC}} = -\frac{1}{\beta}\mathcal{N}\ln\left[1 + \frac{\gamma^2 + 1}{64g^4\gamma^2 d_j^3}M_h^4\right], \quad (188)$$

where the inverse "temperature" $\beta = 1/g^2$, see Eqs. (124) and (129).

We turn to calculate $\langle\bar{\mathcal{A}}_{EC}\rangle_\circ$ in Eq. (178). The mean-field value of $\bar{\mathcal{A}}_{\mathrm{EC}}$ (174) is calculated in Appendix E [see Eq. (E7)],

$$\langle\bar{\mathcal{A}}_{\mathrm{EC}}\rangle_\circ = \sum_{h\in\mathcal{M}}\langle\bar{\mathcal{A}}_h\rangle^h_\circ = \mathcal{N}\frac{\gamma^2 + 1}{32g^4\gamma^2 d_j^3}M_h^4\left[1 + \frac{\gamma^2 + 1}{64g^4\gamma^2 d_j^3}M_h^4\right]^{-1}, \quad (189)$$

where the vacuum expectation value with respect to the *local* mean-field partition function $\bar{Z}_h$ (172) is defined by

$$\langle\cdots\rangle^h_\circ = \frac{1}{\bar{Z}_h}\int_h \mathcal{D}U\mathcal{D}e(\cdots)e^{-\bar{\mathcal{A}}_h}. \quad (190)$$

The mean-field value $\langle\bar{\mathcal{A}}_{\mathrm{EC}}\rangle_\circ$ (189) has discrete values depending on the discrete values $d_j = 4, \cdots$ of the fundamental state $j_{L,R} = 1/2$ and excitation states $j_{L,R} = 3/2, \cdots$, coupling to different fermion spinor states $\psi^j_{L,R}$.

We are in the position to calculate $\langle\mathcal{A}_{\mathrm{EC}}\rangle_\circ$ in Eq. (178). Since there are three vertex fields in the smallest holonomy field $X_h(v, U)$ (120) that constitutes the regularized EC action $\mathcal{A}_{\mathrm{EC}}$ (124), (129), and (131), while there is only one vertex field $v_{\nu\mu}$ in the mean-field action (168)–(170), we assign the vertex field $v_{\mu\nu}$ to the *local* mean-field action (168)–(171) of the 2-simplex h, the vertex fields $v_{\mu\rho}$, $v_{\rho\nu}$ to the *local* mean-field actions of neighboring 2-simplices, and approximate

$$\begin{aligned}&\langle\mathrm{tr}[v_{\nu\mu}(x)U_\mu(x)v_{\mu\rho}(x + a_\mu)U_\rho(x + a_\mu)v_{\rho\nu}(x + a_\nu)U_\nu(x + a_\nu)]\rangle_\circ + \mathrm{H.c.}\\ &= \mathrm{tr}[\langle v_{\nu\mu}U_\mu U_\rho U_\nu v_{\mu\rho}v_{\rho\nu}\rangle^h_\circ] + \mathrm{H.c.}\\ &\approx (\bar{Z}_h)^2\,\mathrm{tr}[\langle v_{\nu\mu}U_\mu U_\rho U_\nu\rangle^h_\circ\langle v_{\mu\rho}\rangle^h_\circ\langle v_{\rho\nu}\rangle^h_\circ] + \mathrm{H.c.}\\ &\approx (\bar{Z}_h)^2\,\mathrm{tr}[(\langle v_{\nu\mu}U_\mu U_\rho U_\nu\rangle^h_\circ + \mathrm{H.c.})\langle v_{\mu\rho}\rangle^h_\circ\langle v_{\rho\nu}\rangle^h_\circ].\end{aligned} \quad (191)$$

where $(v_{\mu\rho}v_{\rho\nu})^\dagger = (v_{\rho\nu}v_{\mu\rho})$. Using Eqs. (170) and (171), we have

$$\langle\mathcal{A}_{\mathrm{EC}}\rangle_\circ \approx \sum_{h\in\mathcal{M}}\frac{(\bar{Z}_h)^2}{4M_h^2}\{\langle\mathrm{tr}[e_\nu\Gamma^h_{\nu\mu}e_\mu - e_\mu\Gamma^h_{\nu\mu}e_\nu]\rangle_\circ\mathrm{tr}[\langle[e_{\mu\rho}]\rangle^h_\circ\langle[e_{\rho\nu}]\rangle^h_\circ]\}. \quad (192)$$

In the last part of Appendix E, we obtain

$$\langle\mathcal{A}_{\mathrm{EC}}\rangle_\circ \approx \mathcal{N}\frac{1}{M_h^2}\left(\frac{1}{\bar{Z}_h}\right)\left(\frac{1}{8g^2}\right)^6(M_h^4)^3\left(\frac{1}{4}\right)\left(\frac{2}{d_j^3}\right)^3\left(\frac{\gamma^2 + 1}{\gamma^2}\right)^3. \quad (193)$$

Putting Eqs. (187), (189), and (193) into the approximate free energy (179), we obtain

$$\mathcal{F}^{\mathrm{app}}_{\mathrm{EC}}(M_h, g, \gamma) = -\ln(1 + y) - \frac{2y}{1 + y} + \chi\frac{y^{5/2}}{(1 + y)}, \quad (194)$$

where

$$y = \frac{\gamma^2 + 1}{64 g^4 \gamma^2 d_j^3} M_h^4, \qquad \chi = 2\sqrt{\frac{\gamma^2 + 1}{64 g^4 \gamma^2 d_j^3}}. \tag{195}$$

In Fig. 6, we plot the approximate free energy (179) as a function of the mean-field value M_h (166) for selected values of the parameter χ (195). The minimal values of the approximate free energy $\mathcal{F}_{\rm EC}^{\rm app}$ (179) locate at the nonvanishing mean-field value $M_h^* \neq 0$, which increases as the parameter χ decreases, namely, the gauge coupling increases. The gauge coupling g and Immirzi parameter γ remain to be determined. These two parameters (g, γ) should be determined at critical points of the second-order phase transition, as discussed in the last section. The mean-field approximation approach adopted here needs to be improved to see whether we can have a critical value χ_c, and for $\chi > \chi_c$ the minimal value of the approximate free energy $\mathcal{F}_{\rm EC}^{\rm app}$ locates at the vanishing mean-field value $M_h^* = 0$. It is usually difficult to study the vicinity of critical points of the second-order phase transition by the mean-field approximation approach.

Considering the case that $\gamma \gg 1$, $d_j = 4$, $g \to 4/3$ for $G_{\rm eff} \to G$ [see Eq. (128) in Sec. III F], and $\chi \approx 0.02$, we have

$$M_h^* > 1, \tag{196}$$

see the curve for $\chi = 0.03$ in Fig. 6, since M_h^* becomes larger as χ decreases. For larger gauge coupling g and higher dimensions d_j of irreducible representations, the values of χ (195) become smaller, and M_h^* becomes larger.

Therefore, the mean-field value of the 2-simplex area (166)

$$\langle S_h \rangle = a^2 M_h^* > a^2 = \frac{8\pi}{m_{\rm Planck}^2}, \tag{197}$$

and the mean-field value of the volume element (167)

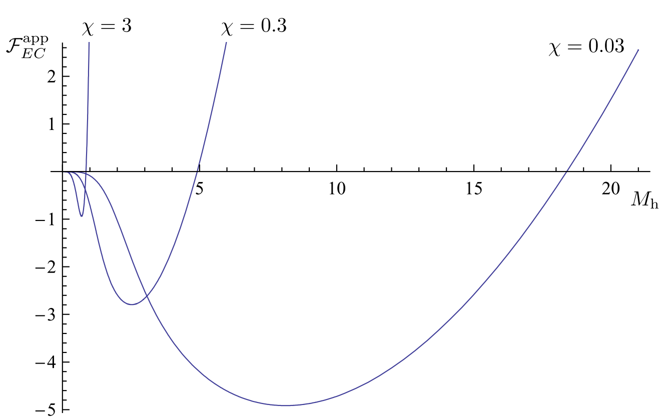

FIG. 6 (color online). In the Planck unit $a = 1$, the approximate free energy (179) as a function of the mean-field value M_h (166) is plotted for selected values $\chi = 0.03, 0.3, 3$. The minimal values of the approximate free energy $\mathcal{F}_{\rm EC}^{\rm app}$ locate at the nonvanishing mean-field value M_h^*. The minimal locations are $M_h^*(\chi = 0.03) \approx 7.9$, $M_h^*(\chi = 0.3) \approx 2.1$, $M_h^*(\chi = 3) \approx 0.8$.

$$\langle dV(x) \rangle = a^4 N_h (M_h^*)^2 > N_h \frac{(8\pi)^2}{m_{\rm Planck}^4}. \tag{198}$$

Equations (197) and (198) indicate that the averaged sizes of 2-simplex, 3-simplex, and 4-simplex, i.e., elements of the simplicial complex, are larger than the Planck length, which is probed by short wavelengths of quantum fields e_μ, U_μ, ψ in strong gauge couplings g. This implies that due to the quantum gravity, the Planck length sets the scale for the minimal separation between two spacetime points [25]. We end this section by noting that the mean-field approximation is not only a poor approximation, but also breaks diffeomorphism and *local* gauge symmetries.

VII. SOME REMARKS

In addition to the Planck length a, the regularized EC action (147) proposed in this article contains three dimensionless parameters: the gauge coupling g; the Immirzi parameter γ and the cosmological parameter λ. In the view of the naive continuum limit, the regularized EC action (147) proposed in this article is not unique. In principle, permitted by the diffeomorphism and *local* gauge invariances, the regularized action (147) is allowed to contain nonlocal high-dimensional ($d > 6$) operators of fields e_μ, U_μ and ψ with extra free parameters. On the other hand, although the regularized EC action (147) approaches to the continuum EC action (21) in the naive continuous limit, it has not been clear yet whether the regularized EC theory is physically sensible. The regularized EC theory is physically sensible, only if only it has a nontrivial continuum limit, where we could possibly explore the relationship to the Minkowski counterpart. Therefore, it is crucial, on the basis of nonperturbative methods and renormalization-group invariance, to find:

(1) the scaling invariant region (nontrivial ultraviolet fix points) $(g_c, \gamma_c, \lambda_c)$, where the singularity in the free energy appears for phase transition occurring, and the physical correlation length ξ of two-point Green-functions of fields is much larger than the Planck length, while the inverse correlation length ξ^{-1} gives the mass scale of low-energy excitations of the "effective continuum theory";

(2) β function $\beta(g)$, i.e., the scale dependence of the gauge coupling g in the vicinity of the nontrivial ultraviolet fix points g_c, and renormalization-group invariant equation

$$\xi = {\rm constant} \cdot a \cdot \exp \int^g dg'/\beta(g'), \quad \xi \gg a, \tag{199}$$

in this scaling invariant region, and "constant" that can only be obtained by nonperturbative methods. And it is a question how Eq. (199) is related to γ_c and λ_c;

(3) an effective action $\mathcal{A}^{\rm eff}_{\rm EC}$ (135), all relevant and renormalizable operators [one-particle irreducible (1PI) functions] with effective dimension-four to obtain an effective low-energy theory in this scaling invariant region.

The gauge-invariant correlation length ξ can be possibly measured by the gauge-invariant two-point correlation function of the holonomy fields $X_h(v, U)$ (120),

$$\langle X_h[v(x), U(x)], X_h^\dagger[v(y), U(y)]\rangle \sim e^{-|x-y|/\xi}, \quad |x-y| \gg \xi, \tag{200}$$

where $|x-y|$ indicates the separation between two holonomy fields $X_h(v, U)$. Actually, Eq. (200) is related to the invariant curvature correlation function [see Eq. (B12)].

Although we have added the bare cosmological term (139) into the regularized action, 1PI functions $\mathcal{A}^{\rm eff}_{\rm EC}$ (135) effectively contain this dimensional operator (139), which is related to the two-point correlation function (200). It is then a question what is the scaling property of this operator in terms of the low-energy scale ξ^{-2}. We speculate that the gauge-invariant correlation length ξ, instead of the Planck length, sets the scale for the nonperturbative renormalized cosmological constant, i.e.,

$$\Lambda_{\rm COSM} \sim \xi^{-2}, \tag{201}$$

which is rather similar to the scale $\Lambda_{\rm QCD}$ calculated in the lattice QCD theory. This would possibly explain why the observed cosmological constant is much smaller than that expected in terms of the Planck scale [see Eq. (199)]. We also speculate that in the pure gravity at strong gauge coupling $g \gg 1$, the scale ξ^{-2} should measure the exponential area-decay law of holonomy fields (134) and (138) for sufficiently large loops

$$\langle X_{\mathcal{C}}(v, U)\rangle \sim e^{-A_{\rm min}(\mathcal{C})/\xi^2}, \qquad A_{\rm min}(\mathcal{C}) \gg \xi^2, \tag{202}$$

where $A_{\rm min}(\mathcal{C})$ is the minimal area, corresponding to the minimal number of 2-simplices h, that can be spanned by the loop $\mathcal{C}$ (see Ref. [26]). The scaling invariant region g_c, scaling law (199) and correlation length ξ are important to study our present Universe (see Ref. [27]).

The effective quadralinear-fermion interactions in the continuum EC theory (38) are originated by integrating over *static* torsion fields and the torsion-free condition is satisfied as required by the equivalence principle. In this sense, quadralinear-fermion interactions are inevitable as long as the interacting between fermion and gravitational fields is included.

The bilinear fermion action (144) introduces a nonvanishing torsion field (145) in the regularized EC theory. The torsion fields (145) are not exactly *static*, however, they are fields only surviving in short distances at the Planck scale, which is due to the quantum gravity [see, for example, the mean-field approximation result (196)–(198)]. The effective quadralinear-fermion interactions (146) is formulated by hand together with a torsion-free bilinear fermion action (144) so that they approach to the fermion action of the continuum EC theory in the continuum limit. In principle, it should be possible to obtain an effective action by solving the discretized Cartan structure [Eq. (46) or Eq. (62)] with the nonvanishing discretized torsion (145), and integrating over torsion fields at short distances, in the same way as (30)–(38) of the continuum EC theory. In this way, one will obtain a complicate effective action of fermion fields with high-order dimensional ($d > 6$) operators. However, we expect that in the continuum limit the relevant operators of fermion fields should be Eq. (146) and its continuum counterpart (40).

On the other hand, due to the no-go theorem [28], the bilinear fermion action (144) has the problem of either fermion doubling or chiral (parity) gauge symmetry breaking, which is inconsistent with the low-energy standard model for particle physics. As discussed, the effective quadralinear-fermion interactions (146) are inevitable, due to mediating very massive torsion fields in short distances at the Planck scale. We expect that in the invariant scaling region of the nontrivial ultraviolet fix points $(g_c, \gamma_c, \lambda_c)$, the quadralinear-fermion interactions should be relevant operators, which not only give a possible resolution to the fermion doubling problem [29,30], but also the compelling dynamics for fermion mass generation [31,32], via the Nambu Jona-Lasinio mechanism [33].

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APPENDIX A

By using Eqs. (56) and (84) and the identity $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+[\hat{A},\hat{B}]/2}$, we calculate $U_{\mu\nu}(x)$ (87)–(89) in the *native continuum limit*: $ag\omega_\mu \ll 1$. Expanding $U_{\mu\nu}(x)$ in powers of $ag\omega_\mu$, we have

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a_\mu) = \exp\left\{iga[\omega_\mu(x)+\omega_\nu(x)] + iga^2\partial_\mu\omega_\nu(x) - \frac{1}{2}(ga)^2[\omega_\mu(x), \omega_\nu(x)] + \mathcal{O}(a^3)\right\}$$
$$= \exp\left\{iga[\omega_\mu(x)+\omega_\nu(x)] + iga^2\partial_\mu\omega_\nu(x) - \frac{i}{2}(ga)^2[\omega^{ae}(x)\wedge\omega^b_e(x)]_{\mu\nu}\sigma_{ab} + \mathcal{O}(a^3)\right\}$$
$$= \exp\{iga\sigma_{AB}G^{AB}_{\mu\nu} + \mathcal{O}(a^3)\}, \tag{A1}$$

where

$$G^{AB}_{\mu\nu} = [\omega^{AB}_\mu(x) + \omega^{AB}_\nu(x)] + a\partial_\mu\omega^{AB}_\nu(x) - \tfrac{1}{2}(ga)[\omega^{Ae}(x)\wedge\omega^B_e(x)]_{\mu\nu}, \tag{A2}$$

and $\mathcal{O}(a^3)$ indicates high-order powers of $ag\omega_\mu$. In Eq. (A1), we use $[\sigma_{ab}, \sigma_{bc}] = i\delta_{bb}\sigma_{ca}$ (no sum with index b), $[\gamma_5, \sigma_{ca}] = 0$ and

$$\omega_{\mu\nu}(x) \equiv [\omega_\mu(x), \omega_\nu(x)] = [\omega^{ae}(x)\wedge\omega^{eb}(x)]_{\mu\nu}[\sigma_{ae}, \sigma_{eb}] = i[\omega^{ae}(x)\wedge\omega^b_e(x)]_{\mu\nu}\sigma_{ab}. \tag{A3}$$

For exchanging $\mu \leftrightarrow \nu$ in Eqs. (A1) and (A2)

$$G^{AB}_{\nu\mu} = [\omega^{AB}_\mu(x) + \omega^{AB}_\nu(x)] + a\partial_\nu\omega^{AB}_\mu(x) - \tfrac{1}{2}(ga)[\omega^{Ae}(x)\wedge\omega^B_e(x)]_{\nu\mu}. \tag{A4}$$

As a result, the curvature $R^{AB}_{\mu\nu}(x)$ (19)

$$aR^{AB}_{\mu\nu}(x) = G^{AB}_{\mu\nu}(x) - G^{AB}_{\nu\mu}(x) = a[\partial_\mu\omega^{AB}_\nu(x) - \partial_\nu\omega^{AB}_\mu(x)] - (ga)[\omega^{Ae}(x)\wedge\omega^B_e(x)]_{\mu\nu}, \tag{A5}$$

where we use

$$[\omega^{Ae}(x)\wedge\omega^B_e(x)]_{\mu\nu} = -[\omega^{Ae}(x)\wedge\omega^B_e(x)]_{\nu\mu}. \tag{A6}$$

APPENDIX B

The properties of the vertex fields $v_{\mu\nu}(x)$ (116) and (117):

$$v_{\mu\nu} = \gamma_5\frac{i}{2}[\gamma_a\gamma_b - \gamma_b\gamma_a]\frac{1}{2}(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu) = \gamma_5\frac{i}{2}(e_\mu e_\nu - e_\nu e_\mu) = \frac{i}{2}\gamma_5(e\wedge e)_{\mu\nu}; \tag{B1}$$

$$v^\dagger_{\mu\nu} = \gamma^\dagger_5\sigma^\dagger_{ab}(e^a\wedge e^b)^\dagger_{\mu\nu} = \gamma_5\sigma_{ab}\tfrac{1}{2}(e^b_\mu e^a_\nu - e^b_\nu e^a_\mu) = -\gamma_5\sigma_{ab}(e^a\wedge e^b)_{\mu\nu} = -v_{\mu\nu} = v_{\nu\mu} \tag{B2}$$

for the case $v_{\mu\nu}(x) = \gamma_5 e_{\mu\nu}(x)$. Equations (B1) and (B2) are the same for the case $v_{\mu\nu}(x) = e_{\mu\nu}(x)$, because of $\gamma^\dagger_5 = \gamma_5$. For the sake of simplifying notations in following calculations, we introduce

$$t^{ab}_{\mu\nu} \equiv (e^a\wedge e^b)_{\mu\nu} = \tfrac{1}{2}(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu),\quad [t^{ab}_{\mu\nu}]^\dagger = -t^{ab}_{\mu\nu}, \tag{B3}$$

$t^{ab}_{\mu\nu} = -t^{ab}_{\nu\mu}$, $t^{ab}_{\mu\nu} = -t^{ba}_{\mu\nu}$ and $e_{\mu\nu} = \sigma_{ab}t^{ab}_{\mu\nu}$.

We calculate the naive continuum limit of Eqs. (120), (122), and (123), in powers of $ga\omega_\mu$. First, at the order $\mathcal{O}(a^0)$, we consider all link fields in Eqs. (120) and (122) to be identity, e.g., $U_\mu(x) \approx 1$, $U_\rho(x+a_\mu) \approx 1$, and $U_\nu(x+a_\nu) \approx 1$. Using Eqs. (121)–(123), (B1), and (B2), we obtain up to order $\mathcal{O}(a^0)$

$$X_h(v, U) + X^\dagger_h(v, U) = \mathrm{tr}[v_{\nu\mu}(x)v_{\mu\rho}(x+a_\mu) \times v_{\rho\nu}(x+a_\nu)] + \mathrm{H.c.} = 0. \tag{B4}$$

Second, at the order $\mathcal{O}(a)$, we consider two link fields in Eqs. (120) and (122) to be identity. The case (1): $U_\nu(x+a_\nu) \approx 1$ and $U_\rho(x+a_\mu) \approx 1$, we have up to order $\mathcal{O}(a)$,

$$X_h(v, U) \approx \mathrm{tr}[v_{\nu\mu}(x)U_\mu(x)v_{\mu\rho}(x+a_\mu)v_{\rho\nu}(x+a_\nu)] \approx \mathrm{tr}[v_{\nu\mu}(x)v_{\mu\rho}(x+a_\mu)v_{\rho\nu}(x+a_\nu)] + iga\omega^{AB}_\mu(x)\,\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}] \times t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu). \tag{B5}$$

for the case $v_{\mu\nu}(x) = \gamma_5 e_{\mu\nu}(x)$. Using Eqs. (120)–(123) and (B4), we have

$$X_h(v, U) + X^\dagger_h(v, U) \approx iga[\omega^{AB}_\mu(x) - \omega^{AB}_\nu(x)] \cdot \mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}]t^{ab}_{\nu\mu}(x) \times t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu). \tag{B6}$$

The case (2): $U_\mu(x+a_\mu) \approx 1$ and $U_\rho(x+a_\mu) \approx 1$, we obtain the result with the replacement $[\omega^{AB}_\mu(x) - \omega^{AB}_\nu(x)] \to [\omega^{AB}_\nu(x) - \omega^{AB}_\mu(x)]$ in Eq. (B6). Taking into account all contributions from these cases, we obtain up to the order $\mathcal{O}(a)$

$$X_h(v, U) + X^\dagger_h(v, U) = 0. \tag{B7}$$

These results are also valid for the case $v_{\mu\nu}(x) = e_{\mu\nu}(x)$, since the calculations of Eqs. (B4)–(B6) without γ_5 are the same.

Third, at the order $\mathcal{O}(a^2)$, we consider one link field in Eqs. (120) and (122) to be identity, e.g., $U_\rho(x+a_\mu) \approx 1$,

$$X_h(v,U) \approx \mathrm{tr}[v_{\nu\mu}(x)U_\mu(x)v_{\mu\rho}(x+a_\mu)v_{\rho\nu}(x+a_\nu) \times U_\nu(x+a_\nu)] \approx \mathrm{tr}[v_{\nu\mu}(x)U_\mu(x)U_\nu(x)v_{\mu\rho}(x+a_\mu) \times v_{\rho\nu}(x+a_\nu)], \quad \text{(B8)}$$

where in the second line, we use Eq. (56), $[\sigma_{ab},\gamma_5]=0$, $[U_\mu(x), v_{\rho\nu}]=\mathcal{O}(a)$, and $U_\nu(x+a_\nu)=U_\nu(x)+\mathcal{O}(a)$. Using Eq. (89) or (A1) for $U_{\mu\nu}(x)\equiv U_\mu(x)U_\nu(x)$ and the result (B4), we have up to $\mathcal{O}(a^2)$

$$X_h(v,U) \approx \mathrm{tr}[v_{\nu\mu}(x)U_{\mu\nu}(x)v_{\mu\rho}(x+a_\mu)v_{\rho\nu}(x+a_\nu)] = iagG^{AB}_{\mu\nu}(x)\,\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}] \times t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu), \quad \text{(B9)}$$

for the case $v_{\mu\nu}(x)=\gamma_5\sigma_{\mu\nu}(x)$. Using the relationships $X^\dagger_h(v,U)=X_h(v,U)|_{\mu\leftrightarrow\nu}$ (121) and (122) and $t^{ab}_{\mu\nu}=-t^{ab}_{\nu\mu}$ (B3), we have

$$X^\dagger_h(v,U) \approx -iagG^{AB}_{\nu\mu}(x)\,\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}] \times t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu). \quad \text{(B10)}$$

As a result, using Eq. (A5) in Appendix A, we obtain up to $\mathcal{O}(a^2)$

$$X_h(v,U)+X^\dagger_h(v,U) \approx ia^2gR^{AB}_{\mu\nu}(x)\,\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}] \times t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu). \quad \text{(B11)}$$

For the case $v_{\mu\nu}(x)=e_{\mu\nu}(x)$, the result is given by Eq. (B11) without γ_5.

In Appendix C, we show the calculations of $\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}]$ and $\mathrm{tr}[\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}]$ in Eq. (B11). Using these results (C4) and (C8), we obtain for the case $v_{\mu\nu}(x)=\gamma_5 e_{\mu\nu}(x)$,

$$X_h(v,U)+X^\dagger_h(v,U) \approx 8a^2gR^{AB}_{\mu\nu}(x)\epsilon_{abAB}t^{ab}_{\nu\mu}(x) \times t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu); \quad \text{(B12)}$$

and for the case $v_{\mu\nu}(x)=e_{\mu\nu}(x)$,

$$X_h(v,U)+X^\dagger_h(v,U) \approx 2i\cdot 8a^2gR^{AB}_{\mu\nu}(x)t^{AB}_{\nu\mu}(x) \times t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu). \quad \text{(B13)}$$

Using Eqs. (119) and (B3), we rewrite the fundamental area (96) and (97) of the 2-simplex $h(x)$ in terms of $t^{cd}_{\mu\rho}(x+a_\mu)$ and $t^{cd}_{\rho\nu}(x+a_\nu)$:

$$S^h_{\mu\rho}(x+a_\mu)=\sigma_{cd}S^{cd}_{\mu\rho}(x+a_\mu), \qquad S^{cd}_{\mu\rho}(x+a_\mu)=-ia^2t^{cd}_{\mu\rho}(x+a_\mu), \quad \text{(B14)}$$

$$S^h_{\rho\nu}(x+a_\nu)=\sigma_{cd}S^{cd}_{\rho\nu}(x+a_\nu), \qquad S^{cd}_{\rho\nu}(x+a_\nu)=-ia^2t^{cd}_{\rho\nu}(x+a_\nu), \quad \text{(B15)}$$

where $S^h_{\mu\rho}(x+a_\mu)=-S^h_{\rho\mu}(x+a_\mu)$ and $S^h_{\rho\nu}(x+a_\nu)=-S^h_{\nu\rho}(x+a_\nu)$. As discussed in Eqs. (103), (96), and (97) [see Sec. III D], three area operators $S^h_{\mu\nu}(x)$, $S^h_{\rho\mu}(x+a_\mu)$ and $S^h_{\nu\rho}(x+a_\nu)$ are identical. Therefore, equivalently to Eqs. (104) and (107), we write the volume element contributed from the 2-simplex $h(x)$ as

$$dV_h \equiv S^{cd}_{\mu\rho}(x+a_\mu)S^{cd\dagger}_{\rho\nu}(x+a_\nu) = a^4t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu) = S^{cd}_{\mu\nu}(x)S^{cd\dagger}_{\mu\nu}(x) = a^4t^{cd}_{\mu\nu}(x)t^{cd}_{\mu\nu}(x), \quad \text{(B16)}$$

where indexes c, d are summed, while indexes μ, ν and ρ are not summed. Using Eq. (C7) in Appendix C, we obtain

$$dV_h(x)=S^2_h(x)=\tfrac{1}{8}\mathrm{tr}[S^h_{\mu\nu}(x)S^{h\dagger}_{\mu\nu}(x)], \quad \text{(B17)}$$

where $S^h_{\mu\nu}(x)=\sigma_{ab}S^{ab}_{\mu\nu}(x)$ and $S^{ab}_{\mu\nu}(x)=-ia^2t^{ab}_{\mu\nu}(x)$. Using Eqs. (B12)–(B17), we can show the regularized Palatini action (124) and Host action (129) approach to their continuum counterparts (22) and (23) in the naive continuum limit $ag\omega_\mu \ll 1$.

APPENDIX C

It can be shown that $\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{cd}\sigma_{ef}]=0$ for $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$ in the four-dimensional space-time. Nonvanishing contributions of the following trace

$$\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}], \quad \text{(C1)}$$

come from the product of two spinor matrices σ's in Eq. (C1) being identical,

$$\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}] \Rightarrow \mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}]. \quad \text{(C2)}$$

In Eq. (B11), as example, we take (i) $\sigma_{cd}\sigma_{ef}=1$ for $c=e$, $d=f$ and (ii) $\sigma_{cd}\sigma_{ef}=-1$ $c=f$, $d=e$,

$$\begin{aligned}\sum_{cdef}[\sigma_{cd}\sigma_{ef}]t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu) &= \sum_{cd}[\sigma_{cd}\sigma_{cd}]t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu) + \sum_{cd}[\sigma_{cd}\sigma_{dc}]t^{cd}_{\mu\rho}(x+a_\mu)t^{dc}_{\rho\nu}(x+a_\nu), \\ &= \sum_{cd}t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu) - \sum_{cd}t^{cd}_{\mu\rho}(x+a_\mu)t^{dc}_{\rho\nu}(x+a_\nu) \\ &= 2\sum_{cd}t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu).\end{aligned} \quad \text{(C3)}$$

Thus, in Eq. (B11) we have

$$\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}]t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu) = 2\,\mathrm{tr}[\gamma_5\sigma_{ab}\sigma_{AB}]t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu) = -8i\epsilon^{abAB}t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu), \quad \text{(C4)}$$

where we use the formula

$$\mathrm{tr}\,(\gamma_5\sigma^{ab}\sigma^{AB}) = \tfrac{1}{2}\mathrm{tr}(\gamma_5\{\sigma^{ab},\sigma^{AB}\}) = -4i\epsilon^{abAB}, \quad \text{(C5)}$$

and Eq. (18). In the same way we calculate Eq. (B11) for other possibilities, e.g., $\sigma_{ab}\sigma_{ef} = 1$ for (i) $a = e$, $b = f$ and (ii) $\sigma_{ab}\sigma_{ef} = -1$ $a = f$, $b = e$. As a result, we obtain Eq. (B12).

Analogous to the discussions for Eq. (C2), nonvanishing contributions to $\mathrm{tr}[\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}]$ come from the product of two spinor matrices σ's being identical,

$$\mathrm{tr}\,[\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}] \Rightarrow \mathrm{tr}[\sigma_{ab}\sigma_{AB}]. \quad \text{(C6)}$$

In Eq. (B11) without γ_5, as example, we take (i) $\sigma_{cd}\sigma_{ef} = 1$ for $c = e$, $d = f$ and (ii) $\sigma_{cd}\sigma_{ef} = -1$ $c = f$, $d = e$, and use formula

$$\mathrm{tr}\,(\sigma^{ab}\sigma^{AB}) = 4(\delta^{aA}\delta^{bB} - \delta^{aB}\delta^{bA}). \quad \text{(C7)}$$

As a result we obtain

$$\begin{aligned}\mathrm{tr}\,[\sigma_{ab}\sigma_{AB}\sigma_{cd}\sigma_{ef}]t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{ef}_{\rho\nu}(x+a_\nu) &= 2\,\mathrm{tr}[\sigma_{ab}\sigma_{AB}]t^{ab}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu)\\ &= 2\cdot 8t^{AB}_{\nu\mu}(x)t^{cd}_{\mu\rho}(x+a_\mu)t^{cd}_{\rho\nu}(x+a_\nu),\end{aligned} \quad \text{(C8)}$$

and Eq. (B11) without γ_5 becomes Eq. (B13).

APPENDIX D

Using the properties (B1) of the vertex field $v_{\mu\nu}(x) = \gamma_5 e_{\mu\nu}(x)$, we have

$$\begin{aligned}\bar{X}_h(v,U) &= \frac{i}{2}\,\mathrm{tr}\gamma_5[e_\nu(x)U_\mu(x)U_\rho(x+a_\mu)U_\nu(x+a_\nu)e_\mu(x)\\ &\quad - e_\mu(x)U_\mu(x)U_\rho(x+a_\mu)U_\nu(x+a_\nu)e_\nu(x)]M_h^2\\ \bar{X}^\dagger_h(v,U) &= \frac{i}{2}\,\mathrm{tr}\gamma_5[e_\mu(x)U_\nu(x)U_\rho(x+a_\nu)U_\mu(x+a_\mu)e_\nu(x)\\ &\quad - e_\nu(x)U_\nu(x)U_\rho(x+a_\nu)U_\mu(x+a_\mu)e_\mu(x)]M_h^2,\end{aligned} \quad \text{(D1)}$$

and

$$\begin{aligned}\bar{X}_h(v,U) + \bar{X}^\dagger_h(v,U) &= \frac{i}{2}M_h^2\,\mathrm{tr}\gamma_5 e_\nu(x)[U_\mu(x)U_\rho(x+a_\mu)U_\nu(x+a_\nu) - U_\nu(x)U_\rho(x+a_\nu)U_\mu(x+a_\mu)]e_\mu(x)\\ &\quad + \frac{i}{2}M_h^2\,\mathrm{tr}\gamma_5 e_\mu(x)[U_\nu(x)U_\rho(x+a_\nu)U_\mu(x+a_\mu) - U_\mu(x)U_\rho(x+a_\mu)U_\nu(x+a_\nu)]e_\nu(x)\\ &= \mathrm{tr}[e_\nu(x)\gamma_5 H_{\nu\mu}(x)e_\mu(x)] - \mathrm{tr}[e_\mu(x)\gamma_5 H_{\nu\mu}(x)e_\nu(x)],\end{aligned} \quad \text{(D2)}$$

where $\gamma_5 e_\mu(x) = -e_\mu(x)\gamma_5$ and the tensor

$$\begin{aligned}H_{\nu\mu}(x) &\equiv \frac{i}{2}M_h^2[U_\nu(x)U_\rho(x+a_\nu)U_\mu(x+a_\mu) - U_\mu(x)U_\rho(x+a_\mu)U_\nu(x+a_\nu)]\\ &= \frac{i}{2}M_h^2[U_\nu(x)U_\rho(x+a_\nu)U_\mu(x+a_\mu) - U^\dagger_\mu(x+a_\mu)U^\dagger_\rho(x+a_\nu)U^\dagger_\nu(x)]\\ &= \frac{i}{2}M_h^2[U_\nu(x)U_\rho(x+a_\nu)U_\mu(x+a_\mu)] + \mathrm{H.c.} = \frac{i}{2}M_h^2[U_\nu(x)U_\rho(x+a_\nu)U^\dagger_\mu(x)] + \mathrm{H.c.},\end{aligned} \quad \text{(D3)}$$

$H_{\nu\mu} = -H_{\mu\nu}$ and $H^\dagger_{\nu\mu} = H_{\nu\mu}$, following the relations $U_\mu(x) = U^\dagger_\mu(x+a_\mu)$, $U^\dagger_\nu(x) = U_\nu(x+a_\nu)$ and $U_\rho(x+a_\mu) = U^\dagger_\rho(x+a_\nu)$. The $H_{\mu\nu}(x)$ is a product of three edge fields $U_\nu(x)$, $U^\dagger_\mu(x)$ and $U_\rho(x+a_\mu)$ of the 2-simplex $h(x)$. For the case $v_{\mu\nu}(x) = e_{\mu\nu}(x)$, the same result can be obtained by the replacement $\gamma_5 \to -1$ in Eq. (D2). The sum of two contributions gives Eqs. (170) and (171) in the main text.

APPENDIX E

For each 2-simplex $h(\mu \neq \nu \neq \rho)$, we have the fundamental area operator $e_\mu \wedge e_\nu \equiv e_\nu e_\mu - e_\mu e_\nu$ [see Eq. (103)] and $\mathrm{tr}(e_\nu e_\mu - e_\mu e_\nu) = 0$, we can rewrite the mean-field action (170) as follows:

$$\begin{aligned}\bar{\mathcal{A}}_h &= \mathrm{tr}(e_\nu e_\mu - e_\mu e_\nu) + \bar{\mathcal{A}}_h\\ &= \mathrm{tr}[e_\nu(I-\Gamma^h_{\nu\mu})e_\mu - e_\mu(I-\Gamma^h_{\nu\mu})e_\nu]\\ &= \mathrm{tr}\left\{\begin{pmatrix} e_\nu & e_\mu\end{pmatrix}\begin{bmatrix} 0 & (I-\Gamma^h_{\nu\mu})\\ -(I-\Gamma^h_{\nu\mu}) & 0\end{bmatrix}\begin{pmatrix} e_\nu\\ e_\mu\end{pmatrix}\right\},\end{aligned} \quad \text{(E1)}$$

where I is the identity matrix. For each single 2-simplex h, we have the integrations

$$\int_h de_\mu de_\nu \exp - \bar{\mathcal{A}}_h = \det{}^{-1}[I - \Gamma^h], \quad \text{(E2)}$$

$$\int_h de_\mu de_\nu (e_\mu e_\nu) \exp - \bar{\mathcal{A}}_h = \frac{1}{2}[I-\Gamma^h]^{-1}_{\mu\nu} \det{}^{-1}[I-\Gamma^h], \tag{E3}$$

$$\int_h de_\mu de_\nu e_{\mu\nu} \exp - \bar{\mathcal{A}}_h = \frac{i}{4}\{[I-\Gamma^h]^{-1}_{\mu\nu} - [I-\Gamma^h]^{-1}_{\nu\mu}\} \times \det{}^{-1}[I-\Gamma^h]. \tag{E4}$$

Using Eqs. (181) and (182), we calculate the mean-field partition function (175)

$$\begin{aligned}\bar{Z}_{\rm EC} &= \prod_{h\in\mathcal{M}} \int_h dU_\mu dU_\nu dU_\rho \det{}^{-1}[I-\Gamma^h] \\ &= \prod_{h\in\mathcal{M}} \int_h dU_\mu dU_\nu dU_\rho \Big[1 + \sum_a \Gamma^h_{aa} + \frac{1}{2}\sum_{a,b}(\Gamma^h_{aa}\Gamma^h_{bb} \\ &\quad + \Gamma^h_{ab}\Gamma^h_{ba}) + \cdots\Big].\end{aligned} \tag{E5}$$

In Eq. (E5), the first term is one due to the formula (183), the second term vanishes due to the formula (184), and nonvanishing contribution, due to Eqs. (184) and (185), comes from the term $\Gamma^h_{ab}\Gamma^h_{ba}$ in the third term. Using Eqs. (171), (184), and (185), we have

$$\begin{aligned}\int_h dU_\mu dU_\nu dU_\rho \frac{1}{2}\sum_{a,b}\Gamma^h_{ab}\Gamma^h_{ba} &= \frac{1}{2}\Big(\frac{1}{8g^2}\Big)^2 M_h^4 \Big(\frac{i}{2}\Big)\Big(\frac{-i}{2}\Big) \\ &\times \int_h dU_\mu dU_\nu dU_\rho \cdot 2\Big[\Big(\gamma_5 - \frac{1}{\gamma}\Big)_{aj}[U_\nu]_{jl}[U_\rho]_{ln}[U^\dagger_\mu]_{nb} \\ &\times \Big(\gamma_5 - \frac{1}{\gamma}\Big)_{bm}[U_\mu]_{mk}[U^\dagger_\rho]_{ki}[U^\dagger_\nu]_{ia}\Big] \\ &= \frac{1}{2}\Big(\frac{1}{8g^2}\Big)^2 M_h^4 \Big(\frac{1}{4}\Big)\frac{2}{d_j^3}\,{\rm tr}\Big[\Big(\gamma_5 - \frac{1}{\gamma}\Big)^2\Big] \\ &= \Big(\frac{1}{8g^2}\Big)^2 M_h^4 \frac{1}{d_j^3}\Big(1 + \frac{1}{\gamma^2}\Big).\end{aligned} \tag{E6}$$

As a result, we obtain the mean-field partition function (186) in the main text.

Using Eq. (E4), we calculate the mean-field value of the mean-field action $\bar{\mathcal{A}}_h$ (170) of the single 2-simplex h,

$$\begin{aligned}\langle\bar{\mathcal{A}}_h\rangle_\circ &= \langle{\rm tr}[e_\nu\Gamma^h_{\nu\mu}e_\mu - e_\mu\Gamma^h_{\nu\mu}e_\nu]\rangle_\circ \\ &= \frac{1}{2\bar{Z}_h}\int_h \mathcal{D}U\,{\rm tr}\Big\{\frac{\Gamma^h_{\nu\mu}}{I-\Gamma^h_{\nu\mu}} - \frac{\Gamma^h_{\nu\mu}}{I-\Gamma^h_{\mu\nu}}\Big\}\det{}^{-1}[I-\Gamma^h] \\ &= \frac{1}{2\bar{Z}_h}\int_h \mathcal{D}U\,{\rm tr}\{2\Gamma^h_{\nu\mu}\Gamma^h_{\nu\mu} + \cdots\}\det{}^{-1}[I-\Gamma^h] \\ &= \frac{1}{\bar{Z}_h}\Big(\frac{1}{8g^2}\Big)^2 M_h^4\Big(\frac{1}{4}\Big)\frac{2}{d_j^3}\,{\rm tr}\Big[\Big(\gamma_5 - \frac{1}{\gamma}\Big)^2\Big] \\ &= \frac{1}{\bar{Z}_h}\Big(\frac{1}{8g^2}\Big)^2 M_h^4 \frac{2}{d_j^3}\Big(\frac{\gamma^2+1}{\gamma^2}\Big),\end{aligned} \tag{E7}$$

which gives Eq. (189) in the main text.

Using Eqs. (E2), (E3), and (E5) and $(\Gamma^h)_{\mu\rho} = -(\Gamma^h)_{\rho\mu}$ [see Eqs. (171) and (D3)], we have

$$\begin{aligned}\langle[e_{\mu\rho}]\rangle^h_\circ &= \frac{i}{4}\frac{1}{\bar{Z}_h}\int_h \mathcal{D}U\{[I-\Gamma^h]^{-1}_{\mu\rho} - [I-\Gamma^h]^{-1}_{\rho\mu}\} \\ &\quad \times \det{}^{-1}[I-\Gamma^h] \\ &= \frac{i}{4}\frac{1}{\bar{Z}_h}\int_h \mathcal{D}U[2\Gamma^h_{\mu\rho} + \cdots]\det{}^{-1}[I-\Gamma^h] \\ &= \frac{i}{4}\frac{2}{\bar{Z}_h}\Big(\frac{1}{8g^2}\Big)^2 M_h^4\Big(\frac{1}{4}\Big)\frac{2}{d_j^3}\Big[\Big(\gamma_5 - \frac{1}{\gamma}\Big)^2\Big],\end{aligned} \tag{E8}$$

and $\langle[e_{\rho\mu}]\rangle^h_\circ = -\langle[e_{\mu\rho}]\rangle^h_\circ$. As a result, Eq. (192) becomes

$$\begin{aligned}\langle\mathcal{A}_{\rm EC}\rangle_\circ &\approx \sum_{h\in\mathcal{M}}\frac{(\bar{Z}_h)^2}{4M_h^2}\{\langle{\rm tr}[e_\nu\Gamma^h_{\nu\mu}e_\mu \\ &\quad - e_\mu\Gamma^h_{\nu\mu}e_\nu]\rangle_\circ\,{\rm tr}[\langle[e_{\mu\rho}]\rangle^h_\circ\langle[e_{\rho\nu}]\rangle^h_\circ]\} \\ &= \sum_{h\in\mathcal{M}}\frac{1}{M_h^2}\Big(\frac{1}{\bar{Z}_h}\Big)\Big(\frac{1}{8g^2}\Big)^6(M_h^4)^3\Big(\frac{1}{4}\Big)\Big(\frac{2}{d_j^3}\Big)^3\Big(\frac{\gamma^2+1}{\gamma^2}\Big) \\ &\quad \times \Big[\Big(\frac{\gamma^2+1}{\gamma^2}\Big)^2 + \frac{4}{\gamma^2}\Big],\end{aligned} \tag{E9}$$

and we obtain Eq. (193) in the main text.

[1] T. Regge, Nuovo Cimento **19**, 558 (1961).
[2] J. A. Wheeler, in *Relativity, Groups and Topology*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964), p. 463.
[3] H. W. Hamber, in *Les Houches Summer School in Theoretical Physics, Session 43: Critical Phenomena, Random Systems, Gauge Theories, Les Houches, France, Aug 1 - Sep 7, 1984*, edited by K. Osterwalder and R. Stora, Les Houches Summer School Proceedings (North-Holland, Amsterdam, 1986); *Quantum Gravitation—The Feynman Path Integral Approach* (Springer Publishing, Berlin and Heidelberg, 2008), ISBN 978-3-540-85292-6.
[4] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, New York, 1973), Chap. 42; J. B. Hartle, J. Math. Phys. (N.Y.) **26**, 804 (1985); N. H. Christ, R. Friedberg, and T. D. Lee, Nucl. Phys. **B202**, 89 (1982); I. T. Drummond, Nucl. Phys. **B273**, 125 (1986); M. Caselle, A. D'Adda, and L. Magnea, Phys. Lett. B **232**, 457 (1989).
[5] F. David, in *Les Houches Summer School on Gravitation and Quantizations, Session 57, Les Houches, France,*

5 Jul - 1 Aug 1992, edited by J. Zinn-Justin and B. Julia (Elsevier, Amsterdam, New York, 1995), p. 679; R. Loll, Living Rev. Relativity **1**, 13 (1998), http://www.livingreviews.org/lrr-1998-13.

[6] L. Smolin, Nucl. Phys. **B148**, 333 (1979); A. Das, M. Kaku, and R. K. Townsend, Phys. Lett. B **81**, 11 (1979); C. L. T. Mannion and J. G. Taylor, Phys. Lett. B **100**, 261 (1981); K. I. Kondo, Prog. Theor. Phys. **72**, 841 (1984); Y. Ne'eman and T. Regge, Phys. Lett. B **74**, 54 (1978); P. Menotti and A. Pelissetto, Phys. Rev. D **35**, 1194 (1987); S. Caracciolo and A. Pelissetto, Nucl. Phys. **B299**, 693 (1988); M. A. Zubkov, Phys. Lett. B **582**, 243 (2004); **638**, 503(E) (2006); **655**, 309 (2007).

[7] R. M. Williams, Classical Quantum Gravity **3**, 853 (1986); T. Piran and R. M. Williams, Phys. Rev. D. **33**, 1622 (1986); M. Bander, Phys. Rev. D. **36**, 2297 (1987); **38**, 1056 (1988); R. M. Williams and P. A. Tuckey, Classical Quantum Gravity **7**, 2055 (1990); J. W. Barret, M. Rocek, and R. M. Williams, Classical Quantum Gravity **16**, 1373 (1999); T. Regge and R. M. Williams, J. Math. Phys. (N.Y.) **41**, 3964 (2000); V. Khatsymovsky, Phys. Lett. B **651**, 388 (2007); **633**, 653 (2006); Mod. Phys. Lett. A **25**, 351 (2010); **25**, 1407 (2010), (references therein).

[8] G. 't Hooft, Found. Phys. **38**, 733 (2008).

[9] S.-S. Xue, Phys. Lett. B **682**, 300 (2009).

[10] S. Weinberg, *Gravitation and Cosmology* (John Wiley & Sons, Inc., New York, 1972), p. 365, ISBN: 978-0-471-92567-5.

[11] A. Ashtekar, J. D. Romano, and R. S. Tate, Phys. Rev. D **40**, 2572 (1989); T. Jacobson, Classical Quantum Gravity **5**, L143 (1988).

[12] H. Kleinert, *Multivalued Fields* (World Scientific, Singapore, 2008), ISBN 978-981-279-171-92.

[13] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. D **36**, 1587 (1987).

[14] J. Fernando Barbero G., Phys. Rev. D **51**, 5498 (1995); **51**, 5507 (1995).

[15] G. Immirzi, Classical Quantum Gravity **14**, L177 (1997).

[16] C. Rovelli and T. Thiemann, Phys. Rev. D **57**, 1009 (1998); C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004); A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).

[17] S. Capoziello, G. Lambiase, and C. Stornaiolo, Phys. Rev. D **59**, 047505 (1999).

[18] A. Perez and C. Rovelli, Phys. Rev. D **73**, 044013 (2006); L. Freidel, D. Minic, and T. Takeuchi, Phys. Rev. D **72**, 104002 (2005).

[19] I. L. Shapiro, Phys. Rep. **357**, 113 (2002).

[20] A. Randono, arXiv:gr-qc/0504010; S. Alexandrov, Classical Quantum Gravity **23**, 1837 (2006); S. Mercuri, Phys. Rev. D **73**, 084016 (2006).

[21] H. C. Ohanian and R. Ruffini, *Gravitation and Spacetime* (W. W. Norton & Company, New York and London, 1994), 2nd ed., p. 311, ISBN 0-393-9651-5.

[22] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967); see also Secs. 2.4 and 6.9 in [3].

[23] R. P. Feynman, *Statistical Mechanics* (Benjamin, New York, 1972), ISBN-10: 0805325085; H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (World Scientific, Singapore, 2004), ISBN 981-238-106-6.

[24] C. Itzykson and J-B. Zuber, *Quantum Field Theory* (McGraw-Hill Inc., New York, 1980), p. 187, ISBN 0-07-032071-3 and Eqs. (4–86) therein.

[25] We recall the ''Planck lattice'', G. Preparata and S.-S. Xue, Phys. Lett. B **264**, 35 (1991); S. Cacciatori, G. Preparata, S. Rovelli, I. Spagnolatti, and S.-S. Xue, Phys. Lett. B **427**, 254 (1998); G. Preparata, R. Rovelli, and S.-S. Xue, Gen. Relativ. Gravit. **32**, 1859 (2000); Ref. [12], a discretized space-time with minimal spacing of the Planck length due to quantum gravity.

[26] H. W. Hamber and R. M. Williams, Phys. Rev. D **76**, 084008 (2007); **81**, 084048 (2010).

[27] H. W. Hamber, *Proceedings of the 12th Marcel Grossmann Meetings, Paris* (World Scientific, Singapore, 2009).

[28] H. B. Nielson and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981); **B193**, 173 (1981).

[29] E. Eichten and J. Preskill, Nucl. Phys. **B268**, 179 (1986); M. Creutz, M. Tytgat, C. Rebbi, and S.-S. Xue, Phys. Lett. B **402**, 341 (1997).

[30] S.-S. Xue, Nucl. Phys. **B486**, 282 (1997); **B580**, 365 (2000); Phys. Lett. B **381**, 277 (1996); **395**, 275 (1997); **402**, 341 (1997); **408**, 299 (1997); Phys. Rev. D **61**, 054502 (2000); **64**, 094504 (2001).

[31] S.-S. Xue, J. Phys. G **29**, 2381 (2003), and references therein; Nucl. Phys. B, Proc. Suppl. **94**, 781 (2001); Phys. Lett. B **665**, 54 (2008).

[32] See also, M. A. Zubkov, arXiv:1003.5473; arXiv:1004.1375.

[33] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).