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Momentum-space representation of the Feynman propagator in Riemann-Cartan spacetime

Yu-Huei Wu*

Center for Mathematics and Theoretical Physics, National Central University, and Department of Physics, National Central University, No. 300, Jhongda Road, Jhongli 320, Taiwan

Chih-Hung Wang†

Department of Physics, Tamkang University, Tamsui, Taipei 25137, Taiwan and Institute of Physics, Academia Sinica, Taipei 115, Taiwan

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We first construct generalized normal coordinates by using autoparallels, instead of geodesics, in an arbitrary Riemann-Cartan spacetime. With the aid of generalized normal coordinates and their associated orthonormal frames, we obtain a momentum-space representation of the Feynman propagator for scalar fields, which is a direct generalization of Bunch and Parker's works to curved spacetime with torsion. We further derive the proper-time representation in n-dimensional Riemann-Cartan spacetime from the momentum-space representation. It leads us to obtain the renormalization of the one-loop effective Lagrangians of free scalar fields by using dimensional regularization. When the torsion tensor vanishes, our resulting momentum-space representation returns to the standard Riemannian results.

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I. INTRODUCTION

General relativity (GR) was developed almost a century ago and has been considered as one of the most successful classical theories of gravity. Nevertheless, GR is established (by hypothesis) in the pseudo-Riemannian (i.e. torsion free) framework, so the conservation law of angular-momentum does not involve intrinsic spin of elementary particles, i.e., there is no spin-orbit coupling. In most of the torsion theories of gravity, e.g., Einstein-Cartan theory and Poincaré gauge theory of gravity (PGT) [1], the intrinsic spin does play a significant role and becomes the source of torsion field. Hence, Riemann-Cartan spacetime, which is characterized by a metric g and a metric-compatible connection ∇, provides a natural geometrical structure to cooperate with the intrinsic spin. Moreover, recent astrophysical observations, e.g., supernova Type Ia observations, indicate that the expansion of the present Universe is in an *accelerating* phase [2]. This is contrary to the prediction of standard cosmological model, which is based on GR plus the known matter fields. Hence, a new cosmological model beyond the standard model is necessary. A recent development on torsion cosmology yields a power-law inflation in the early Universe [3], and also presently accelerating phase without introducing the dark energy [4,5]. These results show that torsion has notable effects on cosmology.

Since torsion and intrinsic spin have direct interactions, spin-polarized bodies are used to detect torsion directly in the laboratory (see the review article [6]). Up to the present, there is no experimental evidence showing the existence of torsion field due to the smallness of torsion-spin coupling [6,7] in the laboratory. However, the cosmological observations, e.g., cosmic microwave background radiation (CMB), provide other possibilities to search for torsion-spin coupling in the early Universe. Instead of looking for a torsion-spin coupling, Dereli and Tucker considered a spinless particle following an *autoparallel* curve in the Brans-Dicke theory with torsion, and then estimated the precession rate of Mercury's orbit [8,9]. Later, the precession rate of a gyroscope following an autoparallel in the Kerr-Brans-Dicke field with torsion had also been calculated [10].

The discovery of the CMB and its anisotropic structure provides us with a window to understand the evolution of our early Universe. It can be expected that the quantum effects will become significant in the very early Universe (near the Planck scale). Since there is no completely satisfactory quantum theory of gravity, a semiclassical approximation, i.e. quantized matter fields in a background classical gravitational field, becomes important to study a region where quantum effects of gravitational field can be neglected.

Quantum field theory in the pseudo-Riemannian structure of spacetime has been extensively investigated [11–13]. The covariant approach to study the renormalization of stress-energy tensor was discussed by using DeWitt-Schwinger proper-time method [12,14] with some regularization methods. It requires to introduce a bi-scalar world-function $\sigma(x, x')$, i.e., one-half the square of the geodesic distance between x and x', and then solve a heat kernel equation in the *normal* neighborhood of a point x', which is defined by the exponential map [15]. This

*yhwu@webmail.phy.ncu.edu.tw
†chwang@phy.ncu.edu.tw, robbin1101@gmail.com

covariant approach had been generalized to Riemann-Cartan spacetime [16,17]. However, it immediately encounters a question: which curve, autoparallel or geodesic, should be used to construct the exponential map? In [16], the DeWitt-Schwinger ansatz was applied to solve a heat kernel equation in Riemann-Cartan spacetime by using *autoparallels*. However, we found that these curves are not autoparallels since the one-half the square of the *autoparallel distance* $\sigma(x, x')$ satisfies the equation $\sigma(x, x') = \frac{1}{2} g^{\mu\nu}\nabla_\mu \sigma \nabla_\nu \sigma$ (see Eq. (3.8) in [16]), which is actually the geodesic equation [12]. It is still nontrivial how to apply autoparallel interval to the DeWitt-Schwinger proper-time representation.

Besides the DeWitt-Schwinger proper-time representation, Bunch and Parker [18] developed a momentum-space representation which is useful for discussing the renormalizability of interacting fields, e.g., $\lambda\phi^4$ theory, in a general pseudo-Riemannian structure of spacetime. By constructing the Riemann-normal coordinates in the normal neighborhood of an original point x', they solved the Feynman Green's function $G(x, x')$ of free scalar and Dirac fields in the large wave number k approximation and also showed the equivalence of momentum-space and proper-time representations.

The method of momentum-space representation can be naturally extended to Riemann-Cartan spacetime. The major difference is that the background field variables are changed from the metric tensor $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ to orthonormal coframes $\{e^a = e^a{}_\mu dx^\mu\}$ and connection 1-forms $\{\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu\}$, so we should construct a local coordinate system $\{x^\mu\}$, where the coefficients of $e^a{}_\mu$ and $\omega^a{}_{b\mu}$ in the Taylor series expansions can be systematically expressed in terms of full curvature, torsion and the covariant derivative ∇_μ at the original point x'. Tucker established Fermi coordinates with associated orthonormal-frames in Riemann-Cartan spacetime [19]. Instead of using geodesics, he used autoparallels $\gamma_v(\lambda)$ to define an exponential map and then the Fermi coordinates can be constructed in the normal neighborhood of a timelike curve. We follow a similar process to construct generalized normal coordinates at a point x'. A detail construction will be presented in Sec. II. A recent investigation on normal frames in a general connection (non metric-compatible) can be found in [20]. By using generalized normal coordinates, we then extend Bunch and Parker's work to Riemann-Cartan spacetime.

In [18], the divergent terms of the one-loop effective action for a free scalar field require one to calculate an approximate solution of Feynman Green's function, $G_i(x, x')$ to fourth-order, though the discussion of renormalizability of $\lambda\phi^4$ theory only needs $G_i(x, x')$ to second-order. Since solving $G_3(x, x')$ and $G_4(x, x')$ involves extremely complicated and tedious calculations in a general Riemann-Cartan spacetime, we will restrict our background torsion to be *totally* antisymmetric in the calculation of the G_3 and G_4. Furthermore, we will first concentrate on the renormalization of the effective action of a free scalar field in this paper.

The discussion of quantum field theory in Riemann-Cartan spacetime has another approach by considering torsion as an extra background field (see the review article [21]). In this approach, the fundamental variables are the components of metric $g_{\mu\nu}$ and torsion $T^\alpha{}_{\mu\nu}$ with respect to a coordinate basis $\{\partial_\mu\}$. The full connection ∇ will be separated into Levi-Civita connection $\tilde{\nabla}$ and a contortion part. Following this approach, the divergent terms of the one-loop effective action of matter fields turn out to be the geometrical invariants associated with the *Riemannian* curvature, torsion and $\tilde{\nabla}$ [21–23]. It is still unclear to us whether our results obtained in this paper are equivalent to theirs. This will require a further algebraic computation.

Sec. II presents a detail construction of generalized normal coordinates with associated orthonormal frames in the general Riemann-Cartan spacetime, and then calculates the expansions of $e^a{}_\mu$ and $\omega^a{}_{b\mu}$ to fifth-order. Hence, these expansion coefficients are expressed in terms of full curvature, torsion and ∇_μ at the original point x'. In Sec. III, we solve the equation of Feynman Green's function in the generalized normal coordinates with the large k approximation, and then obtain the solutions $\bar{G}_i(k)$ up to second-order in general background torsion. In Section III A, the solutions $\bar{G}_i(k)$ to fourth-order are derived in the totally antisymmetric background torsion. Sec. IV shows the equivalence of the proper-time representation and the momentum-space representation in n dimensional Riemann-Cartan spacetime. Since the solutions $\bar{G}_i(k)$ are valid in n dimensional spacetime, we use dimensional regularization to study the renormalization of one-loop effective action. In Appendix , we present the detailed and tedious calculations for writing down the equation of Feynman Green's function in the generalized normal coordinates.

In this paper, we use the units $\hbar = c = 1$, and for n dimensional spacetime, the metric signature is $(-, +, \cdots, +)$. The Greek indices $\alpha, \beta, \gamma \cdots$ are referred to coordinate indices and the Latin indices $a, b, \cdots$ referred to frame indices. Both types of indices run from 0 to $n-1$. The covariant derivative ∇_μ on any tensor components $Z^{a\cdots b}{}_{c\cdots d}$ is defined by $(\nabla Z) \times (e^a, \cdots, e^b, X_c, \cdots, X_d, \partial_\mu)$. Any geometrical object defined by the Levi-Civita connection $\tilde{\nabla}$ will have a $\tilde{}$ on it.

II. GENERALIZED NORMAL COORDINATES

In a general Riemann-Cartan spacetime, the definitions of autoparallels and geodesics are completely different. However, they become equivalent in the pseudo-Riemannian geometry. Autoparallels γ: $\lambda \mapsto \gamma(\lambda)$, which satisfy

$$\nabla_{\gamma'}\gamma' = 0, \tag{1}$$

where γ' denotes the tangent vector of γ, are defined in terms of connection ∇, but geodesics C: $t \mapsto C(t)$, which satisfy

$$\delta \int \sqrt{g(\dot{C}, \dot{C})} dt = 0, \tag{2}$$

are defined in terms of g. It is worth pointing out that autoparallels and geodesics coincide if the background torsion tensor is totally antisymmetric. Since Eq. (1) and (2) both provide unique solutions with given initial values, it is not difficult to see that either autoparallels or geodesics can be used to construct a local coordinate system.

If one uses geodesics to construct local coordinates y^α, i.e., the Riemann-normal coordinates, with origin at point x' in the Riemann-Cartan spacetime, the expansion of the metric components $g_{\mu\nu}$ (i.e. $g(\partial_\mu, \partial_\nu)$) in these coordinates is given by [24]

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3}\tilde{R}_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}\tilde{\nabla}_\gamma \tilde{R}_{\mu\alpha\nu\beta}y^\alpha y^\beta y^\gamma + \ldots, \tag{3}$$

which has the same result as in the pseudo-Riemannian geometry. Here $\tilde{R}_{abcd}$ denotes the components of Riemann curvature. Furthermore, it can be shown that in the expansions of all geometric quantities, e.g. torsion tensor components $T^a{}_{bc}$, the coefficients will only involve $\tilde{R}_{abcd}$ and $\tilde{\nabla}$, instead of the *full* curvature R_{abcd} and connection ∇. Since our background gravitational variables are $\{e^a\}$ and $\{\omega^a{}_b\}$, we should construct a local coordinates, where both $\{e^a\}$ and $\{\omega^a{}_b\}$ can be systematically expanded. It is obvious that the Riemann-normal coordinates are not a proper choice. It leads us to establish the generalized normal coordinates, where the expansion coefficients of $\{e^a\}$ and $\{\omega^a{}_b\}$ will be expressed in terms of the full curvature $R^a{}_{bcd}$, covariant derivative ∇_a, and torsion $T^a{}_{bc}$. Generalized Fermi coordinates have been constructed by using autoparallels and the associated orthonormal coframes in the Riemann-Cartan spacetime [19]. Here, we apply a similar procedure to establish generalized normal coordinates.

Consider an autoparallel γ_v: $\lambda \mapsto \gamma_v(\lambda) \in M$ with its initial values

$$\gamma_v(0) = x', \tag{4}$$

$$\gamma_v'(0) = v, \tag{5}$$

where M denotes an n-dimensional Riemann-Cartan spacetime. Provided $\gamma_v(1)$ exists, the exponential map $\exp_{x'}$: $T_{x'}M \mapsto M$ is then defined in an open neighborhood $\mathcal{U}$ of x' by

$$\exp_{x'}(v) \equiv \gamma_v(1) \in M, \tag{6}$$

where $T_{x'}M$ denotes the tangent space to M at x'. Using $\exp_{x'}$ with an orthonormal frame $\{\hat{X}_a\}$ at x', we obtain the generalized normal coordinates x^α

$$\Psi^\alpha(\exp_{x'} v) = x^\alpha, \tag{7}$$

where Ψ^α is a coordinate chart, and

$$v = \sum_{\alpha=0}^{n-1} \delta^a{}_\alpha x^\alpha \hat{X}_a, \tag{8}$$

where $\delta^a{}_\alpha = \mathrm{diag}(1, \cdots, 1)$. In the following, $\hat{}$ on any tensor field Z denotes $Z|_{x^\alpha=0}$ (i.e. Z at x'). A natural induced coordinate basis $\{\partial_\alpha\}$, by construction, has $\{\hat{\partial}_\alpha = \delta^a{}_\alpha \hat{X}_a\}$.

It will be useful to introduce generalized normal hyperspherical coordinates $\{\lambda, p^\alpha\}$ defined by

$$x^\alpha = \lambda p^\alpha, \tag{9}$$

where λ is the radial coordinate with affine parametrization and p^a are the direction cosines of tangent vectors of autoparallels γ_{∂_α} at x' satisfying

$$\sum_{\alpha=0}^{n-1} p^\alpha p^\alpha = 1. \tag{10}$$

From the inverse relations

$$\lambda^2 = \sum_{\alpha=0}^{n-1} x^\alpha x^\alpha, \tag{11}$$

one has

$$\partial_\lambda = p^\alpha \partial_\alpha, \tag{12}$$

$$\partial_\lambda p^\alpha = 0, \tag{13}$$

and $v = \lambda\hat{\partial}_\lambda$. It should be mentioned that $\hat{Z} = Z|_{\lambda=0}$ denotes the initial value of any tensor field Z in hyperspherical coordinates $\{\lambda, p^\alpha\}$. Using $\{\lambda, p^\alpha\}$, we can parallel transport $\{\hat{X}_a\}$ along autoparallels $\gamma_{\partial_\lambda}$ to set up a field of orthonormal frames $\{X_a\}$ and its dual coframe field $\{e^a\}$ on $\mathcal{U}$.

From the above construction, one has

$$\nabla_{\partial_\lambda} e^a = 0, \tag{14}$$

i.e.,

$$i_{\partial_\lambda}\omega^a{}_b = \omega^a{}_b(\partial_\lambda) = 0, \tag{15}$$

with its initial value $\hat{e}^a = \delta^a{}_\alpha \widehat{dx^\alpha} = \delta^a{}_\alpha p^\alpha \widehat{d\lambda}$. Since ∂_λ are tangent vectors of autoparallels, we further obtain

$$\partial_\lambda(e^a(\partial_\lambda)) = 0. \tag{16}$$

It turns out that $e^a(\partial_\lambda)$ is independent of λ and equal to its initial value p^a. So $\{e^a\}$ in $\{\lambda, p^a\}$ gives

$$e^a = \delta^a{}_\alpha p^\alpha d\lambda + \mathcal{A}^a{}_\mu dp^\mu \tag{17}$$

with the initial values

$$\hat{\mathcal{A}}^a \equiv \hat{\mathcal{A}}^a{}_\mu dp^\mu = 0. \tag{18}$$

Equation (15) indicates that $\omega^a{}_b$ does not contain the $d\lambda$ term, so

$$\omega^a{}_b = \mathcal{C}^a{}_{b\mu} dp^\mu \tag{19}$$

with the initial values

$$\hat{\mathcal{C}}^a{}_b = \hat{\mathcal{C}}^a{}_{b\mu} dp^\mu = \widehat{\omega^a{}_b(\partial_\mu)} dp^\mu = 0. \tag{20}$$

It is known that the Riemann-normal coordinates in the pseudo-Riemannian geometry have a local Minkowski structure (i.e. $g_{\mu\nu}(x') = \eta_{\mu\nu}$, $\tilde{\omega}^\mu{}_\nu(x') = 0$), which is associated with the equivalence principle. Similarly, Eqs. (18) and (20) also represent a local Minkowski structure of spacetime at x' in the Riemann-Cartan spacetime, so the revised version of the equivalence principle has been discussed [25].

Since we have completely constructed the generalized normal coordinates with the associated orthonormal coframes $\{e^a\}$ on $\mathcal{U}$, the next step is to expand the fundamental variables $\{e^a\}$ and $\{\omega^a{}_b\}$ with respect to radial variable λ and then to express their coefficients in terms of the full curvature $\hat{R}^a{}_{bcd}$, torsion $\hat{T}^a{}_{bc}$ and their covariant derivative ∇_α.

We start from the Cartan structure equations defined by the torsion and full curvature [26]:

$$de^a = -\omega^a{}_b \wedge e^b + \mathrm{T}^a, \tag{21}$$

$$d\omega^a{}_b = -\omega^a{}_c \wedge \omega^c{}_b + \mathrm{R}^a{}_b \tag{22}$$

where

$$\mathrm{T}^a = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c \quad \text{and} \quad \mathrm{R}^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d \tag{23}$$

are torsion 2-forms and curvature 2-forms in the coframe field $\{e^a\}$. Substituting Eqs. (17) and (19) into Eqs. (21) and (22) and equating the forms containing $d\lambda \wedge dp^\alpha$ on each side gives ordinary differential equations for $\mathcal{A}^a$ and $\mathcal{C}^a{}_b$:

$$\mathcal{A}'^a = \delta^a{}_\alpha dp^\alpha + \mathcal{C}^a{}_b \delta^b{}_\alpha p^\alpha + T^a{}_{bc} \delta^b{}_\alpha p^\alpha \mathcal{A}^c, \tag{24}$$

$$\mathcal{C}'^a{}_b = R^a{}_{bcd} \delta^c{}_\alpha p^\alpha \mathcal{A}^d, \tag{25}$$

where $'$ denotes the radial derivative ∂_λ. $\mathcal{A}'^a$ and $\mathcal{C}'^a{}_b$ denote $(\partial_\lambda \mathcal{A}^a{}_b) dp^b$ and $\mathcal{C}'^a{}_b = (\partial_\lambda \mathcal{C}^a{}_{bc}) dp^c$, respectively. In the remaining part of Sec. II, we will use the notations $dp^a \equiv \delta^a{}_\alpha dp^\alpha$ and $p^a \equiv \delta^a{}_\alpha p^\alpha$.

We know that the Taylor series representations of $\mathcal{A}^a$ and $\mathcal{C}^a{}_b$ with respect to the radial coordinate λ are

$$\mathcal{A}^a = \hat{\mathcal{A}}^a + \hat{\mathcal{A}}'^a \lambda + \frac{1}{2!} \hat{\mathcal{A}}''^a \lambda^2 + \cdots, \tag{26}$$

$$\mathcal{C}^a{}_b = \hat{\mathcal{C}}^a{}_b + \hat{\mathcal{C}}'^a{}_b \lambda + \frac{1}{2!} \hat{\mathcal{C}}''^a{}_b \lambda^2 + \cdots. \tag{27}$$

It should be mentioned that, for any function f, $\hat{f}'^{\cdots\prime}$ denotes $(\partial_\lambda \cdots \partial_\lambda f)|_{\lambda=0}$. By successively differentiating Eqs. (24) and (25) with respect to λ and then evaluating the results at $\lambda = 0$, one can obtain $\hat{\mathcal{A}}'^{\cdots\prime a}$ and $\hat{\mathcal{C}}'^{\cdots\prime a}{}_b$ in terms of $\hat{R}^a{}_{bcd}$, $\hat{T}^a{}_{bc}$, and their radial derivative ∂_λ. Since the discussion of renormalization of W in terms of the momentum-space representation requires us to calculate $\hat{\mathcal{A}}'^{\cdots\prime a}$ and $\hat{\mathcal{C}}'^{\cdots\prime a}{}_b$ to *fifth*-order, we will present our results to fifth-order of the radial derivative. To first order in λ, one finds

$$\hat{\mathcal{A}}'^a = dp^a, \tag{28}$$

$$\hat{\mathcal{C}}'^a{}_b = 0. \tag{29}$$

The curvature and torsion start to appear at the second order:

$$\hat{\mathcal{A}}''^a = \hat{T}^a{}_{bc} p^b dp^c, \tag{30}$$

$$\hat{\mathcal{C}}''^a{}_b = \hat{R}^a{}_{bcd} p^c dp^d. \tag{31}$$

At the third order:

$$\hat{\mathcal{A}}'''^a = \hat{R}^a{}_{bcd} p^b p^c dp^d + 2\hat{T}'^a{}_{bc} p^b dp^c + \hat{T}^a{}_{bc} \hat{T}^c{}_{de} p^b p^d dp^e, \tag{32}$$

$$\hat{\mathcal{C}}'''^a{}_b = 2\hat{R}'^a{}_{bcd} p^c dp^d + \hat{R}^a{}_{bcd} \hat{T}^d{}_{ef} p^c p^e dp^f, \tag{33}$$

which have one radial derivative of the curvature and torsion. The two radial derivatives of the curvature and torsion start to appear at the fourth order:

$$\begin{aligned}\hat{\mathcal{A}}''''^a = {}& 2\hat{R}'^a{}_{bcd} p^b p^c dp^d + \hat{R}'^a{}_{bcd} \hat{T}^d{}_{ef} p^b p^c p^e dp^f \\ & + 3\hat{T}''^a{}_{bc} p^b dp^c + \hat{T}^a{}_{bc} \hat{R}^c{}_{def} p^b p^d p^e dp^f \\ & + 3\hat{T}'^a{}_{bc} \hat{T}^c{}_{de} p^b p^d dp^e + 2\hat{T}^a{}_{bc} \hat{T}'^c{}_{de} p^b p^d dp^e \\ & + \hat{T}^a{}_{bc} \hat{T}^c{}_{de} \hat{T}^e{}_{fg} p^b p^d p^f dp^g,\end{aligned} \tag{34}$$

$$\begin{aligned}\hat{\mathcal{C}}''''^a{}_b = {}& 3\hat{R}''^a{}_{bcd} p^c dp^d + \hat{R}^a{}_{bcd} \hat{R}^d{}_{efg} p^c p^e p^f dp^g \\ & + 3\hat{R}'^a{}_{bcd} \hat{T}^d{}_{ef} p^c p^e dp^f + 2\hat{R}^a{}_{bcd} \hat{T}'^d{}_{ef} p^c p^e dp^f \\ & + \hat{R}^a{}_{bcd} \hat{T}^d{}_{ef} \hat{T}^f{}_{gh} p^c p^e p^g dp^h.\end{aligned} \tag{35}$$

At the fifth order, it becomes much more complicated and involves three radial derivatives of the curvature and torsion:

$$\begin{aligned}\hat{\mathcal{A}}^{\prime\prime\prime\prime a} &= 3\hat{R}^{\prime\prime a}{}_{bcd}p^b p^c dp^d + 3\hat{R}^{\prime a}{}_{bcd}\hat{T}^d{}_{ef}p^b p^c p^e dp^f + \hat{R}^a{}_{bcd}\hat{R}^d{}_{efg}p^b p^c p^e p^f dp^g + 2\hat{R}^a{}_{bcd}\hat{T}^{\prime d}{}_{ef}p^b p^c p^e dp^f \\ &\quad + \hat{R}^a{}_{bcd}\hat{T}^d{}_{ef}\hat{T}^f{}_{gh}p^b p^c p^e p^g dp^h + 4\hat{T}^{\prime\prime\prime a}{}_{bc}p^b dp^c + 6\hat{T}^{\prime\prime a}{}_{bc}\hat{T}^c{}_{ed}p^b p^e dp^d + 4\hat{T}^{\prime a}{}_{bc}\hat{R}^c{}_{def}p^b p^d p^e dp^f \\ &\quad + 8\hat{T}^{\prime a}{}_{bc}\hat{T}^{\prime c}{}_{de}p^b p^d dp^e + 4\hat{T}^{\prime a}{}_{bc}\hat{T}^c{}_{de}\hat{T}^e{}_{fg}p^b p^d p^f dp^g + 2\hat{T}^a{}_{bc}\hat{R}^{\prime c}{}_{def}p^b p^d p^e dp^f \\ &\quad + \hat{T}^a{}_{bc}\hat{R}^c{}_{def}\hat{T}^f{}_{gh}p^b p^d p^e p^g dp^h + 3\hat{T}^a{}_{bc}\hat{T}^{\prime\prime c}{}_{de}p^b p^d dp^e + 3\hat{T}^a{}_{bc}\hat{T}^{\prime c}{}_{de}\hat{T}^e{}_{fg}p^b p^d p^f dp^g \\ &\quad + \hat{T}^a{}_{bc}\hat{T}^c{}_{de}\hat{R}^e{}_{fgh}p^b p^d p^f p^g dp^h + 2\hat{T}^a{}_{bc}\hat{T}^c{}_{de}\hat{T}^{\prime e}{}_{fg}p^b p^d p^f dp^g + \hat{T}^a{}_{bc}\hat{T}^c{}_{de}\hat{T}^e{}_{fg}\hat{T}^g{}_{hi}p^b p^f p^d p^h dp^i \end{aligned} \tag{36}$$

$$\begin{aligned}\hat{\mathcal{C}}^{\prime\prime\prime\prime a}{}_b &= 4\hat{R}^{\prime\prime\prime a}{}_{bcd}p^c dp^d + 6\hat{R}^{\prime\prime a}{}_{bcd}\hat{T}^d{}_{ef}p^c p^e dp^f + 4\hat{R}^{\prime a}{}_{bcd}\hat{R}^d{}_{efg}p^c p^e p^f dp^g + 8\hat{R}^{\prime a}{}_{bcd}\hat{T}^{\prime d}{}_{ef}p^c p^e dp^f \\ &\quad + 4\hat{R}^{\prime a}{}_{bcd}\hat{T}^d{}_{ef}\hat{T}^f{}_{gh}p^c p^e p^g dp^h + 2\hat{R}^a{}_{bcd}\hat{R}^{\prime d}{}_{efg}p^c p^e p^f dp^g + \hat{R}^a{}_{bcd}\hat{R}^d{}_{efg}\hat{T}^g{}_{hi}p^c p^e p^f p^h dp^i \\ &\quad + 3\hat{R}^a{}_{bcd}\hat{T}^{\prime\prime d}{}_{ef}p^c p^e dp^f + 3\hat{R}^a{}_{bcd}\hat{T}^{\prime d}{}_{ef}\hat{T}^f{}_{gh}p^c p^e p^g dp^h + \hat{R}^a{}_{bcd}\hat{T}^d{}_{ef}\hat{R}^f{}_{ghi}p^c p^e p^g p^h dp^i \\ &\quad + 2\hat{R}^a{}_{bcd}\hat{T}^d{}_{ef}\hat{T}^{\prime f}{}_{gh}p^c p^e p^g dp^i + \hat{R}^a{}_{bcd}\hat{T}^d{}_{ef}\hat{T}^f{}_{gh}\hat{T}^h{}_{ij}p^c p^e p^g p^i dp^j \end{aligned} \tag{37}$$

Although these expressions involve the radial derivative ∂_λ, it can be changed to the covariant derivative $\nabla_{\partial_\lambda}$ by using Eq. (14), e.g.,

$$\begin{aligned}\nabla_{\partial_\lambda} R^a{}_{bcd} &\equiv (\nabla_{\partial_\lambda} R)(e^a, X_b, X_c, X_d) \\ &= \nabla_{\partial_\lambda}(R(e^a, X_b, X_c, X_d)) \equiv \partial_\lambda R^a{}_{bcd}. \end{aligned} \tag{38}$$

Moreover, it is understood that any tensor-field components $Z^{a\cdots b}{}_{c\ldots d}$ satisfy $\hat{Z}^{a\cdots b}{}_{c\ldots d} = \delta^a{}_\alpha \cdots \delta^b{}_\beta \delta^\gamma{}_c \cdots \delta^\delta{}_d \hat{Z}^{\alpha\cdots\beta}{}_{\gamma\ldots\delta}$, so there is no difference of using the Greek or Latin indices for any tensor-field components at the original point x'. In the following, we will adapt the Greek indices on any tensor-field components at x'.

III. MOMENTUM-SPACE REPRESENTATION OF THE FEYNMAN PROPAGATOR OF A SCALAR FIELD

The classical action functional of a scalar field in the pseudo-Riemannian (i.e. torsion free) structure of spacetime is [11]

$$\tilde{S}[g, \phi] = -\frac{1}{2}\int_M d\phi \wedge \star d\phi + (m^2 + \xi\tilde{R})\phi^2 \star 1, \tag{39}$$

where $\star$ is the Hodge map associated with g, m is the scalar field's mass, ξ is an arbitrary real number, and $\tilde{R}$ is the Ricci scalar curvature. Since the background gravitational field is now described by g and ∇ in the Riemann-Cartan spacetime, the basic gravitational variables will be a class of arbitrary local orthonormal 1-form coframes $\{e^a\}$ on spacetime related by $SO(3,1)$ transformation and connection 1-forms $\{\omega^a{}_b\}$, which is a representation of ∇ with respect to $\{e^a\}$. A direct generalization of Eq. (39) to Riemann-Cartan spacetime is

$$S[e^a, \omega^a{}_b, \phi] = -\frac{1}{2}\int_M d\phi \wedge \star d\phi + (m^2 + \xi R)\phi^2 \star 1, \tag{40}$$

where R is the full scalar curvature. Varying S with respect to ϕ, the equations of motion of ϕ can be obtained

$$0 = \frac{\delta S}{\delta \phi} = -d \star d\phi + (m^2 + \xi R)\phi \star 1. \tag{41}$$

The classical stress 3-forms τ_a and spin 3-forms $S_a{}^b$ are defined as

$$\begin{aligned}\tau_a \equiv \frac{\delta S}{\delta e^a} &= \frac{1}{2}(i_a d\phi \wedge \star d\phi + d\phi \wedge i_a \star d\phi \\ &\quad - m^2\phi^2 \star e_a - \xi\phi^2 \mathrm{R}_{bc} \wedge \star e_a{}^{bc}), \end{aligned} \tag{42}$$

$$S_a{}^b \equiv \frac{\delta S}{\delta \omega^a{}_b} = -\frac{1}{2}\xi\phi^2\left(\frac{2d\phi}{\phi} \wedge \star e_a{}^b + \mathrm{T}^c \wedge \star e_{ca}{}^b\right), \tag{43}$$

where $i_a \equiv i_{X_a}$ is the interior derivative, $\{X_a\}$ is the dual basis of $\{e^a\}$, and $e^{a\ldots b}{}_{c\ldots d} \equiv e^a \wedge \ldots \wedge e^b \wedge e_c \wedge \ldots e_d$.

Using the path-integral quantization [11], we obtain the one-loop effective action

$$W[e^a, \omega^a{}_b] = -\frac{i}{2}\int_M \langle x|\ln G|x\rangle \star 1, \tag{44}$$

where $\langle x|G|x'\rangle \equiv G(x, x')$ is the Feynman Green's function, and also the vacuum expectation value of τ_a and $S_a{}^b$ defined by

$$\frac{\delta W}{\delta e^a} = \langle \tau_a \rangle, \qquad \frac{\delta W}{\delta \omega^a{}_b} = \langle S_a{}^b \rangle. \tag{45}$$

From Eq. (40), one can show that the Feynman Green's function of a scalar field satisfies [11,12]

$$\sqrt{|g(x)|}[-\star^{-1} d \star d + m^2 + \xi R]G(x, x') = \delta(x - x'), \tag{46}$$

where $|g(x)| \equiv |\det g_{ab}(x)|$. It is useful to define $\bar{G}(x, x')$ by

$$\bar{G}(x, x') = |g(x)|^{1/4} G(x, x') |g(x')|^{1/4}, \tag{47}$$

and Eq. (46) becomes

$$[(-|g(x)|^{1/4} \star^{-1} d \star d|g(x)|^{-1/4}) + m^2 + \xi R]\bar{G}(x, x') = \delta(x - x'). \tag{48}$$

It is known that, in the coincident limit $x \to x'$, the divergences of $G(x, x')$ and also the effective action W come from the high frequency field behavior [11,27]. In the following, we will use $[Z]$ to denote the coincident limit of any two-point function $Z(x, x')$, i.e., $[Z] = \lim_{x\to x'} Z(x, x')$. These ultraviolet divergences can be obtained by solving Eq. (48) in the generalized normal coordinates with asymptotic expansion in large wave number k.

Equation (48) in the generalized normal coordinates x^α with associated orthonormal coframe $\{e^a\}$ gives

$$(\eta^{\mu\nu} + \overset{(1)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha}x^\alpha + \overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta}x^\alpha x^\beta + \overset{(3)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma + \overset{(4)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma\lambda}x^\alpha x^\beta x^\gamma x^\lambda)\partial_\mu\partial_\nu\bar{G} - m^2\bar{G} + (\overset{(1)}{\mathcal{S}}{}^{\nu} + \overset{(2)}{\mathcal{S}}{}^{\nu}{}_{\alpha}x^\alpha + \overset{(3)}{\mathcal{S}}{}^{\nu}{}_{\alpha\beta}x^\alpha x^\beta + \overset{(4)}{\mathcal{S}}{}^{\nu}{}_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma)\partial_\nu\bar{G} + [(\overset{(2)}{\mathcal{P}} - \xi\hat{R}) + (\overset{(3)}{\mathcal{P}}_\alpha - \xi\widehat{\nabla_\alpha R})x^\alpha + (\overset{(4)}{\mathcal{P}}_{\alpha\beta} - \frac{\xi}{2}\nabla_\beta\widehat{\nabla_\alpha R})x^\alpha x^\beta]\bar{G} = -\delta(x), \tag{49}$$

where the coefficients $\overset{(i)}{\mathcal{F}}{}^{\mu\nu}\ldots$, $\overset{(i)}{\mathcal{S}}{}^{\nu}\ldots$, and $\overset{(i)}{\mathcal{P}}\ldots$ involve the i derivatives of orthonormal coframe. The explicit expressions of these coefficients in terms of $\hat{T}^a{}_{bc}$, $\hat{R}^a{}_{bcd}$, and their covariant derivatives are given in Appendix . We have only retained the coefficients for $i \leqslant 4$ in Eq. (49) since the divergences of W involve the coefficients up to four derivatives of $\{e^a\}$. However, due to the complicated computation of the coefficients $i = 4$, the discussion of renormalization of W will be restricted in *totally antisymmetric* torsion, i.e., $T_{abc} = T_{[abc]}$, where square brackets indicate index antisymmetrization. On the other hand, the divergences of $[G]$, which are used to study the renormalizability of interacting scalar fields, involve the coefficients for $i \leqslant 2$, and these coefficients can be obtained in the general background torsion. More specifically, we will find an approximate solution of $\bar{G}$ up to second-order in general background torsion in the remaining part of Sec. III, and the approximate solution of $\bar{G}$ to fourth-order in the background totally antisymmetric torsion will be presented in Sec. III A.

By making the n-dimensional Fourier transformation, $\bar{G}(x, x')$ in the momentum space yields

$$\bar{G}(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{ik_\alpha x^\alpha}\bar{G}(k), \tag{50}$$

where $\bar{G}(k) = \bar{G}(k; x')$ is assumed to have compact support in the normal neighborhood of x'. We consider the following expansion of $\bar{G}(k)$

$$\bar{G}(k) = \bar{G}_0(k) + \bar{G}_1(k) + \bar{G}_2(k) + \cdots, \tag{51}$$

and

$$\bar{G}_i(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{ik_\alpha x^\alpha}\bar{G}_i(k), \tag{52}$$

where $\bar{G}_i(k)$ involves the coefficients $\overset{(i)}{\mathcal{F}}{}^{\mu\nu}\ldots$, $\overset{(i)}{\mathcal{S}}{}^{\nu}\ldots$, and $\overset{(i)}{\mathcal{P}}\ldots$. For $i = 0$, we have $\overset{(0)}{\mathcal{F}}{}^{\mu\nu}\ldots = \overset{(0)}{\mathcal{S}}{}^{\nu}\ldots = \overset{(0)}{\mathcal{P}}\ldots = 0$. On dimensional ground, $G_i(k)$ is of order $k^{-(2+i)}$ so Eq. (51) corresponds to an asymptotic expansion of $\bar{G}(k)$ in large k [18].

In [18], the discussion of the renormalizability of $\lambda\phi^4$ theory needed the solution $\bar{G}_i$ to second-order, i.e. $i \leqslant 2$, so our calculations on G_i will be considered in general background torsion field, which will be useful for our future investigation on interacting scalar fields.

By substituting Eq. (52) into Eq. (49), the lowest-order equation yields

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G}_0 - m^2 = -\delta(x), \tag{53}$$

which has the Minkowski-space solution

$$\bar{G}_0(k) = \frac{1}{k^2 + m^2}. \tag{54}$$

From Eq. (53), we also know that $\bar{G}_0(x, x')$ is a function of $\eta_{\mu\nu}x^\mu x^\nu \equiv x_\nu x^\nu$, i.e. Lorentz invariant. The equation for $\bar{G}_1(x, x')$ gives

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G}_1 - m^2\bar{G}_1 + \overset{(1)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha}x^\alpha\partial_\mu\partial_\nu\bar{G}_0 + \overset{(1)}{\mathcal{S}}{}^{\nu}\partial_\nu\bar{G}_0 = 0. \tag{55}$$

Substituting the solution $\bar{G}_0(x, x')$ into Eq. (55) and using Eqs. (A8) and (A9), we obtain

$$\bar{G}_1(k) = -\frac{i}{4}\hat{T}_\alpha\partial^\alpha\left(\frac{1}{k^2 + m^2}\right), \tag{56}$$

where $\hat{T}_\alpha = \hat{T}^\beta{}_{\beta\alpha}$ is the trace torsion, and $\partial^\alpha \equiv \partial/\partial k_\alpha$. It turns out that $\bar{G}_1(k) = 0$ in the pseudo-Riemannian geometry, which has been shown in [18]. Using integration by parts, one can show that $[\bar{G}_1] = 0$ (see Sec. IV).

The equation for $\bar{G}_2(x, x')$ gives

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G}_2 - m^2\bar{G}_2 + \overset{(1)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha}x^\alpha\partial_\mu\partial_\nu\bar{G}_1 + \overset{(1)}{\mathcal{S}}{}^{\nu}\partial_\nu\bar{G}_1 + \overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta}x^\alpha x^\beta\partial_\mu\partial_\nu\bar{G}_0 + \overset{(2)}{\mathcal{S}}{}^{\nu}{}_{\alpha}x^\alpha\partial_\nu\bar{G}_0 + (\overset{(2)}{\mathcal{P}} - \xi\hat{R})\bar{G}_0 = 0. \tag{57}$$

By substituting the solutions $\bar{G}_0(x, x')$, $\bar{G}_1(x, x')$, Eqs. (A8)–(A12) into Eq. (57) and integrating by-parts, a straightforward but tedious calculation yields

$$\bar{G}_2(k)=\bigg[\bigg(\frac{1}{6}-\xi\bigg)\hat{R}-\frac{1}{4}\hat{T}_\alpha\hat{T}^\alpha+\frac{1}{3}\widehat{\nabla_\alpha T^\alpha}-\frac{1}{8}\hat{T}_{\alpha\beta\gamma}\hat{T}^{\alpha\beta\gamma}-\frac{1}{6}\hat{T}_{\alpha\beta\gamma}\hat{T}^{\gamma\beta\alpha}\bigg]\frac{1}{(k^2+m^2)^2}-\frac{1}{8}\bigg[\frac{1}{4}\hat{T}_\alpha\hat{T}_\beta-\frac{1}{2}\hat{T}_{\alpha\beta\mu}\hat{T}^\mu+4(\overset{(2)}{\mathcal{F}}{}_\alpha{}^\mu{}_{(\beta\mu)}+\overset{(2)}{\mathcal{F}}{}^\mu{}_{\alpha(\beta\mu)}+2\overset{(2)}{\mathcal{F}}{}_{\alpha\beta}{}^\mu{}_{\mu)}-2\overset{(2)}{\mathcal{S}}{}_{\alpha\beta}\bigg]\partial^\alpha\partial^\beta\bigg(\frac{1}{k^2+m^2}\bigg), \quad (58)$$

where the indices are raised and lowered by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ and round brackets indicate index symmetrization. In Sec. IV, it will be shown that the terms involving $\partial^\alpha\partial^\beta\Big(\frac{1}{k^2+m^2}\Big)$ in Eq. (58) does not contribute to $[\bar{G}_2]$. In pseudo-Riemannian geometry, Eq. (58) reduces to

$$\bar{G}_2(k)=\frac{(\frac{1}{6}-\xi)\hat{\tilde{R}}}{(k^2+m^2)^2}, \quad (59)$$

which is the same as in [18].

A. A special case: Total antisymmetric torsion

In this subsection, we will consider the background torsion to be totally antisymmetric and find the divergences of the effective action W in this restricted background gravitational field. The reason is that the totally antisymmetric torsion plays a significant role for generating inflation in the early Universe [3]. Moreover, since it is necessary to obtain $\bar{G}_4(k)$ for finding the divergences of W, this consideration will largely simplify our calculations.

When $T_{\alpha\beta\gamma}=T_{[\alpha\beta\gamma]}$, the autoparallels and geodesics will coincide, and it gives $\overset{(1)}{\mathcal{F}}{}^{\mu\nu}{}_\alpha=\overset{(1)}{\mathcal{S}}{}^\nu=0$. So Eq. (55) gives a trivial solution $\bar{G}_1(k)=0$. Since $\bar{G}_0(x,x')$ is a function of $x_\mu x^\nu$, and using Eqs. (A14) and (A15), we obtain

$$\overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta}x^\alpha x^\beta\partial_\mu\partial_\nu\bar{G}_0+\overset{(2)}{\mathcal{S}}{}^\nu{}_\alpha x^\alpha\partial_\nu\bar{G}_0\equiv 0. \quad (60)$$

Therefore, Eq. (57) becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G}_2-m^2\bar{G}_2+(\overset{(2)}{\mathcal{P}}-\xi\hat{R})\bar{G}_0=0, \quad (61)$$

which has a solution

$$\bar{G}_2(k)=\bigg[\bigg(\frac{1}{6}-\xi\bigg)\hat{R}+\frac{1}{24}\hat{T}_{\alpha\beta\gamma}\hat{T}^{\alpha\beta\gamma}\bigg]\frac{1}{(k^2+m^2)^2}. \quad (62)$$

Equation (61) indicates that $\bar{G}_2(x,x')$ is Lorentz invariant and hence it is also a function of $x^\alpha x_\alpha$. It follows that $\bar{G}_2(x,x')$ also satisfies Eq. (60). Moreover, by using Eqs. (A17), (A18), (A20), and (A21), a straightforward but tedious calculation gives two more identities

$$\overset{(3)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma\partial_\mu\partial_\nu\bar{G}_0+\overset{(3)}{\mathcal{S}}{}^\nu{}_{\alpha\beta}x^\alpha x^\beta\partial_\nu\bar{G}_0\equiv 0, \quad (63)$$

$$\overset{(4)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma\lambda}x^\alpha x^\beta x^\gamma x^\lambda\partial_\mu\partial_\nu\bar{G}_0+\overset{(4)}{\mathcal{S}}{}^\nu{}_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma\partial_\nu\bar{G}_0\equiv 0. \quad (64)$$

so $\bar{G}_3(x,x')$ and $\bar{G}_4(x,x')$ satisfy

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G}_3-m^2\bar{G}_3+(\overset{(3)}{\mathcal{P}}{}_\alpha-\xi\widehat{\nabla_\alpha R})x^\alpha\bar{G}_0=0, \quad (65)$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G}_4-m^2\bar{G}_4+(\overset{(2)}{\mathcal{P}}-\xi\hat{R})\bar{G}_2+(\overset{(4)}{\mathcal{P}}{}_{\alpha\beta}-\frac{\xi}{2}\widehat{\nabla_\beta\nabla_\alpha R})x^\alpha x^\beta\bar{G}_0=0. \quad (66)$$

By substituting Eq. (A19) into Eq. (65) and integrating by parts, we obtain

$$\bar{G}_3(k)=\frac{i}{2}\bigg[\bigg(\frac{1}{12}-\xi\bigg)\widehat{\nabla_\alpha R}+\frac{1}{12}\bigg(2\widehat{\nabla^\beta R_{(\beta\alpha)}}+2\hat{R}^\gamma_{(\alpha\beta)\lambda}\hat{T}^{\lambda\beta}{}_\gamma+\hat{T}^{\beta\lambda\gamma}\nabla_{(\alpha}\hat{T}_{\lambda)\gamma\beta}+\frac{1}{2}\hat{T}_{\alpha\gamma\beta}\nabla_\lambda\hat{T}^{\lambda\gamma\beta}\bigg)\bigg]\partial^\alpha\frac{1}{(k^2+m^2)^2}, \quad (67)$$

where $\nabla^\alpha\equiv g^{\alpha\beta}\nabla_\beta$. When torsion vanishes, Eq. (67) reduces to

$$\bar{G}_3(k)=\frac{i}{2}\bigg(\frac{1}{6}-\xi\bigg)\tilde{\nabla}_\alpha\hat{\tilde{R}}\partial^\alpha\frac{1}{(k^2+m^2)^2}, \quad (68)$$

where the Bianchi identities have been used. This agrees with the result in [18]. Similarly, substituting Eqs. (A16) and (A22) into Eq. (66) and integrating by-parts yields

$$\bar{G}_4(k)=\bigg[\bigg(\frac{1}{6}-\xi\bigg)\hat{R}+\frac{1}{24}\hat{T}_{\alpha\beta\gamma}\hat{T}^{\alpha\beta\gamma}\bigg]^2\frac{1}{(k^2+m^2)^3}+\frac{2}{3}\bigg(\overset{(4)}{\mathcal{P}}{}^\alpha{}_\alpha-\frac{1}{2}\xi\widehat{\Box R}\bigg)\frac{1}{(k^2+m^2)^3}-\frac{1}{3}\bigg(\overset{(4)}{\mathcal{P}}{}_{\alpha\beta}-\frac{\xi}{2}\widehat{\nabla_\beta\nabla_\alpha R}\bigg)\partial^\alpha\partial^\beta\frac{1}{(k^2+m^2)^2}, \quad (69)$$

where $\Box\equiv\nabla^\alpha\nabla_\alpha$ and

$$\overset{(4)}{\mathcal{P}}{}^{\alpha}{}_{\alpha} = \frac{1}{20}\widehat{\Box R} + \frac{1}{10}\nabla_{(\beta}\widehat{\nabla_{\alpha)}R^{\alpha\beta}} - \frac{1}{72}\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta} - \frac{1}{360}\hat{R}_{\alpha\beta}\hat{R}^{\beta\alpha} - \frac{1}{90}\hat{R}^{\gamma}{}_{\lambda}\nabla_{\alpha}\widehat{T^{\lambda\alpha}}{}_{\gamma} - \frac{1}{30}\hat{R}^{\gamma}{}_{\lambda}\hat{T}^{\lambda\alpha}{}_{\mu}\hat{T}^{\mu}{}_{\alpha\gamma}$$
$$- \frac{3}{10}\nabla_{(\nu}\widehat{\nabla_{\alpha)}T^{\gamma\alpha}}{}_{\lambda}\hat{T}^{\lambda\nu}{}_{\gamma} - \frac{1}{45}\nabla_{\beta}\widehat{T^{\gamma\beta}}{}_{\lambda}\nabla_{\nu}\widehat{T^{\lambda\nu}}{}_{\gamma} + \frac{1}{45}\nabla_{\nu}\widehat{T^{\gamma\beta\lambda}}\nabla^{\nu}\widehat{T_{\gamma\beta\lambda}} - \frac{1}{45}\nabla^{\nu}\widehat{T^{\gamma\beta}}{}_{\lambda}\nabla_{\beta}\widehat{T^{\lambda}}{}_{\nu\gamma}$$
$$+ \frac{11}{180}\hat{R}_{\kappa\nu\beta\gamma}\hat{T}^{\kappa\nu\lambda}\hat{T}^{\beta\gamma}{}_{\lambda} - \frac{11}{180}\hat{R}^{\lambda\nu\kappa\beta}\hat{T}_{\lambda\beta\gamma}\hat{T}_{\nu\kappa}{}^{\gamma} - \frac{1}{180}\hat{R}^{\gamma}{}_{\beta}{}^{\nu}{}_{\lambda}\nabla_{(\nu}\widehat{T^{\lambda\beta}}{}_{\gamma)} + \frac{1}{90}\hat{R}^{\gamma\beta\lambda\alpha}\hat{R}_{\gamma\lambda\beta\alpha}$$
$$+ \frac{1}{90}\hat{R}^{\gamma\beta\lambda\alpha}\hat{R}_{\lambda\alpha\gamma\beta} - \frac{1}{2880}\hat{T}^{\lambda\beta\gamma}\hat{T}_{\lambda\beta\kappa}\hat{T}_{\alpha\nu}{}^{\kappa}\hat{T}^{\alpha\nu}{}_{\gamma} + \frac{1}{1440}\hat{T}^{\gamma\beta}{}_{\nu}\hat{T}^{\nu\lambda\alpha}\hat{T}_{\alpha\beta\kappa}\hat{T}^{\kappa}{}_{\lambda\gamma}. \quad (70)$$

Sec. IV will show that $[\bar{G}_3] = 0$ and the second line of Eq. (69) does not contribute to $[\bar{G}_4]$. When torsion vanishes, Eq. (69) becomes

$$\bar{G}_4(k) = \left[\left(\frac{1}{6} - \xi\right)^2 \hat{\tilde{R}}^2 + \frac{1}{3}\left(\frac{1}{5} - \xi\right)\tilde{\Box}\hat{\tilde{R}} - \frac{1}{90}\hat{\tilde{R}}_{\alpha\beta}\hat{\tilde{R}}^{\alpha\beta} + \frac{1}{90}\hat{\tilde{R}}^{\gamma\beta\lambda\alpha}\hat{\tilde{R}}_{\gamma\beta\lambda\alpha}\right]\frac{1}{(k^2+m^2)^3} + \left[\frac{1}{6}\left(\xi - \frac{3}{20}\right)\tilde{\nabla}_{\alpha}\tilde{\nabla}_{\beta}\hat{\tilde{R}}\right.$$
$$\left. - \frac{1}{120}\tilde{\Box}\widehat{\tilde{R}_{\alpha\beta}} + \frac{1}{90}\hat{\tilde{R}}_{\alpha\lambda}\hat{\tilde{R}}^{\lambda}{}_{\beta} + \frac{1}{270}\hat{\tilde{R}}_{\gamma\lambda}\hat{\tilde{R}}_{\alpha}{}^{\gamma\lambda}{}_{\beta} - \frac{1}{180}\hat{\tilde{R}}^{\gamma\kappa\lambda\alpha}\hat{\tilde{R}}_{\gamma\kappa\lambda\beta}\right]\partial^{\alpha}\partial^{\beta}\frac{1}{(k^2+m^2)^2}, \quad (71)$$

which agrees with the result in [18]. The Feynman propagator can be obtained by giving m^2 an infinitesimal negative imaginary part $i\epsilon$, i.e., $m^2 + i\epsilon$, and take $i\epsilon$ to be zero at the end of calculation. Since our calculations are valid in n dimensions, it is natural to use dimensional regularization to handle the divergences of the Feynman propagator and effective action in the coincident limit.

IV. RENORMALIZATION OF A SCALAR FIELD IN RIEMANN-CARTAN SPACETIME

It is known that the proper-time representation can be derived from the momentum-space representation in the n dimensional pseudo-Riemannian structure of spacetime [18]. We will show that the derivation can be extended to n dimensional Riemann-Cartan spacetime. In the following, we only consider the approximate solution of $G(x, x')$ up to $G_4(x, x')$. Substituting Eqs. (54), (56), (58), (67), and (69) into (57) and integrating by parts yields

$$\bar{G}(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{ik_\alpha x^\alpha}\left[1 - \frac{1}{4}\hat{T}_{\alpha}x^{\alpha} + a_{\alpha\beta}x^{\alpha}x^{\beta} + (a + b_{\alpha}x^{\alpha} + c_{\alpha\beta}x^{\alpha}x^{\beta})\left(-\frac{\partial}{\partial m^2}\right) + c\left(\frac{\partial}{\partial m^2}\right)^2\right]\frac{1}{k^2+m^2}, \quad (72)$$

where

$$a_{\alpha\beta} = \frac{1}{8}\left[\frac{1}{4}\hat{T}_{\alpha}\hat{T}_{\beta} - \frac{1}{2}\hat{T}_{\alpha\beta\mu}\hat{T}^{\mu} + 4(\overset{(2)}{\mathcal{F}}{}_{\alpha}{}^{\mu}{}_{(\beta\mu)} + \overset{(2)}{\mathcal{F}}{}^{\mu}{}_{\alpha(\beta\mu)}) + 2\overset{(2)}{\mathcal{F}}{}_{\alpha\beta}{}^{\mu}{}_{\mu} - 2\overset{(2)}{\mathcal{S}}{}_{\alpha\beta}\right], \quad (73)$$

$$a = \left(\frac{1}{6} - \xi\right)\hat{R} - \frac{1}{4}\hat{T}_{\alpha}\hat{T}^{\alpha} + \frac{1}{3}\widehat{\nabla_{\alpha}T^{\alpha}} - \frac{1}{8}\hat{T}_{\alpha\beta\gamma}\hat{T}^{\alpha\beta\gamma} - \frac{1}{6}\hat{T}_{\alpha\beta\gamma}\hat{T}^{\gamma\beta\alpha}, \quad (74)$$

$$b_{\alpha} = \frac{i}{2}\left[\left(\frac{1}{12} - \xi\right)\widehat{\nabla_{\alpha}R} + \frac{1}{12}\left(2\nabla^{\beta}\widehat{R_{(\beta\alpha)}} + 2\hat{R}_{\gamma(\alpha\beta)\lambda}\hat{T}^{[\lambda\beta\gamma]} + \hat{T}^{[\beta\lambda\gamma]}\nabla_{(\alpha}\hat{T}_{[\lambda)\gamma\beta]} + \frac{1}{2}\hat{T}_{[\alpha\gamma\beta]}\nabla_{\lambda}\hat{T}^{[\lambda\gamma\beta]}\right)\right], \quad (75)$$

$$c_{\alpha\beta} = \frac{1}{3}\left(\overset{(4)}{\mathcal{P}}{}_{\alpha\beta} - \frac{\xi}{2}\nabla_{\beta}\widehat{\nabla_{\alpha}R}\right), \quad (76)$$

$$c = \frac{1}{2}\left[\left(\frac{1}{6} - \xi\right)\hat{R} + \frac{1}{24}\hat{T}_{[\alpha\beta\gamma]}\hat{T}^{[\alpha\beta\gamma]}\right]^2 + \frac{1}{3}\left(\overset{(4)}{\mathcal{P}}{}^{\alpha}{}_{\alpha} - \frac{1}{2}\xi\widehat{\Box R}\right). \quad (77)$$

It should be stressed that the coefficients a and $a_{\alpha\beta}$ are considered in a general background torsion field but the other coefficients b_α, $c_{\alpha\beta}$ and c are considered in a background totally antisymmetric torsion field.

Defining

$$F(x, x'; is) = 1 - \frac{1}{4}\hat{T}_{\alpha}x^{\alpha} + a_{\alpha\beta}x^{\alpha}x^{\beta} + (a + b_{\alpha}x^{\alpha} + c_{\alpha\beta}x^{\alpha}x^{\beta})is + c(is)^2, \quad (78)$$

and using the integral representation

$$(k^2 + m^2 + i\epsilon)^{-1} = \int_0^\infty ids \exp[-is(k^2 + m^2 + i\epsilon)], \quad (79)$$

one then performs $d^n k$ integration in Eq. (72) to obtain (dropping $i\epsilon$)

$$\bar{G}(x, x') = i(4\pi)^{-n/2}\int_0^\infty ids(is)^{-n/2} \times \exp[-im^2 s - (\sigma/2is)]F(x, x'; is), \quad (80)$$

where $\sigma(x, x') = \frac{1}{2}x^{\alpha}x_{\alpha}$ is half the square of the autoparallel distance between x and x'. Since $|g(x')| = 1$ in the generalized normal coordinates, it gives

$$G(x, x') = |g(x)|^{-1/4}\bar{G}(x, x'). \quad (81)$$

By introducing a determinant defined by[1]

$$\triangle(x, x') = -|g(x)|^{-1/2}\det[-\partial_\mu \partial_{\nu'}\sigma]|g(x')|^{-1/2} \quad (82)$$

and noticing that Eq. (82) reduces to $|g(x)|^{-1/2}$ in the generalized normal coordinates, we obtain

$$G(x, x') = \frac{i\triangle(x, x')1/2}{(4\pi)^{n/2}}\int_0^\infty ids(is)^{-n/2} \times \exp[-im^2s - (\sigma/2is)]F(x, x'; is), \quad (83)$$

which may be considered as the proper-time representation in n-dimensional Riemann-Cartan spacetime. When torsion vanishes, Eq. (83) yields a usual expression of the DeWitt-Schwinger proper-time representation in the n-dimensional pseudo-Riemannian structure of spacetime.

It is known that the first $\frac{n}{2}$ terms of Eq. (83) are divergent in the $x \to x'$ limit [11]. If one considers that n can be analytically continued throughout the complex plane, Eq. (83) at $x \to x'$ limit becomes

$$G(x, x) = \frac{i}{(4\pi)^{n/2}}\left[m^2\Gamma\left(-\frac{n}{2}+1\right) + a(x)\Gamma\left(-\frac{n}{2}+2\right) + m^{-2}c(x)\Gamma\left(-\frac{n}{2}+3\right)\right]. \quad (84)$$

When $n \to 4$, Eq. (84) indicates that only the first two terms are divergent.

From Eq. (44), it can be shown that [11]

$$W = -\frac{i}{2}\int_M\left[\lim_{x\to x'}\int_0^\infty ids(is)^{-1}G(x, x')\right]\star 1. \quad (85)$$

By substituting Eq. (83) into Eq. (85), the divergent terms in the four dimensional spacetime yield

$$L_{\rm div} = \lim_{n\to 4}\frac{1}{(32\pi^2)}\left[m^4\Gamma\left(-\frac{n}{2}\right) + m^2a(x)\Gamma\left(-\frac{n}{2}+1\right) + c(x)\Gamma\left(-\frac{n}{2}+2\right)\right]. \quad (86)$$

It turns out that the divergent terms are entirely geometrical and involve only $a(x)$ and $c(x)$. By adding the counterterms, which contain bare coefficients, into the gravitational Lagrangian, the infinite quantities of $L_{\rm div}$ can be absorbed into bare coefficients to obtain renormalized physical quantities. It should be pointed out that, for totally antisymmetric torsion, $a(x)$ and $c(x)$ may be compared to the coefficients b_2 and b_4 (i.e. Eq. (4.2.27) and (4.3.10)) in [16]. It is easy to see that $a(x)$ in totally antisymmetric torsion case, which is referred to Eq. (62), is equivalent to b_2. However, we have not verified the equivalence of $c(x)$ and b_4 yet, since it involves using the Bianchi identities.

[1]Equation (82) returns to the well-known Van Vleck determinant in pseudo-Riemannian geometry.

V. CONCLUSION AND DISCUSSION

We obtain the momentum-space representation of the Feynman propagator of a free massive scalar field in Riemann-Cartan spacetime. Moreover, the proper-time representation in n-dimensional Riemann-Cartan spacetime has been derived from our momentum-space representation. It leads us to find the divergences of the one-loop effective action by using dimensional regularization. It turns out that the divergences of one-loop effective action of the scalar field are purely geometrical and involve full curvature, torsion and their covariant derivatives. It is interesting to notice that though there is no direct coupling between torsion and the scalar field in the classical action, those divergences do contain torsion parts. When torsion vanishes, our momentum-space representation agrees with the results in [18].

It has been demonstrated that the momentum-space representation is useful for studying the renormalizability of interacting fields in the pseudo-Riemannian structure of spacetime [18]. So our current work can be directly applied to study the renormalizability of interacting scalar fields in Riemann-Cartan spacetime. Moreover, finding momentum-space representation of the Feynman propagator for spin 1/2 field in Riemann-Cartan spacetime is straightforward by using the generalized normal coordinates. These considerations will be taken up in our future work.

Our original motivation was to study quantum effects of our inflation model [3] in Riemann-Cartan spacetime. It turns out that our inflation model, which contains quadratic curvature terms, is a subclass of the effective action. Therefore, it might be interesting to find the renormalized stress 3-forms and spin 3-forms, and study these quantum effects in the early Universe. A further investigation on reheating and primordial perturbations will also be studied in the future.

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APPENDIX A: FEYNMAN PROPAGATOR IN THE GENERALIZED NORMAL COORDINATES

In Sec. II, we obtained the orthonormal coframes $\{e^a\}$ and connection 1-forms $\{\omega^a{}_b\}$ in the generalized normal coordinates. To obtain Eq. (48) in the generalized normal coordinates, it is useful to find the metric components $g_{\alpha\beta}$ with respect to $\{dx^\alpha\}$. Using

$$g = \eta_{ab} e^a \otimes e^b = g_{\alpha\beta} dx^\alpha \otimes dx^\beta \tag{A1}$$

and substituting Eqs. (28), (30), and (32) into (A1) gives

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \hat{T}_{(\alpha\beta)\gamma} x^\gamma + \frac{1}{3}\Big[\hat{R}_{\gamma(\alpha\beta)\delta} - 2\widehat{\nabla_\delta T}_{(\alpha\beta)\gamma} + \frac{1}{2}(\hat{T}_{\alpha\gamma\epsilon}\hat{T}^\epsilon{}_{\delta\beta} + \hat{T}_{\beta\gamma\epsilon}\hat{T}^\epsilon{}_{\delta\alpha}) + \frac{3}{4}\hat{T}^\epsilon{}_{\gamma\alpha}\hat{T}_{\epsilon\delta\beta}\Big] x^\gamma x^\delta + \cdots, \tag{A2}$$

which can be used to find the solutions $G_0(x, x')$, $G_1(x, x')$ and $G_2(x, x')$. However, the solutions $G_3(x, x')$ and $G_4(x, x')$ are restricted to the background of totally antisymmetric torsion $T_{\alpha\beta\gamma} = T_{[\alpha\beta\gamma]}$, so substituting Eqs. (28), (30), (32), (34), and (36), into (A1) and considering the torsion field to be totally antisymmetric yields

$$\begin{aligned} g_{\alpha\beta} = {}& \eta_{\alpha\beta} + \frac{1}{3}\Big[\hat{R}_{\gamma(\alpha\beta)\delta} + \frac{1}{4}\hat{T}_{\alpha\gamma\epsilon}\hat{T}^\epsilon{}_{\delta\beta}\Big] x^\gamma x^\delta + \frac{1}{12}\Big[-\widehat{\nabla_\delta R}_{\alpha\gamma\beta\epsilon} + \frac{1}{2}\hat{R}_{\alpha\gamma\epsilon}{}^\kappa\hat{T}_{\kappa\delta\beta} + \frac{1}{2}\hat{T}_{\alpha\gamma\kappa}\widehat{\nabla_\delta T}^\kappa{}_{\epsilon\beta} \\ & - \frac{1}{2}\hat{T}_{\alpha\gamma\kappa}\hat{R}^\kappa{}_{\delta\epsilon\beta} + \alpha \leftrightarrow \beta\Big] x^\gamma x^\delta x^\epsilon + \Big[\frac{1}{120}(-3\widehat{\nabla_\kappa\nabla_\epsilon R}_{\alpha\gamma\beta\delta} + 3\widehat{\nabla_\epsilon R}_{\alpha\gamma\delta\lambda}\hat{T}^\lambda{}_{\kappa\beta} + \hat{R}_{\alpha\gamma\delta\lambda}\hat{R}^\lambda{}_{\epsilon\kappa\beta} \\ & + 2\hat{R}_{\alpha\gamma\delta\lambda}\widehat{\nabla_\epsilon T}^\lambda{}_{\kappa\beta} + \hat{R}_{\alpha\gamma\delta\lambda}\hat{T}^\lambda{}_{\kappa\mu}\hat{T}^\mu{}_{\epsilon\beta} + 9\widehat{\nabla_\epsilon\nabla_\delta T}_{\alpha\gamma\lambda}\hat{T}^\lambda{}_{\kappa\beta}) - \frac{1}{45}\widehat{\nabla_\delta T}_{\alpha\gamma\lambda}\hat{R}^\lambda{}_{\epsilon\kappa\beta} + \frac{1}{90}\widehat{\nabla_\delta T}_{\alpha\gamma\lambda}\widehat{\nabla_\kappa T}^\lambda{}_{\epsilon\beta} \\ & + \frac{1}{360}\widehat{\nabla_\delta T}_{\alpha\gamma\lambda}\hat{T}^\lambda{}_{\epsilon\mu}\hat{T}^\mu{}_{\kappa\beta} - \frac{1}{40}\hat{T}_{\alpha\gamma\lambda}\widehat{\nabla_\kappa R}^\lambda{}_{\delta\epsilon\beta} - \frac{1}{80}\hat{T}_{\alpha\gamma\lambda}\hat{R}^\lambda{}_{\epsilon\kappa\mu}\hat{T}^\mu{}_{\delta\beta} - \frac{1}{80}\hat{T}_{\alpha\gamma\lambda}\hat{T}^\lambda{}_{\epsilon\mu}\hat{R}^\mu{}_{\kappa\delta\beta} \\ & + \frac{1}{720}\hat{T}_{\alpha\gamma\lambda}\hat{T}^\lambda{}_{\epsilon\mu}\hat{T}^\mu{}_{\kappa\nu}\hat{T}^\nu{}_{\delta\beta} + \frac{1}{72}\hat{R}_{\lambda\gamma\delta\alpha}\hat{R}^\lambda{}_{\epsilon\kappa\beta} + \alpha \leftrightarrow \beta\Big] x^\gamma x^\delta x^\epsilon x^\kappa + \cdots, \end{aligned} \tag{A3}$$

where $\alpha \leftrightarrow \beta$ denotes interchange of the indices. It is not difficult to verify that when the torsion field vanishes, Eq. (A3) will return to the well-known result obtained in the pseudo-Riemannian geometry [24].

Since Eq. (48) only involves exterior derivative d acting on $\bar{G}$, it can be expressed in terms of $g_{\alpha\beta}$ and the Christoffel symbol $\tilde{\Gamma}^\alpha{}_{\beta\gamma}$

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\bar{G} + \partial_\alpha g^{\alpha\beta}\partial_\beta\bar{G} - \Big(\frac{1}{2}\partial_\alpha g^{\alpha\beta}\tilde{\Gamma}_{\gamma\beta}{}^\gamma + \frac{1}{4}g^{\alpha\beta}\tilde{\Gamma}_{\gamma\alpha}{}^\gamma\tilde{\Gamma}_{\delta\beta}{}^\delta + \frac{1}{2}\tilde{\Gamma}_{\gamma\beta}{}^\gamma{}_{,\alpha}g^{\alpha\beta}\Big) \times \bar{G} - (m + \xi R)\bar{G} = -\delta(x - x'). \tag{A4}$$

In the generalized normal coordinates, one has the following expansions

$$g^{\mu\nu} = \eta^{\mu\nu} + \overset{(1)}{\mathcal{F}}{}^{\mu\nu}{}_\alpha x^\alpha + \overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta} x^\alpha x^\beta + \overset{(3)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma + \overset{(4)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma\lambda} x^\alpha x^\beta x^\gamma x^\lambda + \cdots, \tag{A5}$$

$$\partial_\mu g^{\mu\nu} = \overset{(1)}{\mathcal{S}}{}^\nu + \overset{(2)}{\mathcal{S}}{}^\nu{}_\alpha x^\alpha + \overset{(3)}{\mathcal{S}}{}^\nu{}_{\alpha\beta} x^\alpha x^\beta + \overset{(4)}{\mathcal{S}}{}^\nu{}_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma + \cdots, \tag{A6}$$

and,

$$-\Big(\frac{1}{2}\partial_\alpha g^{\alpha\beta}\tilde{\Gamma}_{\gamma\beta}{}^\gamma + \frac{1}{4}g^{\alpha\beta}\tilde{\Gamma}_{\gamma\alpha}{}^\gamma\tilde{\Gamma}_{\delta\beta}{}^\delta + \frac{1}{2}\tilde{\Gamma}_{\gamma\beta}{}^\gamma{}_{,\alpha}g^{\alpha\beta}\Big) = \overset{(2)}{\mathcal{P}} + \overset{(3)}{\mathcal{P}}_\alpha x^\alpha + \overset{(4)}{\mathcal{P}}_{\alpha\beta} x^\alpha x^\beta + \cdots. \tag{A7}$$

By substituting Eqs. (A5)–(A7) into Eq. (A4), we then obtain Eq. (49).

Using Eq. (A2), we obtain

$$\overset{(1)}{\mathcal{F}}{}^{\mu\nu}{}_\alpha = \hat{T}^{(\mu\nu)}{}_\alpha, \tag{A8}$$

$$\overset{(1)}{\mathcal{S}}{}^\nu = \frac{1}{2}\hat{T}^\nu, \tag{A9}$$

$$\overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta} = \hat{T}_\epsilon{}^\mu{}_\alpha\hat{T}^{\epsilon\nu}{}_\beta - \frac{1}{3}\Big[\hat{R}_\alpha{}^{(\mu\nu)}{}_\beta - 2\widehat{\nabla_\beta T}^{(\mu\nu)}{}_\alpha + \frac{1}{2} \times (\hat{T}^\mu{}_{\alpha\epsilon}\hat{T}^\epsilon{}_\beta{}^\nu + \hat{T}^\nu{}_{\alpha\epsilon}\hat{T}^\epsilon{}_\beta{}^\mu) + \frac{3}{4}\hat{T}^\epsilon{}_\alpha{}^\mu\hat{T}_{\epsilon\beta}{}^\nu\Big], \tag{A10}$$

$$\overset{(2)}{\mathcal{S}}{}^\nu{}_\alpha = \frac{2}{3}\hat{T}_\epsilon\hat{T}^{\epsilon\nu}{}_\alpha + \frac{3}{4}\hat{T}_\epsilon{}^\mu{}_\alpha\hat{T}^{\epsilon\nu}{}_\mu - \frac{1}{6}(\hat{R}_\alpha{}^\nu + \hat{R}^\nu{}_\alpha - 4\widehat{\nabla_\mu T}^{(\mu\nu)}{}_\alpha - 2\widehat{\nabla_\alpha T}^\nu + \hat{T}^\nu{}_{\mu\epsilon}\hat{T}^\epsilon{}_\alpha{}^\mu + \hat{T}^\mu{}_{\alpha\epsilon}\hat{T}^\epsilon{}_\mu{}^\nu), \tag{A11}$$

$$\overset{(2)}{\mathcal{P}} = -\frac{3}{16}\hat{T}^\alpha\hat{T}_\alpha + \frac{1}{6}\hat{R} + \frac{1}{3}\widehat{\nabla_\alpha T}^\alpha - \frac{1}{24}\hat{T}_{\alpha\beta\gamma}\hat{T}^{\gamma\beta\alpha}. \tag{A12}$$

In the case of totally antisymmetric torsion, one may use Eq. (A3) to obtain

$$\overset{(1)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha} = \overset{(1)}{\mathcal{S}}{}^{\nu} = 0, \tag{A13}$$

$$\overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta} = -\frac{1}{3}\left(\hat{R}_{\alpha\beta}{}^{(\mu\nu)} + \frac{1}{4}\hat{T}^{\mu}{}_{\alpha\epsilon}\hat{T}^{\epsilon}{}_{\beta}{}^{\nu}\right), \tag{A14}$$

$$\overset{(2)}{\mathcal{S}}{}^{\nu}{}_{\alpha} = -\frac{1}{6}\left(\hat{R}_{\alpha}{}^{\nu} + \hat{R}^{\nu}{}_{\alpha} + \frac{1}{2}\hat{T}_{\epsilon\mu\alpha}\hat{T}^{\epsilon\mu\nu}\right), \tag{A15}$$

$$\overset{(2)}{\mathcal{P}} = \frac{1}{6}\left(\hat{R} + \frac{1}{4}\hat{T}_{\epsilon\mu\alpha}\hat{T}^{\epsilon\mu\alpha}\right), \tag{A16}$$

$$\overset{(3)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma} = -\frac{1}{12}\left(-\widehat{\nabla_{\beta}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\gamma}} + \frac{1}{2}\hat{R}^{\mu}{}_{\alpha\gamma}{}^{\kappa}\hat{T}_{\kappa\beta}{}^{\nu} + \frac{1}{2}\hat{T}^{\mu}{}_{\alpha\kappa}\widehat{\nabla_{\beta}T^{\kappa}{}_{\gamma}{}^{\nu}} - \frac{1}{2}\hat{T}^{\mu}{}_{\alpha\kappa}\hat{R}^{\kappa}{}_{\beta\gamma}{}^{\nu} + \mu\leftrightarrow\nu\right), \tag{A17}$$

$$\overset{(3)}{\mathcal{S}}{}^{\nu}{}_{\alpha\beta} = \overset{(3)}{\mathcal{F}}{}^{\nu\mu}{}_{\mu\alpha\beta} + \overset{(3)}{\mathcal{F}}{}^{\nu\mu}{}_{\alpha\mu\beta} + \overset{(3)}{\mathcal{F}}{}^{\nu\mu}{}_{\alpha\beta\mu}, \tag{A18}$$

$$\overset{(3)}{\mathcal{P}}_{\alpha} = \frac{1}{12}\left(\widehat{\nabla_{\alpha}R} + \widehat{\nabla_{\mu}R^{\mu}{}_{\alpha}} + \widehat{\nabla_{\mu}R_{\alpha}{}^{\mu}} + 2\hat{R}_{\mu(\nu\alpha)\kappa}\hat{T}^{\kappa\nu\mu} - \hat{T}^{\mu\nu}{}_{\kappa}\widehat{\nabla_{(\alpha}T^{\kappa}{}_{\nu)\mu}} - \frac{1}{2}\hat{T}^{\mu\alpha\kappa}\widehat{\nabla_{\nu}T_{\kappa}{}^{\nu}{}_{\mu}}\right), \tag{A19}$$

$$\begin{aligned}\overset{(4)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta\gamma\delta} = -\Bigg[&\frac{1}{120}\Big(-3\nabla_{\delta}\widehat{\nabla_{\gamma}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}} + 3\widehat{\nabla_{\gamma}R^{\mu}{}_{\alpha\beta\lambda}}\hat{T}^{\lambda}{}_{\delta}{}^{\nu} + \hat{R}^{\mu}{}_{\alpha\beta\lambda}\hat{R}^{\lambda}{}_{\gamma\delta}{}^{\nu} + 2\hat{R}^{\mu}{}_{\alpha\beta\lambda}\widehat{\nabla_{\gamma}T^{\lambda}{}_{\delta}{}^{\nu}} + \hat{R}^{\mu}{}_{\alpha\beta\lambda}\hat{T}^{\lambda}{}_{\delta\epsilon}\hat{T}^{\epsilon}{}_{\gamma}{}^{\nu}\\ &+ 9\nabla_{\gamma}\widehat{\nabla_{\beta}T^{\mu}{}_{\alpha\lambda}}\hat{T}^{\lambda}{}_{\delta}{}^{\nu}\Big) - \frac{1}{45}\widehat{\nabla_{\beta}T^{\mu}{}_{\alpha\lambda}}\hat{R}^{\lambda}{}_{\gamma\delta}{}^{\nu} + \frac{1}{90}\widehat{\nabla_{\beta}T^{\mu}{}_{\alpha\lambda}}\widehat{\nabla_{\delta}T^{\lambda}{}_{\gamma}{}^{\nu}} + \frac{1}{360}\widehat{\nabla_{\beta}T^{\mu}{}_{\alpha\lambda}}\hat{T}^{\lambda}{}_{\gamma\epsilon}\hat{T}^{\epsilon}{}_{\delta}{}^{\nu}\\ &- \frac{1}{40}\hat{T}^{\mu}{}_{\alpha\lambda}\widehat{\nabla_{\delta}R^{\lambda}{}_{\beta\gamma}{}^{\nu}} - \frac{1}{80}\hat{T}^{\mu}{}_{\alpha\lambda}\hat{R}^{\lambda}{}_{\gamma\delta\epsilon}\hat{T}^{\epsilon}{}_{\beta}{}^{\nu} - \frac{1}{80}\hat{T}^{\mu}{}_{\alpha\lambda}\hat{T}^{\lambda}{}_{\gamma\epsilon}\hat{R}^{\epsilon}{}_{\delta\beta}{}^{\nu} + \frac{1}{720}\hat{T}^{\mu}{}_{\alpha\lambda}\hat{T}^{\lambda}{}_{\gamma\epsilon}\hat{T}^{\epsilon}{}_{\delta\kappa}\hat{T}^{\kappa}{}_{\beta}{}^{\nu}\\ &+ \frac{1}{72}\hat{R}_{\lambda\alpha\beta}{}^{\mu}\hat{R}^{\lambda}{}_{\gamma\delta}{}^{\nu} + \mu\leftrightarrow\nu\Bigg] + \frac{1}{9}\left(\hat{R}_{\alpha(\kappa}{}^{\mu)}{}_{\beta} + \frac{1}{4}\hat{T}_{\kappa\alpha\epsilon}\hat{T}^{\epsilon}{}_{\beta}{}^{\mu}\right)\left(\hat{R}_{\gamma}{}^{(\kappa\nu)}{}_{\delta} + \frac{1}{4}\hat{T}^{\kappa}{}_{\gamma\epsilon}\hat{T}^{\epsilon}{}_{\delta}{}^{\nu}\right),\end{aligned} \tag{A20}$$

$$\overset{(4)}{\mathcal{S}}{}^{\nu}{}_{\alpha\beta\gamma} = \overset{(4)}{\mathcal{F}}{}^{\nu\mu}{}_{\mu\alpha\beta\gamma} + \overset{(4)}{\mathcal{F}}{}^{\nu\mu}{}_{\alpha\mu\beta\gamma} + \overset{(4)}{\mathcal{F}}{}^{\nu\mu}{}_{\alpha\beta\mu\gamma} + \overset{(4)}{\mathcal{F}}{}^{\nu\mu}{}_{\alpha\beta\gamma\mu}, \tag{A21}$$

$$\begin{aligned}\overset{(4)}{\mathcal{P}}_{\alpha\beta} = &\frac{1}{2}\overset{(2)}{\mathcal{S}}{}^{\nu}{}_{\alpha}\overset{(2)}{\mathcal{F}}{}^{\mu}{}_{\mu(\nu\beta)} - \frac{1}{4}\overset{(2)}{\mathcal{F}}{}^{\mu}{}_{\mu}{}^{(\nu}{}_{\alpha)}\overset{(2)}{\mathcal{F}}{}^{\mu}{}_{\mu(\nu\beta)} + \frac{1}{2}\overset{(2)}{\mathcal{F}}{}^{\kappa}{}_{\kappa\mu\nu}\overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta} + \frac{1}{2}\left(\overset{(2)}{\mathcal{F}}{}^{\mu\nu\kappa}{}_{\alpha}\overset{(2)}{\mathcal{F}}_{\mu\nu(\kappa\beta)} + \overset{(2)}{\mathcal{F}}{}^{\mu\nu(\kappa}{}_{\alpha)}\overset{(2)}{\mathcal{F}}_{\mu\nu\beta\kappa}\right)\\ &+ \frac{1}{2}\overset{(2)}{\mathcal{F}}{}^{\mu\nu}{}_{\alpha\beta}\overset{(2)}{\mathcal{F}}_{\mu\nu}{}^{\kappa}{}_{\kappa} + \frac{1}{2}\left(\overset{(4)}{\mathcal{F}}{}^{\mu}{}_{\mu}{}^{\nu}{}_{\nu\alpha\beta} + \overset{(4)}{\mathcal{F}}{}^{\mu}{}_{\mu}{}^{\nu}{}_{\alpha\nu\beta} + \overset{(4)}{\mathcal{F}}{}^{\mu}{}_{\mu}{}^{\nu}{}_{\alpha\beta\nu} + \overset{(4)}{\mathcal{F}}{}^{\mu}{}_{\mu\alpha}{}^{\nu}{}_{\beta\nu} + \overset{(4)}{\mathcal{F}}{}^{\mu}{}_{\mu\alpha}{}^{\nu}{}_{\nu\beta} + \overset{(4)}{\mathcal{F}}{}^{\mu}{}_{\mu\alpha\beta}{}^{\nu}{}_{\nu}\right).\end{aligned} \tag{A22}$$

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