

Neutron stars in $f(R)$ gravity with perturbative constraints

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We study the structure of neutron stars in $f(R)$ gravity theories with perturbative constraints. We derive the modified Tolman-Oppenheimer-Volkov equations and solve them for a polytropic equation of state. We investigate the resulting modifications to the masses and radii of neutron stars and show that observations of surface phenomena alone cannot break the degeneracy between altering the theory of gravity versus choosing a different equation of state of neutron-star matter. On the other hand, observations of neutron-star cooling, which depends on the density of matter at the stellar interior, can place significant constraints on the parameters of the theory.

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I. INTRODUCTION

Recent interest in modified theories of gravity has been spurred by the discovery that the Universe is undergoing accelerated expansion (see, e.g., [1–3]). The simplest solution consistent with these observations posits a cosmological constant Λ . The magnitude of this cosmological constant is significantly less than what was expected, and many undertakings have been made to see if there are plausible alternative explanations [4,5]. Outstanding questions also present themselves in the formation of singularities [6] and the seeming contradiction between quantum mechanics and gravity in the context of black hole thermodynamics [7]. All these suggest that there may yet be much to understand about the nature of gravity at extreme-curvature scales, far removed from our everyday experience.

The two most popular approaches to modifying gravity have been the introduction of an additional scalar field (e.g. [8]), or the related approach of replacing the Einstein-Hilbert action with a general function of the Ricci scalar $f(R)$ (e.g. [4]). Within either framework the additional scalar degree of freedom can be tuned to mimic the cosmological constant, or any type of cosmological evolution at cosmological scales [9].

Despite the premise of such modifications, the nonlinear character of gravitational theories has proven a significant obstacle to introducing new dynamical fields to drive modifications to gravity at the cosmological scale without the same fields reemerging at widely different curvature scales. One such example is the problem of ensuring that $f(R) = R \pm \mu^4/R$ theories pass the current parametrized post-Newtonian (PPN) bounds. When the new field is dynamical, the PPN parameter γ is forced to a value of $1/2$, which is very far from the present experimental

bound [10]. As a result one has to choose a function $f(R)$ only from the class which can adequately suppress the new dynamical field on solar-system scales. The chameleon mechanism [11–13] provides such an alternative.

In addition to the PPN constraints, instabilities related to the functional form of $f(R)$ have also been studied at length. This is especially true for the Dolgov-Kawasaki instability [14], which requires that $\partial^2 f / \partial R^2 > 0$ in order that the effective mass of the equivalent scalar degree of freedom be positive. In the strong-field regime, recent results [15] suggest that this very choice may well prohibit the formation of compact objects above a curvature scale readily observed. However, the fatal curvature singularity may be avoided by the chameleon mechanism [16,17].

Perhaps the source of the instabilities and consistency issues many of these models encounter is the result of treating these modifications as though they are exact. The original motivation behind introducing additional functions of the curvature was to generate a new phenomenology at a specific scale. However, many of the problems encountered by $f(R)$ gravity theories originate at curvature scales far removed from the ones under consideration. An alternative formulation for handling corrections to general relativity (GR) is to view the new terms as only the next to leading order terms in a larger expansion. In this context there is no reason to suspect that the new phenomenology is due to new dynamical fields. The technique for handling a field expansion of this form is well developed [18] and is known as perturbative constraints or order reduction [19].

Gravity with perturbative constraints allows us to explore alternative phenomenologies of gravity while maintaining important consistency conditions including gauge invariance, the assumption that we are approximating a fundamentally second order field theory, and the

conservation of stress energy. Maintaining such constraints while enlarging the space of possible behaviors of gravitation is the goal also of the parametrized post-Friedman approach [20–22].

In previous works [23,24], we have analyzed the effect of treating $f(R)$ models of gravity via perturbative constraints primarily at cosmological scales. In this paper, we examine the ramifications of modifications to gravity in the context of compact objects. We show how the method of perturbative constraints allows for a consistent phenomenology for gravity on both large (Hubble-length perturbations linear in metric variables, but strongly relativistic, $L \sim c/H_0$) and small scales (stellar scales, non-linear in metric perturbations, and strongly relativistic, $GM/rc^2 \sim 1$).

The layout of this work is as follows. In Sec. II, we review the equations of $f(R)$ gravity treated with perturbative constraints. In Sec. III, we derive the modified Tolman–Oppenheimer–Volkov equations and show that the exterior solution is the Schwarzschild–de Sitter metric. In Sec. IV, we demonstrate that such objects are stable and we solve numerically for their mass-radius relation for a polytropic equations of state. Finally in Sec. V we discuss how we can discriminate modifications to gravity from uncertainty in the neutron-star equation of state.

II. PERTURBATIVE CONSTRAINTS

Gravity with perturbative constraints [18] (or order reduction [19]) is a technique for treating equations of motion that appear higher than second order, where the origin of the higher derivatives can be traced to the truncation of an infinite series expansion. Such a situation can arise with nonlocal theories as well as effective field theories.

In the context of $f(R)$ gravity theories, we parametrize the deviation from general relativity by a single parameter α and derive the equation of motion from a covariant action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R - 2\Lambda + \alpha f(R) + \mathcal{O}(\alpha^2)] + S_M(g_{\mu\nu}, \psi), \quad (1)$$

with $G = c = 1$. Here $g_{\mu\nu}$ is the metric, g its determinant, and R the Ricci scalar. We denote any additional terms above order α by $\mathcal{O}(\alpha^2)$. We may not impose any constraints at the level of the action without altering the nature of the variational principle. The resulting field equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda + \alpha[f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\square)f_R] + \mathcal{O}(\alpha^2) = 8\pi T_{\mu\nu}, \quad (2)$$

where $f_R \equiv \partial f / \partial R$.

At zeroth order in α , these equations are second order in the metric; we denote the solution at this order by $g_{\mu\nu}^{(0)}$. We then solve the system for the higher order terms by writing

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)} + \mathcal{O}(\alpha^2). \quad (3)$$

The perturbative consistency of this approach is guaranteed to order n provided $\alpha^{n+1} g_{\mu\nu}^{(n+1)} \ll g_{\mu\nu}^{(0)} + \dots + \alpha^n g_{\mu\nu}^{(n)}$, as we outlined in a previous paper [24]. Note that this condition is not to be understood as requiring the product $\alpha f(R)$ to be necessarily smaller in magnitude than R .

For the purposes of this work it will prove useful to rewrite Eq. (2) using its trace

$$R - \alpha[f_R R - 2f + 3\square f_R] + \mathcal{O}(\alpha^2) = -8\pi T + 4\Lambda. \quad (4)$$

Substituting the Ricci scalar R from the above equation into Eq. (2) gives

$$R_{\mu\nu} - g_{\mu\nu}\Lambda + \alpha[f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(f_R R - f) - (\nabla_\mu \nabla_\nu + \frac{1}{2}g_{\mu\nu}\square)f_R] + \mathcal{O}(\alpha^2) = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T). \quad (5)$$

This is the form of the field equation we will be using. Henceforth we shall understand the equality sign to mean equality up to order α and drop the explicit use of $\mathcal{O}(\alpha^2)$.

III. STARS WITH PERTURBATIVE CONSTRAINTS

The metric of a static, spherically symmetric object can always be written in the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (6)$$

where $B(r) = B^{(0)}(r) + \alpha B^{(1)}(r) + \dots$, $A(r) = A^{(0)}(r) + \alpha A^{(1)}(r) + \dots$, and $B^{(0)}(r)$ and $A^{(0)}(r)$ are the general relativistic metric elements.

For the purpose of this paper we presume the form $f(R) \propto R^{n+1}$ for an integer $n \neq 0, -1$. We shall also assume that the energy-momentum tensor within the star is that of a perfect fluid. Following our previous studies [24] we find it convenient to express the $\mathcal{O}(\alpha)$ correction in terms of the derivative f_R . The first three field equations are

$$\frac{R_{00}}{B} + \alpha f_R \left\{ \frac{R_{00}}{B} + \frac{R}{2} \left(\frac{n}{n+1} \right) - \frac{n}{2A} \left[-\frac{R''}{R} - n \frac{R'}{R^2} + \frac{R^2}{R^2} + \frac{R'}{R} \left(\frac{A'}{2A} - \frac{3B'}{2B} - \frac{2}{r} \right) \right] \right\} = 4\pi(\rho + 3P) - \Lambda, \quad (7)$$

$$\frac{R_{11}}{A} + \alpha f_R \left\{ \frac{R_{11}}{A} - \frac{R}{2} \left(\frac{n}{n+1} \right) - \frac{n}{2A} \left[\frac{3R''}{R} + 3n \frac{R'}{R^2} - 3 \frac{R^2}{R^2} + \frac{R'}{R} \left(\frac{B'}{2B} - \frac{3A'}{2A} + \frac{2}{r} \right) \right] \right\} = 4\pi(\rho - P) + \Lambda, \quad (8)$$

and

$$\frac{R_{22}}{r^2} + \alpha f_R \left\{ \frac{R_{22}}{r^2} - \frac{R}{2} \left(\frac{n}{n+1} \right) - \frac{n}{2A} \left[\frac{R''}{R} + n \frac{R'^2}{R^2} - \frac{R'^2}{R^2} + \frac{R'}{R} \left(\frac{B'}{2B} - \frac{A'}{2A} + \frac{4}{r} \right) \right] \right\} = 4\pi(\rho - P) + \Lambda, \quad (9)$$

where the prime denotes differentiation with respect to r . The fourth field equation is identical to Eq. (9) because of the symmetry of the spacetime. Terms with a factor f_R preceding them are already first order in the small parameter α so all such terms should be evaluated at order $\mathcal{O}(\alpha^0)$, where for example

$$R^{(0)} = 8\pi(\rho - 3P) + 4\Lambda \quad (10)$$

and

$$M^{(0)} = 4\pi \int \rho r^2 dr. \quad (11)$$

In order to motivate the form of the metric element $A(r)$ that we will be using, we first examine the solution exterior to the star.

A. The exterior metric

To solve for the exterior solution to Eq. (5), we require that outside the star $T_{\mu\nu} = 0$. Therefore, at $\mathcal{O}(\alpha^0)$, the exterior metric satisfies

$$R_{\mu\nu}^{(0)} = \Lambda g_{\mu\nu}, \quad (12)$$

where $R_{\mu\nu}^{(0)}$ is the Ricci tensor derived from the metric to $\mathcal{O}(\alpha^0)$. Consequently the Ricci scalar at $\mathcal{O}(\alpha^0)$ is $R^{(0)} = 4\Lambda$.

Note from Eqs. (7)–(9) that the $\mathcal{O}(\alpha)$ correction is multiplied by a term $f_R \propto [(n+1)R^{(0)}]^n$. For $n \geq 1$ such a theory will allow a solution with a Minkowski exterior as well as solutions with $\Lambda \neq 0$, while for $n \leq -2$ the appearance of $R^{(0)}$ in the denominator requires that only solutions with $\Lambda \neq 0$ exist.

In order to calculate the corrections to the vacuum solution at successively increasing orders in α , we first investigate the perturbative term in the field Eq. (5), when the Ricci curvature is constant. At $\mathcal{O}(\alpha)$ the correction term is proportional to

$$f_R^{(0)} R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu} (f_R^{(0)} R^{(0)} - f^{(0)}) \propto (n-1) R_{\mu\nu}^{(0)}, \quad (13)$$

where we evaluated everything explicitly in terms of $R^{(0)}$ and $R_{\mu\nu}^{(0)}$. This last relation shows that, in $f(R)$ theories with $n = 1$, the correction term in the field equation vanishes and hence the exterior solution is identical to GR [25].

We can proceed in the same manner to arbitrary orders in $\mathcal{O}(\alpha^m)$. The result can be formally written as

$$R_{\mu\nu}^{(m)} = g_{\mu\nu}^{(m)} \mathcal{F}(\alpha, \alpha^2, \dots, \alpha^m) \Lambda, \quad (14)$$

where the precise form of the function \mathcal{F} is determined by the choice of the function $f(R)$.

The vacuum equations, therefore, choose a unique solution, the Schwarzschild–de Sitter metric, with

$$A(r) = \left(1 - \frac{2M}{r} - \frac{\bar{\Lambda} r^2}{3} \right)^{-1} \quad (15)$$

and $A(r)B(r) = 1$ [26]. The only difference from the general relativistic exterior metric will be in the value of the effective cosmological constant, which in the case of $f(R)$ gravity is

$$\bar{\Lambda} = \mathcal{F}(\alpha, \alpha^2, \dots, \alpha^m) \Lambda. \quad (16)$$

As a result, the PPN parameters [27] for an arbitrary choice of $f(R)$ will be practically those of general relativity (see also discussion in [25]).

B. Interior solution

In the following, we shall suppress the explicit appearance of Λ in the field equations by the useful redefinitions

$$\rho \rightarrow \rho + \Lambda, \quad P \rightarrow P - \Lambda, \quad M \rightarrow M + \frac{4\pi}{3} \Lambda r^3. \quad (17)$$

Subject to these normalizations and given the form of the exterior solution we shall define

$$A(r) \equiv \left[1 - \frac{2M(r)}{r} \right]^{-1}. \quad (18)$$

We will use this definition to all orders in the small parameter α , with the term $M(r)$ acquiring corrections at each successive order, as it is shorthand for a metric element.

From the form of the elements of the Ricci tensor, and the above definition we obtain

$$\frac{R_{00}}{2B} + \frac{R_{11}}{2A} + \frac{R_{22}}{r^2} = \frac{2M'}{r^2}. \quad (19)$$

Combining this with Eqs. (7)–(9) and evaluating the result to order $\mathcal{O}(\alpha)$ we derive the equation for mass conservation in $f(R)$ gravity with perturbative constraints

$$\frac{dM}{dr} = 4\pi\rho r^2 - \alpha f_R r^2 \left\{ 4\pi\rho - \frac{R}{4} \left(\frac{n}{n+1} \right) - \frac{n}{2A} \left[\frac{R''}{R} + (n-1) \frac{R'^2}{R^2} + \frac{R'}{R} \left(\frac{2}{r} - \frac{A'}{2A} \right) \right] \right\}. \quad (20)$$

The conservation equation $\nabla^\mu T_{\mu\nu} = 0$ gives

$$\frac{B'}{B} = - \frac{2P'}{\rho + P}, \quad (21)$$

which we use in the expression for R_{22} to get

$$\frac{R_{22}}{r^2} = \frac{1}{r^2} \left[\frac{dM}{dr} + \frac{M}{r} + \frac{r}{A} \left(\frac{P'}{\rho + P} \right) \right] \quad (22)$$

and arrive at the equation of hydrostatic equilibrium via Eq. (9)

$$\frac{dP}{dr} = -\frac{A}{r^2}(\rho + P)\left\{M + 4\pi Pr^3 - \alpha f_R r^3 \left[\frac{R}{4} \left(\frac{n}{n+1}\right) + \frac{n}{2A} \frac{R'}{R} \left(\frac{2}{r} + \frac{B'}{2B}\right) + 4\pi P\right]\right\}. \quad (23)$$

Note that in solving Eqs. (20) and (23) in practice the evolution of the density and pressure are determined in terms of the familiar Tolman-Oppenheimer-Volkov equations

$$\frac{dM^{(0)}}{dr} = 4\pi\rho_0 r^2 \quad (24)$$

and

$$\frac{dP_0}{dr} = -\frac{A^{(0)}}{r^2}(\rho_0 + P_0)(M^{(0)} + 4\pi P_0 r^3). \quad (25)$$

Here ρ_0 and P_0 are understood to be the pressure and density evaluated at $\mathcal{O}(\alpha^0)$, whereas we will denote the pressure evolved via Eq. (23) to $\mathcal{O}(\alpha)$ by P_1 .

IV. NUMERICAL MODELS OF NEUTRON STARS

The equations we have derived so far are general and accommodate any choice for the correction $f(R)$ to the Einstein-Hilbert action. However, in constructing numerical models of neutron stars in $f(R)$ theories, we need to specify at this point the particular value of the parameter n we will use.

In order to address concerns for the structure and stability of neutron stars in cosmologically motivated modifications of gravity (see [15,28]), we might consider the case $n = -2$ [i.e., $f(R) = R^{-1}$]. Since the matter density and pressure directly determine the Ricci scalar, we would anticipate such a term to be the leading order correction for small-curvature scales. Unlike theories with additional degrees of freedom, however, and as we would expect given the magnitude of R , the low-curvature corrections lead to no observable differences in the structure of compact objects. Our analysis of stars with these low-curvature corrections demand that the perturbative parameter α not be significantly larger than Λ . Such a small correction leads to no discernible distinction from the predictions of general relativity.

For this reason, we will study below the case with $n = 1$, i.e., gravity theories with $f(R) = R^2$. This represents the next to leading order correction in a high-curvature expansion of the action. It is this regime where we expect the correction to be most noticeable in the case of compact objects.

We choose the polytropic equation of state

$$\rho = \left(\frac{P}{K}\right)^{1/\Gamma} + \frac{P}{\Gamma - 1} \quad (26)$$

for the interior of the neutron star, where Γ is the polytropic index. Realistic neutron-star equations of state can be parametrized by piecewise polytropic equations of state [29,30] with $\Gamma \simeq 1 - 3$. The lower the polytropic index,

the stiffer the associated mass-radius relationship. For this study we have chosen $\Gamma = 9/5$ which is consistent with the constraints on Γ_2 in Ref. [30].

We utilize the same dimensionless variables as in Ref. [31], namely

$$\bar{r} \equiv K^{-0.5/(\Gamma-1)} r, \quad (27)$$

$$\bar{M} \equiv K^{-0.5/(\Gamma-1)} M, \quad (28)$$

$$\bar{\rho} \equiv K^{1/(\Gamma-1)} \rho, \quad (29)$$

$$\bar{P} \equiv K^{1/(\Gamma-1)} P, \quad (30)$$

$$\bar{\alpha} \equiv K^{1/(\Gamma-1)} \alpha. \quad (31)$$

Because of this normalization of the various physical quantities, our results are independent of the normalization K of the polytropic equation of state.

We use a fourth order Runge-Kutte integrator with adaptive step size to solve for the mass M and radius R of the star. We start at the center of the star by specifying its density (and corresponding pressure) there and integrate out to its surface defined where the pressure vanishes.

Figure 1 shows the dependence of the mass of a neutron star on its central density in an $f(R) = R^2$ theory, for different values of the small parameter $\bar{\alpha}$. The central line corresponds to neutron stars in general relativity. As expected, for stable neutron stars the deviation from the general relativistic case becomes significant as the central density of the neutron star increases, since it is the matter density that directly determines the value of the Ricci scalar curvature. Moreover, the sign of the deviation is determined by the sign of the perturbative parameter $\bar{\alpha}$.

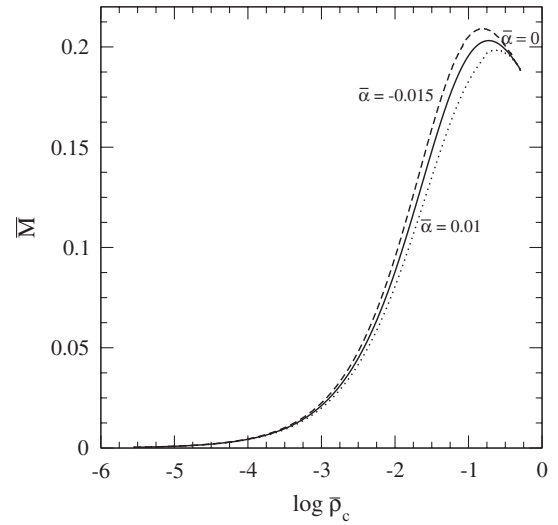


FIG. 1. The mass of a neutron star as a function of its central density in an $f(R) = R^2$ gravity theory for different values of the small parameter $\bar{\alpha}$. The index of the polytropic equation of state was set to $\Gamma = 9/5$.

By properly choosing the sign and magnitude of this parameter, we can cause an increase or a decrease in the maximum mass of stable neutron stars for a particular central density. We can also support stars of a certain mass and radius for a range of central densities and $\bar{\alpha}$.

The maximum allowed magnitude of the deviations from the general relativistic predictions is, of course, constrained by the requirement that the solutions retain their perturbative validity. Though this constraint does not have a ready analytic expression, we can nevertheless explore after the fact the perturbative validity of each stellar model.

In particular, as a measure of the deviations from the general relativistic solution we choose the ratio

$$\xi \equiv \left[\frac{\bar{P}'_1}{\bar{P}'_0} \right] - 1. \quad (32)$$

This ratio varies with radius inside the neutron star. It achieves, however, its highest value at or near the center of the star, where the density (and hence the curvature) is large. Because we require the entire solution to be perturbatively close to the general relativistic one, we will evaluate the ratio ξ at its maximum. A necessary condition for perturbative validity is $\xi < 1$.

Figure 2 shows the maximum ratio ξ as a function of the parameter $\bar{\alpha}$. This figure demonstrates that neutron stars in $f(R) = R^2$ theories can certainly be treated perturbatively as long as $-0.015 < \bar{\alpha} < 0.01$.

Of particular interest from an observational point of view is the mass-radius relation for neutron stars. We show this relation, for the same polytropic equation of state, in Fig. 3. Depending on the value and sign of the parameter α , we obtain stars with larger or smaller radii compared to their general relativistic counterparts of the same gravitational

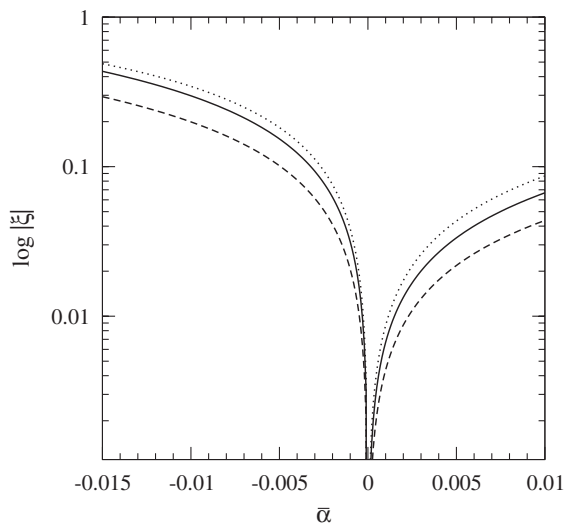


FIG. 2. The ratio ξ [see Eq. (32)] as a function of the parameter $\bar{\alpha}$, for stars with central densities $\log \bar{\rho}_c = -2, -3, -4$ (dotted, solid, and dashed lines, respectively). The ratio ξ measures the degree of perturbative validity of the stellar models. A necessary condition for perturbative validity is $\xi < 1$.

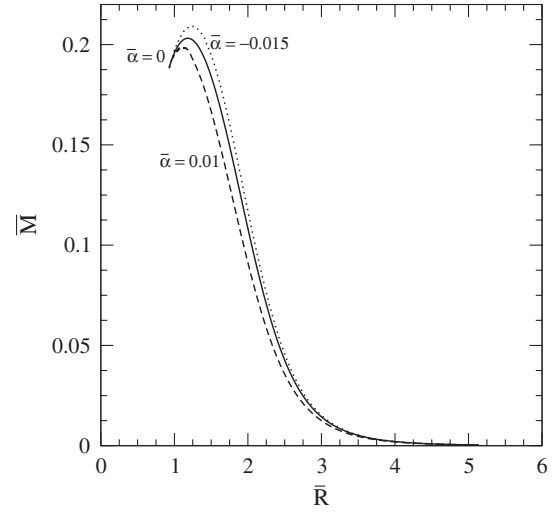


FIG. 3. The mass-radius relation of neutron stars for a polytropic equation of state with $\Gamma = 9/5$, in an $f(R) = \alpha R^2$ gravity for different values of the parameter $\bar{\alpha}$.

mass. The extent of this variation is constrained by perturbative validity, which prevents the onset of dynamical features such as spontaneous scalarization [32].

V. DISCUSSION

The predicted mass-radius relation for neutron stars in $f(R)$ gravity shown above differs from that computed within general relativity. However, very similar deviations in the mass-radius relation can also be obtained within general relativity by simply changing the polytropic index of the equation of state (see [30] for examples). Because the equation of state of neutron-star matter is weakly constrained by current experiments, neutron-star observables that depend only on the mass and radius of the star cannot distinguish between small differences in the equation of state versus small modifications to gravity.

In [33] it was shown that observables that depend also on the effective surface gravity of neutron stars can break, in principle, this degeneracy. In particular it was shown that the Eddington luminosity L_E^∞ of a bursting neutron star depends directly on its effective surface gravity as

$$L_E^\infty \equiv \frac{4\pi m_p r_s}{(1+X)\sigma_T} \left[\frac{z_s(z_s+2)}{(1+z_s)^3} \right] \eta. \quad (33)$$

In this equation, m_p is the mass of the proton, X is the hydrogen mass fraction in the neutron-star atmosphere, σ_T is the Thomson scattering cross section, and

$$z_s = \left(1 - \frac{2M}{R} \right)^{-1} - 1 \quad (34)$$

is the gravitational redshift from the neutron-star surface. The parameter η is the ratio of the effective surface gravity of the neutron star to that calculated in GR, i.e.,

$$\eta \equiv \frac{g_{\text{eff}}}{g_{\text{GR}}} \quad (35)$$

with

$$g_{\text{eff}} \equiv \frac{1}{2\sqrt{A}} \left. \frac{d \ln B}{dr} \right|_{r=R} \quad (36)$$

and

$$g_{\text{GR}} = \frac{1}{2R} \left[\frac{z_s(z_s + 2)}{z_s + 1} \right]. \quad (37)$$

We can calculate easily the value of the parameter η for the $f(R) = R^2$ theory considered here. From the conservation equation (21) we can write

$$g_{\text{eff}} = -\frac{1}{\sqrt{A}} \frac{P'}{(\rho + P)}. \quad (38)$$

We can then evaluate the hydrostatic equilibrium equation (23) to first order in α by noting that

$$\frac{R^{(0)'}}{A^{(0)}} = -8\pi \left(\frac{\partial \rho_0}{\partial P_0} - 3 \right) \frac{(\rho_0 + P_0)}{r^2} (M^{(0)} + 4\pi P_0 r^3). \quad (39)$$

As a result Eq. (38) becomes

$$g_{\text{eff}} = \frac{\sqrt{A^{(1)}}}{r^2} (M^{(1)} + 4\pi P_1 r^3) - \alpha \left\{ 8\pi(\rho_0 + P_0) \sqrt{A^{(0)}} r \right. \\ \times \left[2\pi(\rho_0 - 3P_0) + \frac{2}{r^3} \left(3 - \frac{\partial \rho_0}{\partial P_0} \right) (M^{(0)} + 4\pi r^3 P_0) \right. \\ \left. \left. + \frac{A^{(0)}}{r^4} \left(3 - \frac{\partial \rho_0}{\partial P_0} \right) (M^{(0)} + 4\pi r^3 P_0)^2 \right] \right\} \Big|_{r=R}. \quad (40)$$

At the surface layer of the neutron star $\rho = P = 0$ and hence

$$g_{\text{eff}} = \frac{\sqrt{A^{(1)}}}{R^2} M^{(1)}, \quad (41)$$

which has the same dependence on mass and radius as g_{GR} does. As a result measuring η alone will not suffice to break the degeneracy due to the equation of state.

Nevertheless constraining observationally the cooling rates of neutron stars can offer a discriminant. A neutron star cools both through photon and neutrino emission. The photon luminosity is determined by the temperature at the photosphere, which in turn depends on the density of the photosphere. However neutrino cooling, which depends more sensitively on temperature than photon cooling does, becomes dominant for neutron stars with temperatures above 10^{10} K, and indeed is the primary mechanism of cooling for young neutron stars (see [34] for a detailed

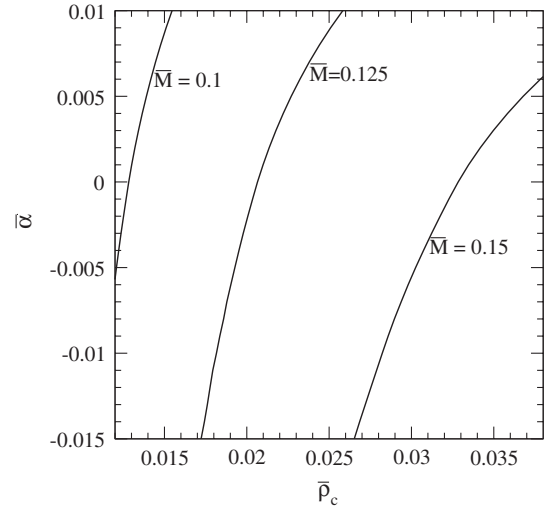


FIG. 4. The central density $\bar{\rho}$ of neutron stars with different masses \bar{M} as a function of the parameter $\bar{\alpha}$. Larger positive deviations from general relativity require larger central densities for the same neutron-star mass and, therefore, lead to shorter cooling times. On the other hand, larger negative deviations require smaller central densities and lead to longer cooling times.

review). The high temperature and low interaction rate make neutrino cooling particularly sensitive to the central density of the neutron star. Figure 4 shows the relation between the parameter $\bar{\alpha}$ and the central density of a neutron star, for three different values of the mass $\bar{M} = 0.15, 0.125,$ and 0.1 . Large positive deviations from general relativity, as measured by the parameter $\bar{\alpha}$ require larger central densities for neutron stars of a given mass, whereas the opposite is true for large negative deviations. As a result, because the cooling time scale scales with central density, observations of the surface temperatures of young neutron star can lead to useful constraints on the deviations from general relativity in an $f(R)$ gravity model, especially if the neutron-star masses are known.

We will study the constraints imposed on $f(R)$ gravity by current measurements of cooling rates of neutron stars in our galaxy in a forthcoming paper.

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