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Gravity, two times, tractors, Weyl invariance, and six-dimensional quantum mechanics

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Fefferman and Graham showed some time ago that four-dimensional conformal geometries could be analyzed in terms of six-dimensional, ambient, Riemannian geometries admitting a closed homothety. Recently, it was shown how conformal geometry provides a description of physics manifestly invariant under local choices of unit systems. Strikingly, Einstein's equations are then equivalent to the existence of a parallel scale tractor (a six-component vector subject to a certain first order covariant constancy condition at every point in four-dimensional spacetime). These results suggest a six-dimensional description of four-dimensional physics, a viewpoint promulgated by the 2 times physics program of Bars. The Fefferman-Graham construction relies on a triplet of operators corresponding, respectively, to a curved six-dimensional light cone, the dilation generator and the Laplacian. These form an $\mathfrak{sp}(2)$ algebra which Bars employs as a first class algebra of constraints in a six-dimensional gauge theory. In this article four-dimensional gravity is recast in terms of six-dimensional quantum mechanics by melding the 2 times and tractor approaches. This parent formulation of gravity is built from an infinite set of six-dimensional fields. Successively integrating out these fields yields various novel descriptions of gravity including a new four-dimensional one built from a scalar doublet, a tractor-vector multiplet and a conformal class of metrics.

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I. INTRODUCTION

Theories with extra dimensions have been heavily scrutinized since the time of Kaluza and Klein [1]. The terminus of this train of thought is string theory which attempts to encode the couplings of four-dimensional theories in the geometry of hidden higher dimensions. A simpler and more generic rationale for further dimensions, however, might follow a line of reasoning similar to Einstein's original identification of time as an additional coordinate, along with a gauge principle—general coordinate invariance—guiding the construction of physical theories in terms of Riemannian geometry.

In this article, we focus on two fairly recent suggestions that physics is inherently six-dimensional. First, motivated by duality and holographic arguments, Bars observed that many seemingly different four-dimensional particle models could be regarded as gauge fixed versions of a single underlying six-dimensional model. In fact, the idea of using six dimensions to describe four-dimensional physics dates back to Dirac [2]. What is notable about Bars' "2 times physics" [3] (see [4] for an overview) is that it aims ultimately to describe *any* physical system; whereas, Dirac's work pertained only to models with conformal symmetry.[1]

The second approach relies on replacing Riemannian geometry with conformal geometry so that physics is described by conformal classes of metrics and all equations are manifestly locally Weyl invariant. This is achieved by utilizing the simple physical principle that no physical quantity can depend on local choices of unit system which implies there must exist a way to write any physical system in a Weyl invariant way [11,12]. Weyl invariance is intimately related to conformal symmetry, and for reasons very similar to those first observed by Dirac, manifest Weyl invariance can be achieved by grouping existing four-dimensional physical quantities in six-dimensional multiplets known as "tractors." This approach relies heavily on tractor calculus [13–15], a mathematical machinery designed for efficiently handling conformal geometries. Not only does the tractor approach identify a simple gauge

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[1]In fact, there is a extensive literature on the handling of four-dimensional conformal theories using six-dimensional methods. Pertinent contributions include Boulanger's conformal tensor calculus [5], the conformal space method of [6], the study of conformal representations of the anti-de Sitter group [7], the conformal higher spin studies [8], the Becchi-Rouet-Stora-Tyutin (BRST) conformal parent action method of [9], and the application to scattering amplitudes in [10].

principle—local unit invariance—for constructing models, it also identifies the additional timelike coordinate in 2 times physics as the choice of scale.

In this article we map out the relationship between the 2 times and tractor approaches, since they are in fact highly complementary, and in doing so present seven different formulations of four-dimensional Einstein gravity,[2] several of which are novel; they are summarized by the action principles (1), (3), (27), (28), (31), (34), and (37). Of these, the action (27) can be viewed as a parent action[3] depending on infinitely many fields living in a six-dimensional spacetime while all other theories are gauge fixed versions of this parent action. This starting point was first proposed by Bars as part of his 2 times description of physics although not precisely as a four-dimensional theory of gravity [16]. This action comes from a BRST quantization of the worldline conformal group gauge symmetries of a 2 times particle model.[4] The operators generating local worldline conformal transformations form the gravity multiplet of the model. Bars' action couples this gravity multiplet to a scalar multiplet which can be viewed as a dilaton. This fits extremely well with the tractor description of gravity in terms of a conformal class of metrics coupled to a scale field—the gauge field for local changes of unit systems.

There is an alternative proposal for a 2 times description of four-dimensional gravity due to Bars [19]. It has the advantage that at least part of the equations for the generators of worldline conformal transformations follow from an action principle. On the other hand, unlike the action (27), it does not make the worldline conformal group $\mathfrak{sp}(2)$ symmetry—a central component of the 2 times setup—manifest. It turns out that the two approaches are in fact equivalent, a fact that follows rapidly using tractor technology.

The tractor approach takes standard four-dimensional physical quantities and groups them in Weyl-multiplets labeled by $SO(d, 2)$ representations[5] known as tractors. These tractors are functions of four-dimensional spacetime. In particular, from the scale field σ (the spacetime dependent Planck's constant), one builds a tractor vector I^M known as the scale tractor. Like any tractor, under Weyl transformations it undergoes a tractor gauge transformation which in turn defines a covariant derivative known as the tractor connection[6] [14]. The beauty of this approach is that the Einstein condition amounts to the scale tractor being parallel with respect to this connection. The length of the scale tractor is therefore parallel for physical geometries and in fact measures the cosmological constant. Upon coupling to matter, it also provides a massive coupling constant. Remarkably, even though the small size of the cosmological constant might seem to make the length of the scale tractor inappropriate for setting particle physics mass scales, including backreaction immediately solves this "cosmological constant hierachy problem" [21]. In fact, parallel scale tractors form the first part of a link between the tractor and 2 times descriptions of gravity.

The link between 2 times physics and tractors is completed by the ambient formulation of tractor calculus developed by [15,22,23]. The main idea underlying ambient tractors relies on the Fefferman-Graham description of four-dimensional conformal geometries in terms of six-dimensional Ricci flat geometries admitting a closed homothety [24]. The latter condition implies that the six-dimensional ambient geometry enjoys a curved null cone with a dilationlike vector field. This allows four-dimensional conformal geometries to be realized as rays in this ambient light cone. Bars' $\mathfrak{sp}(2)$ triplet of worldline conformal group Noether charges can be viewed, respectively, as the defining function for the ambient null cone, dilation generator and the harmonic condition obeyed by the Weyl tensor for a Ricci flat geometry. Essentially taking the old Fefferman-Graham ambient metric construction, alongside with the idea of describing unit invariant four-dimensional physics with conformal geometry leads one directly to Bars' 2 times physics program. Needless to say, this confluence of mathematical and physical technologies is likely to lead to major advances in both fields.

Our paper is organized as follows: In Sec. II, we review how Einstein gravity can be recovered in the tractor framework as a parallel condition on the scale tractor, and we fix conventions and notations. In particular, we define the tractor connection and we introduce the main tractor operators. In Sec. III, we set out the ambient description of tractors and introduce the triplet of $\mathfrak{sp}(2)$ operators underlying the 2 times approach. We discuss the latter in detail in Sec. IV, where we introduce the most general deformation of the flat $\mathfrak{sp}(2)$ algebra which contains an infinite tower of background fields. In Sec. V, we give the main new results based on a detailed analysis of Bars' BRST parent field theory action. By careful gauge choices and identification of the dilaton field, we produce the slew of new descriptions of four-dimensional gravity mentioned above as well as establishing the link between tractor and 2 times

[2]Our results are valid for any spacetime dimensionality, and all formulas will be presented as functions of d, the spacetime dimension. We will, however often use the shorthand "four" to stand for d-dimensional and "six" to stand for $(d+2)$-dimensional.

[3]This use of terminology is slightly looser than that of [9] whose parent action is built from a set of independent auxiliary fields corresponding to all possible derivatives of the fundamental ones.

[4]Massless four-dimensional spinning particles were obtained earlier from six dimensions by Siegel in [17] and further studied in [18].

[5]For example, for a relativistic particle, from the four-velocity v_μ, the component of the four-acceleration a^μ parallel to the four velocity and the vanishing function, one can build a tractor "six-velocity" $V_M = (\frac{v \cdot a}{v \cdot v}, e^\mu_m v_\mu, 0)$ transforming as a multiplet under Weyl transformations according to (5).

[6]In fact, the tractor connection also appears in the Yang-Mills–like construction of conformal supergravity [20].

approaches. In the Appendix we give a succinct tractor analysis of Bars' alternate proposal for a 2 times gravity theory. In our conclusions (Sec. VI), we discuss the six-dimensional quantum mechanical origin of four-dimensional gravity, a candidate master theory generating the $\mathfrak{sp}(2)$ and dilaton dynamics, a framelike formulation of 2 times physics and the relation between the towers of auxiliary fields of the 2 times approach and an unfolding of the full (nonlinear) four-dimensional Einstein's equations.

II. GRAVITY AND PARALLEL SCALE TRACTORS

It is well known that the Einstein-Hilbert gravitational action can be viewed as the gauge fixed version of a conformally improved scalar field theory [25,26]

$$S[\varphi, g] = -\frac{4(d-1)}{(d-2)} \int d^d x \sqrt{-g} \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{1}{8} \times \frac{d-2}{d-1} R \varphi^2 \right], \tag{1}$$

which is invariant under local Weyl rescalings $\Omega(x)$, transforming $\varphi \mapsto \Omega^{(2-d)/2} \varphi$ and

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}. \tag{2}$$

On the one hand this seems a rather trivial observation because choosing the gauge in which φ is constant and equal to κ^{-1}, one recovers the usual gravity action $S(g, \kappa^{-1}) = -\frac{1}{2\kappa^2} \int d^d x \sqrt{-g} R$. To see that this is in fact a statement of fundamental importance, first note that the Weyl transformation (2) defines the equivalence class relation $g_{\mu\nu} \sim \Omega^2 g_{\mu\nu}$ of a conformal class of metrics $[g_{\mu\nu}]$, so that physics can be cast in terms of conformal, rather than Riemannian geometry. Second, note that the Weyl transformation (2) amounts to making local redefinitions of unit systems, which along with general coordinate invariance, is a symmetry that any formulation of physics must enjoy.

So far there is no hint of any six-dimensional quantities. To see these, we attempt to write the Weyl invariant formulation (1) of Einstein-Hilbert gravity as the square of a single vector I^M

$$S[g, \sigma] = \frac{d(d-1)}{2} \int d^d x \frac{\sqrt{-g}}{\sigma^d} I^M I_M. \tag{3}$$

The six-component vector

$$I^M = \begin{pmatrix} \sigma \\ \nabla^m \sigma \\ -\frac{1}{d}[\Delta + \mathsf{P}]\sigma \end{pmatrix}, \tag{4}$$

is called the "scale tractor" and is distinguished by its transformation properties under Weyl transformations. Here the scalar $\sigma = \varphi^{2/(2-d)}$ is simply a relabeling of the dilaton φ so that it has unit Weyl weight

$$\sigma \mapsto \Omega \sigma.$$

The field σ is often called the "scale" since it measures the relative choice of unit system from point to point in spacetime. Also, it is often convenient to work with the Schouten tensor $\mathsf{P}_{\mu\nu}$ which is the trace adjusted Ricci-type tensor, defined by

$$\mathsf{P}_{\mu\nu} = \frac{1}{d-2} \left(R_{\mu\nu} - \frac{1}{2(d-1)} g_{\mu\nu} R \right),$$

and its trace is denoted $\mathsf{P} = \mathsf{P}_\mu^\mu$.

The main features of the action (3) are

(i) It depends on conformal classes of metrics, embedded in the double equivalence class $[g_{\mu\nu}, \sigma] \sim [\Omega^2 g_{\mu\nu}, \Omega\sigma]$. This allows for manifest Weyl invariance while still specifying a canonical metric $g^0_{\mu\nu}$ in the conformal class satisfying $[g_{\mu\nu}, \sigma] \sim [g^0_{\mu\nu}, \kappa^{2/(d-2)}]$.

(ii) The measure $\sqrt{-g}\sigma^{-d}$ is separately Weyl invariant, as is also the square of the scale tractor I^2. This holds because the scale tractor I^M transforms under particular local $SO(d, 2)$ transformations known as tractor gauge transformations.

(iii) Einstein's equations amount to the scale tractor being parallel with respect to the tractor connection, exactly the covariant derivative implied by tractor gauge transformations.

(iv) The "length" of the scale tractor measures the cosmological constant. Hence Ricci flatness implies a lightlike scale tractor.

Let us explain these points and the key ingredients of tractor calculus in more detail.

From the four-dimensional viewpoint, a six-component multiplet (V^+, V^m, V^-) with $m = 0, \ldots, d-1$, forms a weight w tractor vector V^M, $M = +, m, -$, if under Weyl transformations it obeys the tractor gauge transformation:

$$V^M \mapsto \Omega^w U^M{}_N V^N, \qquad U^M{}_N = \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon^m & \delta^m_n & 0 \\ -\frac{\Upsilon^2}{2\Omega} & -\frac{\Upsilon_n}{\Omega} & \frac{1}{\Omega} \end{pmatrix}, \tag{5}$$

where $\Upsilon_\mu = e_\mu{}^m \Upsilon_m = \Omega^{-1} \partial_\mu \Omega$. In Sec. III, we will see that tractors naturally live as six-vectors in a six-dimensional, signature (4, 2) spacetime endowed with a curved light-cone structure. The reduction to four dimensions induces a tractor-covariant connection:

$$\mathcal{D}_\mu = \begin{pmatrix} \partial_\mu & -e_{\mu n} & 0 \\ \mathsf{P}_\mu{}^m & \nabla_\mu{}^m{}_n & e_\mu{}^m \\ 0 & -\mathsf{P}_{\mu n} & \partial_\mu \end{pmatrix}, \tag{6}$$

such that

$$\mathcal{D}_\mu V^M \mapsto \Omega^w U^M{}_N [\mathcal{D}_\mu + w\Upsilon_\mu] V^N.$$

By means of the tractor connection one can construct a weight -1 tractor-vector operator, the so called "Thomas D-operator," which acting on weight w tractors reads

$$D^M = \begin{pmatrix} w(d+2w-2) \\ (d+2w-2)\mathcal{D}^m \\ -(\mathcal{D}_\mu \mathcal{D}^\mu + w\mathsf{P}) \end{pmatrix}. \tag{7}$$

Acting with the Thomas D-operator on the scale σ, we obtain a weight 0 tractor vector, the scale tractor

$$I^M = \frac{1}{d} D^M \sigma,$$

which has components exactly given by (4).

The scale tractor's main importance is twofold: first, in tractor theories it controls the coupling of matter to scale in a Weyl-covariant way [11], parametrizing the breaking of local scale invariance in the $\sigma =$ constant physical gauge. On the other hand, I^M is closely related to gravity itself: remarkably, the gravity-dilaton action (1), can be written entirely in terms of the scale tractor as in (3) where tractor indices are raised and lowered with the $SO(d,2)$ invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta_{mn} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

To see that a tractor-parallel scale tractor, i.e. $\mathcal{D}_\mu I^M = 0$, amounts to Einstein's equations we explicitly compute the tractor derivative of I^M that, once evaluated at the choice of constant scale $\sigma = \sigma_0$, reads

$$\mathcal{D}_\mu I^M|_{\sigma=\sigma_0} = \sigma_0 \begin{pmatrix} 0 \\ \mathsf{P}_\mu{}^m - \frac{1}{d} e_\mu{}^m \mathsf{P} \\ -\frac{1}{d}\partial_\mu \mathsf{P} \end{pmatrix}. \tag{8}$$

Setting this to zero says that the Ricci tensor obeys $R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R$ and the scalar curvature is subject to $R =$ constant, so that $g_{\mu\nu}$ is precisely an Einstein manifold. This happens at the choice of scale $\sigma = \sigma_0$, so we can say that the scale tractor is parallel when the metric is conformally Einstein:

$$\mathcal{D}_\mu I^M = 0 \Leftrightarrow g_{\mu\nu} = \Omega^2 g^0_{\mu\nu}, \quad \text{with} \quad R_{\mu\nu}(g^0) \propto g^0_{\mu\nu}.$$

Moreover, if the scale tractor is parallel then its length squared $I^2 \equiv I^M I_M$ is constant, and proportional to the cosmological constant.

Geometrically, the scale tractor can be viewed as coming from a vector perpendicular to a hypersurface in six dimensions. The intersection of that hypersurface with a (curved) light cone defines a conformal class of metrics on the four-dimensional intersection. This picture relies on a six-dimensional ambient description of tractors which we describe in the next section. Given the significance of the scale tractor I^M, it would be extremely interesting to formulate four-dimensional gravity in terms of an independent six-component vector field. That result is obtained by combining ambient tractors with Bars' 2 times physics proposal and is given in Sec. V.

III. AMBIENT TRACTORS

The importance of six-dimensional spacetimes for describing conformally invariant four-dimensional theories has been clear since the work of Dirac [2]. [Perhaps the simplest motivation for this is that the Minkowski space conformal group $SO(4,2)$ acts naturally on the flat Lorentzian space $\mathbb{R}^{4,2}$.] Weyl invariance ensures rigid conformal symmetry whenever the metric enjoys conformal isometries; this suggests that four-dimensional conformal geometries can be studied in terms of six-dimensional Riemannian geometries. This was shown to be the case by Fefferman and Graham [24] who formulated the problem of constructing conformal invariants in terms of a six-dimensional ambient metric. This idea was extended to the tractor calculus description of conformal geometry in the series of articles [15,22] (see also [23]).

Based on duality and holographic arguments, the 2 times approach of Bars advocates that four-dimensional physics (irrespective of whether it enjoys rigid conformal symmetry or not) can be described using a six-dimensional spacetime. The tractor approach of Gover *et al.* uses the simple principle of invariance under local choices of unit system to argue that four-dimensional physics should be formulated in terms of conformal geometry. Since the latter, in turn, enjoys an ambient six-dimensional formulation, local unit invariance and tractors also support a formulation of four-dimensional physics using a six-dimensional spacetime. In this section we give the main ingredients of the six-dimensional ambient description of tractor calculus.

A four-dimensional conformal manifold equipped with an equivalence class of metrics $[g_{\mu\nu}]$, with equivalence defined by local Weyl transformations

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu},$$

can be viewed as the space of rays in a five dimensional null hypersurface embedded in a six-dimensional Riemannian ambient space with metric G_{MN}. Specializing to the conformally flat case, consider the ambient space $\mathbb{R}^{4,2}$ with the standard flat Lorentzian metric $dX^M \eta_{MN} dX^N$, which enjoys a closed (and therefore hypersurface orthogonal) homothety given by the dilation/Euler operator $X^M \frac{\partial}{\partial X^M}$. The zero locus of the homothetic potential $X^M X_M \equiv X^2$ defines a five dimensional null cone so the space of null rays ξ^M subject to the equivalence relation $\xi^M \sim \Omega \xi^M$ (where $\Omega \in \mathbb{R}^+$) is four-dimensional and determines a (conformally flat) four-dimensional conformal structure. The conformal class of metrics follows by letting $\xi^M(x)$ be a section of the null cone. The ambient metric then pulls back to a four-dimensional metric $ds^2 = d\xi^M d\xi_M$. Choosing a different section $\xi^M(x)$ results in a conformally related metric. For example, in the conformally flat setting, de Sitter, Minkowski, and anti-de Sitter space all inhabit the same conformal class. In this case the tractor connection of (6) is the pullback of the Cartan-Maurer form of $SO(4,2)$ to the conformally flat four-dimensional space time described as a coset $SO(4,2)/P$ where P stabilizes a lightlike ray.

The above flat model of conformal geometry, as the space of lightlike rays in a six dimension ambient space, extends to curved spaces and general conformal structures as follows: A four-dimensional conformal structure determines a Fefferman-Graham ambient metric which admits a hypersurface orthogonal homothety. In the flat case this homothety is generated by the Euler vector field whose components coincide with the standard Cartesian coordinates. In the curved ambient construction, the corresponding homothetic vector field will still be denoted by X^M (which are *not* generally coordinates for which we reserve the notation Y^M). The key identity is then the equation

$$G_{MN} = \nabla_M X_N, \tag{9}$$

where G_{MN} is the ambient metric and ∇ is its Levi-Civita covariant derivative. This condition already suffices to uniquely determine a four-dimensional conformal structure. The symmetric part of (9) implies the homothetic conformal Killing equation while its antisymmetric part says that the 1-form dual to X^M is closed. Indeed this 1-form is exact

$$X_M = \tfrac{1}{2}\nabla_M X^2.$$

Clearly, the ambient metric is the double gradient of the homothetic potential $G_{MN} = \frac{1}{2}\nabla_M \partial_N X^2$. The zero locus of the potential X^2 defines a curved cone, a quotient of which recovers the four-dimensional conformal manifold. Observe that the above identities for the ambient metric imply

$$X^M R_{MNRS} = 0 = (X^T\nabla_T + 2)R_{MN}{}^R{}_S.$$

To ensure uniqueness of the ambient metric for a given four-dimensional structure, Fefferman and Graham require that the ambient metric is formally Ricci flat in any odd dimension (to all orders), and Ricci flat to finite order in the defining function X^2 in even dimensions greater than or equaling four. For our purposes, uniqueness of the underlying four-dimensional conformal structure is all we need, so we will typically work with six-dimensional ambient metrics subject to (9) but need not impose six-dimensional Ricci flatness.

The Rosetta Stone between six-dimensional ambient space operators and the Thomas D-tractor operator (7) on a four-dimensional conformal manifold was first given in [15] and simply reads

$$D_M \equiv \nabla_M(d + 2X^N\nabla_N - 2) - X_M\Delta. \tag{10}$$

The canonical tractor of [14] corresponds to the vector field X^M while tractor weights are eigenvalues of the operator $X^M\nabla_M$. (In [23], it was realized that these operators are related to a momentum space representation of the ambient space conformal group.) Tractor tensors $T^{M_1\cdots M_s}(x)$ (sections of weighted tractor tensor bundles over four-dimensional spacetime) can then be viewed as equivalence classes of six-dimensional ambient space tensors

$$T^{M_1\cdots M_s}(Y) \sim T^{M_1\cdots M_s}(Y) + X^2 U^{M_1\cdots M_s}(Y), \tag{11}$$

subject to a weight constraint

$$X^M\nabla_M T^{M_1\cdots M_s} = wT^{M_1\cdots M_s}. \tag{12}$$

The equivalence relation can also be handled by working with weight $w-2$ ambient space tensors of the form

$$\delta(X^2)T^{M_1\cdots M_s},$$

subject to the constraint $X^2 = 0$. It is not difficult to check that the ambient operator (10) is well defined on equivalence classes defined by the cone condition (11).

The equivalence relation (11) and weight constraint (12) do not define a unique extension of a four-dimensional tractor to the six-dimensional ambient space. For that, one needs to "fix a gauge" for the equivalence relation. A convenient choice is to require that six-dimensional quantities are harmonic. The first example of this is the Ricci flat condition of Fefferman-Graham (because the remaining Weyl part of the ambient Riemann curvature is then harmonic). In fact, it is easily verified that the triplet of operators

$$\left\{X^2, X^M\nabla_M + \frac{d+2}{2}, \Delta\right\}, \tag{13}$$

obey an $\mathfrak{sp}(2)$ Lie algebra. This algebraic fact underlies Bars' 2 times approach described in the next section.

IV. TWO TIMES PHYSICS

A simple starting point for understanding 2 times physics, is the Howe dual pair [27]

$$\mathfrak{sp}(2(d+2)) \supset \mathfrak{sp}(2) \oplus \mathfrak{so}(d,2). \tag{14}$$

This Lie algebra statement—namely that $\mathfrak{sp}(2)$ and $\mathfrak{so}(d,2)$ are maximal cocommutants in $\mathfrak{sp}(2(d+2))$—says that imposing as constraints an $\mathfrak{sp}(2)$ subalgebra of the natural $\mathfrak{sp}(2(d+2))$ algebra acting on a $d+2$ dimensional phase space, leaves a residual $\mathfrak{so}(d,2)$ global symmetry algebra. This latter algebra generates the conformal isometries of d-dimensional Minkowski (or more generally conformally flat) spacetime.

Consider, for example, Bars' approach to the relativistic particle [28,29]. Instead of requiring worldline reparametrization invariance and therefore a four-dimensional Hamiltonian constraint, Bars requires local worldline conformal invariance under $\mathfrak{so}(2,1) \cong \mathfrak{sp}(2)$ which imposes a triplet of first class constraints. In four dimensions a three dimensional constraint algebra would be too constraining, but as is clear from the Fefferman-Graham ambient space construction described above, if this constraint algebra acts in six dimensions as in (13), the null cone and weight constraints perform the reduction to four dimensions leaving a single Hamiltonian constraint just as in the standard approach. By making different gauge choices for the local $\mathfrak{sp}(2)$ symmetry, one can obtain a plethora of four-dimensional models—"holographic shadows"—all encompassed by a single six-dimensional one [30].

The above discussion pertains to single particle models propagating in fixed backgrounds. Our chief interest is a description of four-dimensional field theories and, in particular, four-dimensional gravity. For that, two main ingredients are required. First, we must quantize the underlying particle model so that, in turn, quantum mechanical wave functions can be reinterpreted as quantum fields. Second, we need to write equations of motion for the background fields. Both steps can be achieved in a unified way by working with quantum mechanical operators. (An alternative approach employed heavily by Bars [29,31] is to employ phase space quantization technology [32], but we find working directly with quantum mechanical operators to be more direct.)

Our model, described in detail in the next section, will be built from two multiplets, the first "gravity multiplet" will describe ambiently a conformal class of metrics along with an additional vector field intimately related to the scale tractor of Sec. II. The second "dilaton multiplet" describes the dilaton or scale field (or in other words, a spacetime-varying Planck's constant). Equations of motion for the gravity multiplet have already been proposed by Bars [33]. Classically they amount to a triplet of Hamiltonians $Q_{ij} = Q_{ji}$ $(i, j = 1, 2)$ on a $2(d+2)$ dimensional phase space subject to an $\mathfrak{sp}(2)$ algebra under Poisson brackets

$$\{Q_{ij}, Q_{kl}\} = \varepsilon_{kj}Q_{il} + \varepsilon_{ki}Q_{jl} + \varepsilon_{lj}Q_{ik} + \varepsilon_{li}Q_{jk}. \quad (15)$$

Here one must solve for the Q_{ij} modulo gauge transformations corresponding to canonical transformations

$$Q_{ij} \mapsto Q_{ij} + \{\epsilon, Q_{ij}\}. \quad (16)$$

An elegant solution has been found by Bars [33] by choosing Darboux coordinates $\{P_M, Y^N\} = \delta^N_M$, expanding in powers of the momentum P_M shifted by some vector field $A_M(Y)$, and then partially fixing the gauge invariance (15) so that

$$Q = \begin{pmatrix} X^M G_{MN}(Y) X^N & X^M \tilde{P}_M \\ X^M \tilde{P}_M & \Sigma(Y) + \tilde{P}_M G^{MN}(Y)\tilde{P}_N + H(\tilde{P}, Y) \end{pmatrix}, \quad (17)$$

where

$$\tilde{P}_M \equiv P_M + A_M(Y),$$
$$H(\tilde{P}, Y) \equiv \sum_{k=2}^{\infty} H^{M_1\cdots M_k}(Y)\tilde{P}_{M_1}\cdots\tilde{P}_{M_k}.$$

In addition, this result is intimately connected to ambient tractors, because the algebra (15) requires the metric G_{MN} appearing in (17) to obey the closed homethety condition (9). Moreover, the vector field A_M appearing in $\tilde{P}^M$ obeys

$$X^M F_{MN} \equiv (\mathcal{L}_X + 1)A_N - \nabla_N(X^M A_M) = 0, \quad (18)$$

and the scalar Σ and totally symmetric tensors $H^{M_1\cdots M_k}$ are subject to weight conditions

$$\begin{aligned} (\mathcal{L}_X + 2)\Sigma &\equiv (X^M\nabla_M + 2)\Sigma = 0, \\ (\mathcal{L}_X + 2)H^{M_1\cdots M_k} &\equiv (X^M\nabla_M + 2 - k)H^{M_1\cdots M_k} = 0. \end{aligned} \quad (19)$$

Classically, the tensors $H^{M_1\cdots M_k}$ must also be transverse to the homothetic vector field X^M. The above solution still enjoys residual gauge symmetries of the form (16). The beauty of Bars' solution is that these residual transformations amount to diffeomorphisms of the tensors X^M, G_{MN}, A_M, Σ, and $H^{M_1\cdots M_k}$, Abelian Maxwell gauge transformations of A_M, as well as a certain class of higher rank symmetries of the symmetric tensors $H^{M_1\cdots M_k}$ which we will discuss in detail later.

To quantize the Hamiltonians Q_{ij}, we look for operators acting on wave functions depending on coordinates Y^M. We express these as expansions in the covariant derivatives $\tilde{\nabla}_M = \nabla_M + A_M$. This amounts to a choice of quantum orderings for a basis of all operators acting on wave functions. More precisely, momenta P_M act on wave functions as derivatives ∂_M, but we add subleading ordering terms to higher powers of momenta in order to maintain covariance. We then require that the quantum commutator of the Q_{ij}'s obeys the $\mathfrak{sp}(2)$ algebra

$$[Q_{ij}, Q_{kl}] = \varepsilon_{kj}Q_{il} + \varepsilon_{ki}Q_{jl} + \varepsilon_{lj}Q_{ik} + \varepsilon_{li}Q_{jk}, \quad (20)$$

modulo the quantum symmetry

$$Q_{ij} \mapsto Q_{ij} + [\epsilon, Q_{ij}], \quad (21)$$

whose parameter ϵ is now itself an operator. This system of equations has been proposed by Bars in an equivalent phase space and star product quantization [33]. Quantization necessitates a slight modification of Bars' classical solution to

$$Q = \begin{pmatrix} X^2 & X^M\tilde{\nabla}_M + \frac{d+2}{2} \\ X^M\tilde{\nabla}_M + \frac{d+2}{2} & \Sigma + \tilde{\nabla}^2 + H(\tilde{\nabla}, Y) \end{pmatrix}, \quad (22)$$

with

$$H(\tilde{\nabla}, Y) \equiv \sum_{k=2}^{\infty} H^{M_1\cdots M_k}(Y)\tilde{\nabla}_{M_1}\cdots\tilde{\nabla}_{M_k}.$$

Here the closed homothety, curvature, and weight conditions are unaltered from their classical counterparts (9), (18), and (19), but the transverse conditions on the *symmetric* tensors $H^{M_1\cdots M_k}$ are modified to read

$$2X_M H^{MM_2\cdots M_k} + (k+1)H_M{}^{MM_2\ldots M_k} = 0. \quad (23)$$

From this, we learn iteratively that the trace of H^{MN} vanishes, the trace of H^{MNR} is the part of H^{MN} parallel to X^M etc. More succinctly, the condition (23) just says

$$[X^2, H(\tilde{\nabla}, Y)] = 0.$$

But now let us examine which gauge symmetries respect the quantum solution (22). First, expanding the gauge parameter in powers of $\tilde{\nabla}_M$,

$$\epsilon(\tilde{\nabla}, Y) = -\alpha(Y) + \xi^M(Y)\tilde{\nabla}_M + \varepsilon(\tilde{\nabla}, Y),$$

where all terms of quadratic order and higher are stored in ε, it is easy to verify that the zeroth and first order terms generate Abelian gauge transformations

$$A_M \mapsto A_M + \nabla_M \alpha,$$

and diffeomorphisms with parameter ξ^M. These are desirable symmetries, so we do not want to gauge fix them at this juncture. We still have the higher order gauge freedoms in ε, although these are not completely arbitrary: Requiring $Q_{11} = X^2$ to be inert, the gauge parameter ε obeys the same commutation relation with the homothetic potential as H

$$[X^2, \epsilon] = 0. \tag{24}$$

Furthermore, invariance of Q_{12} implies that

$$[X^M\tilde{\nabla}_M, \varepsilon] = 0.$$

It follows that $\delta Q_{22} \equiv [\varepsilon, \Sigma + \tilde{\nabla}^2 + H]$ obeys the same conditions as H, namely,

$$[X^2, \delta Q_{22}] = 0 = [X^M\tilde{\nabla}_M, Q_{22}] + 2Q_{22}.$$

Now, we define a vector

$$U_M \equiv \nabla_M \Sigma,$$

and note that

$$[\varepsilon, \Sigma] = \frac{1}{2}\varepsilon^{MN}\mathcal{L}_U G_{MN} + \varepsilon^{MN}U_M\tilde{\nabla}_N + \sum_{k=3}^{\infty} k\varepsilon^{M_1\cdots M_k}(U_{M_1}\tilde{\nabla}_{M_2}\cdots\tilde{\nabla}_{M_k})_{\mathrm{W}}, \tag{25}$$

where $(\bullet)_{\mathrm{W}}$ denotes Weyl ordering in the symbols $(U, \tilde{\nabla})$.

We now make the assumption that the vector U_M is nonvanishing. Certainly, the set of vanishing U_M is measure zero (a situation similar to noninvertible metrics among the space of 4×4 matrices). Bars has suggested that models with vanishing U_M might describe a novel "higher spin branch," but we do not pursue this line of argument any further here. With U_M nonvanishing the space of rank two and higher symmetric tensors $U_M\varepsilon^{MM_1\cdots M_k}$ appearing in the summation in formula (25) suffices to gauge away the operators $H(\tilde{\nabla}, Y)$. One might worry that this reintroduces new contributions to Q_{22} at order zero and one in $\tilde{\nabla}$, but we have as yet not used the freedom to choose the first two terms in (25). Clearly, when $U_M \neq 0$, we can choose $\varepsilon^{MN}U_M$ to ensure that Q_{22} has no term linear in $\tilde{\nabla}$. Finally, when U_M is not a conformal Killing vector [notice that (24) implies that ε^{MN} is trace-free], we can try to use the first term in (25) to remove Σ. A generic choice of metric G_{MN} will not admit conformal Killing vectors so we may safely[7] pick a gauge for which $\Sigma = 0$.

Thus, we arrive at our final solution for the quantum Eqs. (20)

$$Q(G_{MN}, A_M) = \begin{pmatrix} X^2 & X^M\tilde{\nabla}_M + \frac{d+2}{2} \\ X^M\tilde{\nabla}_M + \frac{d+2}{2} & \tilde{\nabla}^2 \end{pmatrix}. \tag{26}$$

It is parametrized, modulo diffeomorphisms and $SO(1, 1)$ gauge transformations by a metric G_{MN} and Abelian gauge field A_M subject to the closed homothety and transverse curvature requirements in Eqs. (9) and (18), respectively. This is the gravity mulitplet of our model. It describes spacetime geometry but does not describe gravitational dynamics. From the tractor viewpoint, that requires coupling to scale, or in other words, a dilaton. Therefore, we now describe the coupling of the gravity multiplet to the dilaton multiplet.

V. MAIN RESULTS: GRAVITY

In Sec. II we saw that instead of formulating gravity in terms of an Einstein-Hilbert action functional depending on four-metrics, one could build from the square of the scale tractor I^M an equivalent action depending on the scale (or dilaton) σ and a conformal class of four-dimensional metrics $[g_{\mu\nu}]$. The operator Q of the previous section depended on (i) a six-dimensional metric G_{MN} with closed homothety and (ii) a six-dimensional vector A_M. Since the metric G_{MN} encodes a four-dimensional conformal class of metrics $[g_{\mu\nu}]$, one can hope that the vector A_M is somehow related to the scale tractor and so a theory built from the operator Q could amount to a tractor description of Einstein-Hilbert gravity. For this proposal to work, we still need to couple to a dilaton field, or in other words scalar matter. From a 2 times physics perspective, this coupling should respect the gauge symmetry (21) as well as the $\mathfrak{sp}(2)$ gauge symmetry generated by the operators Q. A coupling to scalars with exactly these symmetries has been computed by Bars using first quantized BRST techniques [16] and reads

$$S(Q, \Omega, \Theta, \Lambda, \Psi) = \frac{2(d-1)}{d-2}\int d^{d+2}Y\sqrt{G}[\Omega Q_{22} + \Theta Q_{12} + \Lambda Q_{11}]\Psi. \tag{27}$$

Our claim is that this action principle, along with the conditions (20) on the operator Q amounts to the tractor description of four-dimensional Einstein-Hilbert gravity.

[7] It is possible that Σ can still be gauged away even if the metric G_{MN} admits conformal Killing vectors $U^M = \nabla^M\Sigma$. We have not analyzed this issue in detail, but it is interesting to note that the condition $\nabla_{(M}U_{N)} \propto G_{MN}$ along with the weight condition (19) for Σ implies that Σ is an eigenstate of the quadratic Casimir of the triplet of operators (13).

The action (27) depends (from a six-dimensional viewpoint) on an infinite set of fields through the operator Q. However it also enjoys infinitely many local symmetries generated by an operator parameter ϵ as well as a local $\mathfrak{sp}(2)$ invariance with local parameters $(\lambda(Y), \theta(Y), \omega(Y))$

$$\begin{aligned} &Q \mapsto Q + [\epsilon, Q], \qquad \Psi \mapsto \Psi + \epsilon\Psi,\\ &\Omega \mapsto \Omega - \epsilon^\dagger\Omega - Q_{11}^\dagger\theta + [Q_{12}^\dagger + 2]\omega,\\ &\Theta \mapsto \Theta - \epsilon^\dagger\Theta + Q_{11}^\dagger\lambda - Q_{22}^\dagger\omega - 4\theta,\\ &\Lambda \mapsto \Lambda - \epsilon^\dagger\Lambda + Q_{22}^\dagger\theta - [Q_{12}^\dagger - 2]\lambda. \end{aligned}$$

Here the dagger operation is the standard adjoint with respect to the six-dimensional measure appearing in (27). We are now ready to verify our claim that (27) is the theory of gravity.

The first step is use the gauge freedom ϵ to reach the gauge (26) for the operator Q. This yields a standard, generally covariant, six-dimensional action depending only on finitely many fields $(G_{MN}, A_M, \Omega, \Theta, \Lambda)$

$$S = \frac{2(d-1)}{d-2}\int d^{d+2}Y\sqrt{G}\Big[\Omega\tilde{\nabla}^2 + \Theta\Big(X^M\tilde{\nabla}_M + \frac{d+2}{2}\Big) + \Lambda X^2\Big]\Psi, \tag{28}$$

with gauge invariance

$$\begin{aligned} &A_M \mapsto A_M + \nabla_M\alpha, \qquad \Psi \mapsto \Psi - \alpha\Psi,\\ &\Omega \mapsto \Omega + \alpha\Omega - X^2\theta - \Big(X^M\tilde{\nabla}_M + \frac{d+2}{2} - 2\Big)\omega,\\ &\Theta \mapsto \Theta + \alpha\Theta + X^2\lambda - \tilde{\nabla}^2\omega - 4\theta,\\ &\Lambda \mapsto \Lambda + \alpha\Lambda + \tilde{\nabla}^2\theta + \Big(X^M\tilde{\nabla}_M + \frac{d+2}{2} + 2\Big)\lambda. \end{aligned} \tag{29}$$

The action (28) is four-dimensional gravity wearing a six-dimensional disguise. To disrobe it further, we use the $SO(1,1)$ gauge symmetry α to choose a gauge

$$X^M A_M = -w, \ \text{which implies}\ X^N\nabla_N A_M = -A_M. \tag{30}$$

Here w is an arbitrary real number. We could equally well have chosen $w = 0$, but we prefer the above since it will imply the most general assignments of tractor weights to the scalar fields. In any case, w will drop out at the end of our computation, and thereby serves as a check on our algebra. Notice that using (18), the potential A_M now has weight -1 with respect to the weight operator $X^M\nabla_M$. Note that the vector A_M still enjoys residual Abelian gauge transformations with weight zero gauge parameter $X^M\nabla_M\alpha = 0$.

We now integrate out the Lagrange multipliers (Θ, Λ) which imposes constraints

$$X^M\nabla_M\Psi = \Big(w - \frac{d}{2} - 1\Big)\Psi, \qquad X^2\Psi = 0.$$

Solving the latter constraint via

$$\Psi = \delta(X^2)\phi, \qquad \phi \sim \phi + X^2\chi,$$

and comparing with (11) and (12), we see that ϕ is a weight $w - \frac{d}{2} + 1$ tractor scalar.

There is still the freedom using the gauge parameter ω to gauge away Ω save for gauge transformations ω in the kernel of $X^M\nabla_M + w + \frac{d}{2} - 1$. Hence, all that remains is the part of Ω of weight $-w - \frac{d}{2} + 1$. The remaining field content along with their weights are summarized in the following table:

Field	Weight
Ω	$-w - \frac{d}{2} + 1$
ϕ	$w - \frac{d}{2} + 1$
A_M	-1

Integrating by parts to ensure no derivatives act on the delta function in Ψ, the action now takes the extremely simple form

$$S = \frac{2(d-1)}{d-2}\int d^{d+2}Y\sqrt{G}\delta(X^2)T, \tag{31}$$

where

$$T = \phi(\nabla^M - A^M)(\nabla_M - A_M)\Omega. \tag{32}$$

Since $T \sim T + X^2 U$, it is a tractor scalar with weight $-d$ (see the above table). We would like to express the action (31) as a four-dimensional integral over tractor-valued objects.[8] To that end we need to express (32) in terms of ambient tractor operators: Using the ambient expression (10) for the Thomas D-operator, we easily derive the following ambient tractor identities:

$$\begin{aligned} &\Delta\Omega - 2A^M\nabla_M\Omega = \frac{1}{w}A^M D_M\Omega,\\ &\nabla^M A_M = \frac{1}{d-2}D_M A^M. \end{aligned} \tag{33}$$

[There is no pole at $w = 0$ in the first identity, as can be easily verified by using the four-dimensional component expression (7) for the Thomas D-operator.] Hence,

$$T = \phi\Big(\frac{1}{w}A^M D_M - \frac{1}{d-2}(D_M A^M) + A^2\Big)\Omega.$$

The beauty of this expression is that $\delta(X^2)T$ now only depends on equivalence classes $A_M \sim A_M + X^2 B_M$, $\Omega \sim \Omega + X^2\Xi$. Therefore *all* fields are now tractor valued.

[8] Bars handles delta-function valued ambient space integrals by developing a calculus for derivative of delta functions [19]. The simple tractor analysis given here, obviates the need for such methods.

Hence, we may replace the ambient space integral (31), with a four-dimensional integral depending on tractors (ϕ, Ω, A_M)

$$S = \frac{2(d-1)}{d-2} \int d^d x \sqrt{-g}\phi \times \left[\frac{1}{w} A^M D_M - \frac{1}{d-2}(D_M A^M) + A^2\right]\Omega. \quad (34)$$

Note that the integrand has weight $-d$, while the metric determinant has weight d under Weyl transformations so this action principle is now manifestly Weyl invariant. Our claim is now that this tractor action is equivalent to the formulation of the Einstein-Hilbert action in terms of the square of the scale tractor (3).

To verify our final claim we must examine the remaining $SO(1, 1)$ gauge symmetry

$$A_M \mapsto A_M + \frac{1}{d-2} D_M \alpha, \qquad \Omega \mapsto \Omega + \alpha\Omega, \qquad \phi \mapsto \phi - \alpha\phi, \quad (35)$$

where the gauge parameter α is a weight zero tractor scalar. Notice that the gauge transformation of A_M respects the condition $X^M A_M = -w$. Now, observe that the action depends only algebraically on the $SO(1, 1)$ gauge field A_M and the pair of fields (ϕ, Ω) form a doublet under this symmetry. Hence, we expect that upon integrating out A_M, only the gauge invariant combination $\phi\Omega$ should survive. This computation can be performed either using component expressions for the tractor quantities in (34) or directly using tractors. In components, one finds that the bottom slot A^- of the gauge field decouples completely from the action and that integrating out the middle slot of A_M sets it equal to the $SO(1, 1)$ current $\frac{1}{2}\nabla_m \log(\Omega/\phi)$. This yields the four-dimensional action for a conformally improved scalar field

$$S = \frac{2(d-1)}{d-2} \int d^d x \sqrt{-g}\varphi\left[\Delta - \frac{d-2}{2}\mathsf{P}\right]\varphi,$$

where φ is the weight $1 - \frac{d}{2}$ scalar field defined by

$$\varphi^2 = \phi\Omega.$$

In other words, it is the *dilaton*. Using the relationship between the dilaton and scale, $\varphi = \sigma^{1-(d/2)}$, we obtain as explained in Sec. II the tractor version of the Einstein-Hilbert action in terms of the square of the scale tractor

$$S = \frac{d(d-1)}{2} \int d^d x \frac{\sqrt{-g}}{\sigma^d} I^M I_M. \quad (36)$$

This completes our demonstration that the $\mathfrak{sp}(2)$ invariant theory (27) amounts to a theory of four-dimensional gravity. We now turn to implications of our results.

VI. CONCLUSIONS AND OUTLOOK

In this article we formulated the Einstein-Hilbert action as a trace

$$S = \mathrm{tr} Q P \quad (37)$$

over quantum mechanical operators Q [as in (26)] and

$$P = \begin{pmatrix} |\Psi\rangle\langle\Lambda| & \frac{1}{2}|\Psi\rangle\langle\Theta| \\ \frac{1}{2}|\Psi\rangle\langle\Theta| & |\Psi\rangle\langle\Omega| \end{pmatrix}.$$

In this formulation, second quantization amounts to integrating over the space of operators Q and P in the path integral. This leads one to wonder whether quantum field theory effects, such as Weyl anomalies, can be understood from this six-dimensional quantum mechanical picture. An advantage of this 2 times approach is that it formulates gravity in terms of a very limited field content: the three components of Q viewed as functions of a 12 dimensional phase space. Weyl and diffeomorphism symmetries are neatly encoded in the algebra (20) and its gauge invariance (21). A pressing question therefore is to compute anomalies in the $\mathfrak{sp}(2)$ symmetry.

Another benefit of the 2 times starting point (27) is that it yields a new tractor formulation of the conformally Einstein condition [see the action (34)]. At the very least, this should have implications for conformal geometry; the triplet of tractor fields (ϕ, Ω, A_M) underlie the scale tractor I^M. This observation deserves further investigation.

Another interesting avenue for further research is whether there exists a framelike formulation of 2 times physics. This is based on the simple observation that the operator (26) can be factorized as

$$Q = \left[\begin{pmatrix} X_M \\ \tilde{\nabla}_M \end{pmatrix}\begin{pmatrix} X^M & \tilde{\nabla}^M \end{pmatrix}\right]_W.$$

The operator $V_i^M = (X^M \tilde{\nabla}^M)$ can then be interpreted as a 2 times frame field, so one could try to impose the Howe dual pair (14) decomposition as equations of motion for fundamental fields V_i^M. This might be particularly interesting when one considers the interpretation of the infinite tower of six-dimensional auxiliary fields appearing in the parent action (27). In particular, one wonders whether these fields solve the problem posed, and partially solved in [34], of finding an unfolding of the full nonlinear Einstein's equations. The relation between these two approaches may be clearer in a framelike formulation, since (unlike unfolding constructions) 2 times models are typically constructed in a metric formulation.

Finally, a gravitational 2 times action principle that simultaneously incorporates the benefits of both actions (27) and (A1)—namely producing the $\mathfrak{sp}(2)$ algebra as equations of motion while maintaining manifest $\mathfrak{sp}(2)$ symmetry—would be very desirable. In fact, once we understand that our work implies that the coupling of the gravity multiplet [built from $\mathfrak{sp}(2)$ generators] to scalars really amounts to a gravity-dilaton coupling, then we can identify yet another action principle proposed by Bars as a candidate model for cosmological four-dimensional Einstein gravity. Bars' proposal is to produce the equations of motion for the operator Q from a Chern-Simons action [31]

$$S_{\rm CS} = \int [Q \star Q + Q \star Q \star Q],$$

(where the Moyal star product $\star$ is employed to produce operator equations of motion from phase space valued fields). Hence, the sum of this action plus the BRST action $S_{\rm BRST}$ in (27)

$$S = S_{\rm CS} + \lambda S_{\rm BRST}, \tag{38}$$

deforms the $\mathfrak{sp}(2)$ relations by dilaton dependent terms (see [31] for explicit formulæ). A simple conjecture, therefore, is that these produce the cosmological constant coupling missing from the action (27). In particular, the relative coefficient λ in the total action (38) could be identified with the cosmological constant.

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APPENDIX: AN ALTERNATIVE SIX-DIMENSIONAL FORMULATION OF GRAVITY

In [19] Bars proposed the following six-dimensional field theory model for gravity coupled to scalar field

$$S = -\frac{1}{2}\int d^{d+2}Y\sqrt{G}[\delta(W)(R(G)\varphi^2 + \alpha(\nabla\varphi)^2 - \lambda\varphi^{2d/(d-2)}) - \delta'(W)((\Delta W - 4)\varphi^2 - \nabla_M W \nabla^M \varphi^2)], \tag{A1}$$

with $\alpha = \frac{4(d-1)}{d-2}$ and for some λ playing the *role* of the cosmological constant. A distinguishing feature of this action is that the homothetic condition and the weight condition on φ follow from its equations of motion; they indeed arise from the field equations for G_{MN} and φ instead of requiring closure of the $\mathfrak{sp}(2)$ algebra. Partially solving those equations, one obtains the following set of relations:

$$W = X^2, \qquad G_{MN} = \nabla_M X_N,$$
$$X^M \nabla_M \varphi = \left(1 - \frac{d}{2}\right)\varphi.$$

Plugging these back in (A1), we get the following model

$$S = -\frac{1}{2}\int d^{d+2}Y\sqrt{G}\delta(X^2)[R(G)\varphi^2 - \alpha\varphi\Delta\varphi - \lambda\varphi^{2d/(d-2)}]. \tag{A2}$$

Now note that, introducing the scale tractor I^M constructed from $\sigma = \varphi^{2/(2-d)}$ in the usual way (see Sec. II), the action (A2) becomes

$$S = -\frac{1}{2}\int d^{d+2}Y\sqrt{G}\delta(X^2)\left[R(G)\varphi^2 + \frac{\alpha}{\sigma}\varphi I^M D_M \varphi - \lambda\varphi^{2d/(d-2)}\right],$$

that in turn, by using the relation $I^M D_M \sigma^k = k(d + k - 1)\sigma^{k-1} I^2$, can be rewritten as

$$S = -\frac{1}{2}\int d^{d+2}Y\sqrt{G}\delta(X^2)\frac{1}{\sigma^d}[R(G)\sigma^2 - d(d-1)I^2 - \lambda]. \tag{A3}$$

Let us observe at this point that, as was shown by Fefferman and Graham in [24], a conformal class of d-dimensional metrics $[g_{\mu\nu}]$ determines a Ricci flat ambient space if d is odd, and a Ricci flat ambient space modulo $(X^2)^{(d-2)/4}$ for even d. Hence, since the action (A3) depends only on the conformal class of metrics $[g_{\mu\nu}]$ and includes the delta function $\delta(X^2)$, we can set to zero the curvature term in (A3). In fact, another way to see this is that we could have chosen a gauge in Sec. IV, where $\Sigma = R(G)$.

Now that the model is completely written in terms of tractor objects it may be directly written in four-dimensional language as

$$S = \frac{d(d-1)}{2}\int d^d x \frac{\sqrt{-g}}{\sigma^d}\left[I^M I_M + \frac{\lambda}{d(d-1)}\right]. \tag{A4}$$

When $\lambda = 0$, this model coincides with (36) demonstrating the equivalence of these two models in that case. The formulation (A1) has the advantage that it includes a cosmological constant and partially imposes the relations (20) as equations of motion coming from a variational principle. Its disadvantage is that the manifest $\mathfrak{sp}(2)$ symmetry is lost. In our conclusions, we speculated that a third model proposed by Bars incorporates the best features of both models (27) and (A1).

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