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Initial conditions for noncanonical inflation

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We investigate the dynamics of homogeneous phase space for single-field models of inflation. Inflationary trajectories are formally attractors in phase space, but since in practice not all initial conditions lead to them, some degree of fine-tuning is required for successful inflation. We explore how the dynamics of noncanonical inflation, which is driven by the potential energy but has additional kinetic terms that are powers of the kinetic energy, can play a role in ameliorating the initial-conditions fine-tuning problem. We present an analysis of inflationary phase space based on the dynamical behavior of the scalar field. This allows us to construct the flow of trajectories, finding that they generically decay towards the inflationary solution at a steeper angle for noncanonical kinetic terms, in comparison to canonical kinetic terms, so that a larger fraction of the initial-conditions space leads to inflation. Thus, noncanonical kinetic terms can be important for removing the initial conditions fine-tuning problem of some small-field inflation models.

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I. INTRODUCTION

The paradigm of cosmological inflation is an elegant solution to the flatness, horizon, and monopole problems of standard big bang cosmology [1]. The simplest constructions of inflation consist of a scalar field (the inflaton ϕ) whose energy density is dominated by potential energy [2–4]. Remarkably, this simple picture also provides a mechanism for the generation of primordial large scale perturbations through the quantum fluctuations of the inflaton field. The spectrum of perturbations is in agreement with cosmological observations for many of the simplest models (see e.g. [5–9]).

However, inflation would lose some of its appeal if special initial conditions were needed for it to occur. For the purposes of studying inflationary initial conditions, it is useful to divide inflationary models into two classes based on the distance $\Delta\phi$ the inflaton travels during the inflationary period: large-field ($\Delta\phi \gg M_p$) inflation and small-field ($\Delta\phi \ll M_p$) inflation [10]. The sensitivity of inflation to initial conditions was studied extensively for chaotic inflation [2] in [11–20], as an example of large-field inflation, and for new inflation [3,4] in [15,16,19–25], as an example of small-field inflation. These studies found that there is a noticeable contrast between the two types of models: the onset of chaotic inflation is insensitive to initial conditions, but the onset of new inflation is very sensitive to initial conditions. Stated more generally, small-field models have an initial conditions fine-tuning problem, while large-field models do not.

These studies assumed the dynamics to be that of a single canonical scalar field. In this paper, we will take the scalar field to have noncanonical dynamics given by the effective Lagrangian

$$\mathcal{L}_{\rm eff} = p(X, \phi), \qquad (1)$$

where $X \equiv -\frac{1}{2}(\partial\phi)^2$ and the Lagrangian is separable into kinetic and potential energies as in (9) and reduces to the canonically normalized case for small X. We impose some physicality constraints, discussed in Sec. II, and focus on this subset of noncanonical Lagrangians to explore the effect of the noncanonical kinetic terms. As argued in [26,27], Lagrangians of this form can arise as effective field theories upon integrating out physics above some intermediate energy scale Λ, including terms that are powers of X/Λ^4 and ϕ/Λ. These Lagrangians are interesting to study because they can lead to inflationary backgrounds which interpolate smoothly between canonical and noncanonical behavior [26]. The modified kinetic terms coming from (1) can modify the dynamics of the inflaton in phase space, potentially changing the sensitivity of small-field inflation to initial conditions.

The inflationary initial conditions problem encompasses both homogeneous and inhomogeneous initial conditions. In this paper, we will assume homogeneity in order to focus on the homogeneous initial conditions problem of small-field inflation. It is an important question how these results generalize to the case of inhomogeneous initial conditions.

Noncanonical kinetic terms have been shown to play an important role in the inflationary initial conditions fine-tuning problem [28], for the specific case of a Dirac-Born-Infeld (DBI)[29] kinetic term.[1] In this paper, we focus on the behavior of the phase-space dynamics in more detail

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[1]In [30], this was investigated in more detail for a specific class of brane inflation models [31–34]. Since DBI inflation does not occur in a controllable regime, the DBI dynamics do not help to alleviate the initial conditions fine-tuning for these models.

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for the general class of noncanonical Lagrangians (1), developing more techniques to understand the dynamics in the region of phase space far from the inflationary solution. Other mechanisms for resolving the initial conditions fine-tuning problem include dynamical fine-tuning of the potential [35,36], tunneling from a false vacuum [37], time-dependent corrections to the potential [38], moduli trapping at enhanced symmetry points [39], and different choices of measure on the space of initial conditions [40,41]. Noncanonical kinetic terms can play a supporting or alternative role to these mechanisms for resolving the initial conditions problem.

In Sec. II, we review the scenario of noncanonical inflation with a Lagrangian of the form (1). The general organizational scheme of the rest of the paper is to examine the phase space and dynamics for canonical and noncanonical inflation in increasing levels of detail. First, in Sec. III, we divide phase space into several dynamical regions that allow us to make general statements about phase-space flow. In Sec. IV, we analytically compute the angles trajectories make in phase space, relative to the horizontal, finding that noncanonical kinetic terms make individual trajectories steeper towards the inflationary attractor. Next, in Sec. V, we focus on the region of phase space with large momentum where most overshooting initial conditions are located. We show that noncanonical kinetic terms can modify the dynamics in this region so that these initial conditions no longer lead to overshooting. In Sec. VI, we study the phase space of specific examples numerically, confirming that canonical small-field models have an initial conditions fine-tuning problem while corresponding noncanonical models do not. In particular, we explicitly compute the size of initial-conditions phase space leading to more than 60 e-folds of inflation for both canonical and noncanonical Lagrangians and show that the latter is much larger than the former. We conclude with some closing discussion in Sec. VII. Computations for an additional noncanonical Lagrangian, the powerlike Lagrangian, not included in the main text are presented in the Appendix.

II. REVIEW OF NONCANONICAL INFLATION

Noncanonical inflation has a generalized scalar field Lagrangian minimally coupled[2] to gravity,

$$S = \int d^4x\sqrt{g_4}\left[\frac{M_p^2}{2}\mathcal{R}_4 + p(X,\phi)\right], \tag{2}$$

where the Lagrangian is a function of $X \equiv -\frac{1}{2}(\partial_\mu \phi)^2$ and ϕ only.[3]

[2]A nonminimal coupling can be written in this form after an appropriate Weyl transformation to Einstein frame [42,43].

[3]As was argued in [26], higher-derivative terms in the scalar field action can be ignored for constructing background noncanonical inflationary solutions.

In this paper, we will be interested in *homogeneous* cosmological backgrounds of the form

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2, \qquad \phi = \phi(t), \tag{3}$$

so that $X = \frac{1}{2}\dot{\phi}^2 > 0$. Noncanonical inflation is described by the inflationary parameters [44]

$$H \equiv \frac{\dot{a}}{a}; \qquad \epsilon \equiv -\frac{\dot{H}}{H^2}; \qquad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon}; \qquad \kappa \equiv \frac{\dot{c}_s}{Hc_s};$$
$$c_s^2 \equiv \left(1 + 2X\frac{\partial^2 p/\partial X^2}{\partial p/\partial X}\right)^{-1}.$$

The parameters ϵ, η are generalizations of the usual slow-roll parameters to a general cosmological background, and the sound speed c_s is the speed at which scalar perturbations travel. The phenomenology of noncanonical inflation, including the scalar power spectrum P_k^ζ, the scalar spectral index n_s, the tensor-to-scalar ratio r, and the (equilateral) non-Gaussianity $f_{NL}^{(\mathrm{equil})}$ is [44,45]

$$P_k^\zeta = \frac{1}{8\pi^2}\frac{H^2}{M_p^2}\frac{1}{c_s\epsilon}\bigg|_{c_s k = aH}; \tag{4}$$

$$n_s - 1 = -2\epsilon - \eta - \kappa; \tag{5}$$

$$r = 16 c_s \epsilon; \tag{6}$$

$$f_{NL}^{(\mathrm{equil})} \sim c_s^{-2}. \tag{7}$$

Not all Lagrangians $p(X, \phi)$ are suitable as physically realistic field theories. In particular, we would like our scalar field to be ghost free with manifestly subluminal speed of perturbations[4] (see however [47] for nonsubluminal cases). These conditions together require [48,49]

$$\frac{\partial p}{\partial X} > 0 \quad \text{and} \quad \frac{\partial^2 p}{\partial X^2} > 0. \tag{8}$$

For simplicity, we will focus on Lagrangians that have nonzero potential energy and take a ‘‘separable’’ form of potential and kinetic terms,

$$p(X,\phi) = q(X,\phi) - V(\phi). \tag{9}$$

This highlights the difference between noncanonical Lagrangians and ‘‘k-inflation’’ [50]; the latter contain no potential energy term, obtaining inflation purely from the kinetic energy of the scalar field. Noncanonical inflation obtains inflation from the potential energy as in standard slow-roll inflation [which is just a limiting case of the Lagrangian (9)], although the potential can in general be steeper than in slow roll [26]. We will constrain the kinetic term to obey the physicality constraints (8) and to reduce to

[4]See [46] for a discussion of the speed of second order perturbations.

a canonically normalized kinetic term $q(X, \phi) \approx X$ for sufficiently small X.

The homogeneous scalar field can be treated as a perfect fluid with pressure $p = p(X, \phi)$ and, under the assumption (9), the energy density separates nicely into kinetic and potential energies:

$$\rho \equiv 2X\frac{\partial p}{\partial X} - p = 2X\frac{\partial q}{\partial X} - q + V(\phi). \tag{10}$$

The Friedmann equation for the scale factor then becomes

$$H^2 = \frac{1}{3M_p^2}\left[2X\frac{\partial q}{\partial X} - q + V(\phi)\right]. \tag{11}$$

The equation of motion for the scalar field can be written compactly in terms of the canonical momentum

$$\Pi \equiv -\frac{\partial p}{\partial \dot\phi} = -\dot\phi\frac{\partial p}{\partial X} = \sqrt{2X}\frac{\partial q}{\partial X}; \tag{12}$$

$$\dot\Pi = -3H\Pi - \frac{\partial p}{\partial \phi}. \tag{13}$$

In defining Π we assumed $\dot\phi < 0$, which is the typical case we will consider below, and added a minus sign to the usual definition of Π. We have chosen the latter convention because we will prefer to work in only one quadrant of phase space where $\phi \geq 0$ and $\dot\phi \leq 0$; the additional minus sign allows us to work in the purely positive quadrant of phase space $\phi \geq 0$ and $\Pi \geq 0$, where there will be no more ambiguities due to minus signs.

For canonical kinetic terms, slow-roll inflation occurs when the potential energy dominates over the kinetic energy, i.e. $\frac{1}{2}\dot\phi^2 \ll V(\phi)$, and the acceleration is small compared to the Hubble friction and driving force, i.e. $\ddot\phi \ll 3H\dot\phi$ and $\ddot\phi \ll V'$. These conditions lead to the slow-roll inflationary solutions $\dot\phi_{\text{slow-roll}} = -V'/3H$. Noncanonical inflation is a straightforward generalization of this [26], with the potential energy dominating the kinetic energy, i.e.

$$V(\phi) \gg 2X\frac{\partial q}{\partial X} - q, \tag{14}$$

and the Hubble friction and driving force[5] dominating over the "acceleration:"

$$\dot\Pi \ll 3H\Pi, V'. \tag{15}$$

The corresponding noncanonical inflationary solutions (for a fixed definition of the inflaton; see [26] for more discussion) are then

$$\Pi_{\text{inf}}(\phi) = \frac{V'}{3H}. \tag{16}$$

We will focus on Lagrangians that can be written as a power series,

$$p(X, \phi) = \sum_{n\geq 0} c_n(\phi)\frac{X^{n+1}}{\Lambda^{4n}} - V(\phi), \tag{17}$$

(with $c_0 = 1$ as discussed below (9) which converges in some finite radius of convergence, i.e. $X/\Lambda^4 \in [0, R(\phi)]$, for $0 < R(\phi) < 1$. Clearly, as $\Lambda \to \infty$ (17) reduces to the usual canonical Lagrangian. As discussed in [26], sufficient conditions for noncanonical inflation to occur for these types of Lagrangians are (a) the derivative of the series with respect to X, $\partial_X p$ diverges at the boundary of the radius of convergence; (b) the dimensionless parameter

$$A \equiv V'/(3H\Lambda^2) \tag{18}$$

is large, $A \gg 1$; and (c) the coefficients c_n satisfy $\Lambda^2\partial_\phi c_n/(3Hc_n) \ll 1$. For simplicity, we will choose the coefficients c_n to be constant throughout this paper so that the latter condition is always satisfied.[6] Condition (b) can be rewritten in terms of the *slow-roll* parameter

$$\epsilon_{\text{SR}} \equiv \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 \tag{19}$$

as $A = (\frac{2}{3}\epsilon_{\text{SR}}\frac{V}{\Lambda^4})^{1/2}$. As discussed in [26], in order to have $A \gg 1$ we need $V/\Lambda^4 \gg 1$. Some typical examples of Lagrangians that satisfy the condition that the derivative of the series diverge at the boundary of the radius of convergence are the DBI Lagrangian [29] and the "geometric series" Lagrangian (where we write the series in closed form for convenience):

$$p(X, \phi)_{DBI} = -\Lambda^4\left(\sqrt{1 - 2\frac{X}{\Lambda^4}} - 1\right) - V(\phi); \tag{20}$$

$$p(X, \phi)_{\text{geo}} = \Lambda^4\left(\frac{1}{1 - \frac{X}{\Lambda^4}} - 1\right) - V(\phi), \tag{21}$$

with $R = \{1/2, 1\}$, respectively. For $X/\Lambda^4 \ll 1$ these Lagrangians consist of a canonical kinetic term plus subleading corrections, while for $X \to R\Lambda^4$ the kinetic terms are far from canonical.

III. GENERAL PHASE-SPACE STRUCTURE

Before we discuss in detail the dynamics of inflationary models in (ϕ, Π) phase space it is helpful to first determine the general structure of this phase space.

[5]In general, noncanonical inflationary solutions occur when the driving force term $\partial p/\partial\phi$ of the scalar field equation of motion (13) is dominated by the driving force from the potential energy, so that $\partial p/\partial\phi \approx -V'$ [26].

[6]As a consequence the radius of convergence will be a constant, i.e. $R(\phi) = R$. It would be interesting to consider how generic this is from an effective field theory point of view.

First, we note that the inflationary solution (16) corresponds to a one-dimensional trajectory in phase space, as shown in Fig. 1. Since the inflationary solution is only valid when the inflationary parameters are small, the trajectory does not extend throughout all of phase space. In [26] it was argued that the dimensionless parameter A (18) keeps track of whether the inflationary solution is predominately canonical (slow roll) or noncanonical. When $A \ll 1$, we have the usual slow-roll inflation, while for $A \gg 1$ inflation is strongly noncanonical. Graphically, this corresponds to the inflationary trajectory being, respectively, below or above the line $\Pi = \Lambda^2$ in phase space. More generally off of the inflationary solution, the noncanonical kinetic terms are important when $X/\Lambda^4 \to R \sim \mathcal{O}(1)$. Since $q' = \partial q/\partial X > 1$ always, we see that the region of phase space where the kinetic terms are noncanonical corresponds to the region with

$$\frac{\Pi}{\Lambda^2} = \sqrt{2\frac{X}{\Lambda^4}q'} \sim \sqrt{2R}q' \gtrsim 1. \tag{22}$$

Correspondingly, the region of phase space below this line is the canonical kinetic term regime.

Clearly, the available phase space must be bounded—our effective description in terms of a single scalar field must break down at the very least for large momenta and potential energies. One way the effective description breaks down is when the Hubble rate of the background exceeds the scale of new physics, i.e. $H > \Lambda$. When this occurs, the high-energy physics can no longer be integrated out and must be included in the dynamics.

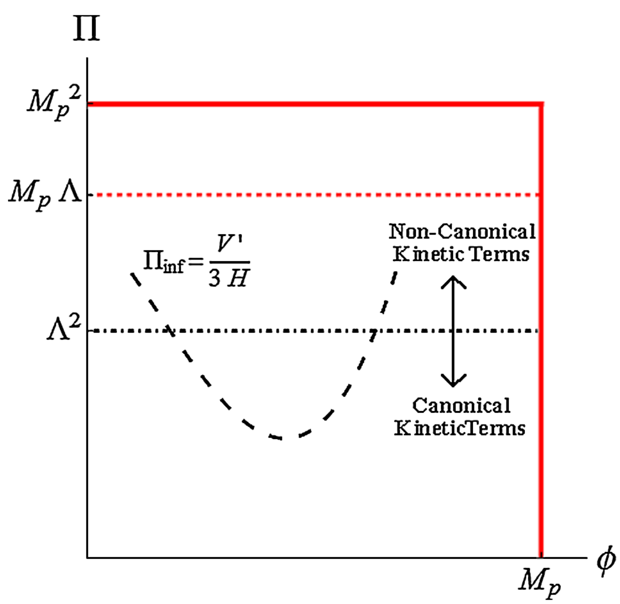

FIG. 1 (color online). Homogeneous phase space is bounded by $H < \Lambda$, resulting in the solid (red) lines for noncanonical Lagrangians and the dashed line for a canonical scalar field. The dot-dashed line corresponds to the boundary between canonical and noncanonical behavior of the kinetic terms.

To derive bounds on the momentum, assume that the energy density is dominated by the kinetic energy (which is true for large momentum):

$$H^2 = \frac{1}{3M_p^2}[2Xq' - q + V] > \frac{2Xq' - q}{3M_p^2} \sim \frac{2Xq'}{3M_p^2} = \frac{\sqrt{2X}\Pi}{3M_p^2}, \tag{23}$$

where we used the definition of the momentum (12).

In the case of a canonical kinetic term, we have $\Pi = \sqrt{2X}$ and the bound $H < \Lambda$ turns into $\Pi < M_p\Lambda$ (this is conservative; if we had chosen the usual $H < M_p$ the initial conditions fine-tuning problem for canonical models would just be worse). The kinetic energy of Lagrangians that lead to noncanonical inflation, $\rho_{\rm kin} \sim \sqrt{2X}\Pi$, tends to grow *slower* with momentum Π than a canonical kinetic energy, $\rho_{\rm kin}^{\rm canon} \sim \Pi^2$. For instance, when the higher powers in X are important and $X/\Lambda^4 \sim R$, power-series Lagrangians have kinetic energies that scale linearly with momentum, i.e. $\rho_{\rm kin}^{\rm power\text{-}series} \sim \sqrt{2R}\Pi$. Because the kinetic energy grows slower in Π, the bound on the momentum from the effective field theory constraint $H < \Lambda$ will be more relaxed than in the canonical case; $\Pi < 3(2R)^{-1/2}M_p^2$. (This is different from the case in [28,30] because these works took the kinetic term to be canonical in deriving the bound.) There is a large range of large momenta $\Lambda^2 < \Pi \lesssim M_p^2$ for which the noncanonical kinetic terms are important in phase space and the effective field theory treatment of the background is still (naively) valid.

There are additional constraints on the validity of the effective field theory that arise from considering the strength of the couplings of the perturbations generated during inflation [51–53]. If these perturbations are too strongly coupled, they can backreact on the (homogeneous) inflating background so that the perturbation spectrum and background evolution are unreliable. We will work in the regime where these constraints are satisfied; when we consider specific examples, in Sec. VI, we will choose our parameters such that these constraints are satisfied throughout the regime of interest. We refer the reader to [26] for a more detailed discussion of these effective field theory constraints.

In principle, the bound $H < \Lambda$ also places restrictions on the allowed range of the scalar field ϕ through the potential energy. In practice, other difficulties arise when ϕ is large, such as ϕ-dependent masses or couplings for other fields becoming important. We are primarily concerned here with inflationary potentials which, for canonical kinetic terms, suffer from an initial conditions fine-tuning problem. As discussed in the introduction, these potentials are almost always of the *small-field* type, so that typical values of the scalar field are small in Planck units $\phi \ll M_p$. We will take $\phi = M_p$ as an upper bound on the field range so that

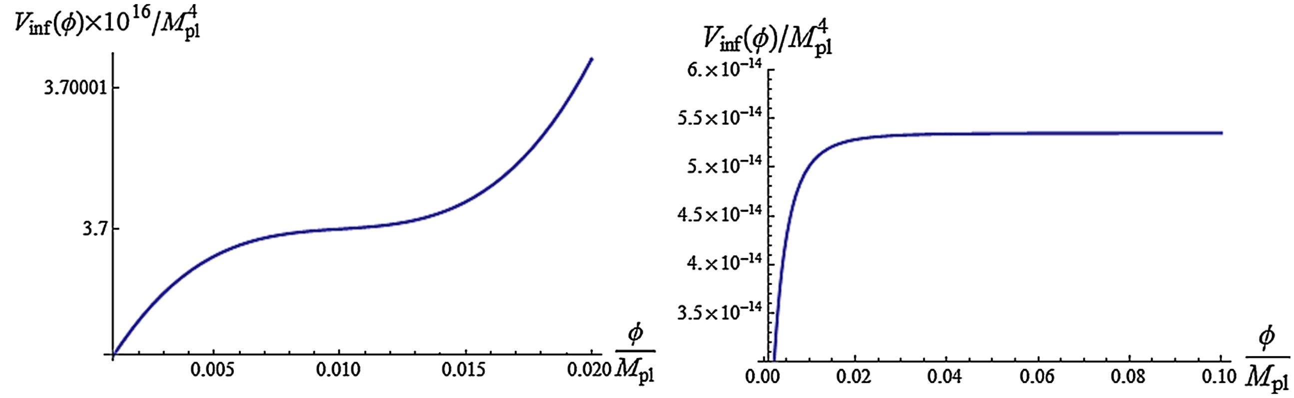

FIG. 2 (color online). Left: An inflection-point type potential (26). Right: A Coulomb-type potential (27) for $n = 4$.

we are always working in the small-field regime. Specific models may have additional stronger bounds on the field range that should be satisfied (such as D-brane inflation [54]), but we will try to be as general as possible here and not restrict ourselves to any one particular model.

The (ϕ, Π) phase space[7] has a general structure, based on the equations of motion

$$\dot{\Pi} = -3H\Pi + V'; \tag{24}$$

$$H^2 = \frac{1}{3M_p^2}\left[2X\frac{\partial q}{\partial X} - q + V(\phi)\right], \tag{25}$$

from which the general qualitative dynamics in phase space can be extracted. In particular, we can divide phase space into several regions of interest, depending on which terms dominate the equations of motion (24) and (25) (see Figs. 3 and 4):

(i) *Region A:* Hubble friction dominates over the driving force from the slope of the potential, $3H\Pi \gg V'$, and kinetic energy dominates over potential energy $(2X\frac{\partial q}{\partial X} - q) \gg V(\phi)$. These conditions are satisfied in the large Π region of phase space. Clearly in this region we have $\dot{\Pi} < 0$, so momentum *decreases* with time.

(ii) *Region B:* The driving force of the potential dominates over the Hubble friction, $3H\Pi \ll V'$, but kinetic energy dominates over potential energy; $(2X\frac{\partial q}{\partial X} - q) \gg V(\phi)$. These conditions are met when the potential has a large slope. In this region we have $\dot{\Pi} > 0$, so momentum *increases* with time.

(iii) *Region C:* Hubble friction dominates, but potential energy dominates over kinetic energy. Trajectories in this region lose momentum since $\dot{\Pi} < 0$; the lower boundary of this region contains the inflationary trajectory.

(iv) *Region D:* In this remaining region, the potential energy driving force dominates over the Hubble friction and the energy density is dominated by the potential energy. Trajectories in this region gain momentum $\dot{\Pi} > 0$; the upper boundary of this region contains the inflationary trajectory.

A similar regional analysis appears in [15,21] for the new inflation and chaotic inflation models (see also [56]). In this paper we will focus instead on small-field inflationary models with two types of potential, chosen to be representative of many of the potentials that arise from modern top-down inflationary model building. The inflection-point type potential,

$$V(\phi)_{\rm inflection} = V_0 + \lambda(\phi - \phi_0) + \beta(\phi - \phi_0)^3, \tag{26}$$

arises in models of D-brane inflation [32–34] and specific closed string inflation models [57,58]. The Coulomb-type potential,

$$V(\phi)_{\rm coulomb} = V_0 - \frac{T}{(\phi + \phi_0)^n}, \tag{27}$$

for $n > 0$ has appeared as a potential for D-brane inflation [59] and its embeddings in warped geometries [31,34]. These potentials are shown in Fig. 2, and are chosen mostly for illustrative purposes; our results, with the exception of the numerical examples of Sec. VI, do not depend on the specific choice of potential.

It is helpful to divide phase space into these regions because it is straightforward to determine the broad structure of the flow of trajectories in each region without solving the numerical equations of motion. In particular, we know that in Regions A and C trajectories tend to lose momentum, while in Regions B and D trajectories tend to gain momentum. Clearly, the boundary between Regions C and D contains the inflationary trajectory, which we will label as Γ. Note that Γ does not encompass the entire boundary between Regions C and D, but rather only a fraction of it. This is because Γ is defined only when ϵ, $|\eta| \ll 1$, while the boundary between Regions C and D exists more generally. The dividing line $\Pi = \Lambda^2$ between

[7] Note that here we will not be using the full (ϕ, Π, H) phase space as in [55], since H is simply set by a constraint equation and is not separately dynamical.

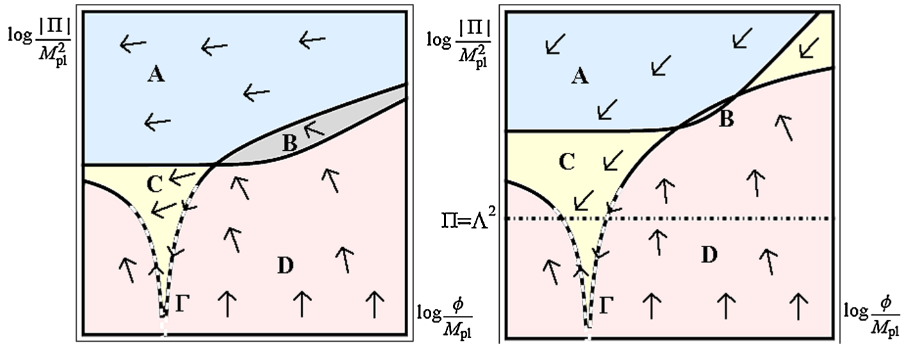

FIG. 3 (color online). A sketch of the boundaries for the Regions A-D of (ϕ, Π) phase space described in the text for an inflection-point type potential (26) with canonical (left) and noncanonical (right) kinetic terms. The inflationary solution $\Pi_{\rm inf}(\phi)$ (16) is the dashed curve Γ. The horizontal dot-dashed line is $\Pi = \Lambda^2$; above this line the noncanonical kinetic term is very far from canonical. Notice how Regions A and B shrink when going to the noncanonical case, as discussed in the text. Arrows denote the general direction of flow of trajectories, and are generally steeper for noncanonical kinetic terms.

the canonical and noncanonical regimes can cut through anywhere on this region plot. For models in which some amount of noncanonical inflation occurs, the dividing line must cut through some part of the inflationary trajectory Γ.

It is straightforward to sketch the boundaries for the regions by finding the inequalities that determine whether a) the energy density is kinetic or potential energy dominated, and b) the Π equation of motion (24) is Hubble friction or driving force dominated, for both the canonical and noncanonical limits. For a canonical kinetic term, the energy density is

$$\rho = \tfrac{1}{2}\Pi^2 + V(\phi), \tag{28}$$

so the condition that the kinetic energy dominates is

$$\Pi > \sqrt{2V}. \tag{29}$$

For a noncanonical kinetic term the energy density is

$$\rho = 2Xq'(X) - q(X) + V(\phi) \sim \sqrt{2X}\Pi + V(\phi) \tag{30}$$

so in the noncanonical limit ($X \longrightarrow \Lambda^4 R$) the condition that the kinetic energy dominates becomes

$$\Pi > \frac{V}{\Lambda^2\sqrt{2R}} = \frac{\sqrt{2V}}{\sqrt{2R}}\left(\frac{V}{2\Lambda^4}\right)^{1/2}. \tag{31}$$

Recalling that noncanonical inflation requires $V/\Lambda^4 \gg 1$ [26], we see that the condition (31) is stronger than the condition (29). Thus, we expect that the region where the kinetic energy dominates will shift to larger momentum for noncanonical kinetic terms.

Now we consider the conditions necessary for Hubble friction to dominate the Π equation of motion (24). This can occur in two subcases: the Hubble friction can dominate while the energy density is dominated by either kinetic or potential energy, corresponding to the boundaries between Regions A and B or C and D, respectively. When the kinetic energy dominates and for a canonical kinetic term, the condition that the Hubble friction dominates can be written in terms of the *slow-roll* parameter (19) as

$$\Pi > \sqrt{2V}\left(\frac{\epsilon_{\rm SR}}{3}\right)^{1/4}. \tag{32}$$

Notice that this implies that Region B, which corresponds to kinetic energy and driving force domination, can only

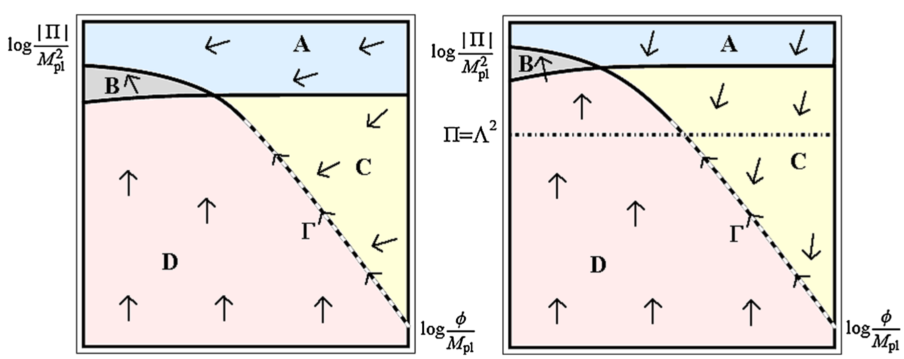

FIG. 4 (color online). The boundaries for the Regions A-D of (ϕ, Π) phase-space described in the text for a Coulomb-type potential (27) with canonical (left) and noncanonical (right) kinetic terms. The differences between the canonical and noncanonical cases are similar to those discussed in Fig. 3.

occur (in the canonical limit) for potentials that have $\epsilon_{SR} > 3$.[8] In the noncanonical limit, these conditions become

$$\Pi > \left(\frac{\epsilon_{SR}^2}{18}\right)^{1/6} \frac{\sqrt{2V}}{(2R)^{1/6}} \left(\frac{V}{\Lambda^4}\right)^{1/6}. \qquad (33)$$

Comparing (31) and (33), we see that the conditions for Region B become even more difficult to satisfy, i.e. $\epsilon_{SR} > \frac{3V}{4R\Lambda^4}$. Now, for the case when the Hubble friction and potential energy dominate, the Hubble friction bound is the same for both canonical and noncanonical kinetic terms (because the Π equation of motion is the same for both):

$$\Pi > \sqrt{2V}\left(\frac{\epsilon_{SR}}{3}\right)^{1/2}. \qquad (34)$$

Combining these bounds (29)–(34) allows us to sketch the boundaries of the Regions A-D. In Figs. 3 and 4 these regions are sketched for both canonical and noncanonical kinetic terms with the inflection point and Coulomb potentials, and we see the qualitative features discussed above.

IV. ANGLES IN PHASE SPACE

In the previous section we saw how phase space has a general structure that can be useful for qualitatively understanding the dynamics of different initial conditions. In particular, our regional analysis allows us to map out where in phase space we expect trajectories to be gaining or losing momentum. However, this analysis is not enough to make judgements about the sensitivity (or lack thereof) of inflation to initial conditions because we do not know the *angle* the trajectory makes in phase space. Let us define the angle of the trajectory in phase space as in Fig. 5 (where the factor of M_p is needed to make the angle dimensionless):

$$\tan\theta = \frac{1}{M_p}\frac{d\Pi}{d\phi} = \frac{1}{M_p}\frac{\dot{\Pi}}{\dot{\phi}}. \qquad (35)$$

All we know from the regional analysis of the previous section is that $\theta > 0$ in Regions A and C and $\theta < 0$ in Regions B and D. However, trajectories in Region A with $\theta \ll 1$ (i.e. nearly horizontal) lead to a very different qualitative picture of phase space than do trajectories with $\theta \sim \frac{\pi}{2}$ (i.e. nearly vertical), so not only the sign but also the *magnitude* of the angle is important.

For a canonical kinetic term the angle can be computed using the equation of motion (24) and the definition of the momentum ($\Pi = -\dot{\phi}$):

$$\tan\theta_{\rm canon} = -\frac{\dot{\Pi}}{\Pi M_p} = \frac{3H\Pi - V'}{\Pi M_p}. \qquad (36)$$

[8] A more precise analysis, not assuming that the kinetic energy dominates, finds that the condition becomes $\epsilon_{SR} > 6$.

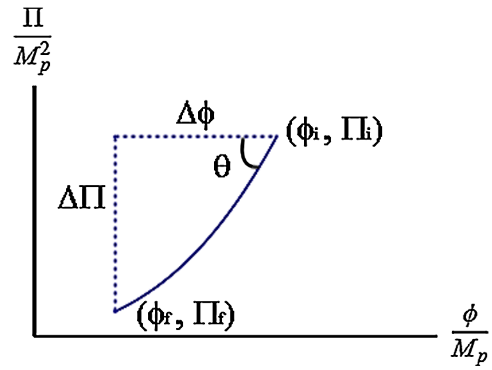

FIG. 5 (color online). A trajectory in phase space makes an angle θ with the horizontal, and traverses a certain distance $(\Delta\phi, \Delta\Pi)$.

It is straightforward to evaluate this angle for the various regions, and we find

$$\tan\theta_{\rm canon} = \begin{cases} \sqrt{\frac{3}{2}}\frac{\Pi}{M_p^2} & \text{Region A} \\ -\frac{\sqrt{2\epsilon_{SR}V}}{\Pi M_p^2} & \text{Region B} \\ \sqrt{3}\frac{\sqrt{V}}{M_p^2} & \text{Region C} \\ -\frac{\sqrt{2\epsilon_{SR}V}}{\Pi M_p^2} & \text{Region D.} \end{cases} \qquad (37)$$

Since $\Pi \ll M_p$, we see that in Region A $\theta_{\rm canon} \ll 1$, so *trajectories are approximately horizontal* here. These horizontal trajectories are overshoot trajectories.

In Region B we have $\sqrt{2V} < \Pi < \sqrt{2V}(\epsilon_{SR}/3)^{1/4}$ so the size of the angle in this region is bounded by

$$(3\epsilon_{SR})^{1/4}\frac{\sqrt{V}}{M_p^2} < |\tan\theta_{\rm canon}| < \epsilon_{SR}^{1/2}\frac{\sqrt{V}}{M_p^2}.$$

Because $V \ll M_p^4$, in Region B we expect $|\theta_{\rm canon}| \lesssim 1$ (recall $\epsilon_{SR} > 3$ in Region B). Similarly, in Region C the angle is small, i.e. $\theta_{\rm canon} \ll 1$. In Region D, it is clear that $\theta_{\rm canon} \to -\pi/2$ as $\Pi \to 0$, so trajectories in this region are mostly vertical. Certainly, by the definition of the trajectory angle (36), we have $\theta \approx 0$ along the inflationary solution (because $\tan\theta = \dot{\Pi}/(\Pi M_p) \sim (H/M_p)\mathcal{O}(\epsilon)$, as discussed in [26], the angle is not exactly horizontal along the inflationary trajectory). The qualitative features of these angles are represented in the left-hand plots of Figs. 3 and 4.

Now let us assume that we have some noncanonical kinetic terms that become important for $\Pi > \Lambda^2$. For simplicity, take the scale Λ such that the transition from canonical to noncanonical cuts through Regions C and D as in Figs. 3 and 4, ensuring Regions A and B are entirely in the noncanonical regime.

Again, using the equation of motion and the definition of the momentum, the trajectory angle in the limit of noncanonical kinetic terms (where $X \to R\Lambda^4$) is

$$\tan\theta_{\text{non-canon}} = -\frac{\dot{\Pi}}{M_p\sqrt{2X}} = \frac{3H\Pi - V'}{\sqrt{2R}M_p\Lambda^2}. \qquad (38)$$

The angles for the different regions in the noncanonical case are then

$$\tan\theta_{\text{non-canon}} = \begin{cases} \frac{\sqrt{3}}{(2R)^{1/4}}\left(\frac{\Pi}{M_p^2}\right)^{3/2}\left(\frac{M_p}{\Lambda}\right) & \text{(A);} \\ -\frac{\sqrt{2\epsilon_{\text{SR}}}V}{\sqrt{2R}M_p^2\Lambda^2} & \text{(B) and (D);} \\ \frac{\sqrt{3V}\Pi}{\sqrt{2R}M_p^2\Lambda^2} & \text{(C).} \end{cases} \qquad (39)$$

Comparing the angle for Region A to the canonical case, we see that the noncanonical angle is larger by a factor of

$$\frac{\tan\theta_{\text{non-canon}}}{\tan\theta_{\text{canon}}} = \frac{\sqrt{2}}{(2R)^{1/4}}\left(\frac{\Pi}{\Lambda^2}\right)^{1/2} \gg 1, \qquad (40)$$

which is larger than 1 because $\Pi \gg \Lambda^2$ in Region A. Thus, noncanonical trajectories in Region A are steeper than their canonical counterparts, an important fact to note for the overshoot problem. A similar comparison for Region B and the parts of Regions C and D that are noncanonical leads to

$$\frac{\tan\theta_{\text{non-canon}}}{\tan\theta_{\text{canon}}} = \frac{\Pi}{\sqrt{2R}\Lambda^2} \gg 1. \qquad (41)$$

Thus, quite generally, *noncanonical trajectories are always steeper* than their canonical counterparts. Of course, the angles in the small Π parts of Regions C and D are the same as in the canonical case discussed above, because the kinetic term is canonical there.

Another way to see how trajectories with noncanonical kinetic terms are generically steeper is to notice that the angle for noncanonical kinetic terms in Regions B-D is just a multiple of the angle for canonical kinetic terms, i.e.

$$\tan\theta_{\text{non-canon}} = \frac{1}{M_p}\frac{\dot{\Pi}}{\dot{\phi}} = \frac{\partial p}{\partial X}\left(\frac{3H\Pi - V'}{\Pi M_p}\right)$$
$$= \frac{\partial q}{\partial X}\tan\theta_{\text{canon}}, \qquad (42)$$

where we used the definition of the momentum (12). As discussed before, the Lagrangians we are considering have $\partial q/\partial X \geq 1$, so $\theta_{\text{non-canon}} \geq \theta_{\text{canon}}$ in Regions B-D in phase space. In Region A, however, (42) becomes

$$\tan\theta_{\text{non-canon}} = \frac{\partial q}{\partial X}\frac{H_{\text{non-canon}}}{H_{\text{canon}}}\tan\theta_{\text{canon}}, \qquad (43)$$

so the comparison of the angles depends both on the size of $\partial_X q$ and $H_{\text{non-canon}}$. In the noncanonical limit, this leads to (40).

V. OVERSHOOT TRAJECTORIES

Trajectories in Region A are a problem for canonical inflation. Although trajectories in this region lose momentum through Hubble friction, as we have seen the angle the trajectory makes with the horizontal is very small (in Planck units). Because of this, in canonical inflation, trajectories in Region A typically remain in Region A and never arrive at the inflationary solution, as in Figs. 3 and 4. (Notice that for the inflection-point potential, even trajectories that start in Region D, with small momentum, evolve towards Region A.) These are *overshoot trajectories*, and require some degree of fine-tuning of the initial conditions of inflation to avoid [60].

Let us make our discussion of overshoot trajectories somewhat more precise. Consider a trajectory that starts at some initial point (ϕ_i, Π_i) in phase space and ends, some time later, at (ϕ_f, Π_f), as in Fig. 5. For any given $\{\Pi_i, \Pi_f\}$, the trajectory moves some distance $\Delta\phi$ in field space. Our primary interest is in estimating the distance the trajectory must travel in order to exit Region A, as a way to better understand the dynamics of the putative overshoot trajectories. Denote the typical distance a trajectory must travel in order to exit Region A as $(\Delta\phi)_A$.

One way of gauging the danger that these trajectories pose to the fine-tuning of initial conditions is to compare $(\Delta\phi)_A$ to the size of field space $(\Delta\phi)_{\text{inf}}$ in which inflation occurs. We can construct an "overshoot parameter,"

$$\alpha \equiv \frac{(\Delta\phi)_A}{(\Delta\phi)_{\text{inf}}}, \qquad (44)$$

that measures the relative size of these distances. Certainly, if a trajectory in Region A only moves some small fraction of the distance over which inflation occurs, so that $\alpha \ll 1$, we do not expect that there will be severe fine-tuning issues. Alternatively, if $\alpha \gg 1$, then a trajectory moves over a distance much larger than the size of the inflationary region, and we expect that fine-tuning of initial conditions will be necessary to avoid these dangerous Region A trajectories.

The parameter α is thus a useful quantity to gauge the severity of large-momentum trajectories for the fine-tuning of initial conditions that lead to inflation. Its advantage over the parameter Σ, whose definition as the ratio of the volume of initial conditions that give rise to more than 60 e-folds of inflation to the total volume of phase space is given in Sec. , is that α can be computed almost completely analytically and is still correlated with the more direct measure of fine-tuning Σ. In addition, as we will see, $(\Delta\phi)_A$ is independent of the details of the potential and is only sensitive to the structure of the kinetic terms, making it ideal for an analysis in which noncanonical kinetic terms play an important role. In contrast, the computation of the fine-tuning parameter Σ is dependent on the details of the potential.

For a canonical kinetic term, we found that in Region A the angle is

$$\left.\frac{d\Pi}{d\phi}\right|_A = \sqrt{\frac{3}{2}}\frac{\Pi}{M_p}.$$

This can integrated to obtain

$$\frac{(\Delta\phi)_A^{\rm canon}}{M_p} = \sqrt{\frac{2}{3}}\log\left(\frac{\Pi_i}{\Pi_f}\right). \tag{45}$$

The maximum value for the initial momentum is, as discussed before, $\Pi_i \lesssim M_p\Lambda$, while the minimum is set by the lower boundary of Region A, i.e. $\Pi_f \sim \sqrt{2V}$. However, since the dependence on the initial and final momenta in (45) is logarithmically weak, we expect a typical trajectory to move an $\mathcal{O}(1)$ distance in Planck units before exiting Region A:

$$(\Delta\phi)_A^{\rm canon} \sim M_p. \tag{46}$$

Notice that this is independent of the type of potential, so it is true for all canonical small- and large-field models.

We now see dynamically, then, why small-field models typically have an overshoot problem but large-field models do not. For small-field models, the size of the field space in which inflation happens is small in Planck units by definition, i.e. $(\Delta\phi)_{\rm inf} \ll M_p \sim (\Delta\phi)_A$, so the overshoot parameter is very large: $\alpha \gg 1$. This is in agreement with the known fine-tuning of initial conditions required in small-field models [15,21]. On the other hand, for large-field models we typically have $(\Delta\phi)_{\rm inf} \sim \mathcal{O}(10\text{–}100) \times M_p$ so $\alpha \ll 1$. Trajectories that start with large momentum typically only move over some small fraction of the size of the inflationary region before losing enough momentum to enter inflation. Large-field models do not require fine-tuning of initial conditions [12–15,61], in agreement with the size of the overshoot parameter $\alpha \ll 1$ for these models.

For a noncanonical kinetic term, we can do a similar integration of the angle in Region A, finding

$$\frac{d\Pi}{d\phi}\bigg|_A = \frac{\sqrt{3}}{(2R)^{1/4}}\left(\frac{\Pi}{M_p^2}\right)^{3/2}\frac{M_p^2}{\Lambda},$$

which gives

$$\frac{(\Delta\phi)_A^{\rm non\text{-}canon}}{M_p} = \frac{(2R)^{1/4}}{2\sqrt{3}}\left[\left(\frac{\Lambda^2}{\Pi_f}\right)^{1/2} - \left(\frac{\Lambda^2}{\Pi_i}\right)^{1/2}\right]. \tag{47}$$

Taking $\Pi_i \gg \Pi_f$, we can drop the last term in (47). The lower boundary of Region A occurs for $\Pi/\Lambda^2 \sim (V/\Lambda^4)(2R)^{-1/2}$, so that we have

$$(\Delta\phi)_A^{\rm non\text{-}canon} \approx \frac{(2R)^{1/2}}{2\sqrt{3}}M_p\sqrt{\frac{\Lambda^4}{V}}. \tag{48}$$

We see that if $V/\Lambda^4 \gg 1$, the noncanonical trajectory moves a much smaller distance than it does with a canonical kinetic term. In particular, compared to the canonical kinetic term, we have

$$\frac{\alpha_{\rm non\text{-}canon}}{\alpha_{\rm canon}} \sim \frac{(2R)^{1/2}}{2\sqrt{3}}\sqrt{\frac{\Lambda 4}{V}} \ll 1, \tag{49}$$

so noncanonical kinetic terms can dramatically improve the degree of overshooting. In the next section, we will see this in more detail by examining a few examples.

VI. EXAMPLES

In this section we will consider some specific examples to see how the effect of the noncanonical kinetic terms is to reduce the overshoot and initial conditions fine-tuning problems. As mentioned before, we will consider two types of characteristic small-field inflation potentials known to have overshoot and initial condition fine-tuning problems, the inflection point (26) and Coulomb (27) potentials,

$$V(\phi)_{\rm inflection} = V_0 + \lambda(\phi - \phi_0) + \beta(\phi - \phi_0)^3;$$
$$V(\phi)_{\rm coulomb} = V_0 - \frac{T}{(\phi + \phi_0)^n}.$$

We will choose the parameters below for these potentials such that, for a canonical kinetic term, more than 60 e-folds of (slow-roll) inflation leads to an acceptable power spectrum in the appropriate observational window, consistent with current observations [5], i.e. $P_\zeta = 2.41 \times 10^{-9}$, $n_s = 0.961$.

$$\text{Inflection}: \begin{pmatrix} V_0 = 3.7 \times 10^{-16}, & \lambda = 1.13 \times 10^{-20} \\ \beta = 1.09 \times 10^{-15}, & \phi_0 = 0.01 \end{pmatrix};$$
$$\text{Coulomb}: \begin{pmatrix} V_0 = 5.35 \times 10^{-14}, & T = \phi_0^4 V_0, \\ n = 4, & \phi_0 = 0.0 \end{pmatrix}.$$

We will choose the scale Λ that controls the strength of the noncanonical kinetic terms to be $\Lambda = \{5 \times 10^{-6}, 5 \times 10^{-5}\}$ for the inflection-point and Coulomb potentials, respectively. These values are chosen so that the background inflationary solution obeys all of the relevant effective field theory constraints and conditions [51–53], but also still allows the noncanonical kinetic terms to play an important role for the dynamics (see [26] for more discussion of these constraints).

As examples, we will compare a canonical Lagrangian to a Lagrangian with a "geometric series"-type kinetic term:

$$p(X, \phi)_{\rm canon} = X - V(\phi); \tag{50}$$

$$p(X, \phi)_{\rm geo} = \Lambda^4\left(\frac{1}{1 - X/\Lambda^4} - 1\right) - V(\phi). \tag{51}$$

The Lagrangians will be compared for the same potential (inflection point or Coulomb) so that the only thing that differs is their kinetic terms.

The phase-space plots shown in Figs. 6 and 7 include the inflationary solution $\Pi_{\rm inf}(\phi)$ (dotted line), sample trajectories (solid red lines) showing the flow in phase space, and boundaries of the regions for the inflection-point and

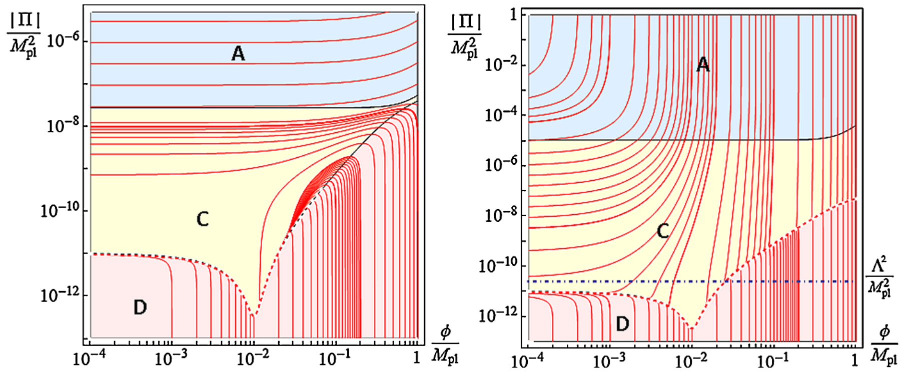

FIG. 6 (color online). Left: Phase-space plot for a canonical Lagrangian (50) with an inflection-point potential (26). Right: Phase-space plot for the noncanonical Lagrangian (51) with the same inflection-point potential. Notice that trajectories that previously were overshoot trajectories for a canonical kinetic term are attracted strongly to the inflationary solution (dotted line). The horizontal dot-dashed line is the value $\Pi = \Lambda^2$; above this the system is noncanonical.

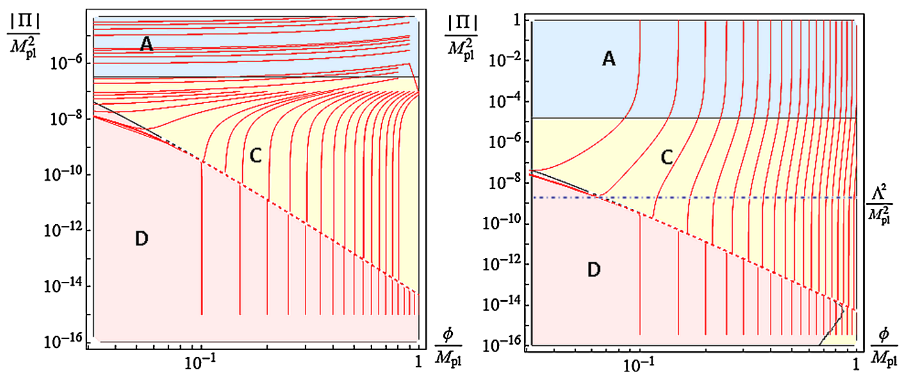

FIG. 7 (color online). Left: Phase-space plot for a canonical Lagrangian (50) with a Coulomb potential (27). Right: Phase-space plot for the noncanonical Lagrangian (51) with the same Coulomb potential. Notice that trajectories that previously were overshoot trajectories for a canonical kinetic term are attracted strongly to the inflationary solution.

Coulomb potentials, respectively, for both the canonical and noncanonical Lagrangians. Notice that the qualitative features of the structure and dynamics of phase space discussed throughout Secs. III and V are evident upon comparison of the canonical and noncanonical Lagrangians:

(i) The upper boundary of the allowed phase space for Π is much larger, by a factor of M_p/Λ.

(ii) The lower boundary of Region A is pushed to larger Π. (Notice that, for these potentials, $\epsilon_{\rm SR} < 3$ in our regime of interest, so Region B does not appear.)

(iii) The angles that trajectories make with the horizontal are generally much steeper.

(iv) Overshoot trajectories in Region A are much less prevalent. In particular, the overshoot parameter $\alpha \sim \infty$ for the canonical Lagrangian (trajectories never exit Region A for the small-field regime we are considering), while $\alpha \lesssim 10^{-3}$ for the noncanonical Lagrangian, putting it on a par with the overshooting found in typical large-field models.

From Figs. 6 and 7, we see that noncanonical kinetic terms modify the dynamics of phase space in such a way that the prevalence of overshooting is reduced.

Initial conditions fine-tuning

It is helpful to make the effects of the noncanonical kinetic terms on the problem of fine-tuning of initial conditions more precise. Consider a standard measure of the amount of fine-tuning of the initial conditions, the "initial conditions fine-tuning" parameter, defined as the fraction of initial-conditions phase space that leads to more than 60 e-folds of inflation,

$$\Sigma = \frac{\int d\phi d\Pi \theta(N_e - 60)}{\int d\phi d\Pi}, \qquad (52)$$

where the number of e-folds is defined as usual, i.e. $N_e \equiv \log\frac{a_e}{a_i} = \int H dt$, and $\theta(x)$ is 1 for $x > 0$ and 0 for $x < 0$. This has been used to parameterize the fine-tuning of initial conditions before [12,13,21,56,62].

We should make it clear that using Σ to quantify the fine-tuning of initial conditions presupposes a flat measure on the (ϕ, Π) phase space, without any volume weighting, for example. The choice of a measure for inflation involves many subtleties; see [40,41] for a discussion. Certainly, for a choice of measure that exponentially disfavors inflation [41] it is unlikely that the modified dynamics we find here will significantly change those conclusions. However, our aim in using Σ is not to present it as a definitive measure of the naturalness of initial conditions, but rather to investigate how sensitive this simple parameter is to changes in the kinetic term. In this way Σ serves as a useful diagnostic of the modified dynamics coming from the noncanonical kinetic term with as few assumptions as possible on the measure of the space of initial conditions.

In Fig. 8 we show the fraction of phase-space initial conditions (in log scale) that gives rise to more than 60 e-folds of inflation for canonical and noncanonical kinetic terms and the two different potentials, inflection point (left) and Coulomb (right). We see, as expected, that for a canonical kinetic term (red shaded region) only a very small region of initial-conditions phase space leads to enough successful inflation, with

$$\Sigma_{\rm canon} \approx \{2.4 \times 10^{-4}, 4.5 \times 10^{-3}\} \tag{53}$$

for the inflection-point and Coulomb potentials, respectively. As expected from Figs. 6 and 7, the region of initial-conditions phase space that leads to enough successful inflation for the noncanonical Lagrangian is *much* larger for both potentials, with

$$\Sigma_{\rm non\text{-}canon} \approx \{0.80, 0.72\} \tag{54}$$

(again for inflection point and Coulomb potentials, respectively). Thus, the noncanonical Lagrangian effectively does not require any initial conditions fine-tuning at all, in contrast to the canonical Lagrangian.

Our results are insensitive to the choice of noncanonical kinetic term. For example, a DBI Lagrangian,

$$p(X, \phi) = \Lambda^4(\sqrt{1 - 2X/\Lambda^4} - 1) - V(\phi), \tag{55}$$

gives very similar results to the geometric-series Lagrangian, as shown in Fig. 9. In the Appendix, we extend our results to a "powerlike" class of Lagrangians, finding that they behave very similarly to the power-series Lagrangians. Thus, our conclusions do not seem to be particularly sensitive to the form of the Lagrangian, only whether or not noncanonical kinetic terms are relevant for inflation.

We see then that noncanonical kinetic terms can drastically reduce the amount of homogeneous initial conditions fine-tuning needed to obtain successful inflation, making these models as "attractive" as large-field models.

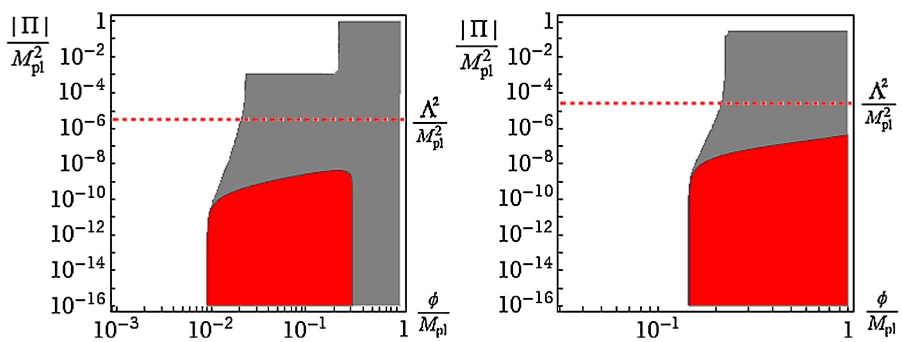

FIG. 8 (color online). Left: The shaded regions represent initial conditions that lead to more than 60 e-folds of inflation for an inflection-point potential (55) with canonical kinetic term (50) [red] and noncanonical kinetic term (51) [gray]. Right: The same, but for a Coulomb potential. Notice that this is a log plot, so that the size of the 60 e-fold region for the canonical Lagrangian is exponentially smaller than that for the noncanonical Lagrangian. The dashed horizontal (red) line denotes the upper boundary of phase space for a canonical kinetic term (see discussion in Sec. III).

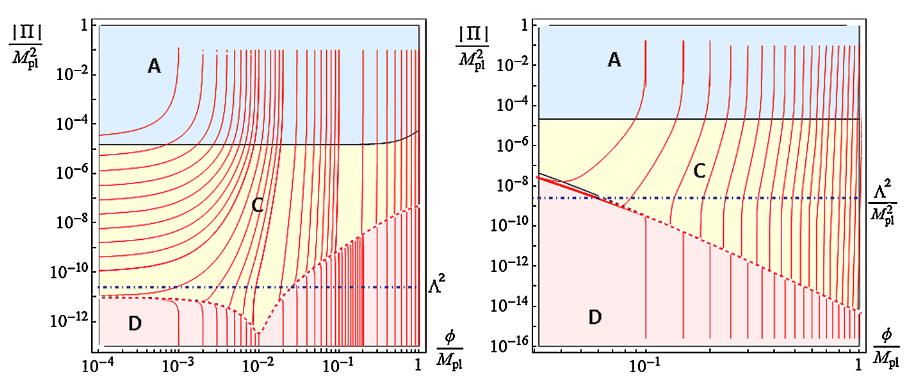

FIG. 9 (color online). Left: Phase-space plot for a DBI Lagrangian (55) with an inflection-point potential (26). Right: Phase-space plot for a DBI Lagrangian with a Coulomb potential (27). Notice that the phase-space dynamics are essentially the same as for the geometric-series Lagrangian in Figs. 6 and 7.

VII. CONCLUSION

In this paper we investigated the dynamics of (homogeneous) phase space for single-field inflationary models with both canonical and noncanonical kinetic terms. Inflationary trajectories are formally attractors in phase space, as discussed in [26], but since in practice not all initial conditions lead to inflation, some degree of fine-tuning is required. We uncovered general features of the phase space of noncanonical kinetic terms that can be useful for understanding inflationary initial conditions.

We divided phase space into several regions depending on the dynamics implied by the equation of motion. Our general regional analysis allowed us to construct a qualitative picture of the flow of trajectories in phase space, as in Figs. 3 and 4. We investigated the angles that trajectories make in phase space, and found that as the kinetic terms become more noncanonical, trajectories become steeper. In particular, trajectories that have a large amount of momentum lose their momentum through Hubble friction *faster* for noncanonical kinetic terms. This effect leads to an amelioration of the overshoot problem for inflation, in which a trajectory with too much kinetic energy will never enter the inflationary phase.

We characterized the prevalence of overshoot trajectories through an "overshoot parameter" α, given by

$$\alpha \equiv \frac{(\Delta\phi)_A}{(\Delta\phi)_{\rm inf}},$$

which is the ratio of the field distance a typical trajectory travels in the large-momentum region (Region A). If $\alpha \ll 1$, then the inflaton only moves a fraction of the size of the inflationary trajectory before shedding its momentum through Hubble friction, so it does not overshoot. If $\alpha \gg 1$, the inflaton must move many times the distance spanned by the inflationary trajectory to shed its momentum through Hubble friction, so it generically overshoots the inflationary regime.

We computed $(\Delta\phi)_A$ for canonical and noncanonical kinetic terms, finding

$$(\Delta\phi)_A^{\rm canon} = \sqrt{\frac{2}{3}} M_p \log\left(\frac{M_p\Lambda}{\sqrt{2V}}\right) \sim \mathcal{O}(1) M_p;$$

$$(\Delta\phi)_A^{\rm non\text{-}canon} = \frac{(2R)^{1/2}}{2\sqrt{3}}\left(\frac{\Lambda^4}{V}\right)^{1/2} M_p \ll M_p.$$

For a typical canonical small-field inflationary model, $(\Delta\phi)_{\rm inf} \ll M_p$ so $\alpha \gg 1$ and we expect a significant overshoot problem. In contrast, for large-field inflation $(\Delta\phi)_{\rm inf} \gtrsim \mathcal{O}(10\text{–}100) M_p$ so $\alpha \ll 1$ and we expect no overshoot problem. These qualitative results are in agreement with the numerical results [12–15,21]. Importantly, the distance traveled while shedding momentum through Hubble friction is significantly smaller for noncanonical kinetic terms. For specific examples, we found that noncanonical kinetic terms decreased the value of α from $\alpha \sim \infty$ for canonical small-field models (corresponding to the case where trajectories never leave Region A) to $\alpha \sim 10^{-3}$ for noncanonical models. Correspondingly, the prevalence of overshoot trajectories is significantly decreased, as can be seen in Figs. 6 and 7.

We also investigated the effect of the noncanonical kinetic terms on the standard initial conditions fine-tuning problem. Because the noncanonical kinetic terms significantly modify the dynamics in the large-momentum region, we found that small-field models with noncanonical kinetic terms do not have an initial conditions fine-tuning problem, even when their canonical counterparts do. In particular, in terms of the fraction Σ of phase space leading to 60 e-folds of inflation, we studied small-field inflationary potentials which required $\Sigma \sim 10^{-4} - 10^{-3}$ for canonical kinetic terms, but $\Sigma \sim 0.8$ for noncanonical kinetic terms.

Of course, not every effective theory for inflation with noncanonical kinetic terms leads to such drastic improvements in the overshoot and initial conditions fine-tuning problems. Certainly, if the energy Λ that controls the scale at which the corrections to the kinetic term become important is too large, we do not expect to see a significant effect. In our analysis, we assumed that some period of noncanonical inflation occurs; this appears to be sufficient to affect the dynamics in phase space, but it is not clear if it is necessary.

It is difficult to make progress on a theory of inflationary initial conditions because we have so little information about the initial conditions; by construction inflation washes out features of the preinflationary era. However, the dynamics of the inflaton, visible through cosmological observables such as the tensor-to-scalar ratio or equilateral non-Gaussianity, may give important clues as to the preinflationary physics. Through the (generalized) Lyth bound [54,63]

$$\frac{\Delta\phi}{M_p} = \int_0^{N_{\rm end}} \sqrt{\frac{r}{8}} \frac{dN_e}{\sqrt{c_s P_X}} \approx \frac{\mathcal{O}(1)}{\sqrt{c_s P_X}}\left(\frac{r}{0.01}\right)^{1/2},$$

we learn that observation of a tensor-to-scalar ratio $r \gtrsim 0.01$ would imply that inflation is a large-field model, and so the standard dynamics of a canonical kinetic term would be enough to drive the (homogeneous) preinflationary universe towards inflation. Similarly, because the size of primordial equilateral non-Gaussianity depends on the deviation of the system from a canonical kinetic term [44] via

$$f_{\rm NL}^{\rm (equil)} \sim c_s^{-2},$$

an observation of equilateral non-Gaussianity would suggest that the dynamics of the noncanonical kinetic terms are important for driving the preinflationary universe towards inflation. It is possible that there are other observable features that may help us refine our ideas about the physics of inflationary initial conditions [37,64].

Certainly, a more robust treatment of inflationary initial conditions must include the effect of inhomogeneities. It is not clear if, or how, noncanonical kinetic terms modify the dynamics of inhomogeneities in the preinflationary universe, questions which it would be interesting to investigate further.

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APPENDIX: POWERLIKE LAGRANGIANS

In the text we considered only the "power-series" type Lagrangians, namely, Lagrangians which lead to noncanonical inflation when expressed as power series within their domain of convergence. As was discussed in [26], there is another class of Lagrangians that give rise to noncanonical inflation *outside* of their domain of convergence when expressed as a power series. It is not clear whether these Lagrangians make sense as effective field theories (EFTs), since we would expect the validity of the EFT to break down at the radius of convergence of the power series. Nevertheless, we consider these Lagrangians in this appendix for completeness, in case they can be seen as sensible EFTs.

The typical example from this class is the "powerlike" Lagrangian

$$p(X,\phi) = \Lambda^4\left[\left(1+\frac{1}{n}\frac{X}{\Lambda^4}\right)^n - 1\right] - V(\phi). \tag{A1}$$

For this Lagrangian, the canonical momentum is

$$\Pi = \sqrt{2X}\left(1+\frac{1}{n}\frac{X}{\Lambda^4}\right)^{n-1}. \tag{A2}$$

In the noncanonical regime, we see that $X/\Lambda^4 \gg 1$ when $\Pi/\Lambda^2 \gg 1$, so $\Pi \sim \Lambda^2$ is the boundary of the noncanonical regime in field space as before. We will restrict the rest of this analysis to this regime, since this is where the phase space will differ from the canonical case.

As before, the boundaries of the Regions A-D are determined by which term in (24) and (25) dominates. The kinetic energy dominates over the potential energy when

$$\Pi > \frac{V^{(n-1/2)/n}}{\Lambda^{2(1-1/n)}(2n)^{(n-1)/2n}} \xrightarrow{n\gg 1} \frac{V(\phi)}{\sqrt{2n}\Lambda^2}, \tag{A3}$$

which reduces to (31) in the large n limit. Similarly, the Hubble friction domination condition, when the energy density is mostly kinetic, becomes (in the large n limit)

$$\Pi > \left(\frac{\epsilon_{SR}^2}{18}\right)^{1/6}\frac{\sqrt{2V}}{(2n)^{1/6}}\left(\frac{V}{\Lambda^4}\right)^{1/6}, \tag{A4}$$

which is the same as (33). The angles in these regions are also modified:

$$\tan\theta = \begin{cases} \frac{\sqrt{3}}{(2n)^{(n-1)/2(2n-1)}}\frac{\Pi^{(3n-2)/(2n-1)}}{M_p^2\Lambda^{(2(n-1)/(2n-1)}}; \\ -\frac{\sqrt{2\epsilon_{SR}}}{(2n)^{(n-1)/(2n-1)}}\frac{V}{M_p^2\Lambda^{4(n-1)/(2n-1)}\Pi^{1/(2n-1)}}; \\ \frac{\sqrt{3V}}{(2n)^{(n-1)/(2n-1)}}\frac{\Pi^{(2n-2)/(2n-1)}}{M_p^2\Lambda^{4(n-1)/(2n-1)}}. \end{cases} \tag{A5}$$

In the large n limit these become

$$\tan\theta \xrightarrow{n\gg 1} \begin{cases} \frac{\sqrt{3}}{(2n)^{1/4}}\frac{\Pi^{3/2}}{\Lambda M_p^2} & \text{(A);} \\ -\frac{\sqrt{2\epsilon_{SR}V}}{\sqrt{2n}M_p^2\Lambda^2} & \text{(B) and (D);} \\ \frac{\sqrt{3V}\Pi}{\sqrt{2n}M_p^2\Lambda^2} & \text{(D).} \end{cases} \tag{A6}$$

We see that these also approach the same form as the power-series Lagrangians in the large n limit. Thus these Lagrangians give very similar results as the power-series Lagrangians considered in the main text.

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