Schwarzschild–anti-de Sitter black holes within isothermal cavity: Thermodynamics, phase transitions, and the Dirichlet problem

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The thermodynamics of Schwarzschild black holes within spherical isothermal cavities in anti–de Sitter (AdS) space is studied for arbitrary dimensions in the semiclassical approximation of the Euclidean path integral of quantum gravity. For such boundary conditions, known classical solutions are a hot AdS and two or no Schwarzschild-AdS depending on whether or not the wall-temperature of the cavity is above or below a minimum value. Earlier work in four dimensions with such boundary conditions showed that the larger and smaller holes have positive and negative specific heats and hence are locally thermodynamically stable and unstable, respectively. The standard area-law of entropy was known to hold too. We derive the area-law for arbitrary dimensions and show that qualitative behavior of local stability remains the same. Then using a careful analysis of the associated Dirichlet boundary-value problem we address global issues. We find that for wall-temperatures above a critical value a phase transition takes hot AdS to the larger Schwarzschild-AdS. The larger hole thus can be globally thermodynamically stable. We find that the smaller the cavity the higher the critical temperature for phase transition is and it always remains above the minimum temperature needed for the classical existence of the holes in that cavity. In the infinite limit of cavity this picture reduces to that considered by Hawking and Page. All these hold for arbitrary dimensions, however the case of five dimensions turns out to be special in that the Dirichlet problem can be solved exactly giving exact analytic expressions for the black-hole masses as functions of boundary variables (cavity-radius and temperature). This makes it possible to compute the on-shell Euclidean action as a function of boundary variables too from which other quantities of interest can be evaluated. In particular, we obtain the minimum temperature (for the holes to exist classically) and the critical temperature (for phase transition) as functions of the cavity-radius in five dimensions.

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I. INTRODUCTION

Following the proposal of the AdS/CFT correspondence [1] which connects string theories on asymptotically anti–de Sitter (AdS) spaces to certain conformal field theories on their boundaries, black holes in AdS spaces have received renewed attention. In particular, the Hawking-Page phase transition from hot AdS space to Schwarzschild-AdS space [2] has been shown by Witten [3,4] to correspond to a phase transition from the confining phase to the deconfining phase in the large N limit of $\mathcal{N} = 4$ Yang-Mills on the boundary of the AdS space. He argued that the strength of the conjecture is demonstrated by the existence of a holographic duality even at nonzero temperature where supersymmetry and conformal invariance are broken.

In the Euclidean picture the boundary at which the conformal field theory is defined is an $S^1 \times S^{n-1}$ which is a codimension-one hypersurface of the Euclideanized AdS and Schwarzschild-AdS at radial infinity. It is the "conformal infinity" of the two spaces in which the radii of S^1 and S^{n-1} are infinite but their ratio is a finite number. However, one can consider a more general, and nontrivial, boundary by taking the $S^1 \times S^{n-1}$ at a finite radial distance so that the S^{n-1} is of finite volume (and assuming, initially, that the S^1-fiber has finite radius as well). Infinite volume for either or both of S^1 and S^{n-1} can occur as limits. Such a boundary in the Euclidean path-integral approach to quantum gravity represents the canonical thermodynamic ensemble with the interpretation that an S^{n-1} cavity of radius α is immersed in an isothermal heat bath of temperature $T = \frac{1}{2\pi\beta}$, where (α, β) are the two radii. To understand the quantum gravitational effects consistent with such a boundary condition, in the Euclidean approach to quantum gravity, one computes the partition function by summing over all the $(n + 1)$-dimensional regular Riemannian geometries which admit the given $S^1 \times S^{n-1}$ geometry as their only boundary, with each geometry contributing to the integral by the exponential of the negative value of its Euclidean action. Such geometries are variedly called bulk or infilling geometries of the (codimension-one) boundary geometry. Since regular classical geometries (whose metrics solve the field equations) extremize the action, one first makes the semiclassical approximation to the path-integral by estimating (tree-level) contributions coming from regular Riemannian solutions of the field equations. In general, depending on the geometry of the given n-boundary there

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may be no, one, or many classical solutions falling within the same or different topological classes. One can then perturb these classical solutions to understand if they are locally stable or not. When multiple solutions exist, especially when they belong to different topological classes, one asks how different solutions dominate the path-integral as one changes the boundary conditions within the permissible range.

A primary objective in such studies therefore is to understand the relationship between the geometries of the possible regular classical bulk solutions and the geometry of the given boundary. Such study falls under the purview of the classical Dirichlet boundary-value problem in Riemannian geometry (see below). Once the relationship between various classical bulk geometries and the boundary geometry are understood, either qualitatively or quantitatively, one can explore their role in quantum gravity using the semiclassical approximation. In particular, if the geometries of the bulk are known as functions of the boundary variables, one can find the on-shell classical action and other quantities of interest purely in terms of boundary data. In the example above, in which the boundary is at infinity (and wherein the only meaningful boundary data is the ratio of the radii of S^1 and S^{n-1}, as we will see later) this can be done exactly for arbitrary dimensions. However, in the case of finite boundaries, where the two radii vary freely, obtaining classical bulk geometries in terms of boundary variables is rather nontrivial for $\Lambda < 0$ and it requires careful analysis to extract semiclassical information as functions of the boundary variables as we will see in this paper. From the holographic point of view, understanding such finite nonconformal boundaries and their relationships with the bulk geometries falls within the study of the stronger form of holography. Indeed a large literature exists where this has been considered in the context of the Randall-Saundrum type scenarios [5] and various related topics (see [6] for a review and references therein).

The canonical ensemble with spherical isothermal cavities for $\Lambda = 0$ was first studied by York [7] for four spacetime dimensions. It was found that apart from hot flat space, classically there are two or no black-hole solutions depending on whether the temperature of the heat bath is above or below a certain value.[1] This is true for any finite radius of the cavity. Of the two solutions, the larger one has a positive specific heat and the smaller one a negative specific heat. Hence, only the larger solution is locally thermodynamically stable. It is possible to find the masses of the two black holes as functions of the cavity-radius and wall-temperature (i.e., in terms of α and β in the Euclidean picture). For sufficiently high temperature the larger mass solution has a more negative action than hot flat space and hence a stable black hole can nucleate from hot flat space in a thermodynamically consistent manner within a cavity of arbitrary radius. In the infinite-volume limit of the cavity only the smaller-mass solution with the negative specific heat survives. Therefore, for $\Lambda = 0$, only in a finite cavity is it possible to have a thermodynamically stable black-hole solution for such boundary conditions. Recently these results have been generalized and discussed for higher dimensions in [8].

However, in the case of a negative cosmological constant, it was known since the work of Hawking and Page [2] that stable Schwarzschild-AdS$_{n+1}$ black holes are possible and one is not required to introduce an isothermal finite cavity, at least for obtaining a stable solution. This perhaps explains the comparative lack of attention in the literature to the finite-cavity case with $\Lambda < 0$ except the studies made in [9–11]. In [9] it is shown that for such a boundary condition, there are two/no Schwarzschild-AdS$_4$ solutions and always an AdS$_4$ solution and—assuming that the standard area-law of black-hole entropy holds for such boundary conditions—the larger and the smaller black hole have positive and negative specific heats, respectively, as in the $\Lambda = 0$ case. Concerning themselves mostly with the local stability, the authors of [10] derived the standard area-law of entropy of Reissner-Nordström-AdS black holes in the grand canonical ensemble (of which Schwarzschild-AdS$_4$ in isothermal cavities is a special case), thus completing the picture in four dimensions. For arbitrary dimensions the only mention of a finite cavity appears in [11] which however ultimately considers the infinite-volume limit to search for the negative modes predicted in [2] for the unstable solution.[2] None of them [9–11] studied the Dirichlet problem in detail to address the global issues like phase transition for such boundary conditions as was done by Hawking and Page in the infinite-cavity limit and for $\Lambda = 0$ in finite cavity by York. The primary objective of this paper is to address global issues in detail for such boundary conditions as well as provide a derivation of the area-law of entropy in arbitrary dimensions from which local stability results follow. Comparing with the flat-space case in four and higher dimensions [7,8], we will see the difficulty comes from the nonvanishing cosmological constant and one cannot reduce the problem to one with a single variable except in the limit of infinite cavity or vanishing cosmological constant.

The goal of this paper is therefore twofold. On the classical side of the picture we will discuss solutions of bulk geometries in terms of boundary data, i.e., the two

[1] Lest the terminology be a source of confusion due to their connotations, in $(n+1)$ dimensions the "cavity" refers to the S^{n-1} base and is not the same as what we called the "boundary" in the Euclideanized picture which is an $S^1 \times S^{n-1}$. Also note that the word "classical" refers to the Riemannian solutions (which embody quantum gravitational effects of the corresponding Lorentzian solutions).

[2] For $\Lambda = 0$ finite-cavity negative modes were studied in [12].

radii (β, α) of $S^1 \times S^{n-1}$. We will see how solutions appear and disappear as α and β are varied. We will then combine this to understand local and global stability and show how a phase transition occurs within the cavity for sufficiently high value of its wall-temperature irrespective of its radius. We will also show that in the infinite-cavity limit how this phase transition reproduces the Hawking-Page phase transition, in which case the phase transition is studied as functions of the Hawking temperature (or equivalently the horizon-radius) of the black-holes instead of the two boundary variables (which is the only way to do this since the local temperature of any finite Schwarzschild-AdS redshifts to zero at infinity, unlike in the flat-space case, see later). We will be as rigorous as possible in our treatment and obtain explicit formulas and relate them to the well-known $\Lambda = 0$ results. Interestingly, we find that a precise quantitative treatment is possible only in five dimensions. This is the most interesting because of its connection with the 4-D world within a holographic context. The results obtained herein can thus be extended in various directions and set the ground for many future investigations.

This paper is arranged in the following way. In Sec. II we discuss the Euclidean AdS_{n+1} and Schwarzschild-AdS_{n+1} metrics and set our conventions. In Sec. III we discuss the Dirichlet boundary-value problem and thermodynamics of the black-hole solutions after briefly discussing the $\Lambda = 0$ case. Phase transitions are studied in Sec. IV. In Sec. V we specialize to five dimensions and show how the above questions can be addressed exactly.

II. EUCLIDEAN AdS_{n+1} AND SCHWARZSCHILD-AdS_{n+1}

The Euclidean AdS_{n+1} and Schwarzschild-AdS_{n+1} metrics have the following forms:

$$ds^2 = V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_{n-1}^2, \tag{2.1}$$

with

$$V(r) = \left(1 + \frac{r^2}{l^2}\right), \tag{2.2}$$

and

$$V(r) = \left(1 - \frac{\mu}{r^{n-2}} + \frac{r^2}{l^2}\right), \tag{2.3}$$

respectively. Both metrics satisfy the Einstein equation with a cosmological constant $\Lambda = -\frac{n(n-1)}{2l^2}$. The quantity μ in Eq. (2.3) gives the mass m of the black hole [13]

$$\mu = \frac{16\pi G m}{(n-1)\mathrm{Vol}(S^{n-1})}. \tag{2.4}$$

For notational convenience we will set $l = 1$ without any loss of generality and denote r as ρ and μ as M in this scale. When there is no chance of confusion we will refer to M as the "mass" of the black hole. The horizon of the Schwarzschild-AdS_{n+1} metric (2.1) is the positive root of the equation

$$V(\rho) \equiv \left(1 - \frac{M}{\rho^{n-2}} + \rho^2\right) = 0 \tag{2.5}$$

and will be denoted by ρ_b. The horizon-radius ρ_b and the mass M are in 1-1 correspondence:

$$M = \rho_b^n + \rho_b^{n-2}. \tag{2.6}$$

The singularity at the horizon can be removed if we give t a periodicity of

$$\Delta t = \frac{2\pi}{\kappa} = \frac{4\pi\rho_b}{n\rho_b^2 + n - 2} \tag{2.7}$$

in which

$$\kappa \equiv \frac{1}{2}V'(\rho_b) = \frac{(n-2)}{2}\frac{M}{\rho_b^{n-1}} + \rho_b = \frac{1}{2}\frac{n\rho_b^2 + n - 2}{\rho_b} \tag{2.8}$$

is the surface gravity. The metric is then well-defined for $\rho_b \leq \rho < \infty$ which includes the horizon which is now the $(n-1)$-dimensional fixed point set of the Killing vector $\partial/\partial t$ and is a regular "bolt" of the metric (which explains the use of subscript in ρ_b) [14]. The periodicity Δt gives the inverse of Hawking temperature of the hole

$$T_H = \frac{1}{4\pi}\frac{n\rho_b^2 + n - 2}{\rho_b}. \tag{2.9}$$

Note that the AdS_{n+1} metric is regular for $0 \leq r \leq \infty$. One therefore is not required to ascribe a certain periodicity to t in order to achieve regularity. In other words, one can choose the periodicity of t arbitrarily as in the case of flat space.

III. BLACK HOLES IN AN ISOTHERMAL CAVITY AND THE DIRICHLET PROBLEM

As mentioned earlier, the first problem that we address in this paper formally falls under the scope of the classical Dirichlet problem in which one seeks one or more regular $(n+1)$-dimensional Riemannian solutions ($\mathcal{M}$, $g_{\mu\nu}$) of the Einstein equations for a given n-boundary (Σ, h_{ij}) such that $\partial\mathcal{M} = \Sigma$ and $g_{\mu\nu}|_{\partial\mathcal{M}} = h_{ij}$ possibly with the condition of regularity. In most physically interesting cases, the Dirichlet problem simplifies as only cohomogeneity one metrics whose principal orbits share the topology and symmetry of the boundary are considered. In the presence of only a possible cosmological constant term, this reduces the Einstein equation to a set of ordinary differential equations to be solved subject to the boundary condition and regularity. However, in cases of high symmetry the general solution of this set and the manifolds over which they can be extended completely or partially may be known in advance. The Dirichlet problem then simplifies to the problem of embedding Σ in known manifolds—the infilling solutions then are the compact regular parts of the manifolds enclosed by Σ. Such solutions provide semiclassical approximations to the path-integral and are the starting point of a quantum treatment. Depending on the boundary data there can be zero, one, or multiple infilling solutions of similar or different topologies even when one assumes a high degree of symmetry for the codimension-one slices.

Because the classical action consists of a bulk and boundary term, if the geometries of the possible bulk solutions are known as functions of the boundary variables one can find the on-shell Euclidean actions purely in terms of the boundary variables. Knowledge of the bulk geometries in relation to the boundary geometry is all that one needs to obtain for semiclassical considerations. However, it may not be possible to obtain the infilling geometries as analytic functions of the boundary data which then requires one to adopt an indirect approach.

For either AdS_{n+1} or Schwarzschild-AdS_{n+1}, with the periodic identification of the t-coordinate, a hypersurface at a constant value of the radial coordinate ρ, say $\rho = \rho_0$, has the trivial product topology $S^1 \times S^{n-1}$ and is endowed with the n-metric h_{ij} given by

$$ds^2_\Sigma = \beta^2 dt^2 + \alpha^2 d\Omega^2_{n-1}. \tag{3.1}$$

If one excises the part of the manifold for which $\rho > \rho_0$ one is left with a nonsingular compact manifold with a boundary which, by construction, provides an infilling solution for (Σ, h_{ij}). Therefore for a given $S^1 \times S^{n-1}$ boundary there are two topologically distinct possibilities for the infillings.

Comparing Eqs. (3.1) and (2.1), one obtains $\rho_0 = \alpha$ trivially for either AdS_{n+1} or Schwarzschild-AdS_{n+1}. For a given $S^1 \times S^{n-1}$ boundary the periodicity of the t coordinate of the AdS_{n+1} metric is given by

$$\sqrt{1+\alpha^2}\Delta t = 2\pi\beta. \tag{3.2}$$

Obviously for a given set (α, β) the periodicity is unique.[3] On the other hand, for Schwarzschild-AdS_{n+1} one needs to find the periodicity via (2.7) by solving the following equation for M first:

$$\sqrt{1 - \frac{M}{\alpha^{n-2}} + \alpha^2} = \beta\kappa(M). \tag{3.3}$$

This is the standard relationship between the local temperature at the cavity-wall and the Hawking temperature

$$T_{\mathrm{loc}} = \frac{T_H}{\sqrt{g_{00}}} \equiv \frac{T_H}{\sqrt{(1 - \frac{M}{\alpha^{n-2}} + \alpha^2)}}. \tag{3.4}$$

[3]This slightly different convention for notation, which makes β the "radius" of the S^1 fiber, would avoid the recurrent appearance of π in the rest of the paper.

Equation (3.3) leads to a complicated algebraic expression in M. Obviously there is no *a priori* obstruction as one can *start* with a Schwarzschild-AdS with known mass and take a constant ρ-slice for which Eq. (3.3) is true trivially. However, it is not clear for what values of α and β black-hole solutions exist and what their possible number and properties are. We will be addressing these questions in the following sections. But before doing that we briefly recapitulate what we know about the $\Lambda = 0$ case which is also a limiting case of the present study (i.e. $l \to \infty$ limit). The results for $\Lambda = 0$ will be a constant reference for judging success of the $\Lambda < 0$ case. The results for $\Lambda = 0$ will also help us understand the limit when the cavity-radius is very small compared to l^2, i.e., when $\alpha \ll 1$.

A. The case of zero cosmological constant

For $\Lambda = 0$, Eq. (3.3) simplifies considerably and can be rewritten as [8]

$$x^n - x^2 + p = 0, \tag{3.5}$$

where $x \equiv M^{1/n-2}/\alpha$ and $p = \frac{(n-2)^2}{4}\frac{\beta^2}{\alpha^2}$. It is easy to see that there are in general two solutions if

$$p \le \left(\frac{2}{n}\right)^{2/n-2}\left(1 - \frac{2}{n}\right). \tag{3.6}$$

For a given α, this sets an upper limit to β, i.e., this gives a lower limit to the temperature of the cavity-wall

$$T \ge T_m \equiv \frac{1}{4\pi}\left(\frac{n}{2}\right)^{1/n-2}\sqrt{n(n-2)}\frac{1}{\alpha}. \tag{3.7}$$

For $T < T_m$ no black-hole solution exists. For $T = T_m$ the two solutions are degenerate with

$$\rho_b = \left(\frac{2}{n}\right)^{1/n-2}\alpha. \tag{3.8}$$

For $T > T_m$, there are always two positive roots of Eq. (3.5). For a given cavity, the higher the wall-temperature the heavier (lighter) is the larger (smaller) solution.

For four dimensions Eq. (3.5) is cubic and is solvable [7]. However, apart from a few other special values of n this is not solvable using ordinary algebraic methods. Fortunately, Eq. (3.5) falls under a special class of algebraic equations—commonly known as trinomial equations in the mathematical literature—which can be solved using higher order hypergeometric functions in one variable, as was first shown by Birkeland [15]. For arbitrary n and p the solutions have been obtained in [8]. Thus the corresponding Riemannian Dirichlet problem is exactly solvable for arbitrary dimension.[4]

[4]Restricting of course within the class of cohomogeneity one metrics whose principal orbits share the symmetry and topology of the boundary.

B. $\Lambda < 0$: Schwarzschild-AdS$_{n+1}$ solutions

For $\Lambda = 0$ the problem reduces to studying Eq. (3.5) for which it is only the squashing α/β that matters. For $\Lambda < 0$, the analogue of Eq. (3.5) reads

$$F := \rho_b^{n+2} + \rho_b^n + \frac{1}{4}\alpha^{n-2}\beta^2 n^2 \rho_b^4 + \frac{1}{2}\alpha^{n-2}[\beta^2 n(n-2) - 2(\alpha^2+1)]\rho_b^2 + \frac{1}{4}\alpha^{n-2}\beta^2(n-2)^2 = 0. \quad (3.9)$$

This is obtained from Eq. (3.3) by simply eliminating the square-root and using Eq. (2.6). Since M and ρ_b are in 1-1 correspondence both are in principle equivalent variables. However, use of ρ_b leads to algebraic simplifications, albeit the resulting equation is still difficult to solve exactly.

Only the positive roots $\rho_b < \alpha$ of Eq. (3.9) will give the mass and hence the geometries of the infilling black holes. Note that Eq. (3.9) cannot be reduced to a one-parameter problem by any redefinition of variables. This is true even for the infinite-cavity limit although a different kind of simplification occurs in this limit. The infinite-cavity limit has been studied exhaustively in [2–4] and will emerge as a special case in our study.

In four dimensions Eq. (3.9) is quintic and hence, ordinary algebraic methods fail to produce analytic solutions for the masses of the two black holes for four dimensions in terms of radicals unlike the $\Lambda = 0$ case found in [7]. One can, however, solve Eq. (3.9) in terms of hypergeometric functions by following Birkeland's general solution of algebraic equations of order n with arbitrary coefficients [15] or by using $\mathcal{A}$-hypergeometric functions [16]. Unlike the $\Lambda = 0$ case, the solutions will be in terms of hypergeometric functions of *several* variables. The variables will be nontrivial functions of α and β. This method would be unsuitable for obtaining direct information as the cavity-radius and temperature are varied and hence will not be used in the rest of the paper. We will return to the issue of explicit solutions in five dimensions in Sec. V. However, as we will see below it is possible to study the infilling solutions and their thermodynamics without requiring explicit solutions.

C. Number of solutions

As mentioned earlier, the local study of Schwarzschild-AdS within finite isothermal cavities was made in four spacetime dimensions in [9]. It was shown there that there are two black-hole solutions with negative and positive specific heats within the cavity. Both survive in the infinite cavity-radius limit unlike the Schwarzschild case in flat space. We will now see that this is true for higher dimensions as well.

It is easy to see that Eq. (3.9) does not admit positive solutions if $\beta^2 \geq \frac{2(\alpha^2+1)}{n(n-2)}$ since no changes of sign occur in the coefficients of the various powers which are all positive in this case. Only the coefficient of ρ_b^2 can be negative and this happens if

$$\beta^2 < \frac{2(\alpha^2+1)}{n(n-2)}. \quad (3.10)$$

There can be up to two positive roots and up to two (for n even) or three (for n odd) negative roots (see, for example, [17]). Note that (3.10) places a necessary, and not a sufficient condition on β (equivalently T). For n even, the quantity $(-1)^{n+2}\beta^2(n-2)^2$ is positive and hence there can never be one positive root and one negative root. The possibility of having a single positive root is thus ruled out (a double-root is counted twice). Since complex roots appear in pairs, similar arguments apply for n odd. Thus, *there will be two positive roots (double root counted twice) or no positive roots*. Only the positive roots which are less than the cavity-radius α can qualify as black-hole solutions inside the cavity. It is easy to check that this would automatically be the case for the two possible positive roots of Eq. (3.9). To see this rewrite Eq. (3.9) in the following form:

$$\beta^2 = \frac{4\rho_b^2(\alpha^n + \alpha^{n-2} - \rho_b^n - \rho_b^{n-2})}{\alpha^{n-2}(n\rho_b^2 + n - 2)^2}. \quad (3.11)$$

β^2 is a single-valued function of ρ_b and is positive if and only if $\rho_b \in (0, \alpha)$. It is continuous and well-defined within this interval (the denominator is strictly positive). This is true for $n \geq 3$ and for any positive value of α. Since the maximum number of positive roots of Eq. (3.9) is two, it immediately follows that β^2 grows from zero (at $\rho_b = 0$) and ends with zero (at $\rho = \alpha$) with *one* "hump" in the middle, i.e., β^2 increases monotonically to a maximum β_m^2 and then decreases monotonically to zero (Fig. 1). Therefore there will be two ρ_b, i.e., two black-hole solutions for a given boundary with specified (α, β) provided β (or T) is equal to or less (greater) than the minimum value needed for the cavity-radius of α. Because β is a continuous function of ρ_b within the interval $(0, \alpha)$ the two black-hole solutions exist for any temperature above the mini-

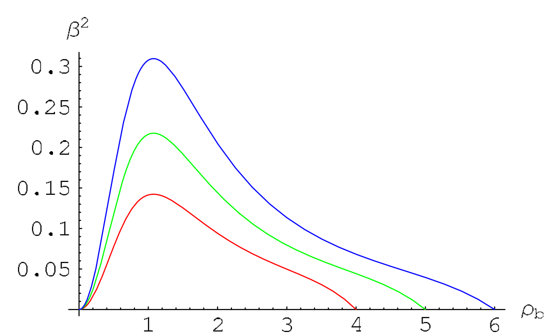

FIG. 1 (color online). β^2 (for $n = 11$) as a function of ρ_b for $\alpha = 4, 5, 6$: there is a unique maximum in each case. Note that the curves corresponding to different values of α do not cross as explained in Sec. III C.

mum value without any discontinuity. This holds for arbitrary cavity-radius. The minimum temperature needed for the solutions to exist is a function of the radius of the cavity and will be discussed in Sec. III D. Note that the shape of the curve for β^2 as a function of ρ_b is not obvious from Eq. (3.11) alone because of the nontriviality of the denominator. The fact that there will be two/no positive roots, as we have shown before, is crucial to the argument for the general number of solutions.

D. The minimum temperature T_m as a function of cavity-radius

For a cavity of fixed radius the minimum temperature T_m required for the black-hole solutions to exist is the one corresponding to the maximum of $\beta \equiv \beta_m$ at which the solutions are degenerate. By differentiating Eq. (3.11) one obtains that at $\beta = \beta_m$

$$n(n-2)\rho_{b_m}^{n+2} + 2(n^2 - 2n - 2)\rho_{b_m}^{n} + n(n-2)\rho_{b_m}^{n-2} + 2n(\alpha^n + \alpha^{n-2})\rho_{b_m}^{2} - 2(n-2)(\alpha^n + \alpha^{n-2}) = 0 \tag{3.12}$$

where ρ_{b_m} is the value corresponding to the maximum (minimum) of β (T) (and does not stand for the minimum/maximum value of ρ_b). One obtains the minimum temperature as a function of the cavity-radius by directly substituting the root of Eq. (3.12) in Eq. (3.11).[5] However, note that, like Eq. (3.9), Eq. (3.12) is not generally solvable using ordinary algebraic methods.

For $n \geq 3$, all terms in Eq. (3.12) are positive except the last. It therefore follows that

$$2n(\alpha^n + \alpha^{n-2})\rho_{b_m}^{2} - 2(n-2)(\alpha^n + \alpha^{n-2}) < 0 \tag{3.13}$$

implying

$$\rho_{b_m} < \sqrt{\frac{n-2}{n}}. \tag{3.14}$$

After dividing Eq. (3.12) by $(\alpha^n + \alpha^{n-2})$ it is easy to see that the larger the value of α the larger is ρ_{b_m}. This accounts for the gradual shift of the peak to the right in Fig. 1 for larger values of the cavity-radius. At the infinite-cavity limit (3.13) is replaced by an equality giving

$$\rho_{b_m} = \sqrt{\frac{n-2}{n}} \tag{3.15}$$

exactly. This therefore gives an absolute upper bound on the horizon-radius, and hence the entropy, the Hawking temperature given by the inverse of Δt. With a finite cavity, one needs to vary the product of the Hawking temperature multiplied by the Tolman redshift factor for which the extremum occurs at a different value of ρ_b. For the finite cavity this value of ρ_b is less than that obtained by varying the Hawking temperature alone which is Eq. (3.15) and gradually increases asymptotically to (3.15) as one increases the cavity-radius.

[5]One can check that Eq. (3.12) has a positive root in the interval $(0,\alpha)$. However, this trivially follows from our previous analysis.

E. Geometry of Eq. (3.9)

The difficulties in obtaining exact solutions of Eq. (3.12) do not preclude us from observing the following general fact : $T_m(\alpha)$ increases with decreasing radius of the cavity. This is because the curve for β as a function of ρ_b for any fixed value α of the cavity-radius completely covers the curve corresponding to a lower value of α. We have already noticed this from Fig. 1. To see that this holds in general consider the converse: if two curves of β (corresponding to two different values of cavity-radius α) as a function of ρ_b meet at some point in the $\beta^2 - \rho_b$ plane, the two values of α should satisfy Eq. (3.9) for the same pair (β^2, ρ_b). Treating Eq. (3.9) as an equation for α, it is easy to see that there can be only one/no positive solution for α for a given (β^2, ρ_b). Therefore nowhere in the $\beta^2 - \rho_b$ plane can two curves corresponding to two different values of α meet. The curves corresponding to lower values of α will be within the envelopes of those corresponding to higher values of α. Therefore the minimum temperature T_m needed for the existence of two black-hole solutions increases with decreasing cavity-radius. Since β (as a function of ρ_b) is continuous, for any arbitrarily higher temperature the two solutions continue to exist.

As derivatives of β^2 with respect to α are smooth, the curves in the $\beta^2 - \rho_b$ plane for various values of α fill in densely. The two-dimensional surface of Eq. (3.9) in the $\alpha^2 - \beta^2 - \rho_b$ space is therefore smooth and continuous. Therefore T_m is a smooth function of the cavity-radius and increases with decreasing value of the radius. It remains to be seen whether it is possible to obtain T_m as a function of the cavity-radius. We will return to this issue in Sec. V A.

F. Infinite-cavity limit

For any finite Schwarzschild-AdS black hole the local temperature at infinity is zero. Therefore the above picture—where one fixes the cavity-radius and its wall-temperature—is not well-defined in the infinite limit of the cavity in the case of Schwarzschild-AdS. In other words, one cannot simply fix a nonzero temperature at infinity and look for a finite Schwarzschild-AdS black-hole solution in the interior. This, however, is possible only in the $\Lambda = 0$ limit. For vanishing temperature at infinity all Schwarzschild-AdS black holes are equally good infilling solutions.

However, the product of the local temperature and the cavity-radius is well-defined in the infinite-cavity limit. This product essentially is the Hawking temperature (2.9) and substitutes for the wall-temperature of the cavity in this limit. One can now study the thermodynamics of black holes in terms of the Hawking temperature alone. This has been done in [2]. For a given Hawking temperature, there are in general two black holes—this corresponds to the doubled-valuedness of the former as one can see from (2.9). The Hawking temperature has a minimum, $T_m = \frac{1}{2\pi}\sqrt{n(n-2)}$, which corresponds to a horizon-radius of $\rho_b = \sqrt{(n-2)/n}$ in conformity with our analysis above. The mass of one hole decreases and the other increases as one raises the Hawking temperature.

One can immediately see the simplifications arising in the infinite limit of the cavity. It is rather trivial to find the masses of the two black holes from the Hawking temperature (2.9) in contrast to the finite-cavity case where one needs to solve (3.9). This leads to a much simplified semiclassical picture that we will see in the next section when we discuss the finite-cavity case.

IV. ON-SHELL ACTIONS, SEMICLASSICAL RESULTS AND PHASE TRANSITION

Our discussion so far was purely classical in that we only discussed infilling Riemannian geometries of the Einstein equations for a prescribed codimension-one boundary (representing a canonical boundary condition in the Lorentzian picture) without reference to their roles in the path integral. This has set the ground for semiclassical considerations as we will discuss now. First, let us briefly recall the facts for the $\Lambda = 0$ case. We have already mentioned that the Dirichlet problem is exactly solvable in this case [7,8]. This enables one to find the classical actions I_E of the infilling black-hole solutions as functions of boundary variables and thus study semiclassical physics. The specific heat of the larger solution is positive and that of the smaller solution is negative. From the action one also finds that the larger mass solution in $(n+1)$-dimensions, which has a positive specific heat, has a lower action than hot flat space, for temperatures [8]

$$T > T_c \equiv \frac{1}{4\pi} \frac{n-2}{\sqrt{(\frac{4(n-1)}{n^2})^{2/(n-2)} - (\frac{4(n-1)}{n^2})^{n/(n-2)}}} \frac{1}{\alpha}. \quad (4.1)$$

The lower mass solution always has a larger action than flat space. Thus above $T_c(\alpha)$ the larger black-hole solution can spontaneously nucleate from hot flat space within the cavity in a thermodynamically consistent manner in which the free energy of the system, $F = I_E/(2\pi\beta)$, does not increase. As one takes the cavity-radius to infinity the larger solution fills in the entire cavity irrespective of its wall-temperature and hence becomes irrelevant and only the smaller solution exists. Therefore, black-hole nucleation is thermodynamically consistent only within a finite cavity. Note that the critical temperature T_c is inversely proportional to the cavity-radius and is higher than the minimum temperature T_m (3.7) needed for the two black-hole solutions to exist classically. We will see below that the situation is similar for anti–de Sitter space as well. To the best of our knowledge, such a study has not been carried out before for four or higher dimensions except the study made in the infinite-cavity limit in [2] and which will reappear as a special case below.

A. Action of the infilling black holes

For an $(n+1)$-dimensional manifold $\mathcal{M}$ with an n-dimensional boundary $\partial\mathcal{M}$, the Euclidean action is [18–21]:

$$I_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{n+1}x\sqrt{g}(R - 2\Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{h}K \quad (4.2)$$

where $g_{\mu\nu}$ is the metric on $\mathcal{M}$ and h_{ij} is the induced n-metric on the boundary, i.e., $g_{ij|_{\partial\mathcal{M}}} = h_{ij}$. K is the trace of the extrinsic curvature K_{ij} of the boundary defined according to the convention that the outward normal to the boundary is positive.

The Einstein equation obtained from this action is $R_{\mu\nu} = \frac{2\Lambda}{n-1} g_{\mu\nu}$. Recall that $\Lambda = -\frac{n(n-1)}{2l^2}$ and that we set $l = 1$. The on-shell action therefore reads

$$I_E = -\frac{1}{16\pi G}\left(-2n\int_{\mathcal{M}}\sqrt{g}d^{n+1}x + 2\int_{\partial\mathcal{M}}\sqrt{h}d^n xK\right). \quad (4.3)$$

The first integral is the $(n+1)$-volume of $\mathcal{M}$ and the second integral geometrically is the rate of change of the n-volume of the boundary $\partial\mathcal{M}$ along the unit outward normal.

For a Schwarzschild-AdS$_{n+1}$ with an $S^1 \times S^{n-1}$ boundary at a constant radial distance it is fairly straightforward to calculate both the above bulk and boundary contributions to the on-shell action. They, respectively, are

$$V_{bh} = \int_{\mathcal{M}}\sqrt{g}d^{n+1}x = \frac{2\pi}{n\kappa}(\alpha^n - \rho_b^n)\mathrm{Vol}(S^{n-1}) \quad (4.4)$$

and

$$\int_{\partial\mathcal{M}} d^n x\sqrt{h}K = \frac{2\pi}{\kappa}\left(n\alpha^n + (n-1)\alpha^{n-2} - \frac{n}{2}M\right)\mathrm{Vol}(S^{n-1}). \quad (4.5)$$

Recalling that $M = \rho_b^n + \rho_b^{n-2}$, one obtains

$$I_{E_{bh}} = \frac{1}{4G}\left(\frac{\rho_b}{n\rho_b^2 + n - 2}\right)((n-2)\rho_b^n + n\rho_b^{n-2} - 2(n-1)(\alpha^n + \alpha^{n-2}))\mathrm{Vol}(S^{n-1}). \quad (4.6)$$

In the convention the $(n+1)$-dimensional Schwarzschild-AdS and AdS metrics have been written, the base manifold S^{n-1} (with the canonical round metric on it) satisfies the $(n-1)$-dimensional Einstein equation with a cosmological constant $(n-1)$. This is what is referred to as the "unit" $(n-1)$-dimensional sphere in the literature. Its volume $\mathrm{Vol}(S^{n-1}) = 2\pi^{n/2}/\Gamma(\frac{n}{2})$.

A few comments are in order. First, note that the action (4.6) is in terms of the horizon-radius. The actions for the two infilling black-hole solutions in terms of the boundary variables (α, β) are obtained from it via substitution once the two solutions of (3.9) are known. This takes us back to the classical Dirichlet problem; explicit expressions for actions can be obtained only when we know the explicit solutions of the infilling geometries. Also note that action (4.6) is finite for finite α. Hence, there is no *a priori* need to do a background subtraction (by taking off the action of AdS) in this case to make the action finite as is needed in the infinite-cavity limit [2]. However, such a subtraction will appear in the next section in the course of determining which of the classical solutions will dominate the path-integral as one varies cavity-radius and wall-temperature.

B. Entropy and specific heats

The canonical entropy of the system is

$$S = \beta\left(\frac{\partial I_{E_{bh}}}{\partial \beta}\right) - I_E \tag{4.7}$$

in which the derivative is evaluated while keeping the area of cavity fixed. The variation of β changes I_E through variation of the mass, or equivalently the horizon-radius ρ_b of an infilling hole, and so

$$S = \beta\left(\frac{\partial I_{E_{bh}}}{\partial \rho_b}\right)\left(\frac{\partial \rho_b}{\partial \beta}\right) - I_E. \tag{4.8}$$

One calculates the above two derivative terms using (4.6) and (3.11). After some lengthy algebra one finds

$$S = \frac{1}{4G}\rho_b^{n-1} \tag{4.9}$$

precisely. This shows that the universal law of black-hole entropy remains valid for such boundary conditions. As mentioned earlier, in four dimensions this result follows from the entropy of Reissner-Nordström-AdS black holes in isothermal cavities derived in [10]. It now immediately follows from the area-law and the study we made in Sec. III D on the variation of the mass of the two holes as functions of wall-temperature that the specific heat

$$C_A = T\frac{\partial S}{\partial T} \tag{4.10}$$

is positive for the larger black hole and negative for the smaller black hole, and hence they are thermodynamically stable and unstable, respectively.

C. Phase transitions between hot AdS_{n+1} and Schwarzschild-AdS_{n+1} within a finite cavity

To see that a phase transition from hot AdS to Schwarzschild-AdS can occur within the cavity, we need to compare the action of the infilling Schwarzschild-AdS_{n+1} solutions with that of hot AdS_{n+1} space which is unique for any given $S^1 \times S^n$ boundary. We do so by computing the actions of the infilling Schwarzschild-AdS_{n+1} solutions in the "background" of the hot unique AdS_{n+1} space, i.e., by subtracting the action of the AdS_{n+1} from that of the black hole (4.6). In the infinite limit of the cavity this calculation simplifies as the boundary terms of the AdS and Schwarzschild-AdS cancel [2,3]. Compared to the $\Lambda = 0$ case, on the other hand, the periodically identified AdS space has a nontrivial Tolman redshift factor. This makes the boundary term more important and the corresponding calculation more delicate than in flat space [8]. One therefore needs to be cautious in subtracting the AdS action from the Schwarzschild-AdS action in the case of the finite cavity. This is done as follows. First calculate the volume and boundary terms for AdS_{n+1}. They, respectively, are

$$V_{\mathrm{AdS}} = \int_{\mathcal{M}} \sqrt{g}\, d^{n+1}x = \frac{\Delta t}{n}\alpha^n \mathrm{Vol}(S^{n-1}) \tag{4.11}$$

and

$$\int_{\partial\mathcal{M}} d^n x\sqrt{h}K_0 = \Delta t(n\alpha^n + (n-1)\alpha^{n-2})\mathrm{Vol}(S^{n-1}) \tag{4.12}$$

where $\Delta t = 2\pi\beta/(\sqrt{1+\alpha^2})$ as given by Eq. (3.2). Since the infilling hot AdS_{n+1} space and the two infilling black holes have the same n-metric on the $S^1 \times S^{n-1}$ boundary, we can replace β using Eq. (3.3), giving

$$\Delta t = \frac{2\pi}{\kappa}\sqrt{\frac{1 - \frac{M}{\alpha^{n-2}} + \alpha^2}{1+\alpha^2}} \tag{4.13}$$

where M is either of the two infilling black-hole masses. The reason for replacing β is to express the action in the same form as the black-hole action. The volume and the boundary terms for AdS_{n+1} then read

$$V_{\mathrm{AdS}} = \int_{\mathcal{M}} \sqrt{g}\, d^{n+1}x = \frac{2\pi}{n\kappa}\sqrt{\frac{1 - \frac{M}{\alpha^{n-2}} + \alpha^2}{1+\alpha^2}}\alpha^n \mathrm{Vol}(S^{n-1}), \tag{4.14}$$

$$\int_{\partial\mathcal{M}} d^n x\sqrt{h}K_0 = \frac{2\pi}{\kappa}\sqrt{\frac{1 - \frac{M}{\alpha^{n-2}} + \alpha^2}{1+\alpha^2}}(n\alpha^n + (n-1)\alpha^{n-2}) \times \mathrm{Vol}(S^{n-1}). \tag{4.15}$$

Note that because periodically identified AdS_{n+1} space has a nontrivial Tolman shift factor unlike hot flat space, the boundary term (4.15) contains contributions from the changes in the volume of cavity as well as the local temperature along the radial direction of the cavity, or equivalently, in the Euclidean language, changes in the volume of the S^{n-1} base as well as the radius of the S^1-fibre along the outward normal at the boundary. The action of a black hole minus that of hot AdS therefore reads

$$I_E = \frac{1}{4G}\left(\frac{\rho_b}{n\rho_b^2 + n - 2}\right)((n-2)\rho_b^n + n\rho_b^{n-2} - 2(n-1)(1-s)(\alpha^n + \alpha^{n-2}))\mathrm{Vol}(S^{n-1}) \quad (4.16)$$

where, for shorthand,

$$s \equiv \sqrt{\frac{1 - \frac{M}{\alpha^{n-2}} + \alpha^2}{1 + \alpha^2}} \leq 1. \quad (4.17)$$

The action for either of the two infilling black holes is found by substituting the corresponding solutions of ρ_b of Eq. (3.9). The actions are then functions of cavity-radius and the wall-temperature, i.e. expressed in terms of boundary data. Therefore once analytic solutions of Eq. (3.9) are known the actions are known exactly.

1. Phase transition

The action (4.16) is a smooth, single-valued function of ρ_b. It is zero for $\rho_b = 0$ and is positive for small ρ_b (compared to α) since the first two terms in the bracket dominate. As one increases ρ_b it grows monotonically to a maximum value and then decreases monotonically to

$$I_E(\alpha) = -\frac{1}{4G}\frac{\alpha}{n-2+n\alpha^2}(n\alpha^n + (n-2)\alpha^{n-2})2\pi^{n/2}/\Gamma\left(\frac{n}{2}\right) \quad (4.18)$$

corresponding to $\rho_b = \alpha$. This is negative definite. The overall behavior is similar to that in flat space [8]. Below (Figs. 2 and 3) we plot I_E for various values of α for $n = 3$. We choose four dimensions here because of its obvious physical importance and also because the case of five dimensions, which is physically interesting from a holographic point of view, will be discussed in detail in Sec. V. We set $G = 1$. This phase transition, however, is in terms of the horizon-radius (equivalently, the mass) of a black hole within the cavity. This shows that within any finite cavity for sufficiently large value of its horizon-radius Schwarzschild-AdS becomes more probable than hot AdS. This phase transition is possible for any value of the cavity-radius and is physical, i.e. the black hole does not engulf the cavity. However, to revert to the canonical language we need to understand how the phase transition takes place as a function of the temperature of the wall instead of horizon-radius (or mass) of the hole while the radius of the cavity is held fixed. This needs care and is not obvious because of the double-valuedness of the infilling black-holes solutions. Can both of the two black holes have larger action than AdS? Does the scenario vary with varying radius of the wall? To answer these questions we need to recall what we have learned about the classical nature of the two solutions within the cavity. It is not difficult to see that it is the larger mass solution that nucleates from hot AdS for sufficiently high temperature of the heat-bath. The action (4.16) of the lower mass solution is always positive thus making it the least probable solution semiclassically. To see this recall that there is an absolute upper bound, $\rho_b < \sqrt{(n-2)/n}$, of the horizon-radius of the smaller black hole. When the cavity-radius is not too small the action (4.16) will always be positive within this range

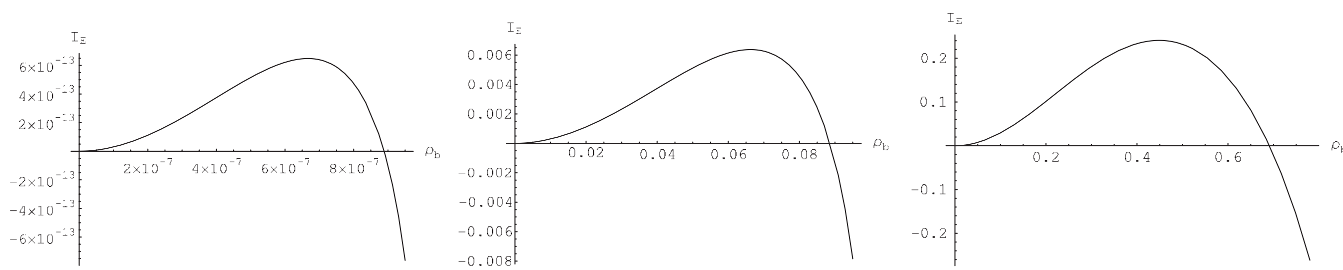

FIG. 2. Actions in four dimensions as functions of ρ_b ($\alpha = 1 \times 10^{-6}$, 1×10^{-1}, 1.0).

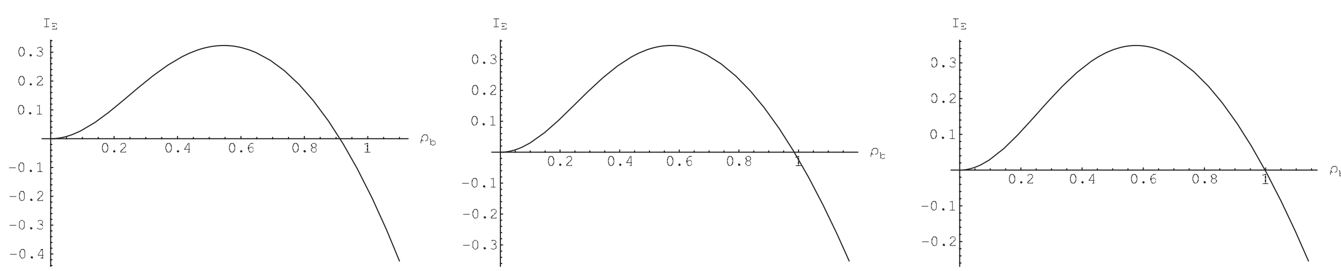

FIG. 3. Actions in four dimensions as functions of ρ_b (α = 2.0, 4.0, 8.0).

(Fig. 3). On the other hand, when the cavity is small the action (4.16) (as well as the overall scenario) approaches the flat-space limit (see Fig. 2 above) and remains positive for the smaller hole as discussed briefly above and in detail in [8]. Therefore the action of the lower mass black hole is always positive irrespective of the cavity-radius and temperature. Only the action of the larger mass hole is negative for $T > T_c$ and remains so for all higher temperatures.

2. Critical temperature

The critical temperature T_c can be found by equating (4.16) to zero and eliminating the square roots. In $(n+1)$ dimensions this gives us an equation

$$(n-2)^2\rho_{bc}^{n+2} + 2n(n-2)\rho_{bc}^{n} + n^2\rho_{bc}^{n-2} + 4(n-1) \times \alpha^{n-2}(1+\alpha^2)\rho_{bc}^2 - 4(n-1)\alpha^{n-2}(1+\alpha^2) = 0 \tag{4.19}$$

where ρ_{bc} is the radius of the larger black hole at the critical temperature.

Once the solution of this equation is found, T_c is determined via (3.11). Note that this equation has an appearance similar to Eq. (3.12). Using the identical arguments used for Eq. (3.12) it is easy to demonstrate that for large value of the cavity-radius ρ_{bc} tends to unity. Also, this value is the upper bound to ρ_{bc}. This is because like Eq. (3.12), the coefficients of various powers in (4.19) do not involve the cavity-radius. As in the case of $\Lambda = 0$, the horizon-radius of the nucleated black hole (or its mass) at the critical temperature increases with the cavity-radius. Therefore this provides us with another example in which a sharp contrast exists between the $\Lambda < 0$ and $\Lambda = 0$ cases. We will come back to this issue in the next section when we discuss five dimensions specifically and in the Conclusion.

D. Infinite-cavity limit

For large α

$$s = 1 - \frac{1}{2}\frac{m}{\alpha^n} + \text{higher order terms.} \tag{4.20}$$

Therefore for $\alpha \to \infty$ one obtains the following action of [2,4]

$$I_E = \frac{1}{16\kappa G}(\rho_b^{n-2} - \rho_b^n)\mathrm{Vol}(S^{n-1}) \tag{4.21}$$

giving the Hawking-Page phase transition at $\rho_b = 1$. This action is certainly much simpler than its finite boundary analogue (4.16).

It is straightforward to check the area-law of entropy and from it the specific heats of the two solutions. Following our discussion in Sec. III F on the Hawking temperature it is easy to see that the specific heat expression (4.10)

$$C_A = T\frac{\partial S}{\partial T} \tag{4.22}$$

essentially reduces to one where T can be replaced by T_H, the Hawking temperature. Thus for any value of the Hawking temperature (above the minimum temperature $\sqrt{n(n-2)}/2\pi$, of course), the larger and the smaller black-hole solutions will have positive and negative specific heats, respectively. The critical (Hawking) temperature in this case is $(n-2)/(2\pi)$, as one can check easily.

V. FIVE DIMENSIONS: EXACT RESULTS

We have already remarked that Eq. (3.9) is a quintic in four dimensions and the degree of this equation increases linearly with dimensionality. However, an observation that we have not made so far is that in odd dimensions Eq. (3.9) is an equation in ρ_b^2. For five and seven dimensions Eq. (3.9) is cubic and quartic, respectively, and can be solved exactly using ordinary algebraic methods. Thus the Dirichlet problem in these dimensions is exactly solvable and one obtains the bulk geometries in terms of the boundary variables. From a holographic perspective five dimensions is special and we will treat it in detail. In this case Eq. (3.9) reads

$$z^3 + (1+4\alpha^2\beta^2)z^2 + \alpha^2[4\beta^2 - (\alpha^2+1)]z + \alpha^2\beta^2 = 0, \tag{5.1}$$

in which we have substituted z for ρ_b^2.

Explicit Solutions: The three roots of (5.1) are given by Cardano's solution of the cubic:

$$z_1 = \frac{1}{6}(P + 12\sqrt{Q})^{1/3} - \frac{R}{6(P+12\sqrt{Q})^{1/3}} - \frac{1}{3}(1+4\alpha^2\beta^2), \tag{5.2}$$

$$z_2 = -\frac{1}{12}(P + 12\sqrt{Q})^{1/3} - \frac{R}{12(P+12\sqrt{Q})^{1/3}} - \frac{1}{3}(1+4\alpha^2\beta^2) - \frac{i\sqrt{3}}{2}\left(\frac{1}{6(P+12\sqrt{Q})^{1/3}} - \frac{R}{(P+12\sqrt{Q})^{1/3}}\right) \tag{5.3}$$

and

$$z_3 = -\frac{1}{12}(P + 12\sqrt{Q})^{1/3} - \frac{R}{12(P+12\sqrt{Q})^{1/3}} - \frac{1}{3}(1+4\alpha^2\beta^2) + \frac{i\sqrt{3}}{2}\left(\frac{1}{6(P+12\sqrt{Q})^{1/3}} - \frac{R}{(P+12\sqrt{Q})^{1/3}}\right) \tag{5.4}$$

where

$$P = -16\alpha^6\beta^2(32\beta^4+9) + 12\alpha^4(16\beta^4 - 12\beta^2 - 3) - 12\alpha^2(5\beta^2+3) - 8 \tag{5.5}$$

and

$$Q = -3\alpha^2(2\alpha^2+1)^2(\alpha^6(4\beta^4+1) - \alpha^4(32\beta^6 - 4\beta^4 + 10\beta^2 - 2) + \alpha^2(13\beta^4 - 10\beta^2 + 1) - 4\beta^2) \tag{5.6}$$

and

$$R = 4\alpha^4(16\beta^4+3) + 4\alpha^2(3-4\beta^2). \tag{5.7}$$

As usual with Cardano's solutions, the three roots of Eq. (5.1) are given in an imaginary form and hence it is not obvious which two are positive. However, since we know that they will be positive/complex together, it is easy to single out the expressions for the two positive roots; they are z_1 and z_2 (see below). These two solutions therefore give the masses of the two black holes, and hence the infilling geometries in terms of the boundary data.

A. The minimum temperature T_m for black holes

In finding various other quantities of interest one still needs to be judicious in choosing the correct algebraic approach as we will see below. First we would like to know T_m (above which the two black-hole solutions classically exist) as function of cavity-radius. This has been discussed qualitatively in Sec. III D for arbitrary dimensions. To find T_m exactly, one is required to find the positive root of (3.12) and then obtain T_m from (3.11). For five dimensions this is possible as (3.12) simplifies to a cubic equation like (5.1). However, the resulting algebraic expression for T_m will not be economical and easy to simplify.

A much nicer algebraic expression follows if one considers Eq. (5.1) directly and makes use of the known algebraic methods. Equation (5.1) admits a negative root and pair of positive or complex roots. These positive roots are in 1-1 correspondence with the two positive roots of Eq. (3.9) (for $n = 4$) and hence all qualitative observations made earlier apply directly. In particular, the positive roots of Eq. (5.1) will appear and disappear simultaneously together with the two positive roots of Eq. (3.9) as β is varied. Because it is cubic and has a negative root, if we ensure that Eq. (5.1) has all real roots then two of them will be positive automatically and will correspond to the two Schwarzschild-AdS$_5$ infilling geometries. The condition for a cubic equation to have all three roots real is well-known (see, for example, [22]). In this case it reads

$$32\alpha^4\beta^6 - \alpha^2(4\alpha^4+4\alpha^2+13)\beta^4 + 2(5\alpha^4+5\alpha^2+2)\beta^2 - \alpha^2(\alpha^2+1)^2 \le 0. \tag{5.8}$$

When the equality of this equation holds, the two positive roots of Eq. (5.1) are degenerate. Therefore the maximum value of β_m is simply the solution of

$$32\alpha^4{\beta_m}^6 - \alpha^2(4\alpha^4+4\alpha^2+13){\beta_m}^4 + 2(5\alpha^4+5\alpha^2+2){\beta_m}^2 - \alpha^2(\alpha^2+1)^2 = 0. \tag{5.9}$$

This is cubic in β_m^2 and admits a unique positive root for any value of α giving

$$T_m = \frac{1}{2\pi}\sqrt{\frac{A^{2/3} - 840\alpha^4 - 856\alpha^2 - 215 + 16\alpha^4 + 32\alpha^6 + 13A^{1/3} + 4\alpha^4A^{1/3} + 4\alpha^2A^{1/3}}{96\alpha^2A^{1/3}}} \tag{5.10}$$

where

$$A = 64\alpha^{12} + 192\alpha^{10} + 8880\alpha^8 + 17440\alpha^6 - 10308\alpha^4 - 18996\alpha^2 - 5291 + 192(2\alpha^2+1)^2\sqrt{3(\alpha^4+\alpha^2+7)^3}. \tag{5.11}$$

This is plotted in Fig. 4. The smaller the cavity the higher is T_m in concordance with our observations in Sec. III (and Fig. 1).

B. Actions, phase transition and critical temperature

The action (4.6) in five dimensions is

$$I_E = \frac{1}{4G}\left(\frac{\sqrt{z}}{2z+1}\right)(z^2+2z-3(\alpha^4+\alpha^2))\mathrm{Vol}(S^3). \tag{5.12}$$

The actions of the two black holes are found by substituting z_1 and z_2 in it. The action in the background of hot AdS space given by (4.16) reads

$$I_E = \frac{1}{4G}\left(\frac{\sqrt{z}}{2z+1}\right)(z^2+2z-3(1-s)(\alpha^4+\alpha^2))\mathrm{Vol}(S^3) \tag{5.13}$$

in which

$$s = \sqrt{\frac{1 - \frac{z^2+z}{\alpha^2} + \alpha^2}{1+\alpha^2}}. \tag{5.14}$$

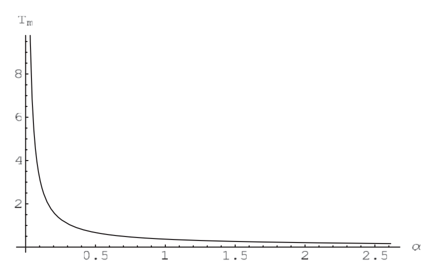

FIG. 4. The minimum temperature T_m in five dimensions needed for the two black holes to exist classically. (For $T = T_m$ the solutions are degenerate.)

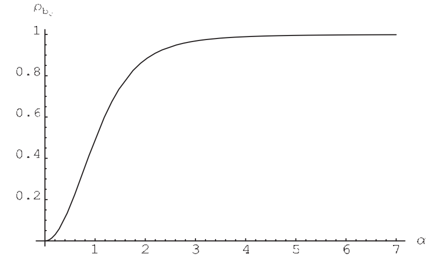

FIG. 5. The horizon-radius of the black hole nucleated precisely at the critical temperature of the cavity as a function of cavity-radius α: its behavior is different from that of the $\Lambda = 0$ case.

Again it is possible to find the actions of the two black holes by substituting z_1 and z_2 directly in (5.13).

Critical Temperature: We have already shown that a phase transition takes the hot AdS to the larger mass Schwarzschild-AdS irrespective of the radius of the cavity if the temperature of the cavity is above a certain critical value. This critical value of temperature T_c is a function of cavity-radius. We found that the radius of the black hole at the critical temperature of a cavity does not increase indefinitely with the cavity-radius as in the case of vacuum. With the simplification arising in the case of five dimensions we are able to treat these issues exactly. First the radius of the black hole at the critical temperature is obtained by equating (5.13) to zero (or just setting $n = 4$ in (4.19)):

$$z_c^3 + 4z_c^2 + (3\alpha^4 + 3\alpha^2 + 4)z - 3(\alpha^4 + \alpha^2) = 0. \tag{5.15}$$

For any value of α this has one positive root. Again this is cubic and hence can be solved exactly. The square-root of the positive solution of Eq. (5.15) ρ_{bc} is plotted in Fig. 5. We do not write the explicit form solution here to save space. The critical radius is thus found as an exact function of the cavity-radius and allows us to find the critical temperature T_c needed for the cavity to undergo a semiclassical phase transition. It is obtained simply by substituting the solutions of Eq. (5.15) in (3.11) for $n = 4$. Again, we do not mention the explicit solution here as the form is not particularly illuminating. We plot T_c as a function of the cavity-radius in Fig. 6.

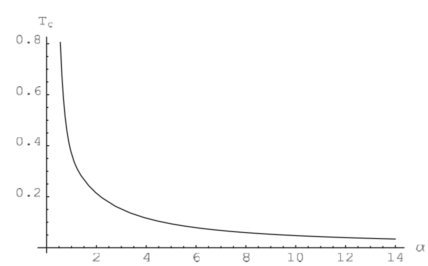

FIG. 6. The critical temperature T_c increases with decreasing radius of the cavity as in the $\Lambda = 0$ case.

VI. CONCLUSION

In this paper we studied the thermodynamics of Schwarzschild black holes in anti–de Sitter space within a cavity immersed in an isothermal bath as functions of the cavity-radius and its wall-temperature, i.e., the temperature of the bath. This led us to consider the associated Dirichlet boundary-value problem in Riemannian geometry for an $S^1 \times S^n$ boundary specified by its two radii (β, α). The connection between the Euclidean and Lorentzian thermodynamic pictures are such that the circumference of the S^1 fiber gives the inverse temperature of the bath. Two topologically distinct infillings are possible—one with a nut (the Euclidean AdS) and the other with a bolt (the Euclidean Schwarzschild-AdS). In the latter case one finds that the infilling geometry is double-valued and exists for a restricted range in the $\alpha - \beta$ plane.[6] We studied the condition on α and β for the existence of such classical infillings and explored the possibility of obtaining their explicit solutions in terms α and β (or T). We then considered which solutions will dominate the path-integral semiclassically as one varies the boundary data by computing their actions and studying them carefully. We found the overall qualitative picture to be similar to that found in flat space [7,8]: For any value of the wall-temperature a unique hot AdS solution exists whereas for wall-temperatures below a minimum value T_m no Schwarzschild-AdS solutions exist. Above this temperature there are two black-hole solutions which continue to exist for all higher temperatures. As one increases the temperature of the cavity the larger hole becomes heavier and the smaller one gets lighter. We found the on-shell action of the holes and showed that the standard area-law of black-hole entropy holds for such boundary conditions from which one can immediately deduce that the larger and the smaller holes have positive and negative specific heats, respectively, as in the flat-space case. At T_m, both of the black holes have a positive action and hence the AdS infilling is most probable and continues to remain so until a critical temperature T_c is reached when the action of the larger hole is zero. For $T > T_c$ the action of the larger hole becomes negative definite. Note that as one increases the temperature of the cavity from T_m, the mass of the smaller hole decreases and its action decreases monotonically (and reaches zero at infinite temperature). But, although its mass increases monotonically with temperature, the action of the larger

[6]In the case of $\Lambda < 0$, for a boundary which is a nontrivial S^1-bundle over S^2 the number of regular bolt-type infillings (Taub-Bolt-AdS infillings) can be as high as ten [23] while the nut-type infillings (self-dual Taub-Nut-AdS infillings) are unique [24]; in the case of the latter explicit solutions can be obtained. In the $\Lambda = 0$ case, such boundaries are studied for arbitrary dimensions in [8] where it is shown that bolt-type infillings are always double-valued with the only exception of Eguchi-Hanson metrics in which case the solutions are unique as in the case of all nut-type infillings.

hole increases only up to a certain positive value before it starts decreasing monotonically to zero at T_c and becomes more negative for increasing temperatures and reaches a fixed value (4.18) at infinite temperature. This picture is the same for an arbitrary value of the cavity-radius and quantitatively approaches the flat-space limit for small radii. Both T_m and T_c decrease for larger value of the cavity-radius as in flat space. However, despite the similarity, we have noted certain differences between the $\Lambda < 0$ and $\Lambda = 0$ cases. As we have seen in Sec. III D, there is an absolute upper bound on the horizon-radius of the smaller hole for $\Lambda < 0$ whereas there is no such absolute limit in the case of $\Lambda = 0$ (Sec. III A)—the upper limit in this case is only on x which translates to the limit (3.8). Another difference that occurs is in the horizon-radius of the larger hole nucleated at the critical temperature. This approaches an absolute value of unity (in the units of the paper) for $\Lambda < 0$. In the flat-space limit this gets larger and larger for higher values of the cavity-radius. From a dimensional point of view, we did not find any qualitative changes either in the local stability results or entropy formula in higher dimensions as compared to four dimensions [9,10]. The analysis of global stability and phase transitions—done here for four and higher dimensions for the first time and which brings the $\Lambda < 0$ case in par with the much-studied $\Lambda = 0$ case—also shows that no qualitative differences exist between four and higher dimensions.

In the infinite limit of the cavity the above study reduces to the study made in [2]. We have shown how this happens and how the only meaningful thermodynamic variable in this limit is the Hawking temperature. In the Euclidean picture, this corresponds to the fact that it is only the ratio of the radii of the fiber and the base of the (infinite boundary) that is meaningful in this limit—this, in fact, is pivotal for the AdS/CFT correspondence studied in [3]. The problem of finding explicit masses of black-hole solutions as functions of the Hawking temperature and evaluation of other thermodynamic quantities simplifies enormously in this limit. Note that the phase transition found in this paper for finite cavity does not follow from the Hawking-Page phase transition by means of logical extrapolation. This is because the larger black-hole solution and the hot AdS solution, which induce the same metric on the finite $S^1 \times S^n$-boundary, do not have the same $S^1 \times S^n$-metric at infinity and if they match near infinity they would not match anywhere at a finite radial distance. That there is only one meaningful variable at infinity is slightly analogous to the situation in the flat-space limit where, although both the radii of the base and the fiber enter into the explicit solutions, it is only the squashing of the two radii that is truly meaningful as one can see from Eq. (3.5). In the case of the finite $S^1 \times S^n$-boundary both of the radii play a nontrivial role for $\Lambda < 0$.

All of the above results were obtained without taking recourse to explicit infilling black-hole solutions for which one needs to solve Eq. (3.9). This equation cannot be simplified by some redefinition of variables and hence solutions are not possible for arbitrary dimensions. However, the case of five dimensions turns out to be rather special in which case this can be reduced to a cubic equation and hence one can solve it exactly using ordinary algebraic methods and thus obtaining the infilling black-hole geometries as exact functions of α and β (or, T). This makes it possible to compute the corresponding actions of the two black holes exactly as functions of the boundary variables. We have found T_m and T_c as exact functions of cavity-radius. The latter is an exact geometric statement for any regular Euclidean Schwarzschild-AdS metric. Other quantities of interest can be computed from the action and the solutions by using their standard definitions. These exact results therefore provide a basis for further dynamical and thermodynamical study and should find applications in brane-world cosmology, holography and other related issues of current interest and are left for future investigations.

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