Boltzmann brains and the scale-factor cutoff measure of the multiverse

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To make predictions for an eternally inflating "multiverse," one must adopt a procedure for regulating its divergent spacetime volume. Recently, a new test of such spacetime measures has emerged: normal observers—who evolve in pocket universes cooling from hot big bang conditions—must not be vastly outnumbered by "Boltzmann brains"—freak observers that pop in and out of existence as a result of rare quantum fluctuations. If the Boltzmann brains prevail, then a randomly chosen observer would be overwhelmingly likely to be surrounded by an empty world, where all but vacuum energy has redshifted away, rather than the rich structure that we observe. Using the scale-factor cutoff measure, we calculate the ratio of Boltzmann brains to normal observers. We find the ratio to be finite, and give an expression for it in terms of Boltzmann brain nucleation rates and vacuum decay rates. We discuss the conditions that these rates must obey for the ratio to be acceptable, and we discuss estimates of the rates under a variety of assumptions.

I. INTRODUCTION

The simplest interpretation of the observed accelerating expansion of the Universe is that it is driven by a constant vacuum-energy density ρ_Λ, which is about 3 times greater than the present density of nonrelativistic matter. While ordinary matter becomes more dilute as the Universe expands, the vacuum-energy density remains the same, and in another 10×10^9 yrs or so the Universe will be completely dominated by vacuum energy. The subsequent evolution of the Universe is accurately described as de Sitter (dS) space.

It was shown by Gibbons and Hawking [1] that an observer in de Sitter space would detect thermal radiation with a characteristic temperature $T_{dS} = H_\Lambda/2\pi$, where

$$H_\Lambda = \sqrt{\frac{8\pi}{3}G\rho_\Lambda} \qquad (1)$$

is the de Sitter Hubble expansion rate. For the observed value of ρ_Λ, the de Sitter temperature is extremely low, $T_{dS} = 2.3 \times 10^{-30}$ K. Nevertheless, complex structures will occasionally emerge from the vacuum as quantum fluctuations, at a small but nonzero rate per unit spacetime volume. An intelligent observer, like a human, could be one such structure. Or, short of a complete observer, a disembodied brain may fluctuate into existence, with a pattern of neuron firings creating a perception of being on Earth and, for example, observing the cosmic microwave background radiation. Such freak observers are collectively referred to as "Boltzmann brains" [2,3]. Of course, the nucleation rate Γ_{BB} of Boltzmann brains is extremely small, its magnitude depending on how one

defines a Boltzmann brain. The important point, however, is that Γ_{BB} is always nonzero.

De Sitter space is eternal to the future. Thus, if the accelerating expansion of the Universe is truly driven by the energy density of a stable vacuum state, then Boltzmann brains will eventually outnumber normal observers, no matter how small the value of Γ_{BB} [4–8] might be.

To define the problem more precisely, we use the term "normal observers" to refer to those that evolve as a result of nonequilibrium processes that occur in the wake of the hot big bang. If our Universe is approaching a stable de Sitter spacetime, then the total number of normal observers that will ever exist in a fixed comoving volume of the Universe is finite. On the other hand, the cumulative number of Boltzmann brains grows without bound over time, growing roughly as the volume, proportional to $e^{3H_\Lambda t}$. When extracting the predictions of this theory, such an infinite preponderance of Boltzmann brains cannot be ignored.

For example, suppose that some normal observer, at some moment in her lifetime, tries to make a prediction about her next observation. According to the theory there would be an infinite number of Boltzmann brains, distributed throughout the spacetime, that would happen to share exactly all her memories and thought processes at that moment. Since all her knowledge is shared with this set of Boltzmann brains, for all she knows she could equally likely be any member of the set. The probability that she is a normal observer is then arbitrarily small, and all predictions would be based on the proposition that she is a Boltzmann brain. The theory would predict, therefore, that the next observations that she will make, if she survives to make any at all, will be totally incoherent,

with no logical relationship to the world that she thought she knew. (While it is of course true that some Boltzmann brains might experience coherent observations, for example, by living in a Boltzmann solar system, it is easy to show that Boltzmann brains with such dressing would be vastly outnumbered by Boltzmann brains without any coherent environment.) Thus, the continued orderliness of the world that we observe is distinctly at odds with the predictions of a Boltzmann-brain-dominated cosmology.[1]

This problem was recently addressed by Page [7], who concluded that the least unattractive way to produce more normal observers than Boltzmann brains is to require that our vacuum should be rather unstable. More specifically, it should decay within a few Hubble times of vacuum-energy domination, that is, in 20×10^9 yrs or so.

In the context of inflationary cosmology, however, this problem acquires a new twist. Inflation is generically eternal, with the physical volume of false-vacuum inflating regions increasing exponentially with time, and "pocket universes" like ours constantly nucleating out of the false vacuum. In an eternally inflating multiverse, the numbers of normal observers and Boltzmann brains produced over the course of eternal inflation are both infinite. They can be meaningfully compared only after one adopts some prescription to regulate the infinities.

The problem of regulating the infinities in an eternally inflating multiverse is known as the measure problem [9], and has been under discussion for some time. It is crucially important in discussing predictions for any kind of observation. Most of the discussion, including the discussion in this paper, has been confined to the classical approximation. While one might hope that someday there will be an answer to this question based on a fundamental principle [10], most of the work on this subject has focused on proposing plausible measures and exploring their properties. Indeed, a number of measures have been proposed [11–27], and some of them have already been disqualified, as they make predictions that conflict with observations.

In particular, if one uses the proper-time cutoff measure [11–15], one encounters the "youngness paradox," predicting that humans should have evolved at a very early cosmic time, when the conditions for life were rather hostile [28]. The youngness problem, as well as the Boltzmann brain problem, can be avoided in the stationary measure [18,27], which is an improved version of the proper-time cutoff measure. However, the stationary measure, as well as the pocket-based measure, is afflicted with a runaway problem, suggesting that we should observe extreme values (either very small or very large) of the primordial density contrast Q [29,30] and the gravitational constant G [31], while these parameters appear to sit comfortably in the middle of their respective anthropic ranges [32,33]. Some suggestions to get around this issue have been described in Refs. [30,33–35]. In addition, the pocket-based measure seems to suffer from the Boltzmann brain problem. The comoving coordinate measure [11,36] and the causal-patch measures [23,24] are free from these problems, but have an unattractive feature of depending sensitively on the initial state of the multiverse. This does not seem to mix well with the attractor nature of eternal inflation: the asymptotic late-time evolution of an eternally inflating universe is independent of the starting point, so it seems appealing for the measure to maintain this property. Since the scale-factor cutoff measure[2] [12–14,16,17,37] has been shown to be free of all of the above issues [38], we consider it to be a promising candidate for the measure of the multiverse.

As we have indicated, the relative abundance of normal observers and Boltzmann brains depends on the choice of measure over the multiverse. This means the predicted ratio of Boltzmann brains to normal observers can be used as yet another criterion to evaluate a prescription to regulate the diverging volume of the multiverse: regulators predicting that normal observers are greatly outnumbered by Boltzmann brains should be ruled out. This criterion has been studied in the context of several multiverse measures, including a causal-patch measure [8], several measures associated with globally defined time coordinates [17,18,27,39,40], and the pocket-based measure [41]. In this work, we apply this criterion to the scale-factor cutoff measure, extending the investigation that was initiated in Ref. [17]. We show that the scale-factor cutoff measure gives a finite ratio of Boltzmann brains to normal observers; if certain assumptions about the landscape are valid, the ratio can be small.[3]

The remainder of this paper is organized as follows. In Sec. II we provide a brief description of the scale-factor cutoff and describe salient features of the multiverse under

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[1]Here we are taking a completely mechanistic view of the brain, treating it essentially as a highly sophisticated computer. Thus, the normal observer and the Boltzmann brains can be thought of as a set of logically equivalent computers running the same program with the same data, and hence they behave identically until they are affected by further input, which might be different. Since the computer program cannot determine whether it is running inside the brain of one of the normal observers or one of the Boltzmann brains, any intelligent probabilistic prediction that the program makes about the next observation would be based on the assumption that it is equally likely to be running on any member of that set.

[2]This measure is sometimes referred to as the volume-weighted scale-factor cutoff measure, but we will define it below in terms of the counting of events in spacetime, so the concept of weighting will not be relevant. The term "volume-weighted" is relevant when a measure is described as a prescription for defining the probability distribution for the value of a field. In Ref. [17], this measure is called the "pseudo-comoving volume-weighted measure."

[3]In a paper that appeared simultaneously with version 1 of this paper, Raphael Bousso, Ben Freivogel, and I-Sheng Yang independently analyzed the Boltzmann brain problem for the scale-factor cutoff measure [42].

the lens of this measure. In Sec. III we calculate the ratio of Boltzmann brains to normal observers in terms of multiverse volume fractions and transition rates. The volume fractions are discussed in Sec. IV, in the context of toy landscapes, and the section ends with a general description of the conditions necessary to avoid Boltzmann brain domination. The rate of Boltzmann brain production and the rate of vacuum decay play central roles in our calculations, and these are estimated in Sec. V. Concluding remarks are provided in Sec. VI.

II. THE SCALE-FACTOR CUTOFF

Perhaps the simplest way to regulate the infinities of eternal inflation is to impose a cutoff on a hypersurface of constant global time [12–16]. One starts with a patch of a spacelike hypersurface Σ somewhere in an inflating region of spacetime, and follows its evolution along the congruence of geodesics orthogonal to Σ. The scale-factor time is defined as

$$t = \ln a, \qquad (2)$$

where a is the expansion factor along the geodesics. The scale-factor time is related to the proper time τ by

$$dt = H d\tau, \qquad (3)$$

where H is the Hubble expansion rate of the congruence. The spacetime region swept out by the congruence will typically expand to unlimited size, generating an infinite number of pockets. (If the patch does not grow without limit, one chooses another initial patch Σ and starts again.) The resulting four-volume is infinite, but we cut it off at some fixed scale-factor time $t = t_c$. To find the relative probabilities of different events, one counts the numbers of such events in the finite spacetime volume between Σ and the $t = t_c$ hypersurface, and then takes the limit $t_c \rightarrow \infty$.

The term ''scale factor'' is often used in the context of homogeneous and isotropic geometries; yet on very large and on very small scales the multiverse may be very inhomogeneous. A simple way to deal with this is to take the factor H in Eq. (3) to be the local divergence of the four-velocity vector field along the congruence of geodesics orthogonal to Σ,

$$H(x) \equiv (1/3) u^{\mu}{}_{;\mu}. \qquad (4)$$

When more than one geodesic passes through a point, the scale-factor time at that point may be taken to be the smallest value among the set of geodesics. In collapsing regions $H(x)$ is negative, in which case the corresponding geodesics are continued unless or until they hit a singularity.

This ''local'' definition of scale-factor time has a simple geometric meaning. The congruence of geodesics can be thought of as representing a ''dust'' of test particles scattered uniformly on the initial hypersurface Σ. As one moves along the geodesics, the density of the dust in the orthogonal plane decreases. The expansion factor a in

Eq. (2) can then be defined as $a \propto \rho^{-1/3}$, where ρ is the density of the dust, and the cutoff is triggered when ρ drops below some specified level.

Although the local scale-factor time closely follows the Friedmann-Robertson-Walker (FRW) scale factor in expanding spacetimes—such as inflating regions and thermalized regions not long after reheating—it differs dramatically from the FRW scale factor as small-scale inhomogeneities develop during matter domination in universes like ours. In particular, the local scale-factor time nearly grinds to a halt in regions that have decoupled from the Hubble flow. It is not clear whether we should impose this particular cutoff, which would essentially include the entire lifetime of any nonlinear structure that forms before the cutoff, or impose a cutoff on some nonlocal time variable that more closely tracks the FRW scale factor.[4]

There are a number of nonlocal modifications of scale-factor time that both approximate our intuitive notion of FRW averaging and also extend into more complicated geometries. One drawback of the nonlocal approach is that no single choice looks more plausible than the others. For instance, one nonlocal method is to define the factor H in Eq. (3) by spatial averaging of the quantity $H(x)$ in Eq. (4). A complete implementation of this approach, however, involves many seemingly arbitrary choices regarding how to define the hypersurfaces over which $H(x)$ should be averaged, so here we set this possibility aside. A second, simpler method is to use the local scale-factor time defined above, but to generate a new cutoff hypersurface by excluding the future light cones of all points on the original cutoff hypersurface. In regions with nonlinear inhomogeneities, the underdense regions will be the first to reach the scale-factor cutoff, after which they quickly trigger the cutoff elsewhere. The resulting cutoff hypersurface will not be a surface of constant FRW scale factor, but the fluctuations of the FRW scale factor on this surface should be insignificant.

As a third and final example of a nonlocal modification of scale-factor time, we recall the description of the local scale-factor cutoff in terms of the density ρ of a dust of test particles. Instead of such a dust, consider a set of massless test particles, emanating uniformly in all directions from each point on the initial hypersurface Σ. We can then construct the conserved number density current J^{μ} for the gas of test particles, and we can define ρ as the rest frame number density, i.e. the value of J^0 in the local Lorentz frame in which $J^i = 0$, or equivalently $\rho = \sqrt{J^2}$. Defining $a \propto \rho^{-1/3}$, as we did for the dust of test particles, we apply the cutoff when the number density ρ drops below some specified level. Since null geodesics are barely perturbed by structure formation, the strong perturbations

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[4]The distinction between these two forms of scale-factor time was first pointed out by Bousso, Freivogel, and Yang in Ref. [42].

inherent in the local definition of scale-factor time are avoided. Nonetheless, we have not studied the properties of this definition of scale-factor time, and they may lead to complications. Large-scale anisotropic flows in the gas of test particles can be generated as the particles stream into expanding bubbles from outside. Since the null geodesics do not interact with matter except gravitationally, these anisotropies will not be damped in the same way as they would be for photons. The large-scale flow of the gas will not redshift in the normal way, either; for example, if the test particles in some region of a FRW universe have a nonzero mean velocity relative to the comoving frame, the expansion of the universe will merely reduce the energies of all the test particles by the same factor, but will not cause the mean velocity to decrease. Thus, the detailed predictions for this definition of scale-factor cutoff measure remain a matter for future study.

The local scale-factor cutoff and each of the three nonlocal definitions correspond to different global-time parametrizations and thus to different spacetime measures. In general, they make different predictions for physical observables; however, with regard to the relative number of normal observers and Boltzmann brains, their predictions are essentially the same. For the remainder of this paper we refer to the generic nonlocal definition of scale-factor time, for which we take FRW time as a suitable approximation. Note that the use of local scale-factor time would make it slightly easier to avoid Boltzmann brain domination, since it would increase the count of normal observers while leaving the count of Boltzmann brains essentially unchanged.

To facilitate later discussion, let us now describe some general properties of the multiverse. The volume fraction f_i occupied by vacuum i on constant scale-factor time slices can be found from the rate equation [43],

$$\frac{df_i}{dt} = \sum_j M_{ij} f_j, \tag{5}$$

where the transition matrix M_{ij} is given by

$$M_{ij} = \kappa_{ij} - \delta_{ij} \sum_r \kappa_{ri}, \tag{6}$$

and κ_{ij} is the transition rate from vacuum j to vacuum i per Hubble volume per Hubble time. This rate can also be written

$$\kappa_{ij} = (4\pi/3) H_j^{-4} \Gamma_{ij}, \tag{7}$$

where Γ_{ij} is the bubble nucleation rate per unit spacetime volume and H_j is the Hubble expansion rate in vacuum j.

The solution of Eq. (5) can be written in terms of the eigenvectors and eigenvalues of the transition matrix M_{ij}.

It is easily verified that each terminal vacuum is an eigenvector with eigenvalue zero. We here define "terminal vacua" as those vacua j for which $\kappa_{ij} = 0$ for all i. Thus the terminal vacua include both negative-energy

vacua, which collapse in a big crunch, and stable zero-energy vacua. It was shown in Ref. [21] that all of the other eigenvalues of M_{ij} have negative real parts. Moreover, the eigenvalue with the smallest (by magnitude) real part is pure real; we call it the "dominant eigenvalue" and denote it by $-q$ (with $q > 0$). Assuming that the landscape is irreducible, the dominant eigenvalue is nondegenerate. In that case the probabilities defined by the scale-factor cutoff measure are independent of the initial state of the multiverse, since they are determined by the dominant eigenvector.[5]

For an irreducible landscape, the late-time asymptotic solution of Eq. (5) can be written in the form[6]

$$f_j(t) = f_j^{(0)} + s_j e^{-qt} + \dots, \tag{8}$$

where the constant term $f_j^{(0)}$ is nonzero only in terminal vacua and s_j is proportional to the eigenvector of M_{ij} corresponding to the dominant eigenvalue $-q$, with the constant of proportionality determined by the initial distribution of vacua on Σ. It was shown in Ref. [21] that $s_j \leq 0$ for terminal vacua, and $s_j > 0$ for nonterminal vacua, as is needed for Eq. (8) to describe a non-negative volume fraction, with a nondecreasing fraction assigned to any terminal vacuum.

By inserting the asymptotic expansion (8) into the differential equation (5) and extracting the leading asymptotic behavior for a nonterminal vacuum i, one can show that

$$(\kappa_i - q) s_i = \sum_j \kappa_{ij} s_j, \tag{9}$$

where κ_j is the total transition rate out of vacuum j,

$$\kappa_j \equiv \sum_i \kappa_{ij}. \tag{10}$$

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[5]In this work we assume that the multiverse is irreducible; that is, any metastable inflating vacuum is accessible from any other such vacuum via a sequence of tunneling transitions. Our results, however, can still be applied when this condition fails. In that case the dominant eigenvalue can be degenerate, in which case the asymptotic future is dominated by a linear combination of dominant eigenvectors that is determined by the initial state. If transitions that increase the vacuum-energy density are included, then the landscape can be reducible only if it splits into several disconnected sectors. That situation was discussed in Appendix A of Ref. [38], where two alternative prescriptions were described. The first prescription (preferred by the authors) leads to initial-state dependence only if two or more sectors have the same dominant eigenvalue q, while the second prescription always leads to initial-state dependence.

[6]M_{ij} is not necessarily diagonalizable, but Eq. (8) applies in any case. It is always possible to form a complete basis from eigenvectors and generalized eigenvectors, where generalized eigenvectors satisfy $(M - \lambda I)^k s = 0$, for $k > 1$. The generalized eigenvectors appear in the solution with a time dependence given by $e^{\lambda t}$ times a polynomial in t. These terms are associated with the nonleading eigenvalues omitted from Eq. (8), and the polynomials in t will not change the fact that they are nonleading.

The positivity of s_i for nonterminal vacua then implies rigorously that q is less than the decay rate of the slowest-decaying vacuum in the landscape:

$$q \leq \kappa_{\min} \equiv \min\{\kappa_j\}. \qquad (11)$$

Since "upward" transitions (those that increase the energy density) are generally suppressed, we can gain some intuition by first considering the case in which all upward transition rates are set to zero. (Such a landscape is reducible, so the dominant eigenvector can be degenerate.) In this case M_{ij} is triangular, and the eigenvalues are precisely the decay rates κ_i of the individual states. The dominant eigenvalue q is then exactly equal to $\kappa_{\min}$.

If upward transitions are included but assumed to have a very low rate, then the dominant eigenvalue q is approximately equal to the decay rate of the slowest-decaying vacuum [44],

$$q \approx \kappa_{\min}. \qquad (12)$$

The slowest-decaying vacuum (assuming it is unique) is the one that dominates the asymptotic late-time volume of the multiverse, so we call it the dominant vacuum and denote it by D. Hence,

$$q \approx \kappa_D. \qquad (13)$$

The vacuum decay rate is typically exponentially suppressed, so for the slowest-decaying vacuum we expect it to be extremely small,

$$q \lll 1. \qquad (14)$$

Note that the corrections to Eq. (13) are comparable to the upward transition rate from D to higher-energy vacua, but for large energy differences this transition rate is suppressed by the factor $\exp(-8\pi^2/H_D^2)$ [45]. Here and throughout the remainder of this paper we use reduced Planck units, where $8\pi G = c = k_B = 1$. We shall argue in Sec. V that the dominant vacuum is likely to have a very low-energy density, so the correction to Eq. (13) is very small even compared to q.

A possible variant of this picture, with similar consequences, could arise if one assumes that the landscape includes states with nearby energy densities for which the upward transition rate is not strongly suppressed. In that case there could be a group of vacuum states that undergo rapid transitions into each other, but very slow transitions to states outside the group. The role of the dominant vacuum could then be played by this group of states, and q would be approximately equal to some appropriately averaged rate for the decay of these states to states outside the group. Under these circumstances q could be much less than $\kappa_{\min}$. An example of such a situation is described in Sec. IV E.

In the asymptotic limit of late scale-factor time t, the physical volumes in the various nonterminal vacua are given by

$$V_j(t) = V_0 s_j e^{(3-q)t}, \qquad (15)$$

where V_0 is the volume of the initial hypersurface Σ and e^{3t} is the volume expansion factor. The volume growth in Eq. (15) is (very slightly) slower than e^{3t} due to the constant loss of volume from transitions to terminal vacua. Note that even though upward transitions from the dominant vacuum are strongly suppressed, they play a crucial role in populating the landscape [44]. Most of the volume in the asymptotic solution of Eq. (15) originates in the dominant vacuum D, and "trickles" to the other vacua through a series of transitions starting with at least one upward jump.

III. THE ABUNDANCE OF NORMAL OBSERVERS AND BOLTZMANN BRAINS

Let us now calculate the relative abundances of Boltzmann brains and normal observers, in terms of the vacuum transition rates and the asymptotic volume fractions.

Estimates for the numerical values of the Boltzmann brain nucleation rates and vacuum decay rates will be discussed in Sec. V, but it is important at this stage to be aware of the kind of numbers that will be considered. We will be able to give only rough estimates of these rates, but the numbers that will be mentioned in Sec. V will range from $\exp(-10^{120})$ to $\exp(-10^{16})$. Thus, when we calculate the ratio $\mathcal{N}^{BB}/\mathcal{N}^{NO}$ of Boltzmann brains to normal observers, the natural logarithm of this ratio will always include one term with a magnitude of at least 10^{16}. Consequently, the presence or absence of any term in $\ln(\mathcal{N}^{BB}/\mathcal{N}^{NO})$ that is small compared to 10^{16} is of no relevance. We therefore refer to any factor f for which

$$|\ln f| < 10^{14} \qquad (16)$$

as "roughly of order one." In the calculation of $\mathcal{N}^{BB}/\mathcal{N}^{NO}$ such factors—although they may be minuscule or colossal by ordinary standards—can be ignored. It will not be necessary to keep track of factors of 2, π, or even 10^{10^8}. Dimensionless coefficients, factors of H, and factors coming from detailed aspects of the geometry are unimportant, and in the end all of these will be ignored. We nonetheless include some of these factors in the intermediate steps below simply to provide a clearer description of the calculation.

We begin by estimating the number of normal observers that will be counted in the sample spacetime region specified by the scale-factor cutoff measure. Normal observers arise during the big bang evolution in the aftermath of slow-roll inflation and reheating. The details of this evolution depend not only on the vacuum of the pocket in question, but also on the parent vacuum from which it nucleated [46]. That is, if we view each vacuum as a local minimum in a multidimensional field space, then the dynamics of inflation, in general, depend on the direction from which the field tunneled into the local minimum. We therefore label pockets with two indices ik, indicating the pocket and parent vacua, respectively.

To begin, we restrict our attention to a single "anthropic" pocket—i.e., one that produces normal observers—which nucleates at scale-factor time $t_{\rm nuc}$. The internal geometry of the pocket is that of an open FRW universe. We assume that, after a brief curvature-dominated period $\Delta\tau \sim H_k^{-1}$, slow-roll inflation inside the pocket gives N_e e-folds of expansion before thermalization. Furthermore, we assume that all normal observers arise at a fixed number N_O of e-folds of expansion after thermalization. (Note that N_e and N_O are both measured along FRW comoving geodesics inside the pocket, which do not initially coincide with, but rapidly asymptote to, the "global" geodesic congruence that originated outside the pocket.) We denote the fixed-internal-time hypersurface on which normal observers arise by $\Sigma^{\rm NO}$, and call the average density of observers on this hypersurface $n_{ik}^{\rm NO}$.

The hypersurface $\Sigma^{\rm NO}$ would have infinite volume, due to the constant expansion of the pocket, but this divergence is regulated by the scale-factor cutoff t_c, because the global scale-factor time t is not constant over the $\Sigma^{\rm NO}$ hypersurface. For the pocket described above, the regulated physical volume of $\Sigma^{\rm NO}$ can be written as

$$V_O^{(ik)}(t_{\rm nuc}) = H_k^{-3} e^{3(N_e + N_O)} w(t_c - t_{\rm nuc} - N_e - N_O), \quad (17)$$

where the exponential gives the volume expansion factor coming from slow-roll inflation and big bang evolution to the hypersurface $\Sigma^{\rm NO}$, and $H_k^{-3} w(t_c - t_{\rm nuc} - N_e - N_O)$ describes the comoving volume of the part of the $\Sigma^{\rm NO}$ hypersurface that is underneath the cutoff. The function $w(t)$ was calculated, for example, in Refs. [39,47], and was applied to the scale-factor cutoff measure in Ref. [48]. Its detailed form will not be needed to determine the answer up to a factor that is roughly of order one, but to avoid mystery we mention that $w(t)$ can be written as

$$w(t) = \frac{\pi}{2} \int_0^{\bar{\xi}(t)} \sinh^2(\xi) d\xi = \frac{\pi}{8}[\sinh(2\bar{\xi}(t)) - 2\bar{\xi}(t)], \quad (18)$$

where $\bar{\xi}(t_c - t_{\rm nuc} - N_e - N_O)$ is the maximum value of the Robertson-Walker radial coordinate ξ that lies under the cutoff. If the pocket universe begins with a moderate period of inflation [$\exp(N_e) \gg 1$] with the same vacuum energy as outside, then

$$\bar{\xi}(t) \approx 2\cosh^{-1}(e^{t/2}). \quad (19)$$

Equation (17) gives the physical volume on the $\Sigma^{\rm NO}$ hypersurface for a single pocket of type ik, which nucleates at time $t_{\rm nuc}$. The number of ik pockets that nucleate between time $t_{\rm nuc}$ and $t_{\rm nuc} + dt_{\rm nuc}$ is

$$dn_{\rm nuc}^{(ik)}(t_{\rm nuc}) = (3/4\pi)H_k^3 \kappa_{ik} V_k(t_{\rm nuc})dt_{\rm nuc}$$
$$= (3/4\pi)H_k^3 \kappa_{ik} s_k V_0 e^{(3-q)t_{\rm nuc}} dt_{\rm nuc}, \quad (20)$$

where we use Eq. (15) to give $V_k(t_{\rm nuc})$. The total number of normal observers in the sample region is then

$$\mathcal{N}_{ik}^{\rm NO} = n_{ik}^{\rm NO} \int^{t_c - N_e - N_O} V_O^{(ik)}(t_{\rm nuc})dn_{\rm nuc}^{(ik)}(t_{\rm nuc})$$
$$\approx n_{ik}^{\rm NO} \kappa_{ik} s_k V_0 e^{(3-q)t_c} \int_0^\infty w(z) e^{-(3-q)z} dz. \quad (21)$$

In the first expression we have ignored the (very small) probability that pockets of type ik may transition to other vacua during slow-roll inflation or during the subsequent period N_O of big bang evolution. In the second line, we have changed the integration variable to $z = t_c - t_{\rm nuc} - N_e - N_O$ (reversing the direction of integration) and have dropped the $\mathcal{O}(1)$ prefactors, and also the factor $e^{q(N_e + N_O)}$, since q is expected to be extraordinarily small. We have kept e^{-qt_c}, since we are interested in the limit $t_c \to \infty$. We have also kept the factor e^{qz} long enough to verify that the integral converges with or without the factor, so we can carry out the integral using the approximation $q \approx 0$, resulting in an $\mathcal{O}(1)$ prefactor that we will drop.

Finally,

$$\mathcal{N}_{ik}^{\rm NO} \approx n_{ik}^{\rm NO} \kappa_{ik} s_k V_0 e^{(3-q)t_c}. \quad (22)$$

Note that the expansion factor $e^{3(N_e + N_O)}$ in Eq. (17) was canceled when we integrated over nucleation times, illustrating the mild youngness bias of the scale-factor cutoff measure. The expansion of the Universe is canceled, so objects that form at a certain density per physical volume in the early Universe will have the same weight as objects that form at the same density per physical volume at a later time, despite the naive expectation that there is more volume at later times.

To compare, we now need to calculate the number of Boltzmann brains that will be counted in the sample spacetime region. Boltzmann brains can be produced in any anthropic vacuum, and presumably in many nonanthropic vacua as well. Suppose Boltzmann brains are produced in vacuum j at a rate $\Gamma_j^{\rm BB}$ per unit spacetime volume. The number of Boltzmann brains $\mathcal{N}_j^{\rm BB}$ is then proportional to the total four-volume in that vacuum. Imposing the cutoff at scale-factor time t_c, this four-volume is

$$\mathcal{V}_j^{(4)} = \int^{t_c} V_j(t)d\tau = H_j^{-1} \int^{t_c} V_j(t)dt$$
$$= \frac{1}{3-q} H_j^{-1} s_j V_0 e^{(3-q)t_c}, \quad (23)$$

where we have used Eq. (15) for the asymptotic volume fraction. By setting $d\tau = H_j^{-1} dt$, we have ignored the time dependence of $H(\tau)$ in the earlier stages of cosmic evolution, assuming that only the late-time de Sitter evolution is relevant. In a similar spirit, we will assume that the Boltzmann brain nucleation rate $\Gamma_j^{\rm BB}$ can be treated as time independent, so the total number of Boltzmann brains nucleated in vacua of type j, within the sample volume, is given by

$$\mathcal{N}_j^{\text{BB}} \approx \Gamma_j^{\text{BB}} H_j^{-1} s_j V_0 e^{(3-q)t_c}, \tag{24}$$

where we have dropped the $\mathcal{O}(1)$ numerical factor.

For completeness, we may want to consider the effects of early universe evolution on Boltzmann brain production, effects which were ignored in Eq. (24). We will separate the effects into two categories: the effects of slow-roll inflation at the beginning of a pocket universe, and the effects of reheating.

To account for the effects of slow-roll inflation, we argue that, within the approximations used here, there is no need for an extra calculation. Consider, for example, a pocket universe A which begins with a period of slow-roll inflation during which $H(\tau) \approx H_{\text{slow roll}} = \text{const}$. Consider also a pocket universe B, which throughout its evolution has $H = H_{\text{slow roll}}$, and which by hypothesis has the same formation rate, Boltzmann brain nucleation rate, and decay rates as pocket A. Then clearly the number of Boltzmann brains formed in the slow-roll phase of pocket A will be smaller than the number formed throughout the lifetime of pocket B. Since we will require that generic bubbles of type B do not overproduce Boltzmann brains, there will be no need to worry about the slow-roll phase of bubbles of type A.

To estimate how many Boltzmann brains might form as a consequence of reheating, we can make use of the calculation for the production of normal observers described above. We can assume that the Boltzmann brain nucleation rate has a spike in the vicinity of some particular hypersurface in the early Universe, peaking at some value $\Gamma_{\text{reheat},ik}^{\text{BB}}$ which persists roughly for some time interval $\Delta\tau_{\text{reheat},ik}^{\text{BB}}$, producing a density of Boltzmann brains equal to $\Gamma_{\text{reheat},ik}^{\text{BB}} \Delta\tau_{\text{reheat},ik}^{\text{BB}}$. This spatial density is converted into a total number for the sample volume in exactly the same way that we did for normal observers, leading to

$$\mathcal{N}_{ik}^{\text{BB,reheat}} \approx \Gamma_{\text{reheat},ik}^{\text{BB}} \Delta\tau_{\text{reheat},ik}^{\text{BB}} \kappa_{ik} s_k V_0 e^{(3-q)t_c}. \tag{25}$$

Thus, the dominance of normal observers is assured if

$$\sum_{i,k} \Gamma_{\text{reheat},ik}^{\text{BB}} \Delta\tau_{\text{reheat},ik}^{\text{BB}} \kappa_{ik} s_k \ll \sum_{i,k} n_{ik}^{\text{NO}} \kappa_{ik} s_k. \tag{26}$$

If Eq. (26) did not hold, it seems likely that we would suffer from Boltzmann brain problems regardless of our measure. We leave numerical estimates for Sec. V, but we will see that Boltzmann brain production during reheating is not a danger.

Ignoring the Boltzmann brains that form during reheating, the ratio of Boltzmann brains to normal observers can be found by combining Eqs. (22) and (24), giving

$$\frac{\mathcal{N}^{\text{BB}}}{\mathcal{N}^{\text{NO}}} \approx \frac{\sum_j H_j^3 \kappa_j^{\text{BB}} s_j}{\sum_{i,k} n_{ik}^{\text{NO}} \kappa_{ik} s_k}, \tag{27}$$

where the summation in the numerator covers only the vacua in which Boltzmann brains can arise, the summation over i in the denominator covers only anthropic vacua, and the summation over k includes all of their possible parent vacua. κ_j^{BB} is the dimensionless Boltzmann brain nucleation rate in vacuum j, related to Γ_j^{BB} by Eq. (7). The expression can be further simplified by dropping the factors of H_j and n_i^{NO}, which are roughly of order one, as defined by Eq. (16). We can also replace the sum over j in the numerator by the maximum over j, since the sum is at least as large as the maximum term and no larger than the maximum term times the number of vacua. Since the number of vacua (perhaps 10^{500}) is roughly of order one, the sum over j is equal to the maximum up to a factor that is roughly of order one. We similarly replace the sum over i in the denominator by its maximum, but we choose to leave the sum over k. Thus we can write

$$\frac{\mathcal{N}^{\text{BB}}}{\mathcal{N}^{\text{NO}}} \sim \frac{\max_j \{\kappa_j^{\text{BB}} s_j\}}{\max_i \{\sum_k \kappa_{ik} s_k\}}, \tag{28}$$

where the sets of j and i are restricted as for Eq. (27).

In dropping n_i^{NO}, we are assuming that $n_i^{\text{NO}} H_i^3$ is roughly of order one, as defined at the beginning of this section. It is hard to know what a realistic value for $n_i^{\text{NO}} H_i^3$ might be, as the evolution of normal observers may require some highly improbable events. For example, it was argued in Ref. [49] that the probability for life to evolve in a region of the size of our observable Universe per Hubble time may be as low as $\sim 10^{-1000}$. But even the most pessimistic estimates cannot compete with the small numbers appearing in estimates of the Boltzmann brain nucleation rate, and hence by our definition they are roughly of order one. Nonetheless, it is possible to imagine vacua for which n_i^{NO} might be negligibly small, but still nonzero. We shall ignore the normal observers in these vacua; for the remainder of this paper we will use the phrase "anthropic vacuum" to refer only to those vacua for which $n_i^{\text{NO}} H_i^3$ is roughly of order one.

For any landscape that satisfies Eq. (8), which includes any irreducible landscape, Eq. (28) can be simplified by using Eq. (9):

$$\frac{\mathcal{N}^{\text{BB}}}{\mathcal{N}^{\text{NO}}} \sim \frac{\max_j \{\kappa_j^{\text{BB}} s_j\}}{\max_i \{(\kappa_i - q) s_i\}}, \tag{29}$$

where the numerator is maximized over all vacua j that support Boltzmann brains, and the denominator is maximized over all anthropic vacua i.

In order to learn more about the ratio of Boltzmann brains to normal observers, we need to learn more about the volume fractions s_j, a topic that will be pursued in the next section.

IV. MINI-LANDSCAPES AND THE GENERAL CONDITIONS TO AVOID BOLTZMANN BRAIN DOMINATION

In this section we study a number of simple models of the landscape, in order to build intuition for the volume fractions that appear in Eqs. (28) and (29). The reader uninterested in the details may skip the pedagogical examples given in Secs. IV A, IV B, IV C, IV D, and IV E, and continue with Sec. IV F, where we state the general conditions that must be enforced in order to avoid Boltzmann brain domination.

A. The *FIB* landscape

Let us first consider a very simple model of the landscape, described by the schematic

$$F \rightarrow I \rightarrow B, \quad (30)$$

where F is a high-energy false vacuum, I is a positive-energy anthropic vacuum, and B is a terminal vacuum. This model, which we call the *FIB* landscape, was analyzed in Ref. [21] and was discussed in relation to the abundance of Boltzmann brains in Ref. [17]. As in Ref. [17], we assume that both Boltzmann brains and normal observers reside only in vacuum I.

Note that the *FIB* landscape ignores upward transitions from I to F. The model is constructed in this way as an initial first step, and also in order to more clearly relate our analysis to that of Ref. [17]. Although the rate of upward transitions is exponentially suppressed relative to the other rates, its inclusion is important for the irreducibility of the landscape, and hence the nondegeneracy of the dominant eigenvalue and the independence of the late-time asymptotic behavior from the initial conditions of the multiverse. The results of this subsection will therefore not always conform to the expectations outlined in Sec. II, but this shortcoming is corrected in the next subsection and all subsequent work in this paper.

We are interested in the eigenvectors and eigenvalues of the rate equation, Eq. (5). In the *FIB* landscape the rate equation gives

$$\dot{f}_F = -\kappa_{IF} f_F, \qquad \dot{f}_I = -\kappa_{BI} f_I + \kappa_{IF} f_F. \quad (31)$$

We ignore the volume fraction in the terminal vacuum as it is not relevant to our analysis. Starting with the ansatz,

$$\mathbf{f}(t) = \mathbf{s} e^{-qt}, \quad (32)$$

we find two eigenvalues of Eqs. (31). These are, with their corresponding eigenvectors,

$$q_1 = \kappa_{IF}, \qquad \mathbf{s}_1 = (1, C), \qquad q_2 = \kappa_{BI},$$
$$\mathbf{s}_2 = (0, 1), \quad (33)$$

where the eigenvectors are written in the basis $\mathbf{s} \equiv (s_F, s_I)$ and

$$C = \frac{\kappa_{IF}}{\kappa_{BI} - \kappa_{IF}}. \quad (34)$$

Suppose that we start in the false vacuum F at $t = 0$, i.e. $\mathbf{f}(t = 0) = (1, 0)$. Then the solution of the *FIB* rate equation, Eq. (31), is

$$f_F(t) = e^{-\kappa_{IF} t}, \qquad f_I(t) = C(e^{-\kappa_{IF} t} - e^{-\kappa_{BI} t}). \quad (35)$$

The asymptotic evolution depends on whether $\kappa_{IF} < \kappa_{BI}$ (case I) or not (case II). In case I,

$$\mathbf{f}(t \rightarrow \infty) = \mathbf{s}_1 e^{-\kappa_{IF} t} \qquad (\kappa_{IF} < \kappa_{BI}), \quad (36)$$

where $\mathbf{s}_1$ is given in Eq. (33), while in case II,

$$\mathbf{f}(t \rightarrow \infty) = (e^{-\kappa_{IF} t}, |C| e^{-\kappa_{BI} t}) \qquad (\kappa_{BI} < \kappa_{IF}). \quad (37)$$

In the latter case, the inequality of the rates of decay for the two volume fractions arises from the reducibility of the *FIB* landscape, stemming from our ignoring upward transitions from I to F.

For case I ($\kappa_{IF} < \kappa_{BI}$), we find the ratio of Boltzmann brains to normal observers by evaluating Eq. (28) for the asymptotic behavior described by Eq. (36):

$$\frac{\mathcal{N}^{\mathrm{BB}}}{\mathcal{N}^{\mathrm{NO}}} \sim \frac{\kappa^{\mathrm{BB}} s_I}{\kappa_{IF} s_F} \sim \frac{\kappa^{\mathrm{BB}}}{\kappa_{IF}} \frac{\kappa_{IF}}{\kappa_{BI} - \kappa_{IF}} \sim \frac{\kappa^{\mathrm{BB}}}{\kappa_{BI}}, \quad (38)$$

where we drop κ_{IF} compared to κ_{BI} in the denominator, as we are only interested in the overall scale of the solution. We find that the ratio of Boltzmann brains to normal observers is finite, depending on the relative rate of Boltzmann brain production to the rate of decay of vacuum I. Meanwhile, in case II (where $\kappa_{BI} < \kappa_{IF}$) we find

$$\frac{\mathcal{N}^{\mathrm{BB}}}{\mathcal{N}^{\mathrm{NO}}} \sim \frac{\kappa^{\mathrm{BB}}}{\kappa_{IF}} e^{(\kappa_{IF} - \kappa_{BI})t} \rightarrow \infty. \quad (39)$$

In this situation, the number of Boltzmann brains overwhelms the number of normal observers; in fact the ratio diverges with time.

The unfavorable result of case II stems from the fact that, in this case, the volume of vacuum I grows faster than that of vacuum F. Most of this I volume is in large pockets that formed very early, and this volume dominates because the F vacuum decays faster than I and is not replenished due to the absence of upward transitions. This leads to Boltzmann brain domination, in agreement with the conclusion reached in Ref. [17]. Thus, the *FIB* landscape analysis suggests that Boltzmann brain domination can be avoided only if the decay rate of the anthropic vacuum is larger than both the decay rate of its parent false vacuum F and the rate of Boltzmann brain production. Moreover, the *FIB* analysis suggests that Boltzmann brain domination in the multiverse can be avoided only if the first of these conditions is satisfied for all vacua in which Boltzmann brains exist. This is a very stringent requirement, since low-energy vacua like I typically have lower decay rates than high-energy vacua (see Sec. V). We shall see,

however, that the above conditions are substantially relaxed in more realistic landscape models.

B. The *FIB* landscape with recycling

The *FIB* landscape of the preceding section is reducible, since vacuum F cannot be reached from vacuum I. We can make it irreducible by simply allowing upward transitions,

$$F \leftrightarrow I \rightarrow B. \qquad (40)$$

This "recycling *FIB*" landscape is more realistic than the original *FIB* landscape, because upward transitions out of positive-energy vacua are allowed in semiclassical quantum gravity [45]. The rate equation of the recycling *FIB* landscape gives the eigenvalue system,

$$-qs_F = -\kappa_{IF}s_F + \kappa_{FI}s_I, \qquad -qs_I = -\kappa_I s_I + \kappa_{IF}s_F, \qquad (41)$$

where $\kappa_I \equiv \kappa_{BI} + \kappa_{FI}$ is the total decay rate of vacuum I, as defined in Eq. (10). Thus, the eigenvalues q_1 and q_2 correspond to the roots of

$$(\kappa_{IF} - q)(\kappa_I - q) = \kappa_{IF}\kappa_{FI}. \qquad (42)$$

Further analysis is simplified if we note that upward transitions from low-energy vacua like ours are very strongly suppressed, even when compared to the other exponentially suppressed transition rates, i.e. $\kappa_{FI} \ll \kappa_{IF}, \kappa_{BI}$. We are interested mostly in how this small correction modifies the dominant eigenvector in the case where $\kappa_{BI} < \kappa_{IF}$ (case II), which led to an infinite ratio of Boltzmann brains to normal observers. To the lowest order in κ_{FI}, we find

$$q \approx \kappa_I - \frac{\kappa_{IF}\kappa_{FI}}{\kappa_{IF} - \kappa_I} \qquad (43)$$

and

$$s_I \approx \frac{\kappa_{IF} - \kappa_I}{\kappa_{FI}} s_F \gg s_F. \qquad (44)$$

The above equation is a consequence of the second of Eqs. (41), but it also follows directly from Eq. (9), which holds in any irreducible landscape. In this case $f_I(t)$ and $f_F(t)$ have the same asymptotic time dependence, $\propto e^{-qt}$, so the ratio $f_I(t)/f_F(t)$ approaches a constant limit, $s_I/s_F \equiv R$. However, due to the smallness of κ_{FI}, this ratio is extremely large. Note that the ratio of Boltzmann brains to normal observers is proportional to R. Although it is also proportional to the minuscule Boltzmann brain nucleation rate (estimated in Sec. V), the typically huge value of R will still lead to Boltzmann brain domination (again, see Sec. V for relevant details). But the story is not over, since the recycling *FIB* landscape is still far from realistic.

C. A more realistic landscape

In the recycling model of the preceding section, the anthropic vacuum I was also the dominant vacuum, while

in a realistic landscape this is not likely to be the case. To see how it changes the situation to have a nonanthropic vacuum as the dominant one, we consider the model

$$A \leftarrow D \leftrightarrow F \rightarrow I \rightarrow B, \qquad (45)$$

which we call the "*ADFIB* landscape." Here, D is the dominant vacuum and A and B are both terminal vacua. The vacuum I is still an anthropic vacuum, and the vacuum F has large, positive vacuum energy. As explained in Sec. V, the dominant vacuum is likely to have very small vacuum energy; hence we consider that at least one upward transition (here represented as the transition to F) is required to reach an anthropic vacuum.

Note that the *ADFIB* landscape ignores the upward transition rate from vacuum I to F; however, this is exponentially suppressed relative to the other transition rates pertinent to I and, unlike the situation in Sec. IV A, ignoring the upward transition does not significantly affect our results. The important property is that all vacuum fractions have the same late-time asymptotic behavior; this property is assured whenever there is a unique dominant vacuum, and all inflating vacua are accessible from the dominant vacuum via a sequence of tunneling transitions. The uniformity of asymptotic behaviors is sufficient to imply Eq. (9), which suggests immediately that

$$\frac{s_I}{s_F} = \frac{\kappa_{IF}}{\kappa_{BI} - q} \approx \frac{\kappa_{IF}}{\kappa_{BI} - \kappa_D} \approx \frac{\kappa_{IF}}{\kappa_{BI}}, \qquad (46)$$

where we used $q \approx \kappa_D \equiv \kappa_{AD} + \kappa_{FD}$, and assumed that $\kappa_D \ll \kappa_{BI}$.

This holds even if the decay rate of the anthropic vacuum I is smaller than that of the false vacuum F.

Even though the false vacuum F may decay rather quickly, it is constantly being replenished by upward transitions from the slowly decaying vacuum D, which overwhelmingly dominates the physical volume of the multiverse. Note that, in light of these results, our constraints on the landscape to avoid Boltzmann brain domination are considerably relaxed. Specifically, it is no longer required that the anthropic vacua decay at a faster rate than their parent vacua. Using Eq. (46) with Eq. (28), the ratio of Boltzmann brains to normal observers in vacuum I is found to be

$$\frac{\mathcal{N}_I^{\mathrm{BB}}}{\mathcal{N}_I^{\mathrm{NO}}} \sim \frac{\kappa_I^{\mathrm{BB}} s_I}{\kappa_{IF} s_F} \sim \frac{\kappa_I^{\mathrm{BB}}}{\kappa_{BI}}. \qquad (47)$$

If Boltzmann brains can also exist in the dominant vacuum D, then they are a much more severe problem. By applying Eq. (9) to the F vacuum, we find

$$\frac{s_F}{s_D} = \frac{\kappa_{FD}}{\kappa_F - q} \approx \frac{\kappa_{FD}}{\kappa_F - \kappa_D} \approx \frac{\kappa_{FD}}{\kappa_F}, \qquad (48)$$

where $\kappa_F = \kappa_{IF} + \kappa_{DF}$, and where we have assumed that $\kappa_D \ll \kappa_F$. The ratio of Boltzmann brains in vacuum D to normal observers in vacuum I is then

$$\frac{\mathcal{N}_D^{\mathrm{BB}}}{\mathcal{N}_I^{\mathrm{NO}}} \sim \frac{\kappa_D^{\mathrm{BB}} s_D}{\kappa_{IF} s_F} \sim \frac{\kappa_D^{\mathrm{BB}}}{\kappa_{FD}} \frac{\kappa_F}{\kappa_{IF}}. \tag{49}$$

Since we expect that the dominant vacuum has very small vacuum energy, and hence a heavily suppressed upward transition rate κ_{FD}, the requirement that $\mathcal{N}_D^{\mathrm{BB}}/\mathcal{N}_I^{\mathrm{NO}}$ be small could be a very stringent one. Note that compared to s_D, both s_F and s_I are suppressed by the small factor κ_{FD}; however, the ratio s_I/s_F is independent of this factor.

Since s_D is so large, one should ask whether Boltzmann brain domination can be more easily avoided by allowing vacuum D to be anthropic. The answer is no, because the production of normal observers in vacuum D is proportional [see Eq. (22)] to the rate at which bubbles of D nucleate, which is not large. D dominates the spacetime volume due to slow decay, not rapid nucleation. If we assume that D is anthropic and restrict Eq. (28) to vacuum D, we find using Eq. (48) that

$$\frac{\mathcal{N}_D^{\mathrm{BB}}}{\mathcal{N}_D^{\mathrm{NO}}} \sim \frac{\kappa_D^{\mathrm{BB}} s_D}{\kappa_{DF} s_F} \sim \frac{\kappa_D^{\mathrm{BB}}}{\kappa_{FD}} \frac{\kappa_F}{\kappa_{DF}}, \tag{50}$$

so again the ratio is enhanced by the extremely small upward tunneling rate κ_{FD} in the denominator.

Thus, in order to avoid Boltzmann brain domination, it seems we have to impose two requirements: (1) the Boltzmann brain nucleation rate in the anthropic vacuum I must be less than the decay rate of that vacuum, and (2) the dominant vacuum D must either not support Boltzmann brains at all, or must produce them with a dimensionless rate κ_D^{BB} that is small even compared to the upward tunneling rate κ_{FD}. If the vacuum D is anthropic then it must support Boltzmann brains, so the domination by Boltzmann brains could be avoided only by the stringent requirement $\kappa_D^{\mathrm{BB}} \ll \kappa_{FD}$.

D. A further generalization

The conclusions of the last subsection are robust to more general considerations. To illustrate, let us generalize the *ADFIB* landscape to one with many low-vacuum-energy pockets, described by the schematic

$$A \leftarrow D \leftrightarrow F_j \rightarrow I_i \rightarrow B, \tag{51}$$

where each high-energy false vacuum F_j decays into a set of vacua $\{I_i\}$, all of which decay (for simplicity) to the same terminal vacuum B. The vacua I_i are taken to be a large set including both anthropic vacua and vacua that host only Boltzmann brains. Equation (9) continues to apply, so Eqs. (46) and (48) are easily generalized to this case, giving

$$s_{I_i} \approx \frac{1}{\kappa_{I_i}} \sum_j \kappa_{I_i F_j} s_{F_j} \tag{52}$$

and

$$s_{F_j} \approx \frac{1}{\kappa_{F_j}} \kappa_{F_j D} s_D, \tag{53}$$

where we have assumed that $q \ll \kappa_{I_i}, \kappa_{F_j}$, as we expect for vacua other than the dominant one. Using these results with Eq. (28), the ratio of Boltzmann brains in vacua I_i to normal observers in vacua I_i is given by

$$\begin{aligned} \frac{\mathcal{N}_{\{I_i\}}^{\mathrm{BB}}}{\mathcal{N}_{\{I_i\}}^{\mathrm{NO}}} &\sim \frac{\max_i \{\kappa_{I_i}^{\mathrm{BB}} s_{I_i}\}}{\max_i \{\sum_j \kappa_{I_i F_j} s_{F_j}\}} \\ &\sim \frac{\max_i \left\{\kappa_{I_i}^{\mathrm{BB}} \frac{1}{\kappa_{I_i}} \sum_j \kappa_{I_i F_j} \frac{1}{\kappa_{F_j}} \kappa_{F_j D} s_D\right\}}{\max_i \{\sum_j \kappa_{I_i F_j} \frac{1}{\kappa_{F_j}} \kappa_{F_j D} s_D\}} \\ &\sim \frac{\max_i \left\{\frac{\kappa_{I_i}^{\mathrm{BB}}}{\kappa_{I_i}} \sum_j \frac{\kappa_{I_i F_j}}{\kappa_{F_j}} \kappa_{F_j D}\right\}}{\max_i \left\{\sum_j \frac{\kappa_{I_i F_j}}{\kappa_{F_j}} \kappa_{F_j D}\right\}}, \end{aligned} \tag{54}$$

where the denominators are maximized over the restricted set of anthropic vacua i (and the numerators are maximized without restriction). The ratio of Boltzmann brains in the dominant vacuum (vacuum D) to normal observers in vacua I_i is given by

$$\frac{\mathcal{N}_D^{\mathrm{BB}}}{\mathcal{N}_{\{I_i\}}^{\mathrm{NO}}} \sim \frac{\kappa_D^{\mathrm{BB}} s_D}{\max_i \{\sum_j \kappa_{I_i F_j} s_{F_j}\}} \sim \frac{\kappa_D^{\mathrm{BB}}}{\max_i \left\{\sum_j \frac{\kappa_{I_i F_j}}{\kappa_{F_j}} \kappa_{F_j D}\right\}}, \tag{55}$$

and, if vacuum D is anthropic, then the ratio of Boltzmann brains in vacuum D to normal observers in vacuum D is given by

$$\frac{\mathcal{N}_D^{\mathrm{BB}}}{\mathcal{N}_D^{\mathrm{NO}}} \sim \frac{\kappa_D^{\mathrm{BB}}}{\sum_j \frac{\kappa_{D F_j}}{\kappa_{F_j}} \kappa_{F_j D}}. \tag{56}$$

In this case our answers are complicated by the presence of many different vacua. We can, in principle, determine whether Boltzmann brains dominate by evaluating Eqs. (54)–(56) for the correct values of the parameters, but this gets rather complicated and model dependent. The evaluation of these expressions can be simplified significantly, however, if we make some very plausible assumptions.

For tunneling out of the high-energy vacua F_j, one can expect the transition rates into different channels to be roughly comparable, so that $\kappa_{I_i F_j} \sim \kappa_{D F_j} \sim \kappa_{F_j}$. That is, we assume that the branching ratios $\kappa_{I_i F_j}/\kappa_{F_j}$ and $\kappa_{D F_j}/\kappa_{F_j}$ are roughly of order one in the sense of Eq. (16). These factors (or their inverses) will therefore be unimportant in the evaluation of $\mathcal{N}^{\mathrm{BB}}/\mathcal{N}^{\mathrm{NO}}$, and may be dropped. Furthermore, the upward transition rates from the dominant vacuum D into F_j are all comparable to one another, as can be seen by writing [45]

$$\kappa_{F_jD} \sim e^{A_{F_jD}} e^{-S_D}, \tag{57}$$

where A_{F_jD} is the action of the instanton responsible for the transition and S_D is the action of the Euclideanized de Sitter four-sphere,

$$S_D = \frac{8\pi^2}{H_D^2}. \tag{58}$$

But generically $|A_{F_jD}| \sim 1/\rho_{F_j}$. If we assume that

$$\left| \frac{1}{\rho_{F_j}} - \frac{1}{\rho_{F_k}} \right| < 10^{14} \tag{59}$$

for every pair of vacua F_j and F_k, then $\kappa_{F_jD} = \kappa_{F_kD}$ up to a factor that can be ignored because it is roughly of order one. Thus, up to subleading factors, the transition rates κ_{F_jD} cancel out[7] in the ratio $\mathcal{N}^{\mathrm{BB}}_{\{I_i\}}/\mathcal{N}^{\mathrm{NO}}$.

Returning to Eq. (54) and keeping only the leading factors, we have

$$\frac{\mathcal{N}^{\mathrm{BB}}_{\{I_i\}}}{\mathcal{N}^{\mathrm{NO}}} \sim \max_i \left\{ \frac{\kappa^{\mathrm{BB}}_{I_i}}{\kappa_{I_i}} \right\}, \tag{60}$$

where the index i runs over all (nondominant) vacua in which Boltzmann brains can nucleate. For the dominant vacuum, our simplifying assumptions[8] convert Eqs. (55) and (56) into

$$\frac{\mathcal{N}^{\mathrm{BB}}_D}{\mathcal{N}^{\mathrm{NO}}} \sim \frac{\kappa^{\mathrm{BB}}_D}{\kappa_{\mathrm{up}}} \sim \kappa^{\mathrm{BB}}_D e^{S_D}, \tag{61}$$

where $\kappa_{\mathrm{up}} \equiv \sum_j \kappa_{F_jD}$ is the upward transition rate out of the dominant vacuum.

Thus, the conditions needed to avoid Boltzmann brain domination are essentially the same as what we found in Sec. IV C. In this case, however, we must require that in any vacuum that can support Boltzmann brains, the

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[7]Depending on the range of vacua F_j that are considered, the bound of Eq. (59) may or may not be valid. If it is not, then the simplification of Eq. (60) below is not justified, and the original Eq. (54) has to be used. Of course one should remember that there was significant arbitrariness in the choice of 10^{14} in the definition of "roughly of order one." It was chosen to accommodate the largest estimate that we discuss in Sec. V for the Boltzmann brain nucleation rate, $\Gamma_{\mathrm{BB}} \sim \exp(-10^{16})$. In considering the other estimates of Γ_{BB}, one could replace 10^{14} by a much larger number, thereby increasing the applicability of Eq. (59).

[8]The dropping of the factor $e^{A_{F_jD}}$ is a more reliable approximation in this case than it was in Eq. (60) above. In this case the factor e^{-S_D} does not cancel between the numerator and denominator, so the factor $e^{A_{F_jD}}$ can be dropped if it is unimportant compared to e^{-S_D}. We of course do not know the value of S for the dominant vacuum, but for our vacuum it is of order 10^{122}, and it is plausible that the value for the dominant vacuum is similar or even larger. Thus as long as $1/\rho_{F_j}$ is small compared to 10^{122}, it seems safe to drop the factor $e^{A_{F_jD}}$.

Boltzmann brain nucleation rate must be less than the decay rate of that vacuum.

E. A dominant vacuum system

In the next to last paragraph of Sec. II, we described a scenario where the dominant vacuum was not the vacuum with the smallest decay rate. Let us now study a simple landscape to illustrate this situation. Consider the toy landscape

$$\begin{array}{ccccc} & F_j & \to & I_i & \to & B \\ & \nearrow & & \searrow & & \\ A \leftarrow D_1 & \longleftrightarrow & & D_2 & \to & A \\ & \searrow & & \nearrow & & \\ & S & \to & A, & & \end{array} \tag{62}$$

where as in Sec. IV D the vacua I_i are taken to include both anthropic vacua and vacua that support only Boltzmann brains. Vacua A and B are terminal vacua and the F_j have large, positive vacuum energies. Assume that vacuum S has the smallest total decay rate.

We have in mind the situation in which D_1 and D_2 are nearly degenerate, and transitions from D_1 to D_2 (and vice versa) are rapid, even though the transition in one direction is upward. With this in mind, we divide the decay rates of D_1 and D_2 into two parts,

$$\kappa_1 = \kappa_{21} + \kappa^{\mathrm{out}}_1, \tag{63}$$

$$\kappa_2 = \kappa_{12} + \kappa^{\mathrm{out}}_2, \tag{64}$$

with $\kappa_{12}, \kappa_{21} \gg \kappa^{\mathrm{out}}_{1,2}$. We assume as in previous sections that the rates for large upward transitions (S to D_1 or D_2, and D_1 or D_2 to F_j) are extremely small, so that we can ignore them in the calculation of q. The rate equation, Eq. (9), then admits a solution with $q \simeq \kappa_D$, but it also admits solutions with

$$q \simeq \frac{1}{2} \left[\kappa_1 + \kappa_2 \pm \sqrt{(\kappa_1 - \kappa_2)^2 + 4\kappa_{12}\kappa_{21}} \right]. \tag{65}$$

Expanding the smaller root to linear order in $\kappa^{\mathrm{out}}_{1,2}$ gives

$$q \simeq \alpha_1 \kappa^{\mathrm{out}}_1 + \alpha_2 \kappa^{\mathrm{out}}_2, \tag{66}$$

where

$$\alpha_1 \equiv \frac{\kappa_{12}}{\kappa_{12} + \kappa_{21}}, \qquad \alpha_2 \equiv \frac{\kappa_{21}}{\kappa_{12} + \kappa_{21}}. \tag{67}$$

In principle, this value for q can be smaller than κ_D, which is the case that we wish to explore.

In this case the vacua D_1 and D_2 dominate the volume fraction of the multiverse, even if their total decay rates κ_1 and κ_2 are not the smallest in the landscape. We can therefore call the states D_1 and D_2 together a dominant vacuum system, which we denote collectively as D. The rate equation [Eq. (9)] shows that

$$s_{D_1} \approx \alpha_1 s_D, \qquad s_{D_2} \approx \alpha_2 s_D, \qquad (68)$$

where $s_D \equiv s_{D_1} + s_{D_2}$, and the equations hold in the approximation that $\kappa_{1,2}^{\text{out}}$ and the upward transition rates from D_1 and D_2 can be neglected. To see that these vacua dominate the volume fraction, we calculate the modified form of Eq. (53):

$$\frac{s_{F_j}}{s_D} \approx \frac{\alpha_1 \kappa_{F_j D_1} + \alpha_2 \kappa_{F_j D_2}}{\kappa_{F_j}}. \qquad (69)$$

Thus the volume fractions of the F_j, and hence also the I_j and B vacua, are suppressed by the very small rate for large upward jumps from low-energy vacua, namely, $\kappa_{F_j D_1}$ and $\kappa_{F_j D_2}$. The volume fraction for S depends on κ_{AD_1} and κ_{AD_2}, but it is maximized when these rates are negligible, in which case it is given by

$$\frac{s_S}{s_D} \approx \frac{q}{\kappa_S - q}. \qquad (70)$$

This quantity can, in principle, be large, if q is just a little smaller than κ_S, but that would seem to be a very special case. Generically, we would expect that since q must be smaller than κ_S [see Eq. (11)], it would most likely be many orders of magnitude smaller, and hence the ratio in Eq. (70) would be much less than 1. There is no reason, however, to expect it to be as small as the ratios that are suppressed by large upward jumps. For simplicity, however, we will assume in what follows that s_S can be neglected.

To calculate the ratio of Boltzmann brains to normal observers in this toy landscape, note that Eqs. (54) and (55) are modified only by the substitution

$$\kappa_{F_j D} \to \bar{\kappa}_{F_j D} \equiv \alpha_1 \kappa_{F_j D_1} + \alpha_2 \kappa_{F_j D_2}. \qquad (71)$$

Thus, the dominant vacuum transition rate is simply replaced by a weighted average of the dominant vacuum transition rates. If we assume that neither of the vacua, D_1 or D_2, are anthropic, and make the same assumptions about magnitudes used in Sec. IV D, then Eqs. (60) and (61) continue to hold as well, where we have redefined κ_{up} by $\kappa_{\text{up}} \equiv \sum_j \bar{\kappa}_{F_j D}$.

If, however, we allow D_1 or D_2 to be anthropic, then new questions arise. Transitions between D_1 and D_2 are, by assumption, rapid, so they copiously produce new pockets and potentially new normal observers. We must recall, however (as discussed in Sec. III), that the properties of a pocket universe depend on both the current vacuum and the parent vacuum. In this case, the unusual feature is that the vacua within the D system are nearly degenerate, and hence very little energy is released by tunnelings within D. For pocket universes created in this way, the maximum particle energy density during reheating will be only a small fraction of the vacuum-energy density. Such a big bang is very different from the one that took place in our

pocket, and presumably much less likely to produce life. We will call a vacuum in the D system "strongly anthropic" if normal observers are produced by tunnelings from within D, and "mildly anthropic" if normal observers can be produced, but only by tunnelings from higher-energy vacua outside D.

If either of the vacua in D were strongly anthropic, then the normal observers in D would dominate the normal observers in the multiverse. Normal observers in the vacua I_i would be less numerous by a factor proportional to the extremely small rate $\bar{\kappa}_{F_j D}$ for large upward transitions. This situation would itself be a problem, however, similar to the Boltzmann brain problem. It would mean that observers like ourselves, who arose from a hot big bang with energy densities much higher than our vacuum-energy density, would be extremely rare in the multiverse. We conclude that if there are any models which give a dominant vacuum system that contains a strongly anthropic vacuum, such models would be considered unacceptable in the context of the scale-factor cutoff measure.

On the other hand, if the D system included one or more mildly anthropic vacua, then the situation is very similar to that discussed in Secs. IV C and IV D. In this case the normal observers in the D system would be comparable in number to the normal observers in the vacua I_i, so they would have no significant effect on the ratio of Boltzmann brains to normal observers in the multiverse. If any of the D vacua were mildly anthropic, however, then the stringent requirement $\kappa_D^{\text{BB}} \ll \kappa_{\text{up}}$ would have to be satisfied without resort to the simple solution $\kappa_D^{\text{BB}} = 0$.

Thus, we find that the existence of a dominant vacuum system does not change our conclusions about the abundance of Boltzmann brains, except insofar as the Boltzmann brain nucleation constraints that would apply to the dominant vacuum must apply to every member of the dominant vacuum system. Probably the most important implication of this example is that the dominant vacuum is not necessarily the vacuum with the lowest decay rate, so the task of identifying the dominant vacuum could be very difficult.

F. General conditions to avoid Boltzmann brain domination

In constructing general conditions to avoid Boltzmann brain domination, we are guided by the toy landscapes discussed in the previous subsections. Our goal, however, is to construct conditions that can be justified using only the general equations of Secs. II and III, assuming that the landscape is irreducible, but without relying on the properties of any particular toy landscape. We will be especially cautious about the treatment of the dominant vacuum and the possibility of small upward transitions, which could be rapid. The behavior of the full landscape of a realistic theory may deviate considerably from that of the simplest toy models.

To discuss the general situation, it is useful to divide vacuum states into four classes. We are only interested in vacua that can support Boltzmann brains. These can be

(1) anthropic vacua for which the total dimensionless decay rate satisfies $\kappa_i \gg q$,
(2) nonanthropic vacua that can transition to anthropic vacua via unsuppressed transitions,
(3) nonanthropic vacua that can transition to anthropic vacua only via suppressed transitions,
(4) anthropic vacua for which the total dimensionless decay rate is $\kappa_i \approx q$.

Here q is the smallest-magnitude eigenvalue of the rate equation [see Eqs. (5)–(8)]. We call a transition "unsuppressed" if its branching ratio is roughly of order one in the sense of Eq. (16). If the branching ratio is smaller than this, it is "suppressed." As before, when calculating $\mathcal{N}^{\mathrm{BB}}/\mathcal{N}^{\mathrm{NO}}$ we assume that factors that are roughly of order one can be ignored. Note that Eq. (11) forbids κ_i from being less than q, so the above four cases are exhaustive.

We first discuss conditions that are sufficient to guarantee that Boltzmann brains will not dominate, postponing until later the issue of what conditions are necessary.

We begin with the vacua in the first class. Very likely all anthropic vacua belong to this class. For an anthropic vacuum i, the Boltzmann brains produced in vacuum i cannot dominate the multiverse if they do not dominate the normal observers in vacuum i, so we can begin with this comparison. Restricting Eq. (29) to this single vacuum, we obtain

$$\frac{\mathcal{N}_i^{\mathrm{BB}}}{\mathcal{N}_i^{\mathrm{NO}}} \sim \frac{\kappa_i^{\mathrm{BB}}}{\kappa_i}, \qquad (72)$$

a ratio that has appeared in many of the simple examples. If this ratio is small compared to 1, then Boltzmann brains created in vacuum i are negligible.

Let us now study a vacuum j in the second class. First note that Eq. (9) implies the rigorous inequality

$$\kappa_i s_i \gtrsim \kappa_{ij} s_j \quad \text{(no sum on repeated indices)}, \qquad (73)$$

which holds for any two states i and j. [Intuitively, Eq. (73) is the statement that, in a steady state, the total rate of loss of the volume fraction must exceed the input rate from any one channel.] To simplify what follows, it will be useful to rewrite Eq. (73) as

$$(\kappa_i s_i) \gtrsim (\kappa_j s_j) B_{j \to i}, \qquad (74)$$

where $B_{j \to i} \equiv \kappa_{ij}/\kappa_j$ is the branching ratio for the transition $j \to i$.

Suppose that we are trying to bound the Boltzmann brain production in vacuum j, and we know that it can undergo unsuppressed transitions

$$j \to k_1 \to \ldots \to k_n \to i, \qquad (75)$$

where i is an anthropic vacuum. We begin by using Eqs. (22) and (24) to express $\mathcal{N}_j^{\mathrm{BB}}/\mathcal{N}_i^{\mathrm{NO}}$, dropping irrelevant factors as in Eq. (28), and then we can iterate the above inequality:

$$\frac{\mathcal{N}_j^{\mathrm{BB}}}{\mathcal{N}_i^{\mathrm{NO}}} \sim \frac{\kappa_j^{\mathrm{BB}} s_j}{\sum_k \kappa_{ik} s_k} \leq \frac{\kappa_j^{\mathrm{BB}} s_j}{\kappa_i s_i} \leq \frac{\kappa_j^{\mathrm{BB}} s_j}{(\kappa_j s_j) B_{j \to k_1} B_{k_1 \to k_2} \cdots B_{k_n \to i}}$$
$$= \frac{\kappa_j^{\mathrm{BB}}}{\kappa_j} \frac{1}{B_{j \to k_1} B_{k_1 \to k_2} \cdots B_{k_n \to i}}, \qquad (76)$$

where again there is no sum on repeated indices, and Eq. (9) was used in the second step on the first line. Each inverse branching ratio on the right of the last line is greater than or equal to 1, but by our assumptions can be considered to be roughly of order one, and hence can be dropped. Thus, the multiverse will avoid domination by Boltzmann brains in vacuum j if $\kappa_j^{\mathrm{BB}}/\kappa_j \ll 1$, the same criterion found for the first class.

The third class—nonanthropic vacua that can only transition to an anthropic state via at least one suppressed transition—presumably includes many states with very low-vacuum-energy density. The dominant vacuum of our toy landscape models certainly belongs to this class, but we do not know of anything that completely excludes the possibility that the dominant vacuum might belong to the second or fourth class. That is, perhaps the dominant vacuum is anthropic, or decays to an anthropic vacuum. If there is a dominant vacuum system, as described in Sec. IV E, then $\kappa_i \gg q$, and the dominant vacua could belong to the first class, as well as to either of classes (2) and (3).

To bound the Boltzmann brain production in this class, we consider two possible criteria. To formulate the first, we can again use Eqs. (75) and (76), but this time the sequence must include at least one suppressed transition, presumably an upward jump. Let us therefore denote the branching ratio for this suppressed transition as B_{up}, noting that B_{up} will appear in the denominator of Eq. (76). Of course, the sequence of Eq. (75) might involve more than one suppressed transition, but in any case the product of these very small branching ratios in the denominator can be called B_{up}, and all the other factors can be taken as roughly of order one. Thus, a landscape containing a vacuum j of the third class avoids Boltzmann brain domination if

$$\frac{\kappa_j^{\mathrm{BB}}}{B_{\mathrm{up}} \kappa_j} \ll 1, \qquad (77)$$

in agreement with the results obtained for the dominant vacua in the toy landscape models in the previous subsections.

A few comments are in order. First, if the only suppressed transition is the first, then $B_{\text{up}} = \kappa_{\text{up}}/\kappa_j$, and the above criterion simplifies to $\kappa_j^{\text{BB}}/\kappa_{\text{up}} \ll 1$. Second, we should keep in mind that the sequence of Eq. (75) is presumably not unique, so other sequences will produce other bounds. All the bounds will be valid, so the strongest bound is the one of maximum interest. Finally, since the vacua under discussion are not anthropic, a likely method for Eq. (77) to be satisfied would be for κ_j^{BB} to vanish, as would happen if the vacuum j did not support the complex structures needed to form Boltzmann brains.

The criterion above can be summarized by saying that if $\kappa_j^{\text{BB}}/(B_{\text{up}}\kappa_j) \ll 1$, then the Boltzmann brains in vacuum j will be overwhelmingly outnumbered by the normal observers living in pocket universes that form in the decay chain starting from vacuum j. We now describe a second, alternative criterion, based on the idea that the number of Boltzmann brains in vacuum j can be compared with the number of normal observers in vacuum i if the two types of vacua have a common ancestor.

Denoting the common ancestor vacuum as A, we assume that it can decay to an anthropic vacuum i by a chain of transitions,

$$A \rightarrow k_1 \rightarrow \ldots \rightarrow k_n \rightarrow i, \tag{78}$$

and also to a Boltzmann-brain-producing vacuum j by a chain

$$A \rightarrow \ell_1 \rightarrow \ldots \rightarrow \ell_m \rightarrow j. \tag{79}$$

From the sequence of Eq. (78) and the bound of Eq. (74), we can infer that

$$(\kappa_i s_i) \geq (k_A s_A) B_{A \rightarrow k_1} B_{k_1 \rightarrow k_2} \cdots B_{k_n \rightarrow i}. \tag{80}$$

To make use of the sequence of Eq. (79) we will want a bound that goes in the opposite direction, for which we will need to require additional assumptions. Starting with Eq. (9), we first require $q \ll \kappa_i$, which is plausible provided that vacuum i is not the dominant vacuum. Next we look at the sum over j on the right-hand side, and we call the transition $j \rightarrow i$ "significant" if its contribution to the sum is within a factor roughly of order one of the entire sum. (The sum over j is the sum over sources for vacuum i, so a transition $j \rightarrow i$ is significant if pocket universes of vacuum j are a significant source of pocket universes of vacuum i.) It follows that for any significant transition $j \rightarrow i$ for which $q \ll \kappa_i$,

$$(\kappa_i s_i) \leq (\kappa_j s_j) Z_{\text{max}} B_{j \rightarrow i} \leq (\kappa_j s_j) Z_{\text{max}}, \tag{81}$$

where Z_{max} denotes the largest number that is roughly of order one. By our conventions, $Z_{\text{max}} = \exp(10^{14})$. If we assume now that all the transitions in the sequence of

Eq. (79) are significant, and that q is negligible in each case, then

$$(\kappa_j s_j) \leq (k_A s_A) Z_{\text{max}}^{m+1}. \tag{82}$$

Using the bounds from Eqs. (80) and (82), the Boltzmann brain ratio is bounded by

$$\frac{\mathcal{N}_j^{\text{BB}}}{\mathcal{N}_i^{\text{NO}}} \sim \frac{\kappa_j^{\text{BB}} s_j}{\sum_k \kappa_{ik} s_k} \leq \frac{\kappa_j^{\text{BB}} s_j}{\kappa_i s_i}$$

$$\leq \frac{Z_{\text{max}}^{m+1}}{B_{A \rightarrow k_1} B_{k_1 \rightarrow k_2} \cdots B_{k_n \rightarrow i}} \frac{\kappa_j^{\text{BB}}}{\kappa_j}. \tag{83}$$

But all the factors on the right are roughly of order one, except that some of the branching ratios in the denominator might be smaller, if they correspond to suppressed transitions. If B_{up} denotes the product of branching ratios for all the suppressed transitions shown in the denominator [i.e., all suppressed transitions in the sequence of Eq. (78)], then the bound reduces to Eq. (77).[9]

To summarize, the Boltzmann brains in a nonanthropic vacuum j can be bounded if there is an ancestor vacuum A that can decay to j through a chain of significant transitions for which $q \ll \kappa_\ell$ for each vacuum, as in the sequence of Eq. (79), and if the same ancestor vacuum can decay to an anthropic vacuum through a sequence of transitions as in Eq. (78). The Boltzmann brains will never dominate provided that $\kappa_j^{\text{BB}}/(B_{\text{up}}\kappa_j) \ll 1$, where B_{up} is the product of all suppressed branching ratios in the sequence of Eq. (78).

Finally, the fourth class of vacua consists of anthropic vacua i with decay rate $\kappa_i \simeq q$, a class which could be empty. For this class, Eq. (29) may not be very useful, since the quantity $(\kappa_i - q)$ in the denominator could be very small. Yet, as in the two previous classes, this class can be treated by using Eq. (76), where in this case the vacuum i can be the same as j or different, although the case $i = j$ requires $n \geq 1$. Again, if the sequence contains only unsuppressed transitions, then the multiverse avoids domination by Boltzmann brains in vacuum i if $\kappa_i^{\text{BB}}/\kappa_i \ll 1$. If upward jumps are needed to reach an anthropic vacuum, whether it is the vacuum i again or a distinct vacuum j, then the Boltzmann brains in vacuum i will never dominate if $\kappa_i^{\text{BB}}/(B_{\text{up}}\kappa_i) \ll 1$.

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[9]Note, however, that the argument breaks down if the sequences in either Eq. (78) or (79) become too long. For the choices that we have made, a factor of Z_{max} is unimportant in the calculation of $\mathcal{N}^{\text{BB}}/\mathcal{N}^{\text{NO}}$, but $Z_{\text{max}}^{100} = \exp(10^{16})$ can be significant. Thus, for our choices we can justify the dropping of $\mathcal{O}(100)$ factors that are roughly of order one, but not more than that. For choices appropriate to smaller estimates of Γ_{BB}, however, the number of factors that can be dropped will be many orders of magnitude larger.

The conditions described in the previous paragraph are very difficult to meet, so if the fourth class is not empty, Boltzmann brain domination is hard to avoid. These vacua have the slowest decay rates in the landscape, $\kappa_i \approx q$, so it seems plausible that they have very low-energy densities, precluding the possibility of decaying to an anthropic vacuum via unsuppressed transitions; in that case Boltzmann brain domination can be avoided if

$$\kappa_i^{\mathrm{BB}} \ll B_{\mathrm{up}} \kappa_i. \tag{84}$$

However, as pointed out in Ref. [42], $B_{\mathrm{up}} \propto e^{-S_D}$ [see Eq. (57)] is comparable to the inverse of the recurrence time, while in an anthropic vacuum one would expect the Boltzmann brain nucleation rate to be much faster than once per recurrence time.

To summarize, the domination of Boltzmann brains can be avoided by, first of all, requiring that all vacuum states in the landscape obey the relation

$$\frac{\kappa_j^{\mathrm{BB}}}{\kappa_j} \ll 1. \tag{85}$$

That is, the rate of nucleation of Boltzmann brains in each vacuum must be less than the rate of nucleation, in that same vacuum, of bubbles of other phases. For anthropic vacua i with $\kappa_i \gg q$, this criterion is enough. Otherwise, the Boltzmann brains that might be produced in vacuum j must be bounded by the normal observers forming in some vacuum i, which must be related to j through decay chains. Specifically, there must be a vacuum A that can decay through a chain to an anthropic vacuum i, i.e.

$$A \rightarrow k_1 \rightarrow \ldots \rightarrow k_n \rightarrow i, \tag{86}$$

where either $A = j$, or else A can decay to j through a sequence

$$A \rightarrow \ell_1 \rightarrow \ldots \rightarrow \ell_m \rightarrow j. \tag{87}$$

In the above sequence we insist that $\kappa_j \gg q$ and that $\kappa_l \gg q$ for each vacuum ℓ_p in the chain, and that each transition must be significant, in the sense that pockets of type ℓ_p must be a significant source of pockets of type ℓ_{p+1}. [More precisely, a transition from vacuum j to i is significant if it contributes a fraction that is roughly of order one to $\sum_j \kappa_{ij} s_j$ in Eq. (9).] For these cases, the bound which ensures that the Boltzmann brains in vacuum j are dominated by the normal observers in vacuum i is given by

$$\frac{\kappa_j^{\mathrm{BB}}}{B_{\mathrm{up}} \kappa_j} \ll 1, \tag{88}$$

where B_{up} is the product of any suppressed branching ratios in the sequence of Eq. (86). If all the transitions in Eq. (86) are unsuppressed, this bound reduces to Eq. (85). If j is anthropic, the case $A = j = i$ is allowed, provided that $n \geq 1$.

The conditions described above are sufficient to guarantee that Boltzmann brains do not dominate over normal observers in the multiverse, but without further assumptions there is no way to know if they are necessary. All of the conditions that we have discussed are quasilocal, in the sense that they do not require any global picture of the landscape of vacua. For each of the above arguments, the Boltzmann brains in one type of vacuum j are bounded by the normal observers in some type of vacuum i that is either the same type or directly related to it through decay chains. Thus, there was no need to discuss the importance of the vacua j and i compared to the rest of the landscape as a whole. The quasilocal nature of these conditions, however, guarantees that they cannot be necessary to avoid the domination by Boltzmann brains. If two vacua j and i are both totally insignificant in the multiverse, then it will always be possible for the Boltzmann brains in vacuum j to overwhelm the normal observers in vacuum i, while the multiverse as a whole could still be dominated by normal observers in other vacua.

We have so far avoided making global assumptions about the landscape of vacua, because such assumptions are generally hazardous. While it may be possible to make statements that are true for the bulk of vacua in the landscape, in this context the statements are not useful unless they are true for *all* the vacua of the landscape. Although the number of vacua in the landscape, often estimated at 10^{500} [50], is usually considered to be incredibly large, the number is nonetheless roughly of order one compared to the numbers involved in the estimates of Boltzmann brain nucleation rates and vacuum decay rates. Thus, if a single vacuum produces Boltzmann brains in excess of required bounds, the Boltzmann brains from that vacuum could easily overwhelm all the normal observers in the multiverse.

Recognizing that our conclusions could be faulty, we can nonetheless adopt some reasonable assumptions to see where they lead. We can assume that the multiverse is sourced by either a single dominant vacuum or by a dominant vacuum system. We can further assume that every anthropic and/or Boltzmann-brain-producing vacuum i can be reached from the dominant vacuum (or dominant vacuum system) by a single significant upward jump, with a rate proportional to e^{-S_D}, followed by some number of significant, unsuppressed transitions, all of which have rates $\kappa_k \gg q$ and branching ratios that are roughly of order one:

$$D \rightarrow k_1 \rightarrow \ldots \rightarrow k_n \rightarrow i. \tag{89}$$

We will further assume that each nondominant anthropic and/or Boltzmann-brain-producing vacuum i has a decay rate $\kappa_i \gg q$, but we need not assume that all of the κ_i are comparable to each other. With these assumptions, the estimate of $\mathcal{N}^{\mathrm{BB}}/\mathcal{N}^{\mathrm{NO}}$ becomes very simple.

Applying Eq. (9) to the first transition of Eq. (89),

$$\kappa_{k_1} s_{k_1} \sim \kappa_{k_1 D} s_D \sim \kappa_{\mathrm{up}} s_D, \tag{90}$$

where we use κ_{up} to denote the rate of a typical transition $D \to k$, assuming that they are all equal to each other up to a factor roughly of order one. Here $\sim$ indicates equality up to a factor that is roughly of order one. If there is a dominant vacuum system, then $\kappa_{k_1 D}$ is replaced by $\bar{\kappa}_{k_1 D} \equiv \sum_\ell \alpha_\ell \kappa_{k_1 D_\ell}$, where the D_ℓ are the components of the dominant vacuum system, and the α_ℓ are defined by generalizing Eqs. (67) and (68).[10] Applying Eq. (9) to the next transition, $k_1 \to k_2$, we find

$$\kappa_{k_2} s_{k_2} = B_{k_1 \to k_2} \kappa_{k_1} s_{k_1} + \ldots \sim \kappa_{k_1} s_{k_1}, \tag{91}$$

where we have used the fact that $B_{k_1 \to k_2}$ is roughly of order one, and that the transition is significant. Iterating, we have

$$\kappa_i s_i \sim \kappa_{k_n} s_{k_n} \sim \kappa_{\mathrm{up}} s_D. \tag{92}$$

Since the expression on the right is independent of i, we conclude that under these assumptions any two nondominant anthropic and/or Boltzmann-brain-producing vacua i and j have equal values of κs, up to a factor that is roughly of order one:

$$\kappa_j s_j \sim \kappa_i s_i. \tag{93}$$

Using Eq. (22) and assuming, as always, that n_{ik}^{NO} is roughly of order one, Eq. (93) implies that any two nondominant anthropic vacua i and j have comparable numbers of ordinary observers, up to a factor that is roughly of order one:

$$\mathcal{N}_j^{\mathrm{NO}} \sim \mathcal{N}_i^{\mathrm{NO}}. \tag{94}$$

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[10] In more detail, the concept of a dominant vacuum system is relevant when there is a set of vacua ℓ that can have rapid transitions within the set, but only very slow transitions connecting these vacua to the rest of the landscape. As a zeroth order approximation one can neglect all transitions connecting these vacua to the rest of the landscape, and assume that $\kappa_\ell \gg q$, so Eq. (9) takes the form

$$\kappa_\ell s_\ell = \sum_{\ell'} B_{\ell \ell'} \kappa_{\ell'} s_{\ell'}.$$

Here $B_{\ell \ell'} \equiv \kappa_{\ell \ell'} / \kappa_{\ell'}$ is the branching ratio within this restricted subspace, where $\kappa_\ell = \sum_{\ell'} \kappa_{\ell' \ell}$ is summed only within the dominant vacuum system, so $\sum_\ell B_{\ell \ell'} = 1$ for all ℓ'. $B_{\ell \ell'}$ is nonnegative, and if we assume also that it is irreducible, then the Perron-Frobenius theorem guarantees that it has a nondegenerate eigenvector υ_ℓ of eigenvalue 1, with positive components. From the above equation $\kappa_\ell s_\ell \propto \upsilon_\ell$, and then

$$\alpha_\ell = \frac{s_\ell}{\sum_{\ell'} s_{\ell'}} = \frac{\upsilon_\ell}{\kappa_\ell \sum_{\ell'} \frac{\upsilon_{\ell'}}{\kappa_{\ell'}}}.$$

The dominant vacuum could conceivably be anthropic, but we begin by considering the case in which it is not. In that case all anthropic vacua are equivalent, so the Boltzmann brains produced in any vacuum j will either dominate the multiverse or not, depending on whether they dominate the normal observers in an arbitrary anthropic vacuum i. Combining Eqs. (9), (22), (24), and (93), and omitting irrelevant factors, we find that for any nondominant vacuum j,

$$\frac{\mathcal{N}_j^{\mathrm{BB}}}{\mathcal{N}_i^{\mathrm{NO}}} \sim \frac{\kappa_j^{\mathrm{BB}} s_j}{\sum_k \kappa_{ik} s_k} \sim \frac{\kappa_j^{\mathrm{BB}} s_j}{\kappa_i s_i} \sim \frac{\kappa_j^{\mathrm{BB}}}{\kappa_j}. \tag{95}$$

Thus, given the assumptions described above, for any nondominant vacuum j, the necessary and sufficient condition to avoid the domination of the multiverse by Boltzmann brains in vacuum j is given by

$$\frac{\kappa_j^{\mathrm{BB}}}{\kappa_j} \ll 1. \tag{96}$$

For Boltzmann brains formed in the dominant vacuum, we can again find out if they dominate the multiverse by determining whether they dominate the normal observers in an arbitrary anthropic vacuum i. Repeating the above analysis for vacuum D instead of vacuum j, using Eq. (92) to relate s_i to s_D, we have

$$\frac{\mathcal{N}_D^{\mathrm{BB}}}{\mathcal{N}_i^{\mathrm{NO}}} \sim \frac{\kappa_D^{\mathrm{BB}} s_D}{\sum_k \kappa_{ik} s_k} \sim \frac{\kappa_D^{\mathrm{BB}} s_D}{\kappa_i s_i} \sim \frac{\kappa_D^{\mathrm{BB}}}{\kappa_{\mathrm{up}}}. \tag{97}$$

Thus, for a single dominant vacuum D or a dominant vacuum system with members D_i, the necessary and sufficient conditions to avoid the domination of the multiverse by these Boltzmann brains is given by

$$\frac{\kappa_D^{\mathrm{BB}}}{\kappa_{\mathrm{up}}} \ll 1 \quad \text{or} \quad \frac{\kappa_{D_i}^{\mathrm{BB}}}{\kappa_{\mathrm{up}}} \ll 1. \tag{98}$$

As discussed after Eq. (84), probably the only way to satisfy this condition is to require that $\kappa_D^{\mathrm{BB}} = 0$.

If the dominant vacuum is anthropic, then the conclusions are essentially the same, but the logic is more involved. For the case of a dominant vacuum system, we distinguish between the possibility of vacua being "strongly" or "mildly" anthropic, as discussed in Sec. IV E. Strongly anthropic means that normal observers are formed by tunneling within the dominant vacuum system D, while mildly anthropic implies that normal observers are formed by tunneling, but only from outside D. Any model that leads to a strongly anthropic dominant vacuum would be unacceptable, because almost all observers would live in pockets with a maximum reheat energy density that is small compared to the vacuum-energy density. With a single anthropic dominant vacuum, or with one or more mildly anthropic vacua within a dominant vacuum

system, the normal observers in the dominant vacuum would be comparable in number (up to factors roughly of order one) to those in other anthropic vacua, so they would have no significant effect on the ratio of Boltzmann brains to normal observers in the multiverse. An anthropic vacuum would also produce Boltzmann brains, however, so Eq. (98) would have to somehow be satisfied for $\kappa_D^{BB} \neq 0$.

V. BOLTZMANN BRAIN NUCLEATION AND VACUUM DECAY RATES

A. Boltzmann brain nucleation rate

Boltzmann brains emerge from the vacuum as large quantum fluctuations. In particular, they can be modeled as localized fluctuations of some mass M, in the thermal bath of a de Sitter vacuum with temperature $T_{dS} = H_\Lambda / 2\pi$ [1]. The Boltzmann brain nucleation rate is then roughly estimated by the Boltzmann suppression factor [6,8],

$$\Gamma_{BB} \sim e^{-M/T_{dS}}, \qquad (99)$$

where our goal is to estimate only the exponent, not the prefactor. Equation (99) gives an estimate for the nucleation rate of a Boltzmann brain of mass M in any particular quantum state, but we will normally describe the Boltzmann brain macroscopically. Thus Γ_{BB} should be multiplied by the number of microstates $e^{S_{BB}}$ corresponding to the macroscopic description, where S_{BB} is the entropy of the Boltzmann brain. Thus we expect

$$\Gamma_{BB} \sim e^{-M/T_{dS}} e^{S_{BB}} = e^{-F/T_{dS}}, \qquad (100)$$

where $F = M - T_{dS} S_{BB}$ is the free energy of the Boltzmann brain.

Equation (100) should be accurate as long as the de Sitter temperature is well defined, which will be the case as long as the Schwarzschild horizon is small compared to the de Sitter horizon radius. Furthermore, we shall neglect the effect of the gravitational potential energy of de Sitter space on the Boltzmann brain, which requires that the Boltzmann brain be small compared to the de Sitter horizon. Thus we assume

$$M/4\pi < R \ll H_\Lambda^{-1}, \qquad (101)$$

where the first inequality assumes that Boltzmann brains cannot be black holes. The general situation, which allows for $M \sim R \sim H_\Lambda^{-1}$, will be discussed in the Appendix and in Ref. [51].

While the nucleation rate is proportional to $e^{S_{BB}}$, this factor is negligible for any Boltzmann brain made of atoms like those in our Universe. The entropy of such atoms is bounded by

$$S \lesssim 3M/m_n, \qquad (102)$$

where m_n is the nucleon mass. Indeed, the actual value of S_{BB} is much smaller than this upper bound because of the complex organization of the Boltzmann brain. Meanwhile, to prevent the Boltzmann brain from being destroyed by pair production, we require that $T_{dS} \ll m_n$. Thus, for these Boltzmann brains the entropy factor $e^{S_{BB}}$ is irrelevant compared to the Boltzmann suppression factor.

To estimate the nucleation rate for Boltzmann brains, we need at least a crude description of what constitutes a Boltzmann brain. There are many possibilities. We argued in the Introduction to this paper that a theory that predicts the domination of Boltzmann brains over normal observers would be overwhelmingly disfavored by our continued observation of an orderly world, in which the events that we observe have a logical relationship to the events that we remember. In making this argument, we considered a class of Boltzmann brains that share exactly the memories and thought processes of a particular normal observer at some chosen instant. For these purposes the memory of the Boltzmann brain can consist of random bits that just happen to match those of the normal observer, so there are no requirements on the history of the Boltzmann brain. Furthermore, the Boltzmann brain need only survive long enough to register one observation after the chosen instant, so it is not required to live for more than about a second. We will refer to Boltzmann brains that meet these requirements as minimal Boltzmann brains.

While an overabundance of minimal Boltzmann brains is enough to cause a theory to be discarded, we nonetheless find it interesting to discuss a wide range of Boltzmann brain possibilities. We will start with very large Boltzmann brains, discussing the minimal Boltzmann brains last.

We first consider Boltzmann brains much like us, who evolved in stellar systems like ours, in vacua with low-energy particle physics like ours, but allowing for a de Sitter Hubble radius as small as a few astronomical units or so. These Boltzmann brains evolved in their stellar systems on a time scale similar to the evolution of life on Earth, so they are in every way like us, except that, when they perform cosmological observations, they find themselves in an empty, vacuum-dominated universe. These "Boltzmann solar systems" nucleate at a rate of roughly

$$\Gamma_{BB} \sim \exp(-10^{85}), \qquad (103)$$

where we have set $M \sim 10^{30}$ kg and $H_\Lambda^{-1} = (2\pi T_{dS})^{-1} \sim 10^{12}$ m. This nucleation rate is fantastically small; we found it, however, by considering the extravagant possibility of nucleating an entire Boltzmann solar system.

Next, we can consider the nucleation of an isolated brain, with a physical construction that is roughly similar

to our own brains. If we take $M \sim 1$ kg and $H_\Lambda^{-1} = (2\pi T_{\mathrm{dS}})^{-1} \sim 1$ m, then the corresponding Boltzmann brain nucleation rate is

$$\Gamma_{\mathrm{BB}} \sim \exp(-10^{43}). \qquad (104)$$

If the construction of the brain is similar to ours, however, then it could not function if the tidal forces resulted in a relative acceleration from one end to the other that is much greater than the gravitational acceleration g on the surface of the Earth. This requires $H_\Lambda^{-1} \gtrsim 10^8$ m, giving a Boltzmann brain nucleation rate

$$\Gamma_{\mathrm{BB}} \sim \exp(-10^{51}). \qquad (105)$$

Until now, we have concentrated on Boltzmann brains that are very similar to human brains. However, a common assumption in the philosophy of mind is that of substrate independence. Therefore, pressing onward, we study the possibility that a Boltzmann brain can be any device capable of emulating the thoughts of a human brain. In other words, we treat the brain essentially as a highly sophisticated computer, with logical operations that can be duplicated by many different systems of hardware.[11]

With this in mind, from here out we drop the assumption that Boltzmann brains are made of the same materials as human brains. Instead, we attempt to find an upper bound on the probability of creation of a more generalized computing device, specified by its information content I_{BB}, which is taken to be comparable to the information content of a human brain.

To clarify the meaning of information content, we can model an information storage device as a system with N possible microstates. $S_{\max} = \ln N$ is then the maximum entropy that the system can have, the entropy corresponding to the state of complete uncertainty of microstates. To store B bits of information in the device, we can imagine a simple model in which 2^B distinguishable macroscopic states of the system are specified, each of which will be used to represent one assignment of the bits. Each macroscopic state can be modeled as a mixture of $N/2^B$ microstates, and hence has entropy $S = \ln(N/2^B) = S_{\max} - B\ln 2$. Motivated by this simple model, one defines the information content of any macroscopic state of entropy S as the difference between $S_{\max}$ and S, where $S_{\max}$ is the maximum entropy that the device can attain. Applying this definition to a Boltzmann brain, we write

$$I_{\mathrm{BB}} = S_{\mathrm{BB,max}} - S_{\mathrm{BB}}, \qquad (106)$$

where $I_{\mathrm{BB}}/\ln 2$ is the information content measured in bits.

As discussed in Ref. [52], the only known substrate-independent limit on the storage of information is the Bekenstein bound. It states that, for an asymptotically flat background, the entropy of any physical system of size R and energy M is bounded by[12]

$$S \leq S_{\mathrm{Bek}} \equiv 2\pi M R. \qquad (107)$$

One can use this bound in de Sitter space as well if the size of the system is sufficiently small, $R \ll H_\Lambda^{-1}$, so that the system does not "know" about the horizon. A possible generalization of the Bekenstein bound for $R = \mathcal{O}(H_\Lambda^{-1})$ was proposed in Ref. [53]; we will study this and other possibilities in the Appendix and in Ref. [51]. To begin, however, we will discuss the simplest case, $R \ll H_\Lambda^{-1}$, so that we can focus on the most important issues before dealing with the complexities of more general results.

Using Eq. (106), the Boltzmann brain nucleation rate of Eq. (100) can be rewritten as

$$\Gamma_{\mathrm{BB}} \sim \exp\left(-\frac{2\pi M}{H_\Lambda} + S_{\mathrm{BB,max}} - I_{\mathrm{BB}}\right), \qquad (108)$$

which is clearly maximized by choosing M as small as possible. The Bekenstein bound, however, implies that $S_{\mathrm{BB,max}} \leq S_{\mathrm{Bek}}$ and therefore $M \geq S_{\mathrm{BB,max}}/(2\pi R)$. Thus

$$\Gamma_{\mathrm{BB}} \leq \exp\left(-\frac{S_{\mathrm{BB,max}}}{R H_\Lambda} + S_{\mathrm{BB,max}} - I_{\mathrm{BB}}\right). \qquad (109)$$

Since $R < H_\Lambda^{-1}$, the expression above is maximized by taking $S_{\mathrm{BB,max}}$ equal to its smallest possible value, which is I_{BB}. Finally, we have

$$\Gamma_{\mathrm{BB}} \leq \exp\left(-\frac{I_{\mathrm{BB}}}{R H_\Lambda}\right). \qquad (110)$$

Thus, the Boltzmann brain production rate is maximized if the Boltzmann brain saturates the Bekenstein bound, with $I_{\mathrm{BB}} = S_{\mathrm{BB,max}} = 2\pi M R$. Simultaneously, we should make $R H_\Lambda$ as large as possible, which means taking our

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[11]Note that the validity of the assumption of substrate independence of mind is not entirely self-evident. Some of us are skeptical of identifying human consciousness with operations of a generic substrate-independent computer, but accept it as a working hypothesis for the purpose of this paper.

[12]In an earlier version of this paper we stated an incorrect form of this bound, and from it derived some incorrect conclusions, such as the statement that the largest Boltzmann brain nucleation rate Γ_{BB} consistent with the Bekenstein bound is attained only when the radius R approaches the Schwarzschild radius R_{Sch}. This in turn led to the conclusion that the maximum rate allowed by the Bekenstein bound is $e^{-2I_{\mathrm{BB}}}$, which can be achieved only if $M^2 = I_{\mathrm{BB}}/(9\pi G)$ and $H_\Lambda^2 = \pi/(3GI_{\mathrm{BB}})$. While these relations hold in the regime we considered, they are not necessary in the general case. With the corrected bound, we find that the maximum nucleation rate is independent of R/R_{Sch} if $R \ll H_\Lambda$ [see Eq. (110)], and otherwise grows with R/R_{Sch} (see the Appendix). However, once one is forced to consider values of $R \gg R_{\mathrm{Sch}}$, then other issues become relevant. How can the system be stabilized against the de Sitter expansion? Can the Bekenstein bound really be saturated for a system with large entropy, especially if it is dilute? In this version of the paper we have added a discussion of these issues. We thank R. Bousso, B. Freivogel, and I. Yang for pointing out the error in our earlier statement of the Bekenstein bound.

assumption $R \ll H_\Lambda^{-1}$ to the boundary of its validity. Thus we write the Boltzmann brain production rate

$$\Gamma_{BB} \lesssim e^{-aI_{BB}}, \tag{111}$$

where $a \equiv (RH_\Lambda)^{-1}$, the value of which is of order a few. In the Appendix we explore the case in which the Schwarzschild radius, the Boltzmann brain radius, and the de Sitter horizon radius are all about equal, in which case Eq. (111) holds with $a = 2$.

The bound of Eq. (111) can be compared to the estimate of the Boltzmann brain production rate, $\Gamma_{BB} \sim e^{-S_{BB}}$, which follows from Eq. (2.13) of Freivogel and Lippert, in Ref. [54]. The authors of Ref. [54] explained that by S_{BB} they mean not the entropy, but the number of degrees of freedom, which is roughly equal to the number of particles in a Boltzmann brain. This estimate appears similar to our result, if one equates S_{BB} to I_{BB}, or to a few times I_{BB}. Freivogel and Lippert describe this relation as a lower bound on the nucleation rate for Boltzmann brains, commenting that it can be used as an estimate of the nucleation rate for vacua with "reasonably cooperative particle physics." Here we will explore in some detail the question of whether this bound can be used as an estimate of the nucleation rate. While we will not settle this issue here, we will discuss evidence that Eq. (111) is a valid estimate for at most a small fraction of the vacua of the landscape, and possibly none at all.

So far, the conditions to reach the upper bound in Eq. (111) are $R = (aH_\Lambda)^{-1} \sim \mathcal{O}(H_\Lambda^{-1})$ and $I_{BB} = S_{\max,BB} = S_{Bek}$. However, these are not enough to ensure that a Boltzmann brain of size $R \sim H_\Lambda^{-1}$ is stable and can actually compute. Indeed, the time required for communication between two parts of a Boltzmann brain separated by a distance $\mathcal{O}(H_\Lambda^{-1})$ is at least comparable to the Hubble time. If the Boltzmann brain can be stretched by cosmological expansion, then after just a few operations the different parts will no longer be able to communicate. Therefore we need a stabilization mechanism by which the brain is protected against expansion.

A potential mechanism to protect the Boltzmann brain against de Sitter expansion is the self-gravity of the brain. A simple example is a black hole, which does not expand when the Universe expands. It seems unlikely that black holes can think,[13] but one can consider objects of mass approaching that of a black hole with radius R. This, together with our goal to keep R as close as possible to H_Λ^{-1}, leads to the following condition:

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[13]The possibility of a black hole computer is not excluded, however, and has been considered in Ref. [52]. Nonetheless, if black holes can compute, our conclusions would not be changed, provided that the Bekenstein bound can be saturated for the near-black hole computers that we discuss. At this level of approximation, there would be no significant difference between a black hole computer and a near-black hole computer.

$$M \sim R \sim H_\Lambda^{-1}. \tag{112}$$

If the Bekenstein bound is saturated, this leads to the following relations between I_{BB}, H_Λ, and M:

$$I_{BB} \sim MR \sim MH_\Lambda^{-1} \sim H_\Lambda^{-2}. \tag{113}$$

A second potential mechanism of Boltzmann brain stabilization is to surround it by a domain wall with a surface tension σ, which would provide pressure preventing the exponential expansion of the brain. An investigation of this situation reveals that one cannot saturate the Bekenstein bound using this mechanism unless there is a specific relation between I_{BB}, H_Λ, and σ [51]:

$$\sigma \sim I_{BB}H_\Lambda^3. \tag{114}$$

If σ is less than this magnitude, it cannot prevent the expansion, while a larger σ increases the mass and therefore prevents saturation of the Bekenstein bound.

Regardless of the details leading to Eqs. (113) and (114), the important point is that both of them lead to constraints on the vacuum hosting the Boltzmann brain.

For example, the Boltzmann brain stabilized by gravitational attraction can be produced at a rate approaching $e^{-aI_{BB}}$ only if $I_{BB} \sim H_\Lambda^{-2}$. For a given value of I_{BB}, say $I_{BB} \sim 10^{16}$ (see the discussion below), this result applies only to vacua with a particular vacuum energy, $\Lambda \sim 10^{-16}$. Similarly, according to Eq. (114), for Boltzmann brains with $I_{BB} \sim 10^{16}$ contained inside a domain wall in a vacuum with $\Lambda \sim 10^{-120}$, the Bekenstein bound on Γ_{BB} cannot be reached unless the tension of the domain wall is incredibly small, $\sigma \sim 10^{-164}$. Thus, the maximal Boltzmann brain production rate $\sim e^{-aI_{BB}}$ saturating the Bekenstein bound cannot be reached unless Boltzmann brains are produced on a narrow hypersurface in the landscape.

This conclusion by itself does not eliminate the danger of a rapid Boltzmann brain production rate, $\Gamma_{BB} \sim e^{-aI_{BB}}$. Given the vast number of vacua in the landscape, it seems plausible that this bound could actually be met. If this is the case, Eq. (111) offers a stunning increase over previous estimates of Γ_{BB}.

Setting aside the issue of Boltzmann brain stability, one can also question the assumption of Bekenstein bound saturation that is necessary to achieve the rather high nucleation rate that is indicated by Eq. (111). Of course black holes saturate this bound, but we assume that a black hole cannot think. Even if a black hole can think, it would still be an open question whether this information processing could make use of a substantial fraction of the degrees of freedom associated with the black hole entropy. A variety of other physical systems are considered in Ref. [55], where the validity of $S_{\max}(E) \leq 2\pi ER$ is studied as a function of energy E. In all cases, the bound is saturated in a limit where $S_{\max} = \mathcal{O}(1)$. Meanwhile, as we shall argue below, the required value of $S_{\max}$ should be greater than 10^{16}.

The present authors are aware of only one example of a physical system that may saturate the Bekenstein bound

and at the same time store sufficient information I to emulate a human brain. This may happen if the total number of particle species with mass smaller than H_Λ is greater than $I_{BB} \gtrsim 10^{16}$. No realistic examples of such theories are known to us, although some authors have speculated about similar possibilities [56].

If Boltzmann brains cannot saturate the Bekenstein bound, they will be more massive than indicated in Eq. (110), and their rate of production will be smaller than $e^{-aI_{BB}}$.

To put another possible bound on the probability of Boltzmann brain production, let us analyze a simple model based on an ideal gas of massless particles. Dropping all numerical factors, we consider a box of size R filled with a gas with maximum entropy $S_{\max} = (RT)^3$ and energy $E = R^3 T^4 = S_{\max}^{4/3}/R$, where T is the temperature and we assume there is not an enormous number of particle species. The probability of its creation can be estimated as follows:

$$\Gamma_{BB} \sim e^{-E/H_\Lambda} e^{S_{BB}} \sim \exp\left(-\frac{S_{\max}^{4/3}}{H_\Lambda R}\right), \qquad (115)$$

where we have neglected the Boltzmann brain entropy factor, since $S_{BB} \leq S_{\max} \ll S_{\max}^{4/3}$. This probability is maximized by taking $R \sim H_\Lambda^{-1}$, which yields

$$\Gamma_{BB} \lesssim e^{-S_{\max}^{4/3}}. \qquad (116)$$

In case the full information capacity of the gas is used, one can also write

$$\Gamma_{BB} \lesssim e^{-I_{BB}^{4/3}}. \qquad (117)$$

For $I_{BB} \gg 1$, this estimate leads to a much stronger suppression of Boltzmann brain production as compared to our previous estimate, Eq. (111).

Of course, such a hot gas of massless particles cannot think—indeed it is not stable in the sense outlined below Eq. (111)—so we must add more parts to this construction. Yet it seems likely that this will only decrease the Boltzmann brain production rate. As a partial test of this conjecture, one can easily check that if instead of a gas of massless particles we consider a gas of massive particles, the resulting suppression of Boltzmann brain production will be stronger. Therefore in our subsequent estimates we shall assume that Eq. (117) represents our next "line of defense" against the possibility of Boltzmann brain domination, after the one given by Eq. (111). One should note that this is a rather delicate issue; see, for example, a discussion of several possibilities to approach the Bekenstein bound in Ref. [57]. A more detailed discussion of this issue will be provided in Ref. [51].

Having related Γ_{BB} to the information content I_{BB} of the brain, we now need to estimate I_{BB}. How much information storage must a computer have to be able to perform all the functions of the human brain? Since no one can write a

computer program that comes close to imitating a human brain, this is not an easy question to answer.

One way to proceed is to examine the human brain, with the goal of estimating its capacities based on its biological structure. The human brain contains $\sim 10^{14}$ synapses that may, in principle, connect to any of $\sim 10^{11}$ neurons [58], suggesting that its information content[14] might be roughly $I_{BB} \sim 10^{15}$–10^{16}. (We are assuming here that the logical functions of the brain depend on the connections among neurons, and not, for example, on their precise locations, cellular structures, or other information that might be necessary to actually construct a brain.) A minimal Boltzmann brain is only required to simulate the workings of a real brain for about a second; but with neurons firing typically at 10 to 100 times a second, it is plausible that a substantial fraction of the brain is needed even for only 1 s of activity. Of course the actual number of required bits might be somewhat less.

An alternative approach is to try to determine how much information the brain processes, even if one does not understand much about what the processing involves.

In Ref. [59], Landauer attempted to estimate the total content of a person's long-term memory, using a variety of experiments. He concluded that a person remembers only about 2 bits/second, for a lifetime total in the vicinity of 10^9 bits. In a subsequent paper [60], however, he emphatically denied that this number is relevant to the information requirements of a "real or theoretical cognitive processor," because such a device "would have so much more to do than simply record new information."

Besides long-term memory, one might be interested in the total amount of information a person receives but does not memorize. A substantial part of this information is visual; it can be estimated by the information stored on high definition DVDs, watched continuously on several monitors over the span of a hundred years. The total information received would be about 10^{16} bits.

Since this number is similar to the number obtained above by counting synapses, it is probably as good an estimate as we can make for a minimal Boltzmann brain. If the Bekenstein bound can be saturated, then the estimated Boltzmann brain nucleation rate for the most favorable vacua in the landscape would be given by Eq. (111):

$$\Gamma_{BB} \lesssim e^{-10^{16}}. \qquad (118)$$

If, however, the Bekenstein bound cannot be reached for systems with $I_{BB} \gg 1$, then it might be more accurate to use instead the ideal gas model of Eq. (117), yielding

$$\Gamma_{BB} \lesssim e^{-10^{21}}. \qquad (119)$$

Obviously, there are many uncertainties involved in the numerical estimates of the required value of I_{BB}.

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[14]Note that the specification of one out of 10^{11} neurons requires $\log_2(10^{11}) = 36.5$ bits.

Our estimate $I_{\mathrm{BB}} \sim 10^{16}$ concerns the information stored in the human brain that appears to be relevant for cognition. It certainly does not include all the information that would be needed to physically construct a human brain, and it therefore does not allow for the information that might be needed to physically construct a device that could emulate the human brain.[15] It is also possible that extra mass might be required for the mechanical structure of the emulator, to provide the analogues of a computer's wires, insulation, cooling systems, etc. On the other hand, it is conceivable that a Boltzmann brain can be relevant even if it has fewer capabilities than what we called the minimal Boltzmann brain. In particular, if our main requirement is that the Boltzmann brain is to have the same "perceptions" as a human brain for just 1 s, then one may argue that this can be achieved using much less than 10^{14} synapses. And if one decreases the required time to a much smaller value required for a single computation to be performed by a human brain, the required amount of information stored in a Boltzmann brain may become many orders of magnitude smaller than 10^{16}.

We find that regardless of how one estimates the information in a human brain, if Boltzmann brains can be constructed so as to come near the limit of Eq. (111), their nucleation rate would provide stringent requirements on vacuum decay rates in the landscape. On the other hand, if

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[15]That is, the actual construction of a brainlike device would presumably require large amounts of information that are not part of the schematic "circuit diagram" of the brain. Thus there may be some significance to the fact that 1×10^9 yrs of evolution on Earth has not produced a human brain with fewer than about 10^{27} particles, and hence of order 10^{27} units of entropy. In counting the information in the synapses, for example, we counted only the information needed to specify which neurons are connected to which, but nothing about the actual path of the axons and dendrites that complete the connections. These are nothing like nearest-neighbor couplings, but instead axons from a single neuron can traverse large fractions of the brain, resulting in an extremely intertwined network [61]. To specify even the topology of these connections, still ignoring the precise locations, could involve much more than 10^{16} bits. For example, the synaptic "wiring" that connects the neurons will, in many cases, form closed loops. A specification of the connections would presumably require a topological winding number for every pair of closed loops in the network. The number of bits required to specify these winding numbers would be proportional to the square of the number of closed loops, which would be proportional to the square of the number of synapses. Thus, the structural information could be something like $I_{\mathrm{struct}} \sim b \times 10^{28}$, where b is a proportionality constant that is probably a few orders of magnitude less than 1. In estimating the resulting suppression of the nucleation rate, there is one further complication: since structural information of this sort presumably has no influence on brain function, these choices would contribute to the multiplicity of Boltzmann brain microstates, thereby multiplying the nucleation rate by $e^{I_{\mathrm{struct}}}$. There would still be a net suppression, however, with Eq. (111) leading to $\Gamma_{\mathrm{BB}} \propto e^{-(a-1)I_{\mathrm{struct}}}$, where a is generically greater than 1. See the Appendix for further discussion of the value of a.

no such physical construction exists, we are left with the less dangerous bound of Eq. (117), perhaps even further softened by the speculations described in Footnote [15]. Note that none of these bounds is based upon a realistic model of a Boltzmann brain. For example, the nucleation of an actual human brain is estimated at the vastly smaller rate of Eq. (105). The conclusions of this paragraph apply to the causal-patch measures [23,24] as well as the scale-factor cutoff measure.

In Sec. III we discussed the possibility of Boltzmann brain production during reheating, stating that this process would not be a danger. We postponed the numerical discussion, however, so we now return to that issue. According to Eq. (26), the multiverse will be safe from Boltzmann brains formed during reheating provided that

$$\Gamma^{\mathrm{BB}}_{\mathrm{reheat},ik} \Delta \tau^{\mathrm{BB}}_{\mathrm{reheat},ik} \ll n^{\mathrm{NO}}_{ik} \qquad (120)$$

holds for every pair of vacua i and k, where $\Gamma^{\mathrm{BB}}_{\mathrm{reheat},ik}$ is the peak Boltzmann brain nucleation rate in a pocket of vacuum i that forms in a parent vacuum of type k, $\Delta \tau^{\mathrm{BB}}_{\mathrm{reheat},ik}$ is the proper time available for such nucleation, and n^{NO}_{ik} is the volume density of normal observers in these pockets, working in the approximation that all observers form at the same time.

Compared to the previous discussion about late-time de Sitter space nucleation, here $\Gamma^{\mathrm{BB}}_{\mathrm{reheat},ik}$ can be much larger, since the temperature during reheating can be much larger than H_Λ. On the other hand, safety from Boltzmann brains requires the late-time nucleation rate to be small compared to the potentially very small vacuum decay rates, while in this case the quantity on the right-hand side of Eq. (120) is not exceptionally small. In discussing this issue, we will consider in sequence three descriptions of the Boltzmann brain: a humanlike brain, a near-black hole computer, and a diffuse computer.

The nucleation of humanlike Boltzmann brains during reheating was discussed in Ref. [27], where it was pointed out that such brains could not function at temperatures much higher than 300 K, and that the nucleation rate for a 100 kg object at this temperature is $\sim \exp(-10^{40})$. This suppression is clearly more than enough to ensure that Eq. (120) is satisfied.

For a near-black hole computer with $I_{\mathrm{BB}} \approx S_{\mathrm{BB,max}} \approx 10^{16}$, the minimum mass is 600 g. If we assume that the reheat temperature is no more than the reduced Planck mass, $m_{\mathrm{Planck}} \equiv 1/\sqrt{8\pi G} \approx 2.4 \times 10^{18}$ GeV $\approx 4.3 \times 10^{-6}$ g, we find that $\Gamma^{\mathrm{BB}}_{\mathrm{reheat}} < \exp(-\sqrt{2I_{\mathrm{BB}}}) \sim \exp(-10^8)$. Although this is not nearly as much suppression as in the previous case, it is clearly enough to guarantee that Eq. (120) will be satisfied.

For the diffuse computer, we can consider an ideal gas of massless particles, as discussed in Eqs. (115)–(117). The system would have approximately S_{max} particles, and a total energy of $E = S^{4/3}_{\mathrm{max}}/R$, so the Boltzmann suppression

factor is $\exp[-S_{\max}^{4/3}/(RT_{\text{reheat}})]$. The Boltzmann brain production can occur at any time during the reheating process, so there is nothing wrong with considering Boltzmann brain production in our Universe at the present time. For $T_{\text{reheat}} = 2.7$ K and $S_{\max} = 10^{16}$, this formula implies that the exponent has magnitude 1 for $R = S_{\max}^{4/3} T_{\text{reheat}}^{-1} \approx$ 200 light-years. Thus, the formula suggests that diffuse-gas-cloud Boltzmann brains of radius 200 light-years can be thermally produced in our Universe, at the present time, without suppression. If this estimate were valid, then Boltzmann brains would almost certainly dominate the Universe.

We argue, however, that the gas clouds described above would have no possibility of computing, because the thermal noise would preclude any storage or transfer of information. The entire device has energy of order $E \approx T_{\text{reheat}}$, which is divided among approximately 10^{16} massless particles. The mean particle energy is therefore 10^{16} times smaller than that of the thermal particles in the background radiation, and the density of Boltzmann brain particles is 10^{48} times smaller than the background. To function, it seems reasonable that the diffuse computer needs an energy per particle that is at least comparable to the background, which means that the suppression factor is $\exp(-10^{16})$ or smaller. Thus, we conclude that for all three cases, the ratio of Boltzmann brains to normal observers is totally negligible.

Finally, let us also mention the possibility that Boltzmann brains might form as quantum fluctuations in stable Minkowski vacua. String theory implies at least the existence of a 10D decompactified Minkowski vacuum; Minkowski vacua of lower dimension are not excluded, but they require precise fine-tunings for which motivation is lacking. While quantum fluctuations in Minkowski space are certainly less classical than in de Sitter space, they still might be relevant. The possibility of Boltzmann brains in Minkowski space has been suggested by Page [5,6,40]. If Γ_{BB} is nonzero in such vacua, regardless of how small it might be, Boltzmann brains will always dominate in the scale-factor cutoff measure as we have defined it. Even if Minkowski vacua cannot support Boltzmann brains, there might still be a serious problem with what might be called "Boltzmann islands." That is, it is conceivable that a fluctuation in a Minkowski vacuum can produce a small region of an anthropic vacuum with a Boltzmann brain inside it. The anthropic vacuum could perhaps even have a different dimension than its Minkowski parent. If such a process has a nonvanishing probability to occur, it will also give rise to Boltzmann brain domination in the scale-factor cutoff measure. These problems would be shared by all measures that assign an infinite weight to stable Minkowski vacua. There is, however, one further complication which might allow Boltzmann brains to form in Minkowski space without dominating the multiverse. If one speculates about Boltzmann brain production in Minkowski space, one may equally well speculate about spontaneous creation of inflationary universes there, each of which could contain infinitely many normal observers [62]. These issues become complicated, and we will make no attempt to resolve them here. Fortunately, the estimates of thermal Boltzmann brain nucleation rates in de Sitter space approach zero in the Minkowski space limit $\Lambda \to 0$, so the issue of Boltzmann brains formed by quantum fluctuations in Minkowski space can be set aside for later study. Hopefully the vague idea that these fluctuations are less classical than de Sitter space fluctuations can be promoted into a persuasive argument that they are not relevant.

B. Vacuum decay rates

One of the most developed approaches to the string landscape scenario is based on the KKLT construction [63]. In this construction, one begins by finding a set of stabilized supersymmetric anti–de Sitter (AdS) and Minkowski vacua. After that, an uplifting is performed, e.g. by adding a $\overline{D3}$-brane at the tip of a conifold [63]. This uplifting makes the vacuum-energy density of some of these vacua positive (AdS $\to$ dS), but, in general, many vacua remain AdS, and the Minkowski vacuum corresponding to the uncompactified 10D space does not become uplifted. The enormous number of vacua in the landscape appears because of the large number of different topologies of the compactified space, and the large number of different fluxes and branes associated with it.

There are many ways in which our low-energy dS vacuum may decay. First of all, it can always decay into the Minkowski vacuum corresponding to the uncompactified 10D space [63]. It can also decay to one of the AdS vacua corresponding to the same set of branes and fluxes [64]. More generally, decays occur due to the jumps between vacua with different fluxes, or due to the brane-flux annihilation [54,65–71], and may be accompanied by a change in the number of compact dimensions [72–74]. If one does not take into account vacuum stabilization, these transitions are relatively easy to analyze [65–67]. However, in the realistic situations where the moduli fields are determined by fluxes, branes, etc., these transitions involve a simultaneous change of fluxes and various moduli fields, which makes a detailed analysis of the tunneling quite complicated.

Therefore, we begin with an investigation of the simplest decay modes due to the scalar field tunneling. The transition to the 10D Minkowski vacuum was analyzed in Ref. [63], where it was shown that the decay rate κ is always greater than

$$\kappa \gtrsim e^{-S_D} = \exp\left(-\frac{24\pi^2}{V_{\text{dS}}}\right). \qquad (121)$$

Here S_D is the entropy of dS space. For our vacuum, $S_D \sim 10^{120}$, which yields

$$\kappa \gtrsim e^{-S_D} \sim \exp(-10^{120}). \qquad (122)$$

Because of the inequality in Eq. (121), we expect the slowest-decaying vacua to typically be those with very small vacuum energies, with the dominant vacuum-energy density possibly being much smaller than the value in our Universe.

The decay to AdS space (or, more accurately, a decay to a collapsing open universe with a negative cosmological constant) was studied in Ref. [64]. The results of Ref. [64] are based on the investigation of Bogomol'nyi-Prasad-Sommerfield (BPS) and near-BPS domain walls in string theory, generalizing the results previously obtained in $\mathcal{N} = 1$ supergravity [75–78]. Here we briefly summarize the main results obtained in Ref. [64].

Consider, for simplicity, the situation where the tunneling occurs between two vacua with very small vacuum energies. For the sake of argument, let us first ignore the gravitational effects. Then the tunneling always takes place, as long as one vacuum has higher vacuum energy than the other. In the limit when the difference between the vacuum energies goes to zero, the radius of the bubble of the new vacuum becomes infinitely large, $R \to \infty$ (the thin-wall limit). In this limit, the bubble wall becomes flat, and its initial acceleration, at the moment when the bubble forms, vanishes. Therefore, to find the tension of the domain wall in the thin-wall approximation one should solve an equation for the scalar field describing a static domain wall separating the two vacua.

If the difference between the values of the scalar potential in the two minima is too small, and at least one of them is AdS, then the tunneling between them may be forbidden because of the gravitational effects [79]. In particular, all supersymmetric vacua, including all KKLT vacua prior to the uplifting, are absolutely stable even if other vacua with lower energy density are available [80–83].

It is tempting to make a closely related but opposite statement: nonsupersymmetric vacua are always unstable. However, this is not always the case. In order to study tunneling while taking account of supersymmetry (SUSY), one may start with two different supersymmetric vacua in two different parts of the Universe and find a BPS domain wall separating them. One can show that if the superpotential does not change its sign on the way from one vacuum to the other, then this domain wall plays the same role as the flat domain wall in the no-gravity case discussed above: it corresponds to the wall of the bubble that can be formed once the supersymmetry is broken in either of the two minima. However, if the superpotential does change its sign, then only a sufficiently large supersymmetry breaking will lead to the tunneling [64,75].

One should keep this fact in mind, but since we are discussing a landscape with an extremely large number of vacua, in what follows we assume that there is at least one direction in which the superpotential does not change its sign on the way from one minimum to another. In what

follows we describe tunneling in one such direction. Furthermore, we assume that at least some of the AdS vacua to which our dS vacuum may decay are uplifted much less than our vacuum. This is a generic situation, since the uplifting depends on the value of the volume modulus, which takes different values in each vacuum.

In this case the decay rate of a dS vacuum with low-energy density and broken supersymmetry can be estimated as follows [64,84]:

$$\kappa \sim \exp\left(-\frac{8\pi^2 \alpha}{3m_{3/2}^2}\right), \qquad (123)$$

where $m_{3/2}$ is the gravitino mass in that vacuum and α is a quantity that depends on the parameters of the potential. Generically one can expect $\alpha = \mathcal{O}(1)$, but it can also be much greater or much smaller than $\mathcal{O}(1)$. The mass $m_{3/2}$ is set by the scale of SUSY breaking,

$$3m_{3/2}^2 = \Lambda_{\text{SUSY}}^4, \qquad (124)$$

where we recall that we use reduced Planck units, $8\pi G = 1$. Therefore the decay rate can also be represented in terms of the SUSY-breaking scale Λ_{SUSY}:

$$\kappa \sim \exp\left(-\frac{24\pi^2 \alpha}{\Lambda_{\text{SUSY}}^4}\right). \qquad (125)$$

Note that in the KKLT theory, Λ_{SUSY}^4 corresponds to the depth of the AdS vacuum before the uplifting, so that

$$\kappa \sim \exp\left(-\frac{24\pi^2 \alpha}{|V_{\text{AdS}}|}\right). \qquad (126)$$

In this form, the result for the tunneling looks very similar to the lower bound on the decay rate of a dS vacuum, Eq. (121), with the obvious replacements $\alpha \to 1$ and $|V_{\text{AdS}}| \to V_{\text{dS}}$.

Let us apply this result to the question of vacuum decay in our Universe. Clearly, the implications of Eq. (125) depend on the details of SUSY phenomenology. The standard requirement that the gaugino mass and the scalar masses are $\mathcal{O}(1)$ TeV leads to the lower bound

$$\Lambda_{\text{SUSY}} \gtrsim 10^4 - 10^5 \text{ GeV}, \qquad (127)$$

which can be reached, e.g., in the models of conformal gauge mediation [85]. This implies that for our vacuum

$$\kappa_{\text{our}} \gtrsim \exp(-10^{56}) - \exp(-10^{60}). \qquad (128)$$

Using Eq. (99), the Boltzmann brain nucleation rate in our Universe exceeds the lower bound of the above inequality only if $M \lesssim 10^{-9}$ kg.

On the other hand, one can imagine universes very similar to ours except with much larger vacuum-energy densities. The vacuum decay rate of Eq. (123) exceeds the Boltzmann brain nucleation rate of Eq. (99) when

$$\left(\frac{m_{3/2}}{10^{-2} \text{ eV}}\right)^2 \left(\frac{M}{1 \text{ kg}}\right) \left(\frac{H_\Lambda^{-1}}{10^8 \text{ m}}\right) \gtrsim 10^9 \alpha. \tag{129}$$

Note that $H_\Lambda^{-1} \sim 10^8$ m corresponds to the smallest de Sitter radius for which the tidal force on a 10 cm brain does not exceed the gravitational force on the surface of the Earth, while $m_{3/2} \sim 10^{-2}$ eV corresponds to $\Lambda_{\text{SUSY}} \sim 10^4$ GeV. Thus, it appears the decay rate of Eq. (123) allows for Boltzmann brain domination.

However, we do not really know whether the models with low Λ_{SUSY} can successfully describe our world. To mention one potential problem, in models of string inflation there is a generic constraint that during the last stage of inflation one has $H \lesssim m_{3/2}$ [86]. If we assume the second and third factors of Eq. (129) cannot be made much less than unity, then we only require $m_{3/2} \gtrsim \mathcal{O}(10^2)$ eV to avoid Boltzmann brain domination. While models of string inflation with $H \lesssim 100$ eV are not entirely impossible in the string landscape, they are extremely difficult to construct [87]. If instead of $\Lambda_{\text{SUSY}} \sim 10^4$ GeV one uses $\Lambda_{\text{SUSY}} \sim 10^{11}$ GeV, as in models with gravity mediation, one finds $m_{3/2} \sim 10^3$ GeV, and Eq. (129) is easily satisfied.

These arguments apply when supersymmetry violation is as large as or larger than in our Universe. If supersymmetry violation is too small, atomic systems are unstable [88], the masses of some of the particles will change dramatically, etc. However, the Boltzmann computers described in the previous subsection do not necessarily rely on laws of physics similar to those in our Universe (in fact, they seem to require very different laws of physics). The present authors are unaware of an argument that supersymmetry breaking must be so strong that vacuum decay is always faster than the Boltzmann brain production rate of Eq. (118).

On the other hand, up to this point we have used the estimates of the vacuum decay rate that were obtained in Refs. [64,84] by investigation of the transition where only moduli fields changed. As we have already mentioned, the description of a more general class of transitions involving the change of branes or fluxes is much more complicated. Investigation of such processes, performed in Refs. [54,68,69], indicates that the process of vacuum decay for any vacuum in the KKLT scenario should be rather fast,

$$\kappa \gtrsim \exp(-10^{22}). \tag{130}$$

The results of Refs. [54,68,69], like the results of Refs. [64,84], are not completely generic. In particular, the investigations of Refs. [54,68,69] apply to the original version of the KKLT scenario, where the uplifting of the AdS vacuum occurs due to $\overline{D3}$-branes, but not to its generalization proposed in Ref. [89], where the uplifting is achieved due to D7-branes. Nor does it apply to the recent version of dS stabilization proposed in Ref. [90].

Nevertheless, the results of Refs. [54,68,69] show that the decay rate of dS vacua in the landscape can be quite large. The rate $\kappa \gtrsim \exp(-10^{22})$ is much greater than the expected rate of Boltzmann brain production given by Eq. (105). However, it is just a bit smaller than the bosonic gas Boltzmann brain production rate of Eq. (119) and much smaller than our most dangerous upper bound on the Boltzmann brain production rate, given by Eq. (118).

VI. CONCLUSIONS

If the observed accelerating expansion of the Universe is driven by constant vacuum-energy density and if our Universe does not decay in the next 20×10^9 yrs or so, then it seems cosmology must explain why we are "normal observers"—who evolve from nonequilibrium processes in the wake of the big bang—as opposed to "Boltzmann brains"—freak observers that arise as a result of rare quantum fluctuations [2–4,7,8]. Put in experimental terms, cosmology must explain why we observe structure formation in a residual cosmic microwave background, as opposed to the empty, vacuum-energy dominated environment in which almost all Boltzmann brains nucleate. As vacuum-energy expansion is eternal to the future, the number of Boltzmann brains in an initially finite comoving volume is infinite. However, if there exists a landscape of vacua, then rare transitions to other vacua populate a diverging number of universes in this comoving volume, creating an infinite number of normal observers. To weigh the relative number of Boltzmann brains to normal observers requires a spacetime measure to regulate the infinities.

Recently, the scale-factor cutoff measure was shown to possess a number of desirable attributes, including avoiding the youngness paradox [28] and the Q (and G) catastrophe [29–31], while predicting the cosmological constant to be measured in a range including the observed value, and excluding values more than about a factor of 10 larger and smaller than this [38]. The scale-factor cutoff does not itself select for a longer duration of slow-roll inflation, raising the possibility that a significant fraction of observers like us measure cosmic curvature significantly above the value expected from cosmic variance [48]. In this paper, we have calculated the ratio of the total number of Boltzmann brains to the number of normal observers, using the scale-factor cutoff.

The general conditions under which Boltzmann brain domination is avoided were discussed in Sec. IV F, where we described several alternative criteria that can be used to ensure safety from Boltzmann brains. We also explored a set of assumptions that allow one to state conditions that are both necessary and sufficient to avoid Boltzmann brain domination. One relatively simple way to ensure safety from Boltzmann brains is to require two conditions: (1) in any vacuum, the Boltzmann brain nucleation rate must be less than the decay rate of that vacuum, and (2) for any anthropic vacuum j with a decay rate $\kappa_j \approx q$, and for any

nonanthropic vacuum j, one must construct a sequence of transitions from j to an anthropic vacuum; if the sequence includes suppressed upward jumps, then the Boltzmann brain nucleation rate in vacuum j must be less than the decay rate of vacuum j times the product of all the suppressed branching ratios B_{up} that appear in the sequence. The condition (2) might not be too difficult to satisfy, since it will generically involve only states with very low-vacuum-energy densities, which are likely to be nearly supersymmetric and therefore unlikely to support the complex structures needed for Boltzmann brains or normal observers. Condition (2) can also be satisfied if there is no unique dominant vacuum, but instead a dominant vacuum system that consists of a set of nearly degenerate states, some of which are anthropic, which undergo rapid transitions to each other, but only slow transitions to other states. Condition (1) is perhaps more difficult to satisfy. Although nearly supersymmetric string vacua can, in principle, be long-lived [63,64,75–78], with decay rates possibly much smaller than the Boltzmann brain nucleation rate, recent investigations suggest that other decay channels may evade this problem [54,68,69]. However, the decay processes studied in [54,63,64,68,69,75–78] do not describe some of the situations which are possible in the string theory landscape, and the strongest constraints on the decay rate obtained in [54] are still insufficient to guarantee that the vacuum decay rate is always smaller than the fastest estimate of the Boltzmann brain production rate, Eq. (118).

One must emphasize that we are discussing a rapidly developing field of knowledge. Our estimates of the Boltzmann brain production rate are exponentially sensitive to our understanding of what exactly the Boltzmann brain is. Similarly, the estimates of the decay rate in the landscape became possible only five years ago, and this subject certainly is going to evolve. Therefore we will mention here two logical possibilities which may emerge as a result of the further investigation of these issues.

If further investigation will demonstrate that the Boltzmann brain production rate is always smaller than the vacuum decay rate in the landscape, the probability measure that we are investigating in this paper will be shown not to suffer from the Boltzmann brain problem. Conversely, if one believes that this measure is correct, the fastest Boltzmann brain production rate will give us a rather strong lower bound on the decay rate of the metastable vacua in the landscape. We expect that similar conclusions with respect to the Boltzmann brain problem should be valid for the causal-patch measures [23,24].

On the other hand, if we do not find a sufficiently convincing theoretical reason to believe that the vacuum decay rate in all vacua in the landscape is always greater than the fastest Boltzmann brain production rate, this would motivate the consideration of other probability measures, where the Boltzmann brain problem can be solved even if the probability of their production is not strongly suppressed.

In any case, our present understanding of the Boltzmann brain problem does not rule out the scale-factor cutoff measure, but the situation remains uncertain.

APPENDIX: BOLTZMANN BRAINS IN SCHWARZSCHILD–DE SITTER SPACE

As explained in Sec. VA, Eq. (100) for the production rate of Boltzmann brains must be reexamined when the Boltzmann brain radius becomes comparable to the de Sitter radius. In this case we need to describe the Boltzmann brain nucleation as a transition from an initial state of empty de Sitter space with horizon radius H_Λ^{-1} to a final state in which the dS space is altered by the presence of an object with mass M. Assuming that the object can be treated as spherically symmetric, the space outside the object is described by the Schwarzschild–de Sitter (SdS) metric [91][16]:

$$ds^2 = -\left(1 - \frac{2GM}{r} - H_\Lambda^2 r^2\right)dt^2 + \left(1 - \frac{2GM}{r} - H_\Lambda^2 r^2\right)^{-1}$$
$$\times \, dr^2 + r^2 d\Omega^2. \tag{A1}$$

The SdS metric has two horizons, determined by the positive zeros of g_{tt}, where the smaller and larger are called R_{Sch} and R_{dS}, respectively. We assume the Boltzmann brain is stable but not a black hole, so its radius satisfies $R_{Sch} < R < R_{dS}$. The radii of the two horizons are given by

$$R_{Sch} = \frac{2}{\sqrt{3}H_\Lambda} \cos\left(\frac{\pi + \xi}{3}\right),$$
$$R_{dS} = \frac{2}{\sqrt{3}H_\Lambda} \cos\left(\frac{\pi - \xi}{3}\right), \tag{A2}$$

where

$$\cos\xi = 3\sqrt{3}GMH_\Lambda. \tag{A3}$$

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[16]We restore $G = 1/8\pi$ in this appendix for clarity.

This last equation implies that for a given value of H_Λ, there is an upper limit on how much mass can be contained within the de Sitter horizon:

$$M \le M_{\max} = (3\sqrt{3}GH_\Lambda)^{-1}. \tag{A4}$$

Equations (A2) and (A3) can be inverted to express M and H_Λ in terms of the horizon radii:

$$\frac{1}{H_\Lambda^2} = R_{\mathrm{Sch}}^2 + R_{\mathrm{dS}}^2 + R_{\mathrm{Sch}}R_{\mathrm{dS}}, \tag{A5}$$

$$M = \frac{R_{\mathrm{dS}}}{2G}(1 - H_\Lambda^2 R_{\mathrm{dS}}^2) \tag{A6}$$

$$= \frac{R_{\mathrm{Sch}}}{2G}(1 - H_\Lambda^2 R_{\mathrm{Sch}}^2) \tag{A7}$$

We relate the Boltzmann brain nucleation rate to the decrease in total entropy ΔS caused by the nucleation process,

$$\Gamma_{\mathrm{BB}} \sim e^{-\Delta S}, \tag{A8}$$

where the final entropy is the sum of the entropies of the Boltzmann brain and the de Sitter horizon. For a Boltzmann brain with entropy S_{BB}, the change in entropy is given by

$$\Delta S = \frac{\pi}{G}H_\Lambda^{-2} - \left(\frac{\pi}{G}R_{\mathrm{dS}}^2 + S_{\mathrm{BB}}\right). \tag{A9}$$

Note that for small M one can expand ΔS to find

$$\Delta S = \frac{2\pi M}{H_\Lambda} - S_{\mathrm{BB}} + \mathcal{O}(GM^2), \tag{A10}$$

giving a nucleation rate in agreement with Eq. (100).[17]

To find a bound on the nucleation rate, we need an upper bound on the entropy that can be attained for a given size and mass. In flat space the entropy is believed to be bounded by Bekenstein's formula, Eq. (107), a bound which should also be applicable whenever $R \ll R_{\mathrm{dS}}$. More general bounds in de Sitter space have been discussed by Bousso [53], who considers bounds for systems that are allowed to fill the de Sitter space out to the horizon $R = R_{\mathrm{dS}}$ of an observer located at the origin. For small mass M, Bousso argues that the tightest known bound on S is the D bound, which states that

$$S \le S_D \equiv \frac{\pi}{G}\left(\frac{1}{H_\Lambda^2} - R_{\mathrm{dS}}^2\right) = \frac{\pi}{G}(R_{\mathrm{Sch}}^2 + R_{\mathrm{Sch}}R_{\mathrm{dS}}), \tag{A11}$$

where the equality of the two expressions follows from Eq. (A5). This bound can be obtained from the principle that the total entropy cannot increase when an object disappears through the de Sitter horizon. For larger values

of M, the tightest bound (for $R = R_{\mathrm{dS}}$) is the holographic bound, which states that

$$S \le S_{\mathrm{H}} \equiv \frac{\pi}{G}R_{\mathrm{dS}}^2. \tag{A12}$$

Bousso suggests the possibility that these bounds have a common origin, in which case one would expect that there exists a valid bound that interpolates smoothly between the two. Specifically, he points out that the function

$$S_m \equiv \frac{\pi}{G}R_{\mathrm{Sch}}R_{\mathrm{dS}} \tag{A13}$$

is a candidate for such a function. Fig. 1 shows a graph of the holographic bound, the D bound, and the m bound [Eq. (A13)] as a function of $M/M_{\max}$. While there is no reason to assume that S_m is a rigorous bound, it is known to be valid in the extreme cases where it reduces to the D and holographic bounds. In between it might be valid, but in any case it can be expected to be valid up to a correction of order one. In fact, Fig. 1 and the associated equations show that the worst possible violation of the m bound is at the point where the holographic and D bounds cross, at $M/M_{\max} = 3\sqrt{6}/8 = 0.9186$, where the entropy can be no more than $(1 + \sqrt{5})/2 = 1.6180$ times as large as S_m.

Here we wish to carry the notion of interpolation one step further, because we would like to discuss in the same formalism systems for which $R \ll R_{\mathrm{dS}}$, where the Bekenstein bound should apply. Hence we will explore the consequences of the bound

$$S \le S_I \equiv \frac{\pi}{G}R_{\mathrm{Sch}}R, \tag{A14}$$

which we will call the interpolating bound. This bound agrees exactly with the m bound when the object is allowed
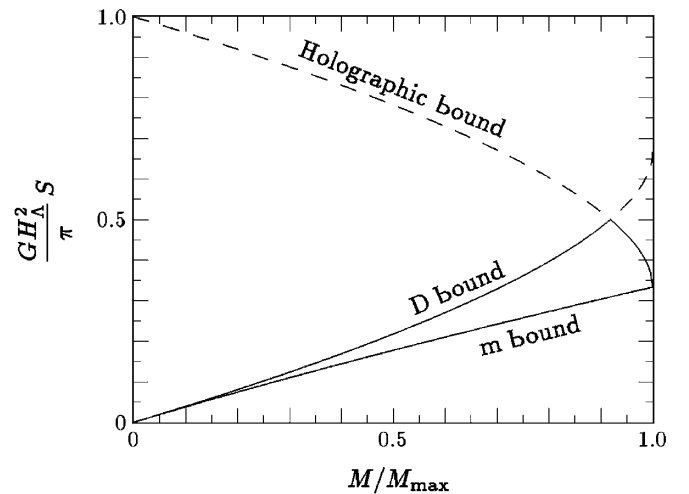

FIG. 1. The graph shows the holographic bound, the D bound, and the m bound for the entropy of an object that fills de Sitter space out to the horizon. The holographic and D bounds are each shown as broken lines in the region where they are superseded by the other. Although the m bound looks very much like a straight line, it is not.

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[17]We thank Lenny Susskind for explaining this method to us.

to fill de Sitter space, with $R = R_{dS}$. Again we have no grounds to assume that the bound is rigorously true, but we do know that it is true in the three limiting cases where it reduces to the Bekenstein bound, the D bound, and the holographic bound. The limiting cases are generally the most interesting for us in any case, since we wish to explore the limiting cases for Boltzmann brain nucleation. For parameters in between the limiting cases, it again seems reasonable to assume that the bound is at least a valid estimate, presumably accurate up to a factor of order one. We know of no rigorous entropy bounds for de Sitter space with R comparable to R_{dS} but not equal to it, so we do not see any way at this time to do better than the interpolating bound.

Proceeding with the I bound of Eq. (A14), we can use Eq. (106) to rewrite Eq. (A9) as

$$\Delta S = \frac{\pi}{G}(H_\Lambda^{-2} - R_{dS}^2) - S_{\text{BB,max}} + I_{\text{BB}}, \qquad (A15)$$

which can be combined with $S_{\text{BB,max}} \leq S_I$ to give

$$\Delta S \geq \frac{\pi}{G}(H_\Lambda^{-2} - R_{dS}^2 - R_{\text{Sch}}R) + I_{\text{BB}}, \qquad (A16)$$

which can then be simplified using Eq. (A5) to give

$$\Delta S \geq \frac{\pi}{G}R_{\text{Sch}}(R_{\text{Sch}} + R_{dS} - R) + I_{\text{BB}}. \qquad (A17)$$

To continue, we have to decide what possibilities to consider for the radius R of the Boltzmann brain, which is related to the question of Boltzmann brain stabilization discussed after Eq. (111). If we assume that stabilization is not a problem, because it can be achieved by a domain wall or by some other particle physics mechanism, then ΔS is minimized by taking R at its maximum value, $R = R_{dS}$, so

$$\Delta S \geq \frac{\pi}{G}R_{\text{Sch}}^2 + I_{\text{BB}}. \qquad (A18)$$

ΔS is then minimized by taking the minimum possible value of R_{Sch}, which is the value that is just large enough to allow the required entropy, $S_{\text{BB,max}} \geq I_{\text{BB}}$. Using again the I bound, one finds that saturation of the bound occurs at

$$\xi_{\text{sat}} = 3\sin^{-1}\left(\frac{\sqrt{1 - 3\tilde{I}}}{2}\right), \qquad (A19)$$

where

$$\tilde{I} \equiv \frac{I_{\text{BB}}}{S_{dS}} = \frac{GH_\Lambda^2}{\pi}I_{\text{BB}} \qquad (A20)$$

is the ratio of the Boltzmann brain information to the entropy of the unperturbed de Sitter space. Note that $\tilde{I}$ varies from zero to a maximum value of $1/3$, which occurs in the limiting case for which $R_{\text{Sch}} = R_{dS}$. The saturating value of the mass and the corresponding values of the Schwarzschild radius and de Sitter radius are given by

$$M_{\text{sat}} = \frac{\tilde{I}\sqrt{1 + \tilde{I}}}{2GH_\Lambda}, \qquad (A21)$$

$$R_{\text{Sch,sat}} = \frac{\sqrt{1 + \tilde{I}} - \sqrt{1 - 3\tilde{I}}}{2H_\Lambda}, \qquad (A22)$$

$$R_{dS,\text{sat}} = \frac{\sqrt{1 - 3\tilde{I}} + \sqrt{1 + \tilde{I}}}{2H_\Lambda}. \qquad (A23)$$

Combining these results with Eq. (A18), one has for this case ($R = R_{dS}$) the bound

$$\frac{\Delta S}{I_{\text{BB}}} \geq \frac{1 + \tilde{I} - \sqrt{1 + \tilde{I}}\sqrt{1 - 3\tilde{I}}}{2\tilde{I}}. \qquad (A24)$$

As can be seen in Fig. 2, the bound on $\Delta S/I_{\text{BB}}$ for this case varies from 1, in the limit of vanishing $\tilde{I}$ (or equivalently, the limit $H_\Lambda \to 0$), to 2, in the limit $R_{\text{Sch}} \to R_{dS}$.

The limiting case of $\tilde{I}_{\text{BB}} \to 0$, with a nucleation rate of order $e^{-I_{\text{BB}}}$, has some peculiar features that are worth mentioning. The nucleation rate describes the nucleation of a Boltzmann brain with some particular memory state, so there would be an extra factor of $e^{I_{\text{BB}}}$ in the sum over all memory states. Thus, a single-state nucleation rate of $e^{-I_{\text{BB}}}$ indicates that the total nucleation rate, including all memory states, is not suppressed at all. It may seem strange that the nucleation rate could be unsuppressed, but one must keep in mind that the system will function as a Boltzmann brain only for very special values of the memory state. In the limiting case discussed here, the "Boltzmann brain" takes the form of a minor perturbation of the degrees of freedom associated with the de Sitter entropy $S_{dS} = \pi/(GH_\Lambda^2)$.
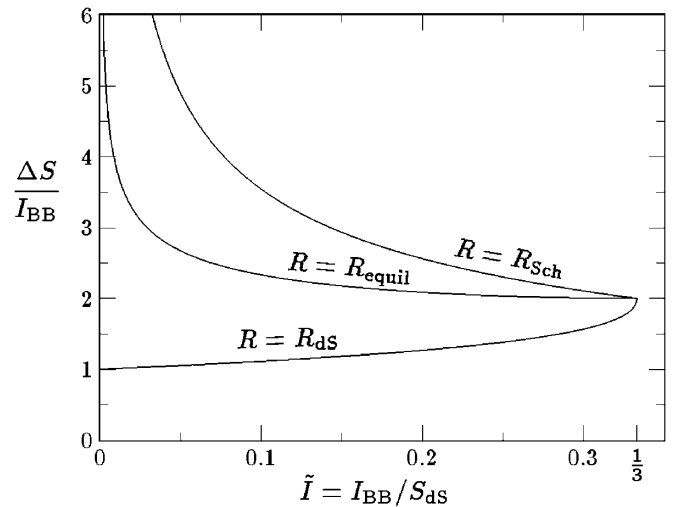

FIG. 2. The graph shows the ratio of ΔS to I_{BB}, where the nucleation rate for Boltzmann brains is proportional to $e^{-\Delta S}$. All curves are based on the I bound, as discussed in the text, but they differ by their assumptions about the size R of the Boltzmann brain.

As a second possibility for the radius R, we can consider the case of strong gravitational binding, $R \to R_{\mathrm{Sch}}$, as discussed following Eq. (111). For this case the bound (A17) becomes

$$\Delta S \geq \frac{\pi}{G} R_{\mathrm{Sch}} R_{\mathrm{dS}} + I_{\mathrm{BB}}. \qquad (A25)$$

[Interestingly, if we take $I = 0$ ($S_{\mathrm{BB}} = S_{\mathrm{max}}$) this formula agrees with the result found in Ref. [92] for black hole nucleation in de Sitter space.] With $R = R_{\mathrm{Sch}}$ the saturation of the I bound occurs at

$$\xi_{\mathrm{sat}} = \frac{\pi}{2} - 3\sin^{-1}\left(\frac{\sqrt{3\tilde{I}}}{2}\right). \qquad (A26)$$

The saturating value of the mass and the corresponding values of the Schwarzschild radius and de Sitter radius are given by

$$M_{\mathrm{sat}} = \frac{\sqrt{\tilde{I}}(1 - \tilde{I})}{2GH_\Lambda}, \qquad (A27)$$

$$R_{\mathrm{Sch,sat}} = \frac{\sqrt{\tilde{I}}}{H_\Lambda}, \qquad (A28)$$

$$R_{\mathrm{dS,sat}} = \frac{\sqrt{4 - 3\tilde{I}} - \sqrt{\tilde{I}}}{2H_\Lambda}. \qquad (A29)$$

Using these relations to evaluate ΔS from Eq. (A25), one finds

$$\frac{\Delta S}{I_{\mathrm{BB}}} = \frac{\sqrt{4 - 3\tilde{I}} + \sqrt{\tilde{I}}}{2\sqrt{\tilde{I}}}, \qquad (A30)$$

which is also plotted in Fig. 2. In this case ($R = R_{\mathrm{Sch}}$) the smallest ratio $\Delta S/I_{\mathrm{BB}}$ is 2, occurring at $\tilde{I} = 1/3$, where $R_{\mathrm{Sch}} = R_{\mathrm{dS}}$. For smaller values of $\tilde{I}$ the ratio becomes larger, blowing up as $1/\sqrt{\tilde{I}}$ for small $\tilde{I}$. Thus, the nucleation rates for this choice of R will be considerably smaller than those for Boltzmann brains with $R \approx R_{\mathrm{dS}}$, but this case would still be relevant in cases where Boltzmann brains with $R \approx R_{\mathrm{dS}}$ cannot be stabilized.

Another interesting case, which we will consider, is to allow the Boltzmann brain to extend to $R = R_{\mathrm{equil}}$, the point of equilibrium between the gravitational attraction of the Boltzmann brain and the outward gravitational pull of the de Sitter expansion. This equilibrium occurs at the stationary point of g_{tt}, which gives

$$R_{\mathrm{equil}} = \left(\frac{GM}{H_\Lambda^2}\right)^{1/3}. \qquad (A31)$$

Boltzmann brains within this radius bound would not be pulled by the de Sitter expansion, so relatively small mechanical forces will be sufficient to hold them together.

Again ΔS will be minimized when the I bound is saturated, which in this case occurs when

$$\xi_{\mathrm{sat}} = \frac{\pi}{2} - 3\sin^{-1}\left[\frac{\sqrt{1 - 2A(\tilde{I})}}{2}\right], \qquad (A32)$$

where

$$A(\tilde{I}) \equiv \sin\left[\frac{\sin^{-1}(1 - 27\tilde{I}^3)}{3}\right]. \qquad (A33)$$

The saturating value of the mass and the Schwarzschild and de Sitter radii are given by

$$M_{\mathrm{sat}} = \frac{\sqrt{3}[1 + A(\tilde{I})]\sqrt{1 - 2A(\tilde{I})}}{9GH_\Lambda}, \qquad (A34)$$

$$R_{\mathrm{Sch,sat}} = \frac{\sqrt{1 - 2A(\tilde{I})}}{\sqrt{3}H_\Lambda}, \qquad (A35)$$

$$R_{\mathrm{dS,sat}} = \frac{\sqrt{3}[\sqrt{3(3 + 2A(\tilde{I}))} - \sqrt{1 - 2A(\tilde{I})}]}{6H_\Lambda}. \qquad (A36)$$

The equilibrium radius itself is given by

$$R_{\mathrm{equil,sat}} = \frac{[1 - 2A(\tilde{I})]^{1/6}[1 + A(\tilde{I})]^{1/3}}{\sqrt{3}H_\Lambda}. \qquad (A37)$$

Using these results with Eq. (A17), ΔS is found to be bounded by

$$\frac{\Delta S}{I_{\mathrm{BB}}} = \frac{\sqrt{3(1 - 2A(\tilde{I}))(3 + 2A(\tilde{I}))} - 2A(\tilde{I}) + 1}{6\tilde{I}}, \qquad (A38)$$

which is also plotted in Fig. 2. As one might expect it is intermediate between the two other cases. Like the $R = R_{\mathrm{Sch}}$ case, however, the ratio $\Delta S/I_{\mathrm{BB}}$ blows up for small $\tilde{I}$, in this case behaving as $(2/\tilde{I})^{1/4}$.

In summary, we have found that our study of tunneling in Schwarzschild–de Sitter space confirms the qualitative conclusions that were described in Sec. VA. In particular, we have found that if the entropy bound can be saturated, then the nucleation rate of a Boltzmann brain requiring information content I_{BB} is given approximately by $e^{-aI_{\mathrm{BB}}}$, where a is of order a few, as in Eq. (111). The coefficient a is always greater than 2 for Boltzmann brains that are small enough to be gravitationally bound. This conclusion applies whether one insists that they be near-black holes, or whether one merely requires that they be small enough so that their self-gravity overcomes the de Sitter expansion. If, however, one considers Boltzmann brains whose radius is allowed to extend to the de Sitter horizon, then Fig. 2 shows that a can come arbitrarily close to 1. However, one must remember that the $R = R_{\mathrm{dS}}$ curve on Fig. 2 can be reached only if several barriers can be overcome. First, these objects are large and diffuse, becoming more and more diffuse as $\tilde{I}$ approaches zero and a approaches 1. There is no known way to saturate the entropy bound for such diffuse systems, and Eq. (117) shows that an ideal gas

model leads to $a \sim I_{BB}^{1/3} \gg 1$. Furthermore, Boltzmann brains of this size can function only if some particle physics mechanism is available to stabilize them against the de Sitter expansion. A domain wall provides a simple example of such a mechanism, but Eq. (114) indicates that the domain wall solution is an option only if a domain wall

exists with tension $\sigma \sim I_{BB} H_\Lambda^3$. Thus, it is not clear how close a can come to its limiting value of 1. Finally, we should keep in mind that it is not clear if any of the examples discussed in this appendix can actually be attained, since black holes might be the only objects that saturate the entropy bound for $S \gg 1$.

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