Imperfect fluids, Lorentz violations, and Finsler cosmology

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We construct a cosmological toy model based on a Finslerian structure of space-time. In particular, we are interested in a specific Finslerian Lorentz violating theory based on a curved version of Cohen and Glashow's very special relativity. The osculation of a Finslerian manifold to a Riemannian manifold leads to the limit of relativistic cosmology, for a specified observer. A modified flat Friedmann-Robertson-Walker cosmology is produced. The analogue of a zero energy particle unfolds some special properties of the dynamics. The kinematical equations of motion are affected by local anisotropies. Seeds of Lorentz violations may trigger density inhomogeneities to the cosmological fluid.

I. INTRODUCTION

The idea of Lorentz violations (LV) in modern physics can be traced back to the studies of Dirac in the late 1950s [1]. Contemporary research problems of high energy physics lead to the formulation of various quantum gravitational (QG) theories which inherit local anisotropies in most circumstances. A sensible approach to this direction is the study of the extensions of general relativity (GR)'s extensions, where new space-time symmetries are introduced. Noteworthy formalizations in this direction are extradimensional physics and the noncommutativity of space-time geometry (see e.g. [2,3]). In this framework, new physical phenomena emerge in long range distances that may resolve questions of modern theoretical physics. A common feature in QG is the prediction of a modified mass-shell condition for elementary particles. Moreover, these departures from Lorentz invariance predict a vacuum refractive index and corrections at the threshold energy. The most debated effects are the time delay of light rays, which depends on the energy of photons (see e.g. [4]), and threshold anomalies reported from astrophysical observations [5].

This phenomenology was recently associated with a velocity dependent geometry called Finsler. In particular, Girelli *et al.* [6] argued that several structures, like the "rainbow metric" [7] and other alternative scenarios of deformed special relativity–like [8] models, can be approached by a Finslerian perspective. The same formalism seems to be compatible with the propagation of rays in

Horava-Lifshitz gravity [9]. Another case where Finsler-like structures appear is the D-particle recoil example, where the effective four-dimensional metric depends on phase space coordinates [10–12]. Similar scenarios from a different point of view on stringlike theories have been discussed in [13]. We also mention the correlation of birefringence optics to Finslerian space-times [14]. The aforementioned phenomenology suggests that Finsler geometry could play a fundamental role in modern QG theories.

Another case on Lorentz violations is the minimalistic approach of very special relativity (VSR). The construction of VSR is based on a proper subgroup of the Poincaré group. An induced *ether* moving with the speed of light simulates a null spurionic vector field. In the context of VSR the introduction of a neutrino mass requires no additional states and needs no violation of leptonic number [15]. However, departures from special relativity and *CPT* invariance are difficult to detect due to the null nature of "ether" [16]. The above construction is incorporated into the Finslerian framework again, after considering a curved version of VSR, namely, general very special relativity (GVSR) [17].

GVSR is manipulated to build a cosmological toy model. We use a similar approach to [18], with the essential difference that the null character of the spurion is preserved in alliance with the original GVSR theory. A better insight about the effects of Finsler geometry to gravitational physics is achieved by the osculation of a Finslerian manifold to a Riemannian one. We consider that the physical geometry is represented by a Finslerian space-time, while gravitational geometry is described by a Riemann structure, following [19]. Therefore, the Friedmann-Robertson-Walker (FRW) metric is invoked

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to the Finslerian background to study deviations from the standard cosmological picture. In this framework, an observer falls on a peculiar nongeodesic congruence with respect to the FRW comoving motion.

This paper is organized as follows: After a short introduction to Finsler geometry we outline the model of GVSR and some of its phenomenological consequences. In Sec. III we describe the useful tool of the osculating Riemannian space and its implications to gravitational physics. In Sec. IV we apply the osculating process to the model of GVSR using the FRW metric for a flat universe. Sections V and VI are devoted to the construction of the equations of motion and continuity, relying on the induced space-time symmetries. In Sec. VII we present the solutions for the scale factor, flux, and anisotropic pressure, while in Sec. VIII we discuss the implied long range modifications depicted by a modified FRW potential. Besides, we attempt to relate the dynamical behavior to some large scale observables. It seems that exotic matter or lower values for the energy density of the cosmic fluid is required to generate late time acceleration. Sections IX and X further analyze the kinematical properties of the model, to achieve some level of insight about the evolution of the medium. Finally, we highlight for future development that departures from the safe harbor of Riemann geometry may trigger density perturbations, leaving artifacts of Lorentz violations.

II. FINSLER GEOMETRY AND VERY SPECIAL RELATIVITY

Finsler geometry is a generalization of Riemann geometry, where all the geometrical structures depend on the element $(x^i, \dot{x}^i)$ rather than the position coordinate solely. The line element is defined by a norm $F(x, \dot{x})$ over the tangent bundle $TM\backslash\{0\}$, where M is the base manifold. The $F(x, \dot{x})$ is a homogeneous function of first degree with respect to $\dot{x}$, such that the integral of the arc length $\int F(x, \dot{x})d\tau$ is independent of the parameter τ. Finsler geometry is strictly discriminated from Riemann geometry after dropping the quadratic restriction over the metric function $F(x, \dot{x})$ [20], i.e.

$$F^2(x, \dot{x}) = f_{\mu\nu}(x, \dot{x})\dot{x}^\mu \dot{x}^\nu. \tag{1}$$

Using Euler's theorem we can calculate the Finsler metric

$$f_{\mu\nu}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2}{\partial \dot{x}^\mu \partial \dot{x}^\nu}(x, \dot{x}), \tag{2}$$

which is homogeneous of zero degree with respect to $\dot{x}$. The definition (2) is reduced to a Riemannian metric when the metric tensor depends solely on the position x^i, indicating that (1) is a quadratic form in $\dot{x}^j$. Therefore, Finsler geometry can be considered as a natural generalization of Riemann geometry.

The unit sphere $I_x = \{F(x, \dot{x}) = 1, \dot{x}\epsilon T_x M\} \subset T_x M$ is called the *indicatrix* and defines a three-dimensional locus

in every tangent space $T_x M$. In case of a Riemannian space the indicatrix is an ellipsoid as a result of the quadratic restriction. However, (1) implies that the tangent spaces of a Finsler space are not equipped with ellipsoidal unit balls as in Riemann geometry, generating local anisotropies of space-time. Therefore a geometrical property should arise to describe this "distortion" of the indicatrix, called color (see e.g. [21]). In particular, a Riemannian space is considered entirely "white," while in most cases a Finsler space possesses different color patterns over the manifold.

The lack of quadratic restriction appears in some phenomenological quantum gravitational theories as a consequence of Lorentz symmetry breaking [6–14]. Thus, in such a scenario, quantities which measure the color and its variations are directly related to Lorentz violations. These space-times can be characterized as *colorful curved manifolds* providing a way to study gravitational phenomena under the hypothesis of Lorentz violations [22].

An intriguing case of Lorentz symmetry breaking, where Finsler geometry turns up, is Cohen and Glashow's VSR [15]. The Lorentz violations are generated by the $ISIM(2)$ subgroup of Lorentz transformations. Gibbons *et al.* investigated a deformation of VSR called GVSR [17]. Among the deformations of $ISIM(2)$ there is an 1-parameter family called the $DISIM_b(2)$. This deformation group leaves invariant the Finslerian line element,

$$ds = (\eta_{ij}dx^i dx^j)^{(1-b)/2}(n_k dx^k)^b, \tag{3}$$

proposed by Bogoslovsky (see [23], and references therein) for the study of local anisotropies. The entity n^k is a null spurionic vector field that determines the direction of the "etheral" motion's 4-velocity and can be selected as $n^i = (1, 0, 0, 1)$ [15,17,24]. The signature of the Minkowski metric is set to $\eta_{ij} = \text{diag}(+1, -1, -1, -1)$ throughout this paper. The line element (3) determines a particle's mass tensor

$$m_{ij} = (1 - b)m(\delta_{ij} + bn_i n_j), \tag{4}$$

which indicates the Machian nature of the theory [23].

The symmetry of the line element (3) indicates the Lagrangian of a free-moving particle

$$\mathcal{L} = m(\eta_{ij}\dot{x}^i \dot{x}^j)^{(1-b)/2}(n_k \dot{x}^k)^b. \tag{5}$$

The above Lagrangian implies the particle's action integral $I = \int \mathcal{L}(x, \dot{x})d\tau$ and the canonical momentum $p_k = \frac{\partial \mathcal{L}}{\partial \dot{x}^k}$ [25]. Using the first order degree homogeneity of the Lagrangian with respect to $\dot{x}$, we can construct the generalized mass-shell condition

$$f_{\mu\nu}(x, \dot{x})p^\mu p^\nu = m^2 \tag{6}$$

since the tangent and cotangent bundles define equivalent geometrical frameworks [6]. Condition (6) can be reformed to the following convenient expression:

$$\eta^{ij}p_ip_j = m^2(1-b^2)\left(\frac{n^ip_i}{m(1-b)}\right)^{2b/(b+1)}. \quad (7)$$

This relation reflects the colored nature of the space-time manifold. The parameter b is restricted by various experiments to the order of $|b| < 10^{-10}$ (*ether drift experiment*) and $|b| < 10^{-26}$ (*anisotropy of inertia*) [17,26].

The metric (2) of Finsler geometry does not describe completely the geometric properties of the underlying manifold as in the Riemannian case. Further information must be supplied concerning the (non)linear connection. We refer to two main branches of formalism among others, characterized by Cartan's and Chern's connections. Chern's approach introduced a connection which gave a complete system of local invariants ensuring that two Finsler structures differ by a change of coordinates (see, for example, [21]). Nevertheless, it is an almost metrical ansatz. On the other hand, Cartan's connection is purely metrical. Physical implications of the aforementioned perspectives suggest a nonconservation of energy and momentum apart from a subclass of Finsler spaces called Berwald spaces (see e.g. [13,27], and references therein). However, it is rather obscure whether departures from relativistic invariance anymore guarantee the energy-momentum conservation and/or metricity of space-time. In the following sections we use a method of great simplicity, initially developed for the purpose of comparing various covariant derivatives of Finsler geometry, *the process of osculation*. In this approach, a purely Riemannian metric is defined in a local subregion (see next section) and GR's machinery is valid for the induced Levi-Civita connection.

Before we proceed to the limit of osculation, we investigate the mass-shell condition (7) which may provide observational motivations as we expect modified dispersion relations and an influence on threshold energies. These phenomenological concepts drew some attention due to the possibility of constraining various QG models with current astrophysical data (time of flight differences between photons of different energies, TeV-γ, and ultra-high energy cosmic rays (UHECR) threshold anomalies) [5]. In order to demonstrate some physical implications of the modified mass-shell condition (7), we will roughly compute the threshold energy of $p + \gamma \rightarrow p + \pi$ for the Finslerian background (3). The threshold anomalies have also been studied for a Finsler-Randers space [28]. For the sake of simplicity, we will assume energy and momentum conservation

$$E_1 + \epsilon = E_2 + E_3, \qquad p_1 - q = p_2 + p_3, \quad (8)$$

where E_1 and p_1 refer to a high energy particle colliding with a photon (ϵ, q) and E_2, p_2, E_3, and p_3 represent the energy and momentum of the produced particles. The non-Lorentz invariant relation (7), for a further assumption of massless photons and small departures from Lorentz invariance, reads for a photon and the i particle,

$$q = \epsilon,$$
$$E_i^2 - p_i^2 = m_i^2\left[1 - 2\ln\left(\frac{E_i - p_i\beta_i}{m_i}\right)b + O(b^2)\right], \quad (9)$$

where $\beta_i = \cos\theta_i$ and θ_i denotes the angle between the particle's spatial momentum and the spatial part of the null spurion. The dependence of (9) on the parameter β_i reflects the local anisotropic structure of the Finslerian space-time. The same particle with different orientation possesses a different energy component. Working only on the lab frame, we can sidestep the definition of different local frames in our LV theory [5]. The limit of high energy particles leads to the following formula for the threshold energy:

$$E_1 \simeq \frac{(m_2 + m_3)^2 - m_1^2}{4\epsilon} - \frac{(m_2 + m_3)^2}{2\epsilon}$$
$$\times \ln\left[\frac{E_1(1 - \beta) + \epsilon(1 + \beta)}{m_2 + m_3}\right]b, \quad (10)$$

where β stands for the angle between the total 3-momentum of the produced particles and the spurion's spatial preferred direction. We remark that the LV effect to the threshold energy fades out parallel to the spatial direction ($\beta = 1$). The above particle's production is expected between soft photons of the cosmic microwave background (CMB) and UHECRs. In general, we do not expect to observe UHECRs above the well-known GZK cutoff $\sim 5 \times 10^{19}$ eV [29]. Particles with energy values above this cutoff should be absorbed by the CMB background. However, cosmic rays with energies up to $\sim 3 \times 10^{20}$ eV have been reported. A possible explanation is that the threshold of the photon-pion process is at larger energies. In that spirit we can constrain various parameters coming from QG scenarios. For our case the value of b is a constraint up to $b \sim -0.1/(27.7 + \ln(1 - \beta))$. This is not a tight constraint for our parameter, but it clearly depicts that locally a Finsler space is directional dependent. Nevertheless, the reader should keep in mind that energy-momentum conservation is not necessarily valid in a Finslerian background. Also, nontrivial physics is involved due to the angle dependence of the mass-shell condition. Last but not least, a procalike Lagrangian is invariant under $DISIM_b(2)$, and therefore massive photons should be embodied to the calculations.

III. THE PROCESS OF OSCULATION

Let U be a region of the space-time Finsler manifold F_4 parametrized by the local coordinates x^μ. In this neighborhood the velocity vector field u^μ can be picked up to be x^μ dependent, $\dot{x}^\mu = u^\mu(x)$. Thus this local region of F_n possesses a Riemannian metric

$$g_{\mu\nu}(x) = f_{\mu\nu}(x, u(x)) \quad (11)$$

and leads to the construction of the *osculating Riemannian space*. In other words, the process of osculation relates the

velocity field uniquely to the space-time points, implying the limit of relativistic cosmology [30,31]. The osculation of a Finslerian manifold determines a purely Riemannian space-time for a specified selection of $u^\mu(x)$. In case of a general fluid the comoving observers live on a different Riemann space from the one of a tilted observer. In such a scenario the link between different families of observers belongs to the context of Finsler geometry [32,33].

Consider an early time of the universe where Lorentz violations and chaotic motion govern the cosmic fluid. Finsler geometry is a candidate physical geometry of this state. As the universe evolves, Lorentz violations and chaotic motion are expected to fade out, enabling the introduction of the observer's 4-velocity with respect to space-time coordinates. Hence, the definition of an osculating Riemannian space is possible, using the observer's 4-velocity field. A post big-bang Riemannian cosmology rises up where seeds of the Finslerian era can survive. The induced Einstein field equations for the u^μ observer are

$$G_{\mu\nu}(x, u(x)) = 8\pi G T_{\mu\nu}(x, u(x)), \quad (12)$$

where $G_{\mu\nu}$ is the standard Einstein tensor coming from the Riemann metric (11). The energy-momentum tensor represents a general imperfect fluid which can be expressed into its irreducible parts (for a recent review see [34])

$$T_{\mu\nu} = \mu u_\mu u_\nu - P h_{\mu\nu} + 2q_{(\mu} u_{\nu)} + \pi_{\mu\nu}, \quad (13)$$

where $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ represents the projective tensor, $\mu = T_{\alpha\beta} u^\alpha u^\beta$ is the energy density, $P = -T_{\alpha\beta} h^{\alpha\beta}/3$ is the isotropic pressure of the fluid coming from the equilibrium pressure and bulk viscosity, $q_\mu = h_\mu^\rho T_{\rho\sigma} u^\sigma$ is the total energy flux vector, and $\pi_{\alpha\beta} = h_{<\alpha}{}^\gamma h_{\beta>}{}^\delta T_{\gamma\delta}$ is the symmetric, trace-free anisotropic stress tensor [35].

IV. OSCULATING GENERAL VERY SPECIAL RELATIVITY

We can construct our cosmological geometrical machinery by introducing the metric function $F(x, u)$ directly from the line element (3). All tensorial items of Finsler geometry obey the linear transformation law $B_\mu = \frac{\partial x^i}{\partial x^\mu} B_i$ relating different coordinate systems. The metric function $F(x, u)$ in any coordinate system is rewritten as

$$F(x, u) = \left(\eta_{ij} \frac{\partial x^i}{\partial x^\mu} \frac{\partial x^j}{\partial x^\nu} u^\mu u^\nu \right)^{b-1} \left(n_i \frac{\partial x^i}{\partial x^\rho} u^\rho \right)^b$$
$$= (a_{\mu\nu} u^\mu u^\nu)^{b-1} (n_\rho u^\rho)^b, \quad (14)$$

where $a_{\mu\nu}$ denotes a Riemannian metric and Greek indices represent an arbitrary coordinate system. In general, the parameter b can be considered x dependent ($b \equiv b(x)$) preserving the 1-homogeneity of (14) with respect to u. The groups that leave (14) invariant for different values of b are not isomorphic, leading to geometrical phase transitions [17,26].

The substitution of relation (14) into (2) implies the Finslerian metric tensor to the explicit form

$$f_{\mu\nu}(x, u) = (1-b)L^{-b}N^{2b} a_{\mu\nu} - 2b(1-b)L^{-b-1}N^{2b} u_\mu u_\nu$$
$$+ 2b(1-b)L^{-b}N^{2b-1} n_{(\mu} u_{\nu)}$$
$$+ b(2b-1)L^{1-b}N^{2b-2} n_\mu n_\nu, \quad (15)$$

where $L = a_{\mu\nu} u^\mu u^\nu$ and $N = n_\rho u^\rho$. We remark that (15) is a disformal relation between the two metric tensors $f_{\mu\nu}$ and $a_{\mu\nu}$. Note that for a Lorentz violating background, $f_{\mu\nu}$ plays the role of physical geometry while $a_{\mu\nu}$ represents the gravitational "potential" [19].

We construct a cosmological model by inserting a flat FRW metric and the observer's 4-velocity $u^\mu = (1, 0, 0, 0)$ into (15). If Lorentz violations "dilute" to thermal energy and entropy, this setup recovers the classical FRW limit. The process of osculation leads to a Riemannian space-time, where the observer's rest frame lies on a tilted non-geodesic congruence.

The cosmological model is investigated during a geometrical phase where b is a constant of first order disturbance upon the FRW metric. Therefore, the osculating Riemannian line element, by virtue of (14), is reduced to

$$ds^2 = dt^2 - a^2(t)[(1-b)(dx^2 + dy^2)$$
$$- (1 - b + b/a^4(t))dz^2] + 2b/a(t)dtdz, \quad (16)$$

where the null spurion has been transformed to $n^\mu = (1, 0, 0, 1/a(t))$. The form of the metric (16) indicates that the background geometry includes anisotropy, since the space-time expansion is of different rates at different directions. Because of the independence of the metric components to the spatial coordinates, all the invariant quantities are only functions of time. The off-diagonal terms of (16) imply that the $u^\mu = (1, 0, 0, 0)$ observer will measure an energy flux component. This effective peculiar motion with respect to the FRW limit is owed to a 4-acceleration vector $A^\mu = u^\nu \nabla_\nu u^\mu$ of Lorentz violating origin.

Since (16) is of a Riemannian nature, we use the standard formulas for the connection and curvature. The Ricci tensor is directly calculated as

$$R_{00} = -3\frac{\ddot{a}}{a} + 2a^{-5}\ddot{a}b - 6a^{-6}\dot{a}^2 b,$$

$$R_{11} = R_{22} = a\ddot{a}(1-b) + 2\dot{a}^2(1-b)$$
$$- (2a^{-4}\dot{a}^2 + 2\dot{a}^2 + a\ddot{a})b,$$

$$R_{33} = a\ddot{a}(1-b) + 2\dot{a}^2(1-b) + (4a^{-4}\dot{a}^2 - a^{-3}\ddot{a})b,$$

$$R_{03} = -\frac{b}{a^3}(a\ddot{a} + 2\dot{a}^2). \quad (17)$$

The Einstein tensor $G_{\mu\nu}$, combined to the field Eqs. (12), recasts the anisotropic irreducible parts of the energy-momentum tensor (13) to the following form:

$$q_\mu \propto h_\mu{}^\alpha R_{\alpha\beta} u^\beta, \qquad \pi_{\mu\nu} \propto h_{<\mu}{}^\alpha h_{\nu>}{}^\beta R_{\alpha\beta}. \quad (18)$$

The flux and the anisotropic pressure (18), expressed in terms of purely geometrical quantities, reflect the general relativistic interpretation of gravity where space-time curvature determines the motion of matter.

V. MODIFIED FRIEDMANN EQUATIONS

The osculating Riemannian approach defines a cosmological *toy model* for a tilted observer, enriching the picture of the ''standard'' cosmology. The presence of flux, anisotropic pressure, and peculiar motion reflects the assumed Finslerian background. This process relates the generated Lorentz violations to large scale structure dynamics. Using the expressions (18), we introduce a relativistic total fluid (13) in alliance with the anisotropic metric (16)

$$q_\mu = (0, 0, 0, Q(t)),$$
$$\pi_\mu{}^\nu = \mathrm{diag}(0, \Pi(t), \Pi(t), -2\Pi(t)), \quad (19)$$

where $Q(t)$ and $\Pi(t)$ are considered unknown functions. For weak deviations from the FRW cosmology $Q(t)$, $\Pi(t)$ can be of first order [36].

The standard calculation of the Einstein field equations (12) for a general fluid leads to the following equations of motion for the scale factor

$$\frac{\dot{a}^2}{a^2} - \frac{4}{3} a^{-6} \dot{a}^2 b = \frac{8\pi G}{3} \mu, \quad (20)$$

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + 2(2a^{-6}\dot{a}^2 - a^{-5}\ddot{a})b = -8\pi G[P - \Pi - \Pi b], \quad (21)$$

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + (2a^{-5}\ddot{a} + a^{-6}\dot{a}^2)b$$
$$= -8\pi G[P + ba^{-4}P + 2\Pi + 2\Pi b], \quad (22)$$

$$\left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a}\right)b = 8\pi G(Qa + \mu b), \quad (23)$$

where b is considered constant and small. We have also made the approximations bQ and $b\dot{Q} \approx 0$ since q_μ is of first order. This system of differential equations will be investigated with the aid of the linear dependent continuity equations.

The relation (20) is a modified Friedmann equation with an extra term $\Lambda(t)/3 = 4a^{-6}\dot{a}^2b/3$, analogous to b. The sign of b will determine the effect of this extra geometrodynamical quantity, as we will briefly demonstrate in Sec. VIII. The term $\Lambda(t)/3$ acts as an ''effective'' time-dependent cosmological constant, which tends fast to zero for an expanding universe. However, at earlier times this Lorentz violating ''contribution'' can crucially affect the dynamics. In case b varies at different geometrical phases, someone must include derivatives of b to describe this general structure.

VI. THE CONTINUITY EQUATION

The definition of the u^μ frame validates the conservation law $\nabla_\nu T^{\mu\nu} = 0$, where the covariant derivative comes from the osculating metric $g_{\mu\nu}(x)$ (11). The nonzero components of the energy-momentum tensor's divergence lead to a set of two differential equations; the timelike part provides the energy density formula

$$\dot\mu + 3\frac{\dot{a}}{a}(\mu + P)\left(1 - \frac{2}{3}a^{-4}b\right) = 0 \quad (24)$$

and the spacelike part the momentum density conservation

$$\dot{Q} + 3Q\frac{\dot{a}}{a} = -2(P + \mu)a^{-2}\dot{a}b - (\dot{P} + \dot{\mu})a^{-1}b, \quad (25)$$

where all the quadratic terms of b and Q have been dropped out. The ordinary differential equation (24) gives back the solution

$$\mu(t) = \mu_0 a(t)^{-3(1+w)} \exp\left(-\frac{1+w}{2}a(t)^{-4}b\right), \quad (26)$$

where we have applied the equation of state $w = P/\mu$ and μ_0 is an integration constant. After plugging (26) into (25), a direct calculation determines Q

$$Q(t) = -\mu_0(w + 1)\exp\left[-\tfrac{1}{2}b(1 + w)a(t)^{-4}\right]$$
$$\times a(t)^{-3w-4}b + Q_0 a(t)^{-3}. \quad (27)$$

In the special case $w = -1$, the energy density evolves as in the FRW model while the energy flux decays as the standard dust limit.

VII. SOLUTIONS FOR THE SCALE FACTOR AND THE ANISOTROPIC PRESSURE

We can recast the system of the ordinary differential equations (20)–(23) to a more convenient form, using the linear dependent continuity equations (24) and (25). The solution for the scale factor $a(t)$ can be directly provided by the modified Friedmann equation (20). After substituting the energy density (26) and (20) we derive

$$\frac{\dot{a}^2}{a^2} - \frac{4}{3}a^{-6}\dot{a}^2b = \frac{8\pi G}{3}\mu_0 a^{-3(w+1)}$$
$$\times \left(1 - \frac{w+1}{2}a^{-4}b\right) + O(b^2), \quad (28)$$

where all the quadratic terms of b have been omitted since we are interested in a first order approximation of the unknown parameter. We present the analytical solutions of (28).

Solutions for $a(t)$, $w \neq -1$

The integration of (28) for $w \neq -1$ implies the solution

$$t \propto \frac{4}{w+1}a^{3(w+1)/2} + a^{(3w-5)/2}b. \tag{29}$$

The second term of (29) reflects the contribution of the underline Finslerian theory to the expansion dynamics. Given an expanding phase, the first term of (29) increases faster compared to the second one, dominating the solution. When b is set to zero we recover the flat FRW solution.

Solutions for $a(t)$, $w = -1$

In the $w = -1$ case the Friedmann equation is simplified to the form

$$3\frac{\dot{a}^2}{a^2} - 4a^{-6}\dot{a}^2 b = 8\pi G\mu_0. \tag{30}$$

The differential equation (30) is integrated to the logarithmic solution

$$t \propto 6\ln(a) + a^{-4}b, \tag{31}$$

indicating a rapid expansion.

The anisotropic pressure $\Pi(t)$

After calculating the scale factor $a(t)$ we can subtract (21) and (22) and derive the following expression for the anisotropic pressure $\Pi(t)$:

$$\Pi(t) = -\tfrac{1}{3}\mu_0 w a(t)^{-3(w+1)}b + (8\pi G)^{-1}$$
$$\times \left(a^{-6}\dot{a}^2 - \tfrac{4}{3}a^{-5}\ddot{a}\right)b + O(b^2). \tag{32}$$

We illustrate the evolution of anisotropic pressure (32) for indicative values of w at first order approach

$$\Pi(t) \propto \begin{cases} \frac{b}{t^{14/3}}, & w = 0, & \text{matter} \\ -\frac{\mu_0}{t^2}b + \frac{21}{4t^4}b, & w = 1/3, & \text{radiation} \\ -e^{-(2/3)t}b, & w = -1, & \text{dark energy} \end{cases}. \tag{33}$$

The anisotropic pressure fades out as t grows up. Nevertheless, $\Pi(t)$ tends to infinity for ordinary matter at early times, while at the exotic case ($w = -1$) behaves exponentially.

VIII. MODIFIED POTENTIAL FOR A ZERO ENERGY PARTICLE

The properties of the model can be further investigated using the zero energy particle approach. The Friedman equation of motion (20) can be written in a form that represents the conservation of a particle's kinetic and potential energy. Substituting the energy density solution (26) in (20) we retrieve

$$V(a) \propto -\frac{a^{-3w-1}}{1 - \tfrac{4}{3}a^{-4}b}\exp\left(-\frac{1+w}{2}a^{-4}b\right)$$
$$= -a^{-1-3w} + \frac{3w-5}{6}a^{-5-3w}b + O(b^2). \tag{34}$$

The first term of (34) refers to the classical FRW potential, while the second one is due to macroscopic consequences of the assumed local anisotropic structure. We restrict our investigation for $w < 5/3$ which ensures that the behavior of the potential depends solely on the sign of b. Consider an ingoing point particle falling into the potential (34), which represents a collapsing universe. As we approach $a \to 0$, the kinetic energy of the particle diverges significantly from the FRW limit, since the a^{-6} term dominates the dynamics; away from the initial singularity the LV effect fades out.

Positive values of b guarantee that the point particle hits the initial singularity faster than the FRW case, since the potential decays at a more rapid rate (see Fig. 1). On the other hand, negative values of b assure an extremal point

$$a_* = 6^{-1/4}\sqrt{\frac{(5-3w)(5+3w)}{1+3w}}(-b)^{1/4}, \tag{35}$$

where the accelerating contraction turns to a decelerating phase for $-1/3 < w < 5/3$ (see Fig. 2). As the particle falls farther down, it is finally bounced by the potential at the point $V(a_{bc}) = 0$,

$$a_{bc} = \left(\frac{5-3w}{6}\right)^{1/4}(-b)^{1/4}. \tag{36}$$

When the particle reaches this turning point, the decelerating contraction is reverted to an accelerating expansion until it crosses the extremal a_* again. The following decelerating phase recovers gradually the FRW model with $\Lambda = 0$.

Using the solution (29) we can roughly estimate the duration of the accelerating phase between the bouncing and the extremal point
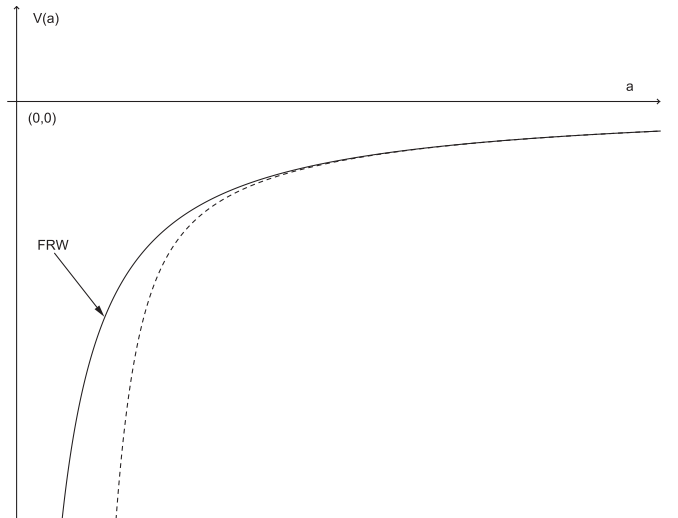

FIG. 1. Free scale potential for the analogous zero energy one-dimensional dynamics with $b > 0$, $w > -1/3$; the continuous line is the FRW potential ($b = 0$). Note that the model's potential approaches $-\infty$ faster than the classical one.
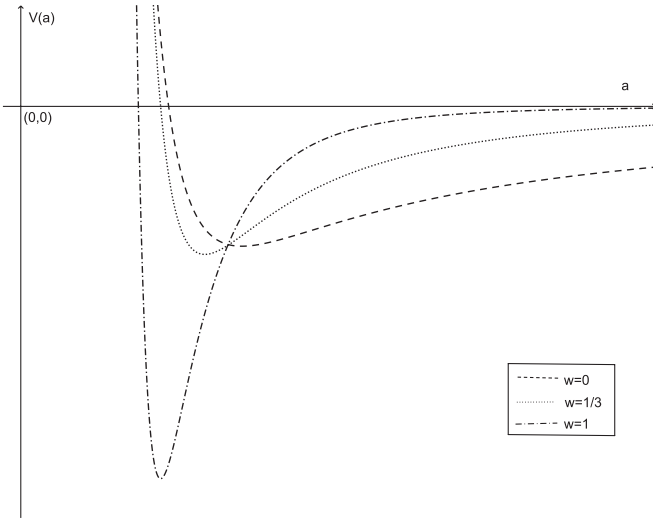
FIG. 2. Free scale potential for $b < 0$. The zero energy particle approaching from $a \to \infty$ accelerates until it crosses the minimum of the potential $V(a_*)$; then it decelerates until it hits the potential at the bouncing point. The model recovers FRW behavior for large values of a.

$$\Delta t \sim \left(\frac{8\pi G}{3} \mu_0 \right)^{-1/2} (-b)^{3(w+1)/8}, \qquad (37)$$

where μ_0 is the energy density for $t = t_0$. If μ_0 corresponds to the present matter density distribution, close to the critical density $\mu_{\mathrm{cr}} \approx 10^{-29}$ gr cm^{-3}, then approximately $\Delta t \approx 10^{14}$ sec for the *ether drift experiment* and $\Delta t \approx 10^8$ sec for the *anisotropy of inertia*. Thus, according to estimations of the age of old stars $\sim 3 \times 10^{17}$ [37] our model, starting from a_{bc}, has entered a decelerating phase. This conclusion is not in alliance with the observed cosmic acceleration from the Type 1a supernova data. Exotic matter with $w < -1/3$ must be assumed for reproducing the desired expanding behavior. An alternative way to introduce late time acceleration in the present epoch is the adoption of the present energy density distribution to lower values. In particular, the values, $\mu_0 \sim 10^{-32}$ gr cm^{-3} for *ether drift experiment* and $\mu_0 \sim 10^{-42}$ gr cm^{-3} for the *anisotropy of inertia*, secure that the model accelerates until today. In case of clusters that are large enough to be representatives of the overall mass density, only the limit of the *ether drift experiment* is close, yet less than the up to date density measurements [38].

IX. INVESTIGATION OF KINEMATICAL QUANTITIES

The 4-velocity u^μ of the observer's rest frame introduces a "slit" between space and time. The vector u^μ determines the projective tensor $h_{\mu\nu}$, which acts as the three-dimensional metric of the observer's instantaneous rest space for a hypersurface orthogonal congruence.

Let a congruence $x^\mu(\tau)$ consist of nongeodesics, where the tangent vector field is the 4-velocity $u^\mu = (1, 0, 0, 0)$ and the line element (16) implies the shear vorticity and expansion tensors. The tensor field

$$\nabla_\beta u_\alpha = \tfrac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} + A_\alpha u_\beta \qquad (38)$$

measures the failure of deviation vector ξ^α to be parallel transported along the congruence. The expansion scalar θ is defined as $\theta = \nabla^\alpha u_\alpha$ and measures the expansion of a volume element dV along u^μ, the shear tensor $\sigma_{\alpha\beta} = \nabla_{\langle\beta} u_{\alpha\rangle} = h_{(\alpha}{}^\rho h_{\beta)}{}^\sigma \nabla_\rho u_\sigma - \tfrac{1}{3}\theta h_{\alpha\beta}$ represents deformations in the shape of dV, while the vorticity tensor $\omega_{\alpha\beta} = \nabla_{[\beta} u_{\alpha]}$ expresses orientation changes of dV. After expanding for small values of b we can derive the following approximation series:

$$\theta = 3\frac{\dot{a}}{a} - 2\dot{a}a^{-5}b + O(b^2) \qquad (39)$$

and

$$\sigma_{\alpha\beta} = \mathrm{diag}\left[0, -\tfrac{2}{3}a^{-3}\dot{a}, -\tfrac{2}{3}a^{-3}\dot{a}, \tfrac{4}{3}a^{-3}\dot{a} \right]b + O(b^2) \quad (40)$$

while

$$\omega_{\mu\nu} = 0. \qquad (41)$$

Thus the hypersurface orthogonal condition holds for the congruence. Additionally, the nonvanishing sheer tensor reflects the kinematical anisotropies of the fluid. An interesting point is the presence of the acceleration vector $A^\mu = u^\nu(\nabla_\nu u^\mu)$ with the nonzero component

$$A^\mu = (0, 0, 0, a^{-4}\dot{a})b + O(b^2). \qquad (42)$$

The presence of acceleration represents nongravitational phenomena. Therefore, the fundamental observer is not moving along geodesics since Lorentz violations contribute substantially to the gravitational theory. This is a direct result from the distortion of the line element (3), coming from the b parameter.

X. RAYCHAUDHURI EQUATION AND FOCUSING THEOREM

The fluid's volume evolution is retrieved by the timelike part of the Ricci identities which lead to the well-known Raychaudhuri equation. In case of a hypersurface orthogonal geodesic congruence and ordinary matter the Raychaudhuri equation implies the formation of a singularity due to the attractive nature of gravity. However, a nongeodesic congruence may avoid the caustic since the external forces may resist the collapse. Given a nongeodesic congruence, the Raychaudhuri equation reads

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - 2(\sigma^2 - \omega^2)$$
$$- R_{\mu\nu}u^\mu u^\nu + D^\mu A_\mu - A_\mu A^\mu, \qquad (43)$$

where $D_\mu A_\nu = h_\mu{}^\alpha h_\nu{}^\beta \nabla_\alpha A_\beta$ denotes the spatial gradient of a vector.

The line element (16) validates the hypersurface orthogonality of the u^μ congruence, $\omega_{\mu\nu} = 0$. Thus, the only quantities that may offer a positive contribution in the right-hand side of (43) are related to 4-acceleration A^μ. In particular, the square magnitude of the 4-acceleration always resists the collapse (assists the expansion) since it is a spacelike vector, while $D^\mu A_\mu$ depends on the state of the expansion. The quantity $D^\mu A_\mu$ always keeps a positive sign for a universe at a decelerating phase. This positive sign is still ensured for an accelerating phase if $\ddot{a} < 4\dot{a}a^2$. The effect of Lorentz violations wins over the attractive nature of gravity if the following condition holds:

$$D^\mu A_\mu - A_\mu A^\mu > R_{\mu\nu}u^\mu u^\nu + 2\sigma^2. \qquad (44)$$

A first order approximation of the additional terms in (43) imply $D^\mu A_\mu - A_\mu A^\mu = O(b^2)$. However, at an earlier stage of the expansion, where the Lorentz violations are stronger, the accelerating terms may dictate over the negative ones preventing the collapse of the fluid to a geometrical singularity. On the other hand, if the values of b^2 are comparable to the cosmological constant, the $D^\mu A_\mu - A_\mu A^\mu$ term may give rise to the dark energy scenario of an almost empty self-accelerating universe.

XI. LORENTZ VIOLATIONS AS A SOURCE OF INHOMOGENEITIES

We consider a space-time where the universe is regarded as a single imperfect fluid, a direct result of the process of osculation. Spatial inhomogeneities in the cosmic fluid are detected by the following dimensionless, gauge-invariant quantities [39,40]:

$$\Delta_\alpha = \frac{a(t)}{\mu} D_\alpha \mu, \qquad (45)$$

$$\mathcal{Z}_\alpha = a(t)D_\alpha \theta, \qquad (46)$$

which both vanish in spatially homogeneous space-times. The tensors (45) and (46) do not vanish even in case they are zero for a specific value of the time coordinate. They are considered as some of the key sources of the density perturbations [34]. Despite the sole dependence on time of μ and θ their spatial gradient does not vanish since $h_a{}^0 \neq 0$. The evolution of (45) and (46) at first order is described by the differential equations [34,41]

$$\dot{\Delta}_{\langle \kappa \rangle} - \frac{P}{\mu}\theta\Delta_\kappa + \left(1 + \frac{P}{\mu}\right)Z_\kappa + \sigma^\nu{}_\kappa \Delta_\nu$$
$$= \frac{a\theta}{\mu}\left(\dot{q}_{\langle \kappa \rangle} + \frac{4}{3}\theta q_\kappa + \sigma_{\kappa\lambda}q^\lambda\right) \qquad (47)$$

and

$$\dot{Z}_{\langle \kappa \rangle} + \tfrac{2}{3}\theta Z_\kappa + +\tfrac{1}{2}\mu\Delta_\kappa + \tfrac{3}{2}aD_\kappa P$$
$$= -a[\tfrac{1}{3}\theta^2 + \tfrac{1}{2}(\mu - 3P)]A_\kappa, \qquad (48)$$

where flux and acceleration act as sources of perturbations. Hence, this mechanism indicates that Lorentz violations may generate inhomogeneities. A direct calculation gives back the nonvanishing components

$$\Delta_3 = 3b(1 + w)\frac{\dot{a}}{a} + O(b^2),$$
$$Z_3 = -3b\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + O(b^2), \qquad (49)$$

where $a(t)$ is the FRW solution since the leading terms of (49) are proportional to b. Note that, in case of $w = -1$, the density inhomogeneities are zero.

XII. DISCUSSION

The osculation of a Finslerian manifold generates a Riemannian cosmological *toy model* for a specific 4-velocity u^μ. The rest frame of the fundamental observer lies on a tilted nongeodesic congruence. The peculiar velocity u^μ defines the energy-momentum tensor of an imperfect fluid, given an almost FRW metric of zero spatial curvature. This construction holds for $u^\mu = (1, 0, 0, 0)$ to retrieve the standard FRW limit of the comoving observer, in case of vanishing Lorentz violations. The resulting line element of space-time describes an anisotropic expanding (contracting) medium. Therefore, an imperfect fluid with flux and anisotropic pressure is ''injected '' into space to support this anisotropy.

The field equations lead to a modified Friedmann equation of motion with an effective varying cosmological constant proportional to b. The spatial curvature is considered $k = 0$ in accordance with the astrophysical observations. The Einstein field equations, combined with the conservation laws for the flux and energy density, provide analytical solutions for the scale factor, anisotropic pressure, energy density, and flux. However, the expressions for the scale factor are given in the form $t \equiv t(a)$. The model's differential equations add a first order exponential factor to the FRW's standard energy density while the flux and anisotropic pressure tend to zero for an increasing $a(t)$. We remark that the flux vector does not directly vanish even if the parameter b is set to zero.

Furthermore, we investigate the model's behavior using the zero energy particle with a modified potential, in relation to the FRW case. The parameter b plays an influential role, since it determines substantial properties of the dynamics. In particular, for $b > 0$ a contracting universe will hit the initial singularity faster than the FRW. On the other hand, if $b < 0$ a richer scenario occurs; an accelerating contraction leads to the potential's local minimum, turning to a decelerating contraction until it bounces back at some turning point. After the particle's reflection to the potential, an accelerating phase takes place until we cross the extremal point. Finally, a decelerating expansion governs the evolution due to the attractive nature of gravity, recovering the FRW limit with $\Lambda = 0$. In order to

reproduce late time acceleration two possibilities rise up: assume exotic matter with $w < -1/3$, or a "lighter" universe with $\mu_0 < \mu_{crit}$ today. Nevertheless, the time-dependent effect to the kinematics, coming from the LV parameter, gives birth to a direct discrimination to the Λ cold dark matter structure since the extra term can be addressed as an effective varying cosmological constant.

One key geometrical quantity to "translate" observational data is the luminosity distance $d_L(a)$. The form of this function depends on the physical hypotheses taken into account. In that sense, the interpretation of the observations is model dependent. Violating the symmetries of space-time in large distances, $d_L(a)$ might be crucially affected. A first question is how sensitive this function can be in terms of any extra parameters introduced by the model. If our model is not sensitive to the parameter b, one could accurately determine whether or not the effective varying cosmological constant plays a role in the expansion dynamics. Even in such a case, observations suggest that a constant Λ term is completely consistent with the current data. On the other hand, because of many theoretical motivations, there has been an extensive amount of work investigating whether improved observational scenarios could measure a time-dependent Λ behavior. These surveys are mainly directed by baryon acoustic oscillations, SNIa, Lensing, and CMB data [42].

In GR ordinary matter always falls along geodesics at the presence of gravity alone. The study of gravitational collapse leads to the formulation of singularity theorems. However, other phenomena in nature impose nongeodesic motion (for a recent review see [34]). In this arena the question of a caustic singularity must be revisited. A characteristic example is the collapse of an ideal MHD fluid where magnetic tension may prevent the formation of a caustic [43]. The present phenomenological model points out that the osculation of a Finsler space to a Riemannian one leads to nongeodesic motion. The observer's 4-acceleration vector mimics Lorentz violations for our effective theory. Therefore, Raychaudhuri's equation implies that the formation of a singularity depends on the magnitude of the parameter b which gives birth to Lorentz violations.

The presence of flux at the energy-momentum tensor is a direct consequence of the reduced noncomoving motion, since $g_{0i} \neq 0$. The existence of flux combined with the peculiar motion acts as a source of inhomogeneities. As a result, relics of Lorentz violations are encoded to density perturbations. In the context of Finsler geometry, this mechanism demonstrates a possible correlation of Lorentz violations to CMB physics, pointing out some possible future developments in the field.

Further research would include the construction of a spatially curved modified FRW model and the backreaction of Lorentz violations to spatial curvature. Also, the concept of geometrical phase transitions via Finslerian geometrical structures is of some interest. In the framework of GVSR we can express these phase transitions by setting b dependent on the space-time coordinates. The study of the b evolution may shed light to the question, "why is b so small?" Finally, a vital task still remains: how a consistent modified gravitational theory can be achieved using the whole machinery of Finsler geometry [44].

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