

Boltzmann hierarchy for the cosmic microwave background at second order including photon polarization

M. Beneke and C. Fidler

Institut für Theoretische Teilchenphysik und Kosmologie, RWTH Aachen University, D-52056 Aachen, Germany

(Received 15 March 2010; published 7 September 2010)

Non-Gaussianity and B -mode polarization are particularly interesting features of the cosmic microwave background, as—at least in the standard model of cosmology—their only sources to first order in cosmological perturbation theory are primordial, possibly generated during inflation. If the primordial sources are small, the question arises how large is the non-Gaussianity and B -mode background induced in second order from the initially Gaussian and scalar perturbations. In this paper we derive the Boltzmann hierarchy for the microwave background photon phase-space distributions at second order in cosmological perturbation theory including the complete polarization information, providing the basis for further numerical studies. As an aside we note that the second-order collision term contains new sources of B -mode polarization and that no polarization persists in the tight-coupling limit.

DOI: [10.1103/PhysRevD.82.063509](https://doi.org/10.1103/PhysRevD.82.063509)

PACS numbers: 98.80.-k, 98.70.Vc

I. INTRODUCTION

The anisotropies of the cosmic microwave background provide an abundant source of information, unrivaled in precision, on the early history of the Universe. Since most of the photons originate from the time of decoupling, when the inhomogeneities of the Universe were small, the anisotropies should be well described in linear perturbation theory around the Friedmann-Robertson-Walker background. Temperature anisotropies [1,2] and E -mode polarization anisotropies [1,3,4] have been detected, and are found in agreement with the standard model, in which the source of the anisotropies consists of a Gaussian, adiabatic, and nearly scale-invariant spectrum of primordial density perturbations.

The polarization pattern of the background radiation is of great interest for the following reason. In contrast to the E mode, B -mode polarization is not sourced by scalar density perturbations in the linear order. Thus, a detection of B -mode polarization would point directly to primordial vector or, more likely, tensor fluctuations (gravitational waves) in a very early phase of the cosmological evolution. So far, however, B -mode polarization has not been observed, which together with the shape of the temperature perturbation spectrum indicates some suppression of the tensor relative to scalar perturbations. Similarly, deviations from Gaussian statistics constrain inflation models and are therefore intensively investigated (see, e.g., the reviews [5,6]).

The absence of non-Gaussianity and B -mode polarization when the primordial fluctuations are purely Gaussian and scalar holds, however, only in linear perturbation theory. Thus, if a small non-Gaussian or B -mode signal is observed, the question arises whether its origin is truly primordial, or whether it might be a second-order effect. While such an effect would naturally be expected at tensor-to-scalar ratios of order 10^{-5} (or $f_{\text{NL}} \sim 1$ for non-

Gaussianity), which is the size of perturbations in the microwave background, only a full second-order calculation can tell whether there are no enhancements. Such enhancements may reach a level relevant to observations, since the planned CMBPol experiment is sensitive to tensor-to-scalar ratios of order 10^{-3} [7]. Indeed, several second-order sources of B -mode polarization are already known. The most important is the weak-lensing effect, reviewed in [8], which converts E polarization to B polarization as the photons travel through the inhomogeneous Universe [9]. The inhomogeneities and the E -mode polarization are both at least of first order, so the resulting effect is at least of second order in perturbation theory. Weak lensing becomes large at small scales, and at large values of the perturbation wave-vector k perturbation series breaks down. The usual treatment of weak lensing therefore avoids cosmic perturbation theory by considering the small deflection angles of the photon trajectories. Another effect that has been estimated is B -mode polarization from gravitational time delay [10] and from sources proportional to second-order vector and tensor metric perturbations, which are themselves generated from the product of scalar perturbations [11]. A full treatment of B -mode polarization at second order should, however, be based on the Boltzmann equation solution for the radiation phase-space density matrix. Previous second-order calculations considered the collision term [12] and radiation transfer function [13] for unpolarized radiation only. In this paper, we derive the complete Boltzmann hierarchy at second order under the assumption that there are no first-order vector and tensor perturbations by extending the results of [12,13] to the photon polarization density matrix. This allows us to identify all sources of B -mode polarization at this order. The polarized equations are presented in a form suitable for numerical evaluation. Numerical results for B polarization will be presented in a follow-up paper [14]. Numerical

results on non-Gaussianity based on second-order Boltzmann equations have appeared recently in [15,16].

Most of the results of this paper have been obtained in the thesis work [17]. In the meantime, the polarized second-order Boltzmann equations have been derived independently in [18]. Our result is derived in a different formalism, allowing for an independent check of the results. We provide expressions for the Boltzmann hierarchy pertaining to the phase-space distribution functions not integrated over frequency, which have not been given explicitly before. In addition, we include a self-contained derivation of the polarized collision term from the quantum-mechanical time evolution of the photon density matrix, which differs from the collision term used in [18]. A detailed comparison is provided in the main body of the paper.

The outline of the paper is as follows. In the remainder of this section we set up our index conventions. Next, in Sec. II we derive the Boltzmann equation from the quantum-mechanical time evolution of the photon density matrix and discuss the differences to the existing literature. Its expansion to second order is presented in Secs. III and IV, first for the propagation of polarized radiation (the left-hand side of the Boltzmann equation), then for the collision term (the right-hand side). These sections are rather technical, and further technical details are collected in the Appendix. The main result, the Boltzmann hierarchy for the second-order intensity and photon polarization phase-space distributions, is summarized in the separate Sec. V. A full analysis of these equations is beyond the scope of this paper. However, in Sec. VI we discuss the sources for B -mode polarization, including a new source in the collision term that converts intensity directly into B -mode polarization. We also analyze the tight-coupling regime.

While we reproduce the presence of a second-order intensity quadrupole already found from the unpolarized equations [19], we do not confirm the effect discussed in [20], which is based on B -mode generation from an E -mode and intensity quadrupole in tight-coupling. Section VII summarizes our conclusions.

A. Index and metric conventions

General coordinate indices will be denoted by Greek letters μ, ν, \dots ranging from 0 to 3, indices referring to tensors in the local inertial frame (tetrad frame) by capital Latin letters $A, B, \dots = 0, 1, 2, 3$ from the beginning of the alphabet. Spatial indices, ranging from 1 to 3, in the tetrad frame are assigned Latin letters i, j, \dots . We also need small Latin letters $a, b, \dots = 1, 2$ to denote the basis of polarization vectors. The signature convention for the space-time metric is $(+, -, -, -)$. Spatial indices in the tetrad frame are contracted with the three-dimensional Euclidean metric and no distinction is made between upper and lower spatial indices. With this convention $v^A w_A = v^0 w^0 - v^i w^i = v^0 w^0 - v^i w_i$, etc. In this paper we assume that the background universe is flat. We then also use Latin letters i, j, \dots to denote the spatial general coordinate indices with the same convention regarding their contraction. In general, it will be clear from the context whether i, j, \dots refers to the tetrad or general coordinate system. Since confusion might arise for the momentum, we denote the covariant momentum $dx^\mu/d\lambda$ by capital P^μ , related to the momentum p^A in the local inertial frame by $P^\mu = [e_A]^\mu p^A$.

The perturbed flat-space Robertson-Walker metric with conformal time denoted by η and coordinates $x^\mu = (\eta, x^i)$ is parametrized as

$$ds^2 = a^2 \left((1 + 2A) d\eta^2 + 2B_i d\eta dx^i - [(1 + 2D)\delta_{ij} + 2E_{ij}] dx^i dx^j \right), \quad (1)$$

where $a(\eta)$ denotes the scale factor. The space-time dependent perturbations $X = A, D, B_i, E_{ij}$ will be expanded into first-order, second-order, etc. terms according to $X = X^{(1)} + X^{(2)} + \dots$. We assume that the vector and tensor perturbations contained in B_i, E_{ij} are smaller than the scalar perturbations, so we formally treat them as second order.

We choose the conformal Newtonian gauge such that B_i is a transverse vector and E_{ij} a transverse, traceless tensor. In this case, when there are only scalar perturbations at first order, we have $B_i^{(1)} = E_{ij}^{(1)} = 0$. Then the tetrad components $[e_A]^\mu$ are given to second order by

$$\begin{aligned} [e_0]^0 &= \frac{1}{a} \left(1 - A^{(1)} - A^{(2)} + \frac{3}{2} A^{(1)2} + \frac{1}{2} U_i^{(1)} U_i^{(1)} \right), \\ [e_0]^i &= \frac{U_i}{a}, \\ [e_k]^0 &= \frac{1}{a} \left(U_k^{(1)} + U_k^{(2)} - B_k^{(2)} + (D^{(1)} - A^{(1)}) U_k^{(1)} \right), \\ [e_k]^i &= \frac{1}{a} \left(\delta_{ik} \left(1 - D^{(1)} - D^{(2)} + \frac{3}{2} D^{(1)2} \right) - E_{ik}^{(2)} - \frac{1}{2} U_i^{(1)} U_k^{(1)} \right). \end{aligned} \quad (2)$$

We denote by $[e^A]_\mu$ the inverse of the tetrad, such that $[e^A]_\mu[e_B]^\mu = \delta_B^A$ and $[e^A]_\mu[e_A]^\nu = \delta_\mu^\nu$. The local inertial frame can be parametrized in terms of the observer three-velocity U_i and a set of angles θ_k , which define the orientation of the local inertial coordinate axes relative to those of x^i . Above we have aligned the local coordinate axes with those of the general coordinate system to set the angles $\theta_k = 0$. In the following we also choose the observer rest frame $U_i = 0$, which coincides to first order with another common frame choice $U_i = B_i$.

II. BOLTZMANN EQUATION FOR THE POLARIZATION DENSITY MATRIX

In this section we briefly review notation and definitions applying to photon polarization. We then derive an expression for the propagation and collision term in the Boltzmann equation for the polarization density matrix, which serves as the starting point for the expansion to second order in perturbations.

A. Photon polarization phase-space distribution

We assume that the polarized radiation ensemble can be described by a single-particle phase-space distribution matrix $\hat{f}_{\mu\nu}(x^\lambda, q^i)$, such that $\hat{\epsilon}^\mu \hat{\epsilon}^{\nu*} \hat{f}_{\mu\nu}(x^\lambda, q^i) d^3 p / (2\pi)^3$ denotes the number density of photons with momentum \mathbf{p} and polarization $\hat{\epsilon}^\mu$. We regard $\hat{f}_{\mu\nu}$ as a function of the comoving momentum $q^i = ap^i$. The unperturbed Bose-Einstein distribution $\hat{f}_{\mu\nu}^{(0)}$ is then independent of conformal time in the expanding homogeneous Universe.

The phase-space distribution is a Hermitian matrix, related to the expectation value $\langle A_\mu(x) A_\nu(y) \rangle$ of the radiation field. We adopt Lorenz gauge $A^\mu{}_{;\mu} = 0$ for the photon field. It then follows that

$$P^\mu \hat{f}_{\mu\nu}(x^\lambda, q^i) = P^\nu \hat{f}_{\mu\nu}(x^\lambda, q^i) = 0, \quad (3)$$

and that $\hat{f}_{\mu\nu}$ is parallel-transported in the absence of collisions. Thus

$$\frac{\mathcal{D}}{\mathcal{D}\lambda} \hat{f}_{\mu\nu} = \hat{C}_{\mu\nu}[\hat{f}]. \quad (4)$$

Here $\mathcal{D}/\mathcal{D}\lambda$ denotes the covariant derivative along a photon trajectory $x^\mu(\lambda)$, and $\hat{C}_{\mu\nu}[\hat{f}]$ is the collision term. The phase-space distribution in the local inertial frame is related to $\hat{f}_{\mu\nu}$ by

$$\hat{f}^{\mu\nu} = [e_A]^\mu [e_B]^\nu \hat{f}^{AB}, \quad (5)$$

where $[e_A]^\mu$ are the space-time dependent tetrad vectors.

The phase-space distribution matrix $\hat{f}_{\mu\nu}$ is not unique, since Lorenz gauge allows the gauge transformations $\hat{f}_{\mu\nu} \rightarrow \hat{f}_{\mu\nu} + \alpha_\mu P_\nu + \beta_\nu P_\mu$ with arbitrary α_μ, β_ν . To obtain a physical distribution function, we decompose the photon four-momentum into

$$P^\mu = E[e_0]^\mu - [e_i]^\mu p^i = E(u^\mu - n^\mu), \quad (6)$$

where $u^\mu = [e_0]^\mu$ is the four-velocity of the locally inertial observer, E the energy of the photon as seen by this observer, and

$$n^\mu = [e_i]^\mu \frac{p^i}{E} \quad (7)$$

the photon three-momentum direction, which satisfies $u_\mu n^\mu = 0$ and $n_\mu n^\mu = -1$. We define

$$p_{\mu\nu} = -g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu, \quad (8)$$

which projects on the components transverse to the observer velocity and photon direction:

$$u^\mu p_{\mu\nu} = u^\nu p_{\mu\nu} = n^\mu p_{\mu\nu} = n^\nu p_{\mu\nu} = 0. \quad (9)$$

We now define the physical phase-space distribution matrix [21]

$$f_{\mu\nu} = p_\mu{}^{\mu'} p_\nu{}^{\nu'} \hat{f}_{\mu'\nu'}, \quad (10)$$

which is orthogonal to P^μ , u^μ , and n^μ , and contains no residual gauge ambiguity. The corresponding projected distribution function in the observer rest frame (local inertial frame) is effectively a three-by-three matrix, since $f^{00} = f^{0i} = f^{i0} = 0$ in this frame. It is orthogonal to the photon propagation direction n^i as expected for the tensor describing the two physical transverse polarizations.

Applying $\mathcal{D}/\mathcal{D}\lambda$ to (10), we obtain from (4)

$$p_\mu{}^{\mu'} p_\nu{}^{\nu'} \frac{\mathcal{D}}{\mathcal{D}\lambda} f_{\mu'\nu'} = p_\mu{}^{\mu'} p_\nu{}^{\nu'} \hat{C}_{\mu'\nu'}[\hat{f}], \quad (11)$$

where we use that $p_{\mu\nu}$ and $\hat{f}_{\mu\nu}$ are orthogonal to P^μ . This provides a closed set of equations for the physical phase-space distribution matrix provided the projected collision term $C_{\mu\nu}[\hat{f}] \equiv p_\mu{}^{\mu'} p_\nu{}^{\nu'} \hat{C}_{\mu'\nu'}[\hat{f}]$ on the right-hand side depends only on f rather than \hat{f} . It will be convenient to choose a polarization basis consisting of two vectors ϵ_a^μ ($a = 1, 2$) orthogonal to P^μ and u^μ , in terms of which

$$f^{\mu\nu} = \sum_{a,b} f_{ab} \epsilon_a^\mu \epsilon_b^\nu. \quad (12)$$

B. Propagation of polarized photons

In the following we shall consider the Boltzmann equation for f_{ab} in a polarization basis defined in the observer rest frame. To this end, we multiply (11) by $\epsilon_a^\mu \epsilon_b^{\nu*}$, insert (12), and calculate

$$\epsilon_a^\mu \epsilon_b^{\nu*} p_\mu{}^{\mu'} p_\nu{}^{\nu'} \frac{\mathcal{D}}{\mathcal{D}\lambda} \sum_{c,d} f_{cd} \epsilon_c^{\mu'} \epsilon_d^{\nu'}. \quad (13)$$

Before giving the result, we need to discuss the choice of polarization basis. If $\hat{\epsilon}^\mu$ denotes the direction of the photon field-amplitude in Lorenz gauge, then $\hat{\epsilon}^\mu$ is parallel-transported along the photon path [22]. This also holds

for the projected vector $\epsilon^\mu = p^\mu{}_{\nu'} \hat{\epsilon}^{\nu'}$ in the sense that for the polarization vector corresponding to a fixed photon momentum $p^\mu{}_{\nu'} \mathcal{D}\epsilon^\nu / \mathcal{D}\lambda = 0$. We could therefore use a basis of two parallel-transported polarization vectors ϵ_a^μ . In this case, the corresponding vectors ϵ_a^A in the observer rest frame, given by $\epsilon_a^\mu = [e_A]^\mu \epsilon_a^A$, depend on the space-time point x in addition to momentum q . The transverse polarization basis axes are continuously rotated as the photon propagates through space-time.

It will be convenient to instead choose rigid basis vectors ϵ_a^A in the observer rest frame that do not depend on space-time but only on momentum. In this case the covariant derivative along the path acting on a polarization vector is

$$\epsilon_{a\mu}^* \frac{\mathcal{D}\epsilon_c^\mu}{\mathcal{D}\lambda} = [e^B]_\mu [e_A]^\mu{}_{;\nu} P^\nu \epsilon_{aB} \epsilon_c^A + \epsilon_{aA} \frac{dq^i}{d\lambda} \frac{\partial \epsilon_c^A}{\partial q^i}, \quad (14)$$

where the first term on the right-hand side arises, since ϵ_c^μ is not parallel-transported in the rigid basis, and the second

is due to the fact that the photon momentum also changes along the path. The conditions $u_\mu \epsilon_a^\mu = P_\mu \epsilon_a^\mu = 0$ require $\epsilon_a^0 = 0$ and $\mathbf{n} \cdot \boldsymbol{\epsilon}_a = 0$ in the observer rest frame. We adopt a spherical coordinate system with

$$\begin{aligned} \mathbf{n} &= (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta) \\ \mathbf{e}_\theta &= (\cos\theta \cos\varphi, \cos\theta \sin\varphi, -\sin\theta) \\ \mathbf{e}_\varphi &= (-\sin\varphi, \cos\varphi, 0) \end{aligned} \quad (15)$$

the standard basis vectors on the sphere. The polarization basis is then taken to consist of the two circular polarization vectors

$$\boldsymbol{\epsilon}_\pm = -\frac{1}{\sqrt{2}}(\mathbf{e}_\theta \pm i\mathbf{e}_\varphi). \quad (16)$$

Evaluating (13) and identifying $x^0 = \eta$ with conformal time results in the Boltzmann equation

$$\begin{aligned} \frac{\partial f_{ab}}{\partial \eta} + \frac{1}{P^0} \frac{dx^i}{d\lambda} \frac{\partial f_{ab}}{\partial x^i} + \frac{1}{P^0} \frac{dq^i}{d\lambda} \left(\frac{\partial f_{ab}}{\partial q^i} + \epsilon_{ak} \frac{\partial \epsilon_c^{k*}}{\partial q^i} f_{cb} + \epsilon_{bk}^* \frac{\partial \epsilon_c^k}{\partial q^i} f_{ac} \right) \\ + [e^i]_\mu [e_k]^\mu{}_{;\nu} \frac{P^\nu}{P^0} (\epsilon_{ai} \epsilon_c^{*k} f_{cb} + \epsilon_{bi}^* \epsilon_c^k f_{ac}) = \frac{1}{P^0} C_{ab}[f], \end{aligned} \quad (17)$$

with $C_{ab}[f] = \epsilon_a^\mu \epsilon_b^{\nu} C_{\mu\nu}[f]$ denoting the (projected) collision term in the polarization basis. Its expression is given in (39) below. The terms to the left of the equality sign in the second line would vanish had we chosen a basis of parallel-transported polarization vectors. The extra terms in the rigid basis are equivalent to similar terms that appear in [23].

For unpolarized radiation $f_{ab} = \delta_{ab} f$, and the terms in (17) that depend on the polarization vectors explicitly vanish. This follows from

$$\epsilon_{ak} \frac{\partial \epsilon_b^{*k}}{\partial q^i} + \epsilon_{bk}^* \frac{\partial \epsilon_a^k}{\partial q^i} = \frac{\partial}{\partial q^i} (\epsilon_{ak} \epsilon_b^{*k}) = 0, \quad (18)$$

since $\epsilon_{ak} \epsilon_b^{*k} = \delta_{ab}$, and from

$$[e^i]_\mu [e_k]^\mu{}_{;\nu} (\epsilon_{ai} \epsilon_b^{*k} + \epsilon_{bi}^* \epsilon_a^k) = \epsilon_{ai} \epsilon_b^{*k} ([e^i]_\mu [e_k]^\mu)_{;\nu} = 0, \quad (19)$$

since $[e^i]_\mu [e_k]^\mu = \delta_k^i$. Thus (17) reduces to the standard equation for unpolarized radiation for diagonal phase-space density matrices, as should be the case.

For the same reason, the explicitly polarization-vector dependent terms are at least of second order in perturbations around the equilibrium distribution in the expanding homogeneous Universe. This is due to the fact that $dq^i/d\lambda$ and $[e^i]_\mu [e_k]^\mu{}_{;\nu}$ are both first order in perturbations. Hence to first order we may set f_{ab} in the polarization-vector dependent terms equal to the unperturbed distribution $f_{ab}^{(0)}$. But the unperturbed distribution is diagonal, so the terms vanish (at first order) as shown above.

C. Simplification at second order

We now show that the terms to the left of the equality sign in the second line of (17) vanish even at second order in perturbations, provided that there are no first-order vector and tensor perturbations. Thus, under these assumptions, there is no difference between the rigid and the parallel-transported polarization basis at second order.

It follows from Sec. II B that the second-order contribution is the product of

$$\epsilon_{ai} \epsilon_c^{*k} f_{cb}^{(1)} + \epsilon_{bi}^* \epsilon_c^k f_{ac}^{(1)} \quad (20)$$

and

$$\begin{aligned} [[e^i]_\mu [e_k]^\mu{}_{;\nu}]^{(1)} \left[\frac{P^\nu}{P^0} \right]^{(0)} &= \frac{1}{2} \left(\partial_i B_k^{(1)} - \partial_k B_i^{(1)} \right) \\ &\quad - n_i \left(\partial_i E_{kl}^{(1)} - \partial_k E_{il}^{(1)} \right) + \dots, \end{aligned} \quad (21)$$

where the superscript in brackets indicates the order at which the expression is evaluated, and the ellipses denote terms proportional to q_i or q_k which vanish when contracted with the polarization vectors in (20) since $\mathbf{q} \propto \mathbf{n}$. To obtain this expression we used the tetrads from (A1), which do not assume a particular gauge choice.

If there are no first-order vector and tensor perturbations, $B_i^{(1)}$ and $E_{ij}^{(1)}$ are zero in conformal Newtonian gauge, and expression (21) immediately vanishes, leading to the desired simplification. More generally, in an arbitrary gauge

$B_i^{(1)}$ can be expressed as the gradient of a scalar function in the absence of vector modes, hence the curl of $B_i^{(1)}$ appearing in (21) is zero. Likewise, $E_{ij}^{(1)} = (\partial_i \partial_j - \delta_{ij} \partial^2) E^{(1)}$ for some function $E^{(1)}$ in the absence of first-order vector and tensor modes. Then,

$$n_l \left(\partial_i E_{kl}^{(1)} - \partial_k E_{il}^{(1)} \right) = (n_i \partial_k - n_k \partial_i) \partial^2 E^{(1)}, \quad (22)$$

which vanishes when contracted with the polarization vectors in (20), since $n_i \epsilon_{ai} = 0$.

$$A(\xi) = \sum_{a=\pm} \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2p^0} \left(e^{-ip \cdot \xi} a_a(p) \epsilon_a(p) + e^{ip \cdot \xi} a_a^\dagger(p) \epsilon_a^*(p) \right). \quad (23)$$

We choose the two circular polarization vectors ϵ_\pm as basis vectors. The creation and annihilation operators satisfy the standard commutation relation

$$\begin{aligned} [a_a(p), a_b^\dagger(p')] &= \delta_{ab} (2\pi)^3 2p^0 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \\ &\equiv \delta_{ab} \delta(\mathbf{p} - \mathbf{p}'). \end{aligned} \quad (24)$$

The one-particle density matrix is defined by the expectation value $\langle a_b^\dagger(p) a_a(p') \rangle$. Spatial homogeneity implies that

$$\langle a_b^\dagger(p) a_a(p') \rangle = \delta(p - p') \rho_{ab}(t, p). \quad (25)$$

We identify $\rho_{ab}(t, p)$ with the phase-space distribution function $f_{ba}(x^\lambda, q^i = ap^i)$. Indeed, since the number operator is

$$\hat{N} = \sum_{a=\pm} \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2p^0} a_a^\dagger(p) a_a(p), \quad (26)$$

we obtain from (25)

$$2p^0 (2\pi)^3 \delta^{(3)}(0) \frac{d}{dt} \rho_{ab}(t, p) = i \langle [H_I(t), D_{ab}(t, p)] \rangle - \int_0^t dt' \langle [H_I(t-t'), [H_I(t'), D_{ab}(t, p)]] \rangle. \quad (29)$$

If H_I were the electron-photon interaction of quantum electrodynamics H_{QED} , we would have to expand to the fourth order in the interaction to recover the Compton scattering collision term. Instead we derive an effective Compton scattering interaction vertex assuming that the electron propagates freely between the two elementary electron-photon interactions in the Compton process. Thus we define $H_I(t)$ through the relation

$$(-i) \int dt H_I(t) = \frac{(-i)^2}{2} \int d^4 x d^4 y T(H_{\text{QED}}(x) H_{\text{QED}}(y)), \quad (30)$$

with the understanding that a pair of electron fields is contracted in the expression on the right-hand side. After a short calculation we obtain [25]

D. Collision term

To obtain the collision term $C_{ab}[f]$ we consider the quantum time evolution of the one-particle density matrix following the formalism developed in [24] for neutrino flavor-mixing in a medium. The formalism was applied to photon polarization and Thomson scattering in [25]. A more general treatment elucidating some of the approximations involved in the truncation of the hierarchy of n -particle density matrices implicit in this formalism can be found in [26].

In the local inertial frame with coordinates ξ the photon field operator is expanded in the form

$$N = \langle \hat{N} \rangle = V \int d^3 \mathbf{p} \text{tr} \rho(t, p), \quad (27)$$

confirming the interpretation of $\rho_{ab}(t, p)$ as phase-space polarization density matrix. The spatial dependence of $f_{ba}(x^\lambda, q^i)$ can be neglected for the calculation of the collision term, since each scattering event is local on the cosmological scales over which $f_{ba}(x^\lambda, q^i)$ varies. The flip in the order of polarization indices follows from the definitions (12) and (25) and the fact that $\sum_a a_a(p) \epsilon_a(p)$ is independent of the choice of polarization basis.

The time evolution of the density matrix is obtained from the Heisenberg equation for the operator $D_{ab}(p) = a_b^\dagger(p) a_a(p)$. Starting from

$$\frac{d}{dt} D_{ab} = i[H, D_{ab}], \quad (28)$$

going to the interaction picture and splitting the Hamiltonian into the free and interaction part H_I , we obtain to second order in the interaction [24]

$$\begin{aligned}
H_I(t) &= \sum_{a,a',s,s'} \int [dp][dp'] [dq][dq'] (2\pi)^3 \delta^{(3)}(\mathbf{q}' + \mathbf{p}' - \mathbf{q} - \mathbf{p}) e^{it(q'^0 + p'^0 - q^0 - p^0)} \\
&\quad \times M(pa; qs \rightarrow p'a'; q's') \alpha_{s'}^\dagger(q') a_{a'}^\dagger(p') a_a(p) \alpha_s(q).
\end{aligned} \tag{31}$$

Here α , α^\dagger denote electron annihilation and creation operators, and $[dp] = d^3\mathbf{p}/((2\pi)^3 2p^0)$ is the phase-space integration measure. The matrix element for the $\gamma(p, a) + e^-(q, s) \rightarrow \gamma(p', a') + e^-(q', s')$ Compton scattering process reads

$$M(pa; qs \rightarrow p'a'; q's') = e^2 \bar{u}(q', s') \left[\not{\epsilon}_{a'}^*(p') \frac{\not{q}' + \not{p}' + m_e}{(q+p)^2 - m_e^2} \not{\epsilon}_a(p) + \not{\epsilon}_a(p) \frac{\not{q}' - \not{p}' + m_e}{(q-p')^2 - m_e^2} \not{\epsilon}_{a'}^*(p') \right] u(q, s). \tag{32}$$

We note that

$$M(pa; qs \rightarrow p'a'; q's') = M^*(p'a'; q's' \rightarrow pa; qs). \tag{33}$$

To avoid confusion let us also note that in this subsection q stands for an electron momentum and not for the comoving photon momentum.

The first-order term $\langle [H_I(t), D_{ab}(t, p)] \rangle$ in (29) involves the forward Compton scattering matrix element, and it is

straightforward to show that this term vanishes. The second-order term is more complicated. It results in expectation values of four photon annihilation and creation operators, since the interaction generates correlations. To proceed we assume that n -particle correlations can be expressed in terms of one-particle correlations, such that, for example

$$\begin{aligned}
\langle a_{a'}^\dagger(q') a_a(q) a_{b'}^\dagger(p') a_b(p) \rangle &= \delta(q-p') \delta_{ab'} \langle a_{a'}^\dagger(q') a_b(p) \rangle + \langle a_{a'}^\dagger(q') a_{b'}^\dagger(p') a_a(q) a_b(p) \rangle \\
&\rightarrow \delta(q-p') \delta_{ab'} \langle a_{a'}^\dagger(q') a_b(p) \rangle + \langle a_{a'}^\dagger(q') a_a(q) \rangle \langle a_{b'}^\dagger(p') a_b(p) \rangle + \langle a_{a'}^\dagger(q') a_b(p) \rangle \langle a_{b'}^\dagger(p') a_a(q) \rangle \\
&= \delta(q-p') \delta(q'-p) \rho_{ba'}(p) [\delta_{ab'} + \rho_{ab'}(q)] + \delta(q-q') \delta(p-p') \rho_{aa'}(q) \rho_{bb'}(p).
\end{aligned} \tag{34}$$

This amounts to the assumption that on average multiparticle correlations built up in a collision decay rapidly in the time interval before the next collision. The corresponding expressions for the electrons are simpler, since we further assume that the electrons are unpolarized and that their phase-space density $g_e(q)$ is sufficiently small for quadratic terms in g_e to be negligible. Thus

$$\langle \alpha_{s'}^\dagger(q') \alpha_s(q) \alpha_{r'}^\dagger(p') \alpha_r(p) \rangle \rightarrow \delta(q-p') \delta_{r's} \langle \alpha_{s'}^\dagger(q') \alpha_r(p) \rangle = \delta(q-p') \delta(q'-p) \delta_{sr'} \delta_{rs'} \frac{1}{2} g_e(q'). \tag{35}$$

Note that $g_e(q)$ is the density summed over both electron polarizations. After working out the expectation value of the second-order term in (29) one ends up with the time integral

$$\int_0^t dt' e^{\pm it'(q'^0 + p'^0 - q^0 - p^0)}. \tag{36}$$

If the interaction time scale is much shorter than the average time between collisions the upper limit may be taken to infinity and supplying the appropriate $i\epsilon$ prescription, we obtain

$$\pm i\text{PV} \frac{1}{q'^0 + p'^0 - q^0 - p^0} + \pi \delta(q'^0 + p'^0 - q^0 - p^0). \tag{37}$$

The imaginary principal-value term should be discarded, since it corresponds to a self-energy contribution. Putting everything together, substituting $\rho_{ba} \rightarrow f_{ab}$ in the last step, we obtain from (29)

$$2p^0 \frac{d}{dt} f_{ab}(p) = 2C_{ab}[f], \tag{38}$$

where the collision term is given by

$$\begin{aligned}
C_{ab}[f] &= \frac{1}{4} \int \frac{d\mathbf{p}'}{(2\pi)^3 2p'^0} \frac{d\mathbf{q}}{(2\pi)^3 2q^0} \frac{d\mathbf{q}'}{(2\pi)^3 2q'^0} (2\pi)^4 \delta^{(4)}(q+p-q'-p') |\bar{M}|_{\lambda\lambda';\omega\omega'}^2 \\
&\quad \times \left\{ g_e(q') f_{\lambda'\omega'}(p') \left[\delta_{a\lambda} (\delta_{\omega b} + f_{\omega b}(p)) + \delta_{\omega b} (\delta_{a\lambda} + f_{a\lambda}(p)) \right] \right. \\
&\quad \left. - g_e(q) \left[\delta_{a\lambda} f_{\omega b}(p) + \delta_{\omega b} f_{a\lambda}(p) \right] (\delta_{\lambda'\omega'} + f_{\lambda'\omega'}(p')) \right\}.
\end{aligned} \tag{39}$$

Here we introduced the electron-spin averaged square of the Compton amplitude

$$|\bar{M}|_{\lambda\lambda';\omega\omega'}^2 = \frac{1}{2} \sum_{s,s'} M(p\lambda; qs \rightarrow p'\lambda'; q's') M^*(p\omega; qs \rightarrow p'\omega'; q's'). \quad (40)$$

The collision term (39) for the polarized phase-space density is the expression that must be used on the right-hand side of the Boltzmann equation (17). It takes an intuitive form with a gain and loss term and the expected Bose enhancement factors. Taking the trace in ab , and averaging the matrix element over polarizations, we recover the standard unpolarized collision term. Equation (39) differs from [25], where it is stated that the terms quadratic in the photon phase-space density cancel exactly in the evaluation of the double commutator in (29). It also differs from the collision term used in [18], which is based on [27]. The differences are located in the structure of the loss term from [27] and the Bose enhancement factors added in [18]. The loss term in [27] is not derived as in the present paper but based on a certain ansatz, which is checked for initial and final pure photon polarization states, and then argued to hold in general due to the superposition principle. However, the loss term ansatz in [27] is nonlinear in the phase-space distribution invalidating the superposition principle, and we suspect that this leads to the discrepancy with our result. Nevertheless, it turns out that the differences do not affect the final result in Sec. V below after the

expansion to second order, at least for the frequency-integrated phase-space distributions considered in [18]. The reason for this is the simple polarization dependence of the Thomson scattering cross section and the fact that the terms quadratic in the photon phase-space densities will be seen to not contribute to the second-order equations for the frequency-integrated distributions. Differences between the present calculation and [18] from the form of the collision term would however be expected at the next order.

E. Fourier transformation and multipole expansion

It is more convenient for the perturbation expansion to work with Fourier-transformed and multipole-expanded functions. We define

$$A(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} A(\mathbf{k}). \quad (41)$$

At second order we encounter products of functions, whose Fourier transform is a convolution. Below we use the shorthand notation

$$A(\mathbf{k}_1)B(\mathbf{k}_2) \equiv \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) A(\mathbf{k}_1)B(\mathbf{k}_2). \quad (42)$$

For the multipole representation we write the comoving momentum as $\mathbf{q} = q\mathbf{n}$ and then define

$$f_{ab}(\boldsymbol{\eta}, \mathbf{k}, \mathbf{q}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (-i)^l \sqrt{\frac{4\pi}{2l+1}} f_{ab,lm}(\boldsymbol{\eta}, \mathbf{k}, q) Y_{lm}^s(\mathbf{n}), \quad (43)$$

$$f_{ab,lm}(\boldsymbol{\eta}, \mathbf{k}, q) = i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega Y_{lm}^{s*}(\mathbf{n}) f_{ab}(\boldsymbol{\eta}, \mathbf{k}, q\mathbf{n}). \quad (44)$$

Here $Y_{lm}^s(\mathbf{n})$ denotes the spin-weighted spherical harmonics. We collect their definition and some basic relations for these functions in Appendix A 2.

We adopt the circular polarization basis (16) such that under a rotation of the coordinate system around the direction of photon propagation with rotation angle $\Delta\Psi$ the polarization basis vectors transform according to

$$\boldsymbol{\epsilon}'_{a=\pm} = e^{\pm i\Delta\Psi} \boldsymbol{\epsilon}_{a=\pm}, \quad (45)$$

i.e. the circular polarization vectors $\boldsymbol{\epsilon}_{\pm}$ have spin $s = \pm 1$ as they should. Since the polarization-basis independent phase-space distribution

$$f^{ij} = \sum_{ab} \boldsymbol{\epsilon}_a^{i*} \boldsymbol{\epsilon}_b^j f_{ab} \quad (46)$$

is invariant under basis rotations, it follows that f_{++} and f_{--} are spin-zero ($s = 0$) objects that do not transform, while

$$f'_{\pm\mp} = e^{\pm 2i\Delta\Psi} f_{\pm\mp}. \quad (47)$$

Thus, f_{+-} has spin 2 and f_{-+} has spin -2 . The corresponding values of s must be used in (43) and (44).

Instead of the phase-space densities of the photon helicity states, one may also parametrize f_{ab} in terms of the four real Stokes parameters. The relation in the circular basis is

$$f_{ab} = \begin{pmatrix} f_{++} & f_{+-} \\ f_{-+} & f_{--} \end{pmatrix} = \begin{pmatrix} f_I - f_V & f_Q - if_U \\ f_Q + if_U & f_I + f_V \end{pmatrix}. \quad (48)$$

The multipole decomposition for the Stokes parameter distribution functions f_X reads

$$\begin{aligned}
f_{I,lm} &= i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega Y_{lm}^*(\mathbf{n}) f_I(\mathbf{n}), \\
f_{V,lm} &= i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega Y_{lm}^*(\mathbf{n}) f_V(\mathbf{n}), \\
f_{E,lm} \pm i f_{B,lm} &= i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega Y_{lm}^{\mp 2*}(\mathbf{n}) [f_Q(\mathbf{n}) \pm i f_U(\mathbf{n})].
\end{aligned} \tag{49}$$

The quantity f_I provides the photon density averaged over the two helicity states, and f_V is related to the degree of circular polarization of the radiation plasma. We shall include f_V in the set of second-order equations, but since there are no sources of circular polarization in the standard cosmological scenario, it is usually of little interest. Our main concern are the off-diagonal components of the photon phase-space density, which are decomposed in (49) into the E and B polarization modes. The conversion between the two sets of phase-space distributions follows from

$$f_{X,lm} = U_{X:[ab]} f_{ab,lm}. \tag{50}$$

Interpreting $[ab]$ as a single index taking the values $++$,

$$\begin{aligned}
\frac{\partial f_{ab}}{\partial q^i} + \epsilon_{ak} \frac{\partial \epsilon_c^{k*}}{\partial q^i} f_{cb} + \epsilon_{bk}^* \frac{\partial \epsilon_c^k}{\partial q^i} f_{ac} &= \frac{\partial f_{ab}}{\partial q^i} + \frac{2ie_{\varphi i}}{q \tan\theta} \begin{pmatrix} 0 & f_{+-} \\ -f_{-+} & 0 \end{pmatrix}_{ab} \\
&= \sum_{l,m} (-i)^l \sqrt{\frac{4\pi}{2l+1}} \left\{ Y_{lm}^s \frac{\partial f_{ab,lm}}{\partial q} n^i + \frac{1}{\sqrt{2}} \frac{f_{ab,lm}}{q} (\epsilon_-^i \bar{\delta}_s Y_{lm}^s + \epsilon_+^i \bar{\delta}_s Y_{lm}^s) \right\}.
\end{aligned} \tag{54}$$

This form makes explicit that each term carries definite spin, such that $s = 0$ for the diagonal elements and $s = \pm 2$ for the off-diagonals. The derivatives on the spin-weighted spherical harmonics can be easily taken using (A10).

III. EXPANSION OF THE PHOTON PROPAGATION TERM TO SECOND ORDER

We now turn to the expansion of the Boltzmann equation (17) to second order in perturbations. Implementing the simplification of the polarization-dependent terms derived in Sec. II C, we obtain the first- and second-order equations

$$\left[\frac{\partial}{\partial \eta} + \frac{q^i}{aE} \frac{\partial}{\partial x^i} \right] f_{ab}^{(1)} + \left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(1)} \frac{q^i}{q} \frac{\partial f_{ab}^{(0)}}{\partial q} = \left[\frac{1}{P^0} C_{ab}[f] \right]^{(1)}, \tag{55}$$

$$\left[\frac{\partial}{\partial \eta} + \frac{q^i}{aE} \frac{\partial}{\partial x^i} \right] f_{ab}^{(2)} + \left[\frac{P^i}{P^0} \right]^{(1)} \frac{\partial f_{ab}^{(1)}}{\partial x^i} + \left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(1)} \left(\frac{\partial f_{ab}^{(1)}}{\partial q^i} + \epsilon_{ak} \frac{\partial \epsilon_c^{k*}}{\partial q^i} f_{cb}^{(1)} + \epsilon_{bk}^* \frac{\partial \epsilon_c^k}{\partial q^i} f_{ac}^{(1)} \right) + \left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(2)} \frac{q^i}{q} \frac{\partial f_{ab}^{(0)}}{\partial q} = \left[\frac{1}{P^0} C_{ab}[f] \right]^{(2)}. \tag{56}$$

Here we used that

$$\frac{1}{P^0} \frac{dx^i}{d\lambda} = \frac{P^i}{P^0} = \frac{q^i}{aE} \tag{57}$$

at zeroth order in the perturbation expansion. In this section we keep the energy and momentum distinct so that the

---, +- , -+ in this order, and with $X = I, V, E, B$, the matrix $U_{X:[ab]}$ and its inverse read

$$\begin{aligned}
U_{X:[ab]} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2i} & \frac{1}{2i} \end{pmatrix}, \\
U_{[ab];X}^{-1} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{pmatrix}.
\end{aligned} \tag{51}$$

We note the relations

$$U_{[ab];X}^{-1} = U_{[ba];X}^{-1*} = 2U_{X:[ab]}^*. \tag{52}$$

In terms of multipoles the momentum derivative terms in the first line of the Boltzmann equation (17) can be written in a simple form. First, from (16) we calculate

$$\epsilon_{bk}^* \frac{\partial \epsilon_c^k}{\partial q^i} = \mp \frac{i}{q \tan\theta} e_{\varphi i} \delta_{bc}, \tag{53}$$

where the upper (lower) sign holds for $b = c = +$ ($b = c = -$). Then, making use of (A9), we obtain

results also apply to the propagation of massive particles. For photons we may use $E = |\mathbf{p}| = |\mathbf{q}|/a = q/a$ and $q^i/q = n^i$ to simplify the equations. Equation (55) reproduces the Boltzmann equation in the linear approximation with the familiar free-streaming term on the left-hand side. The Fourier transformation converts $\partial/\partial x^i \rightarrow ik^i$ in the

free-streaming terms. However, the second-order equation also contains products of two Fourier-transformed functions, which are to be interpreted as convolutions according to (42). Thus, for instance,

$$\begin{aligned} \left[\frac{P^i}{P^0} \right]^{(1)} \frac{\partial f_{ab}^{(1)}}{\partial x^i} &\rightarrow \left[\frac{P^i}{P^0} \right]^{(1)} (\mathbf{k}_1) i k_2^j f_{ab}^{(1)}(\mathbf{k}_2) \\ &= \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} \left[\frac{P^i}{P^0} \right]^{(1)} (\mathbf{k} - \mathbf{k}') i k'^j f_{ab}^{(1)}(\mathbf{k}'), \end{aligned} \quad (58)$$

in the Fourier transform of (56). In this section we work out the multipole transformation of the left-hand side of (56). The more complicated transformation of the collision term is derived in Sec. IV.

A. Covariant momentum and momentum derivative

The expression of the covariant momentum in terms of the comoving momentum required to evaluate (56) is

$$\frac{dp^i}{d\lambda} = \frac{d([e^i]_{\mu} P^{\mu})}{d\lambda} = \frac{\partial [e^i]_{\mu}}{\partial x^{\nu}} P^{\nu} P^{\mu} + [e^i]_{\mu} (-\Gamma_{\nu\rho}^{\mu} P^{\nu} P^{\rho}) = [e^i]_{\mu;\nu} P^{\mu} P^{\nu}. \quad (62)$$

Then

$$\frac{1}{P^0} \frac{dq^i}{d\lambda} = \frac{dq^i}{d\eta} = \frac{da}{d\eta} p^i + \frac{a}{P^0} \frac{dp^i}{d\lambda} = H_c q^i + a [e^i]_{\mu;\nu} \frac{P^{\mu} P^{\nu}}{P^0}, \quad (63)$$

where $H_c = a^{-1} da/d\eta$ denotes the conformal Hubble parameter. The previous expression vanishes at zeroth order in the perturbations. Its perturbation expansion can be calculated from (59) and (60) and the explicit expressions for the inverse tetrad vectors. The first- and second-order terms required for (56) read

$$\left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(1)} = -aE \partial^i A^{(1)} - q^i \dot{D}^{(1)} + \frac{q^j q^k}{aE} \left(\delta_{jk} \partial^i D^{(1)} - \delta_{ij} \partial^k D^{(1)} \right) \quad (64)$$

$$\begin{aligned} \frac{q^i}{q} \left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(2)} &= -\frac{aE}{q} q^i \partial^i A^{(2)} - q \dot{D}^{(2)} - \frac{q^i q^j}{q} \dot{E}_{ij}^{(2)} + \frac{aE}{q} q^i \dot{B}_i^{(2)} + \frac{(a^2 E^2 - q^2)}{qaE} q^i H_c B_i^{(2)} \\ &\quad + \frac{aE}{q} q^i \partial^i A^{(1)} (A^{(1)} + D^{(1)}) + 2q D^{(1)} \dot{D}^{(1)}. \end{aligned} \quad (65)$$

The dot denotes a derivative with respect to conformal time and $\partial^i = \partial/\partial x^i$. The term proportional to H_c in the first line of (65) vanishes for photons and massless propagating particles in general.

B. Multipole transformation and spherical basis

The general procedure to obtain the multipole decomposition of (55) and (56) is as follows. First we insert the representation (43) for the phase-space distributions. Then the direction vector \mathbf{n} and polarization vectors are written in terms of spherical harmonics according to

$$n^i = \sum_m \xi_m^i \sqrt{\frac{4\pi}{3}} Y_{1m}, \quad n^i n^j = \chi_0^{ij} \sqrt{4\pi} Y_{00} + \sum_m \chi_{2m}^{ij} \sqrt{\frac{4\pi}{5}} Y_{2m}, \quad \epsilon_+^i = \sum_m \xi_m^i \sqrt{\frac{4\pi}{3}} Y_{1m}^{+1}, \quad \epsilon_-^i = -\sum_m \xi_m^i \sqrt{\frac{4\pi}{3}} Y_{1m}^{-1}, \quad (66)$$

which defines ξ_m^i (for $m = 0, \pm 1$), $\chi_0^{ij} = \frac{1}{3} \delta^{ij}$ and the trace-free tensors χ_{2m}^{ij} (for $m = 0, \pm 1, \pm 2$). Explicit expressions are provided in Appendix A 3. The multiplication of these objects with Cartesian vectors and tensors, respec-

obtained from $P^{\mu} = [e_A]^{\mu} p^A$. Under the assumptions made in this paper (no first-order vector and tensor perturbations, conformal Newtonian gauge, observer frame; see Sec. IA), we find

$$P^0 = \frac{E}{a} \left(1 - A + \frac{3A^2}{2} - \frac{q^i B_i}{aE} \right) + \dots, \quad (59)$$

$$P^i = \frac{q^i}{a^2} \left(1 - D + \frac{3D^2}{2} \right) - \frac{q^k}{a^2} E_{ki} + \dots, \quad (60)$$

where the ellipses denote corrections of the third order in perturbations. Hence,

$$\left[\frac{P^i}{P^0} \right]^{(1)} = \frac{q^i}{aE} (A^{(1)} - D^{(1)}). \quad (61)$$

The change of comoving momentum $dq^i/d\lambda$ along the particle trajectory follows from the geodesic equation. We have

tively, projects on the components of the corresponding vectors and tensors in the spherical basis. For vectors V and traceless symmetric tensors T we define the components in the spherical basis by

$$V_{[0]} = iV_3, \quad V_{[\pm 1]} = \mp \frac{i}{\sqrt{2}}(V_1 \mp iV_2)$$

$$T_{[0]} = -\frac{3}{2}T_{33}, \quad T_{[\pm 1]} = \pm\sqrt{2}(T_{13} \mp iT_{23}) \quad (67)$$

$$T_{[\pm 2]} = -\frac{1}{\sqrt{6}}(T_{11} - T_{22} \mp 2iT_{12}).$$

Then

$$\xi_m^i V_i = (-i)V_{[m]}$$

$$\chi_{2m}^{ij} T_{ij} = -\alpha_m T_{[m]} \quad (\text{no sum over } m), \quad (68)$$

with $\alpha_0 = \frac{2}{3}$, $\alpha_{\pm 1} = \frac{1}{\sqrt{3}}$, and $\alpha_2 = 1$. At this point, we can use the product formula for the spin-weighted spherical harmonics (A5) to express any term in terms of a sum of single harmonics. The result of these manipulations is integrated with

$$L \equiv i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega Y_{lm}^{s*}(\mathbf{n}), \quad (69)$$

which projects (55) and (56) on the lm multipole component. The final step consists of transforming from the ab

$$\begin{aligned} \frac{i\mathbf{q} \cdot \mathbf{k}}{aE} f_{ab}^{(2)} &= \frac{i\mathbf{q}}{aE} \sum_{m_2} \xi_{m_2}^i \sqrt{\frac{4\pi}{3}} Y_{1m_2} k^i \sum_{l_1, m_1} (-i)^{l_1} \sqrt{\frac{4\pi}{2l_1+1}} f_{ab, l_1 m_1}^{(2)}(\mathbf{k}) Y_{l_1 m_1}^s \\ &= \sum_{m_2=-1}^1 \frac{qk^{[m_2]}}{aE} \sum_{l_1, m_1} (-i)^{l_1} \sqrt{\frac{4\pi}{2l_1+1}} \sum_{L=|l_1-1|}^{l_1+1} \sum_{S, M=-L}^L \frac{\sqrt{2l_1+1}}{\sqrt{2L+1}} \begin{pmatrix} l_1 & 1 & L \\ -s & 0 & -S \end{pmatrix} \begin{pmatrix} l_1 & 1 & L \\ m_1 & m_2 & M \end{pmatrix} Y_{LM}^S f_{ab, l_1 m_1}^{(2)}(\mathbf{k}). \end{aligned} \quad (72)$$

Applying the multipole transformation operator L from (69) to this expression sets $L = l$ and $M = m$. Interchanging orders of summations according to

$$\sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{L=|l_1-1|}^{l_1+1} \sum_{M=-L}^L = \sum_{L=0}^{\infty} \sum_{l_1=|L-1|}^{L+1} \sum_{M=-L}^L \sum_{m_1=-l_1}^{l_1} \quad (73)$$

yields the final result

$$L \left[\frac{i\mathbf{q} \cdot \mathbf{k}}{aE} f_{ab}^{(2)} \right] = \sum_{m_2=-1}^1 \frac{qk^{[m_2]}}{aE} \sum_{l_1=|l-1|}^{l+1} \sum_{m_1=-l_1}^{l_1} i^{l-l_1} \begin{pmatrix} l_1 & 1 & l \\ -s & 0 & -s \end{pmatrix} \begin{pmatrix} l_1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} f_{ab, l_1 m_1}^{(2)}(\mathbf{k}). \quad (74)$$

Here we used that the first Clebsch-Gordan coefficient sets $S = s$ to eliminate the sum over S . The second one implies $m_2 = m - m_1$. Recall that s takes the value 0 when $ab = ++, --$ and $s = \pm 2$ for $ab = \pm\mp$. Thus, in the first case only $l_1 = l \pm 1$ contribute to the sum, while for the off-diagonal terms $l_1 = l$ is also nonzero. Equation (74) reproduces the standard first-order free-streaming term, in which one usually aligns \mathbf{k} with the three-direction implying $k^{[\pm 1]} = 0$ and $k^{[0]} = ik$, which simplifies the expression.

The free-streaming term is diagonal in the circular polarization basis, but the equations for the two off-diagonal components are slightly different, which leads to a mixing of E and B polarization in the Stokes parameter basis. The difference arises from

helicity polarization basis to the $X = I, V, E, B$ components of the phase-space distribution matrix.

C. Free-streaming term

We first consider the three space-time derivative terms in (56), which after Fourier transformation read

$$\frac{\partial f_{ab}^{(2)}}{\partial \eta}, \quad \frac{i\mathbf{q} \cdot \mathbf{k}}{aE} f_{ab}^{(2)}, \quad \left[\frac{P^i}{P^0} \right]^{(1)}(\mathbf{k}_1) ik_2^i f_{ab}^{(1)}(\mathbf{k}_2). \quad (70)$$

The multipole transformation of the time derivative is trivial since

$$L \left[\frac{\partial f_{ab}^{(2)}}{\partial \eta} \right] = \frac{\partial}{\partial \eta} f_{ab, lm}^{(2)}(\mathbf{k}). \quad (71)$$

For the transformation of the second term we follow the procedure described in Sec. III B. The manipulations are the same as for the corresponding term in the first-order equation (55), and we discuss them here only to illustrate the general method.

Inserting the expansion of $q^i = qn^i$ and $f_{ab}^{(2)}$ in spherical harmonics gives

$$\begin{pmatrix} l_1 & 1 & l \\ 2 & 0 & 2 \end{pmatrix} = (-1)^{l_1+1-l} \begin{pmatrix} l_1 & 1 & l \\ -2 & 0 & -2 \end{pmatrix}, \quad (75)$$

i.e. when $l_1 + 1 - l$ is odd, which happens precisely for the terms with $l_1 = l$ present only for $s = \pm 2$. To express the equations in the $IVEB$ basis in a compact form we introduce the matrices $H_{XY}(l)$ with

$$H_{XY}(l) = \delta_{XY} \quad (\text{for } l \text{ even}),$$

$$H_{XY}(l) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad (\text{for } l \text{ odd}) \quad (76)$$

and define

$$F_X = \begin{cases} 0 & X = I, V \\ -2 & X = E, B \end{cases} \quad (77)$$

Taking linear combinations of (74) according to (50) we obtain in the *IVEB* basis

$$L \left[\frac{i\mathbf{q} \cdot \mathbf{k}}{aE} f_X^{(2)} \right] = \sum_{m_2=-1}^1 \frac{qk^{[m_2]}}{aE} \sum_{l_1=|l-1|}^{l+1} \sum_{m_1=-l_1}^{l_1} i^{l-l_1} \begin{pmatrix} l_1 & 1 & l \\ F_X & 0 & F_X \end{pmatrix} \begin{pmatrix} l_1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} \sum_Y H_{XY}^* (l_1 + 1 - l) f_{Y,l_1 m_1}^{(2)}(\mathbf{k}). \quad (78)$$

The sum over Y encodes the mixing between the E - and B -mode polarization. Since the H matrices are block-diagonal in IV and EB , and equal to the identity matrix in the IV sector, the sum is redundant for $X = I, V$. The equations are decoupled and identical for $X = I$ and $X = V$. Nevertheless, the notation introduced above is convenient in order to present the results in the *IVEB* basis without having to resort to multiple equations for the different cases.

The third term in the list (70) requires no further work, since using (61)

$$L \left[\left[\frac{P^i}{P^0} \right]^{(1)}(\mathbf{k}_1) i k_2^i f_{ab}^{(1)}(\mathbf{k}_2) \right] = \left(A^{(1)} - D^{(1)} \right)(\mathbf{k}_1) L \left[\frac{i\mathbf{q} \cdot \mathbf{k}_2}{aE} f_{ab}^{(1)}(\mathbf{k}_2) \right]. \quad (79)$$

A convolution of the two mode momenta in the sense of (42) is implied. The application of the L operator gives as final result the expression (74) with $k^{[m_2]} \rightarrow k_2^{[m_2]}$ and $f_{ab,l_1 m_1}^{(2)}(\mathbf{k}) \rightarrow f_{ab,l_1 m_1}^{(1)}(\mathbf{k}_2)$, or the corresponding result (78) in the *IVEB* basis.

The generation of B polarization from E polarization through free-streaming requires propagation through an inhomogeneous universe, and is thus a second-order effect, known as time-delay induced B polarization [10]. The time-delay effect is contained in the above equations through the off-diagonal terms $H_{BE}^*(\pm 1) = -i$. The relevant product of Clebsch-Gordan coefficients is

$$\begin{pmatrix} l & 1 & l \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} l & 1 & l \\ m_1 & m_2 & m \end{pmatrix} = \delta_{m_1, m-m_2} \times \frac{1}{l(l+1)} \times \begin{cases} 2m & m_2 = 0 \\ \mp \sqrt{2(l+1 \mp m)(l \pm m)} & m_2 = \pm 1 \end{cases} \quad (80)$$

In first order in perturbation theory we can always align the mode vector \mathbf{k} such that only $m_2 = 0$ contributes. Then, using (80) in the first-order analogue of (78) shows that EB mixing occurs at first order only when $m \neq 0$, which implies the well-known result that no B polarization is induced, when there are no vector or tensor perturbations. At second order (79) contains a convolution over all wave vectors, and the sum over m_2 always extends over $m_2 = 0, \pm 1$. It follows from (79) and (80) that EB mixing occurs through free-streaming, when the first-order scalar perturbations $A^{(1)}$ or $D^{(1)}$ do not vanish.

To summarize the result of this subsection: the second-order space-time derivative terms (free-streaming terms) in the Boltzmann equation, Fourier- and multipole-transformed, are given in the *IVEB* basis by

$$\begin{aligned} \frac{\partial}{\partial \eta} f_{X,lm}^{(2)}(\mathbf{k}) + \sum_{m_2=-1}^1 \sum_{l_1=|l-1|}^{l+1} \sum_{m_1=-l_1}^{l_1} i^{l-l_1} \begin{pmatrix} l_1 & 1 & l \\ F_X & 0 & F_X \end{pmatrix} \begin{pmatrix} l_1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} \\ \times \sum_Y H_{XY}^* (l_1 + 1 - l) \left[\frac{qk^{[m_2]}}{aE} f_{Y,l_1 m_1}^{(2)}(\mathbf{k}) + \left(A^{(1)} - D^{(1)} \right)(\mathbf{k}_1) \frac{qk_2^{[m_2]}}{aE} f_{Y,l_1 m_1}^{(1)}(\mathbf{k}_2) \right]. \end{aligned} \quad (81)$$

D. Momentum-derivative terms

We now turn to the multipole decomposition of the two terms involving $dq^i/d\lambda$ in (56). With the help of (64) and (54) the first one can be written as

$$\begin{aligned} \left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(1)} \left(\frac{\partial f_{ab}^{(1)}}{\partial q^i} + \epsilon_{ak} \frac{\partial \epsilon_c^{k*}}{\partial q^i} f_{cb}^{(1)} + \epsilon_{bk}^* \frac{\partial \epsilon_c^k}{\partial q^i} f_{ac}^{(1)} \right) \\ = (-1) \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} (-i)^{l_1} \sqrt{\frac{4\pi}{2l_1+1}} \left[\frac{aE}{q} i\mathbf{n} \cdot \mathbf{k}_1 A^{(1)}(\mathbf{k}_1) + \dot{D}^{(1)}(\mathbf{k}_1) \right] Y_{l_1 m_1}^s q \frac{\partial}{\partial q} f_{ab,l_1 m_1}^{(1)}(\mathbf{k}_2) \\ + \left[\frac{aE}{q} i k_1^i A^{(1)}(\mathbf{k}_1) - \frac{q}{aE} i k_1^i D^{(1)}(\mathbf{k}_1) \right] \frac{1}{\sqrt{2}} \left(\epsilon_-^i \delta_s Y_{l_1 m_1}^s + \epsilon_+^i \bar{\delta}_s Y_{l_1 m_1}^s \right) f_{ab,l_1 m_1}^{(1)}(\mathbf{k}_2), \end{aligned} \quad (82)$$

where we used that ϵ_{\pm} is orthogonal to \mathbf{q} , and $\mathbf{n} \cdot \mathbf{q} = qn^2 = q$. Next we express \mathbf{n} and the polarization vectors in terms of spherical harmonics according to (66) to write

$$i\mathbf{n} \cdot \mathbf{k}_1 = \sum_{m_2=-1}^1 k_1^{[m_2]} \sqrt{\frac{4\pi}{3}} Y_{1m_2}, \quad ik_1^i (\epsilon_-^i \delta_s Y_{l_1 m_1}^s + \epsilon_+^i \delta_s Y_{l_1 m_1}^s) = - \sum_{m_2=-1}^1 k_1^{[m_2]} \sqrt{\frac{4\pi}{3}} \left([L_1]_s^+ Y_{1m_2}^{-1} Y_{l_1 m_1}^{s+1} + [L_1]_s^- Y_{1m_2}^{+1} Y_{l_1 m_1}^{s-1} \right) \quad (83)$$

after taking the derivatives on the spin-weighted spherical harmonics using (A10) in the second equation. The remaining steps are straightforward. We eliminate the products of spherical harmonics with (A5) and apply the multipole transformation operator (69) to obtain

$$\begin{aligned} L \left[\text{left-hand side of Eq. (82)} \right]_{ab} &= -\dot{D}^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{ab,lm}^{(1)}(\mathbf{k}_2) + \sum_{m_2=-1}^1 \sum_{l_1=|l-1|}^{l+1} \sum_{m_1=l_1}^{l_1} i^{l-l_1} \begin{pmatrix} l_1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} \\ &\times \left\{ \begin{pmatrix} l_1 & 1 & l \\ -s & 0 & -s \end{pmatrix} \left(-\frac{aE}{q} \right) k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{ab,l_1 m_1}^{(1)}(\mathbf{k}_2) \right. \\ &+ \frac{1}{\sqrt{2}} \left\{ [L_1]_s^+ \begin{pmatrix} l_1 & 1 & l \\ -(s+1) & 1 & -s \end{pmatrix} + [L_1]_s^- \begin{pmatrix} l_1 & 1 & l \\ -(s-1) & -1 & -s \end{pmatrix} \right\} \\ &\left. \times k_1^{[m_2]} \left[\frac{aE}{q} A^{(1)}(\mathbf{k}_1) - \frac{q}{aE} D^{(1)}(\mathbf{k}_1) \right] f_{ab,l_1 m_1}^{(1)}(\mathbf{k}_2) \right\}. \quad (84) \end{aligned}$$

As for the free-streaming terms the equations for the off-diagonal terms are slightly different, which implies conversion of E into B polarization and vice versa. The last two lines in the previous equation, which originate from the derivative of the first-order photon perturbation with respect to the direction of the photon momentum, correspond precisely to the weak-lensing effect [8]. If $\kappa(s)$ denotes the expression in curly brackets in the third line of (84), the relation $\kappa(-2) = (-1)^{l_1+1-l} \kappa(2)$ holds, and because of the similarity with (75) the same matrix H_{XY} appears in the transformation to the Stokes parameters. The final result for this term in the $IVEB$ basis reads

$$\begin{aligned} L \left[\text{left-hand side of Eq. (82)} \right]_X &= -\dot{D}^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{X,lm}^{(1)}(\mathbf{k}_2) + \sum_{m_2=-1}^1 \sum_{l_1=|l-1|}^{l+1} \sum_{m_1=-l_1}^{l_1} i^{l-l_1} \begin{pmatrix} l_1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} \\ &\times \left\{ \begin{pmatrix} l_1 & 1 & l \\ F_X & 0 & F_X \end{pmatrix} \left(-\frac{aE}{q} \right) k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) \sum_Y H_{XY}^*(l_1 + 1 - l) q \frac{\partial}{\partial q} f_{Y,l_1 m_1}^{(1)}(\mathbf{k}_2) \right. \\ &+ \frac{1}{\sqrt{2}} \left\{ [L_1]_{F_X}^- \begin{pmatrix} l_1 & 1 & l \\ F_X - 1 & 1 & F_X \end{pmatrix} + [L_1]_{F_X}^+ \begin{pmatrix} l_1 & 1 & l \\ F_X + 1 & -1 & F_X \end{pmatrix} \right\} \\ &\left. \times k_1^{[m_2]} \left[\frac{aE}{q} A^{(1)}(\mathbf{k}_1) - \frac{q}{aE} D^{(1)}(\mathbf{k}_1) \right] \sum_Y H_{XY}^*(l_1 + 1 - l) f_{Y,l_1 m_1}^{(1)}(\mathbf{k}_2) \right\}. \quad (85) \end{aligned}$$

The other momentum-derivative term at second order can be written as

$$\left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(2)} \frac{q^i}{q} \frac{\partial f_{ab}^{(0)}}{\partial q} = \left[X + Y_i \frac{q^i}{q} - \dot{E}_{ij}^{(2)} \frac{q^i q^j}{q^2} \right] \delta_{ab} q \frac{\partial f_I^{(0)}}{\partial q}, \quad (86)$$

where X and Y_i represent the q -independent and linear terms in q^i in (65), respectively. We also used that the unperturbed photon phase-space distribution is unpolarized. Since the only dependence on the direction of \mathbf{q} in this term arises from the factors of q^i in square brackets, it contributes only to $l = 0, 1, 2$. In the quadratic term we write

$$\dot{E}_{ij}^{(2)} \frac{q^i q^j}{q^2} = \sqrt{\frac{4\pi}{5}} \sum_{m_2=-2}^2 \chi_{2m_2}^{ij} Y_{2m_2} \dot{E}_{ij}^{(2)} = -\sqrt{\frac{4\pi}{5}} \sum_{m_2=-2}^2 \alpha_{m_2} \dot{E}_{[m_2]}^{(2)} Y_{2m_2}, \quad (87)$$

employing the definitions (66) and (68) and the tracelessness of E_{ij} . The remainder of the calculation is straightforward, resulting in

$$\begin{aligned}
L \left[\left[\frac{1}{P^0} \frac{dq^i}{d\lambda} \right]^{(2)} \frac{q^i}{q} \frac{\partial f_{ab}^{(0)}}{\partial q} \right] &= \left\{ [-\dot{D}^{(2)}(\mathbf{k}) + 2D^{(1)}(\mathbf{k}_1)\dot{D}^{(1)}(\mathbf{k}_2)]\delta_{l0} \right. \\
&+ \frac{aE}{q} \left[-ik^{[m]}A^{(2)}(\mathbf{k}) + ik_1^{[m]}A^{(1)}(\mathbf{k}_1) \left(A^{(1)}(\mathbf{k}_2) + D^{(1)}(\mathbf{k}_2) \right) + \dot{B}_{[m]}^{(2)}(\mathbf{k}) + H_c \left(1 - \frac{q^2}{a^2 E^2} \right) B_{[m]}^{(2)}(\mathbf{k}) \right] \delta_{l1} \\
&\left. - \alpha_m \dot{E}_{[m]}^{(2)}(\mathbf{k}) \delta_{l2} \right\} \delta_{ab} q \frac{\partial f_l^{(0)}}{\partial q}. \tag{88}
\end{aligned}$$

In the *IVEB* polarization basis the multipole transform of this term takes the same form with the replacement $\delta_{ab} \rightarrow \delta_{Xl}$.

Our final result for the Boltzmann hierarchy for the multipole moments $f_{X,lm}^{(2)}(\mathbf{k})$ of the Stokes parameter phase-space densities at second order is given by the sum of (81), (85), and (88) excluding the collision term that we consider in the following section. These expressions remain valid in the case of massive particles with mass M , for which

$$E = \sqrt{M^2 + \frac{q^2}{a^2}}. \tag{89}$$

For photons the simplification $aE/q = 1$ can be applied and the term proportional to H_c in the second line of (88) vanishes.

IV. EXPANSION OF THE COLLISION TERM TO SECOND ORDER

In this section we compute the expansion of the collision term in the Boltzmann hierarchy for the multipole moments $f_{X,lm}^{(2)}(\mathbf{k})$. This is done in two steps. First we expand (39) to second order. Then we apply the operator (69) that converts to equations for the multipole moments. Our treatment follows [12] extended to the polarized phase-space distributions.

A. Nonrelativistic expansion

The cosmic background photons that we see have mostly last scattered around the time of recombination, when the temperature of the Universe was less than 1 eV. Polarization of the cosmic microwave background is generated at this time or later. The electrons on which the photons scatter are therefore highly nonrelativistic with

thermal velocities

$$\frac{|\mathbf{q}|}{m_e} \sim \sqrt{\frac{T_e}{m_e}} \approx 10^{-3}. \tag{90}$$

We therefore perform an expansion of the Compton scattering matrix element in the electron momentum and consider the expansion parameter (90) of the same order as the cosmological perturbations. Note that in this subsection \mathbf{q} and \mathbf{q}' refer to the electron momentum and not the comoving photon momentum.

The electrons are in local thermal equilibrium and sufficiently dilute to be described by the Maxwell-Boltzmann distribution

$$g_e(\mathbf{q}) = n_e \left(\frac{2\pi}{m_e T_e} \right)^{3/2} e^{-((\mathbf{q} - m_e \mathbf{v}_e)^2 / 2m_e T_e)}. \tag{91}$$

Here T_e , \mathbf{v}_e , and n_e denote the local electron temperature, bulk velocity, and number density of free electrons, i.e. electrons not bound in hydrogen or helium. If x_e denotes the ionization fraction and ρ_b the baryon density, then n_e is given by

$$n_e = n_e^{(0)} \left(1 + \left[\frac{\delta \rho_b}{\rho_b} \right]^{(1)} + \left[\frac{\delta x_e}{x_e} \right]^{(1)} + \dots \right) \tag{92}$$

to first order in perturbations. A complete account of the collision term to second order therefore requires a calculation of the recombination history that goes beyond the homogeneous Universe to obtain the perturbations in the ionization fraction. We refer to [28] for a discussion of this issue. In our equations we keep n_e as an overall factor without expanding it for the time being.

The integral over \mathbf{q}' in (39) is eliminated by the three-momentum delta function, which sets $\mathbf{q}' = \mathbf{q} + \mathbf{p} - \mathbf{p}'$. This allows us to expand

$$\begin{aligned}
g(\mathbf{q}') &= g(\mathbf{p} + \mathbf{q} - \mathbf{p}') = g(\mathbf{q}) \left[1 - \frac{(\mathbf{p} - \mathbf{p}')(\mathbf{q} - m_e \mathbf{v})}{m_e T} - \frac{(\mathbf{p} - \mathbf{p}')^2}{2m_e T} + \frac{1}{2} \left(\frac{(\mathbf{p} - \mathbf{p}')(\mathbf{q} - m_e \mathbf{v})}{m_e T} \right)^2 + \dots \right], \\
\delta(p^0 + q^0 - p'^0 - q'^0) &= \delta(p + E(\mathbf{q}) - p' - E(\mathbf{p} + \mathbf{q} - \mathbf{p}')) \\
&= \delta(p - p') + \frac{(\mathbf{p} - \mathbf{p}')\mathbf{q}}{m_e} \frac{\partial \delta(p - p')}{\partial p'} + \frac{(\mathbf{p} - \mathbf{p}')^2}{2m_e} \frac{\partial \delta(p - p')}{\partial p'} + \frac{1}{2} \left(\frac{(\mathbf{p} - \mathbf{p}')\mathbf{q}}{m_e} \right)^2 \frac{\partial^2 \delta(p - p')}{\partial p'^2} + \dots, \tag{93}
\end{aligned}$$

where $p = |\mathbf{p}|$, $p' = |\mathbf{p}'|$. The expansion is based on the observation that $p, p' \sim T$ while $|\mathbf{q}| \sim (m_e T_e)^{1/2}$ and that the difference of electron energies

$$E(\mathbf{q}) - E(\mathbf{q}') = \frac{\mathbf{q}^2}{2m_e} - \frac{\mathbf{q}'^2}{2m_e} = -\frac{\mathbf{q}(\mathbf{p} - \mathbf{p}')}{m_e} - \frac{(\mathbf{p} - \mathbf{p}')^2}{2m_e} \sim \frac{T_e^{3/2}}{m_e^{1/2}} \ll T_e. \quad (94)$$

The terms neglected in (93) are therefore of third order in the expansion parameter (90). Inserting these expansions into the collision term the zeroth-order terms cancel, so that the collision term begins at first order as it should. It is therefore sufficient to expand the Compton matrix element to first order. The result of expanding (32) and (40) can be written in the form

$$|\bar{M}|_{\lambda\lambda';\omega\omega'}^2 = 24\pi m_e^2 \sigma_T \left(S_{0,\lambda\lambda'\omega\omega'} + \frac{q^i}{m_e} S_{1,\lambda\lambda'\omega\omega'}^i + O\left(\frac{q^2}{m_e^2}\right) \right) \quad (95)$$

with

$$S_{0,\lambda\lambda'\omega\omega'} = \boldsymbol{\epsilon}_\lambda(\mathbf{p}) \cdot \boldsymbol{\epsilon}_{\lambda'}^*(\mathbf{p}') \boldsymbol{\epsilon}_\omega^*(\mathbf{p}) \cdot \boldsymbol{\epsilon}_{\omega'}(\mathbf{p}'), \quad (96)$$

$$S_{1,\lambda\lambda'\omega\omega'}^i = \boldsymbol{\epsilon}_\lambda(\mathbf{p}) \cdot \boldsymbol{\epsilon}_{\lambda'}^*(\mathbf{p}') \left\{ \boldsymbol{\epsilon}_\omega^*(\mathbf{p}) \frac{\boldsymbol{\epsilon}_{\omega'}(\mathbf{p}') \cdot \mathbf{p}}{p} + \boldsymbol{\epsilon}_{\omega'}^i(\mathbf{p}') \frac{\boldsymbol{\epsilon}_\omega^*(\mathbf{p}) \cdot \mathbf{p}'}{p'} \right\} + \boldsymbol{\epsilon}_\omega^*(\mathbf{p}) \cdot \boldsymbol{\epsilon}_{\omega'}(\mathbf{p}') \left\{ \boldsymbol{\epsilon}_{\lambda'}^i(\mathbf{p}') \frac{\boldsymbol{\epsilon}_\lambda^*(\mathbf{p}) \cdot \mathbf{p}}{p} + \boldsymbol{\epsilon}_{\lambda'}^*(\mathbf{p}') \frac{\boldsymbol{\epsilon}_\lambda(\mathbf{p}) \cdot \mathbf{p}'}{p'} \right\}, \quad (97)$$

and $\sigma_T = 8\pi\alpha^2/(3m_e^2)$ the Thomson scattering cross section. At this point the integrand is polynomial in \mathbf{q} except for $g_e(\mathbf{q})$ so that the integral over \mathbf{q} in (39) can be expressed in terms of the moments of the electron distribution:

$$\int \frac{d\mathbf{q}}{(2\pi)^3} g_e(\mathbf{q}) \times \{1; q^i; q^i q^j\} = n_e \times \{1; m_e \mathbf{v}_e^i; m_e T_e \delta^{ij} + m_e^2 \mathbf{v}_e^i \mathbf{v}_e^j\}. \quad (98)$$

B. Expansion of $C_{ab}[f]$

It is straightforward to insert the nonrelativistic expansions discussed above into the collision term (39) and to perform the integrations over the incoming and scattered electron momentum. It is convenient to express the results in terms of the coefficient of the gain term in (39) and the difference of the gain and loss terms, given by

$$\begin{aligned} G_{\lambda\lambda'\omega\omega'}^{(i)} &= f_{\lambda'\omega'}^{(i)}(p') \left[\delta_{a\lambda} (\delta_{\omega b} + f_{\omega b}(p)) + \delta_{\omega b} (\delta_{a\lambda} + f_{a\lambda}(p)) \right], \\ GL_{\lambda\lambda'\omega\omega'}^{(i)} &= 2\delta_{a\lambda} \delta_{\omega b} f_{\lambda'\omega'}^{(i)}(p') - \delta_{\lambda'\omega'} \left[\delta_{a\lambda} f_{\omega b}^{(i)}(p) + \delta_{\omega b} f_{a\lambda}^{(i)}(p) \right] \end{aligned} \quad (99)$$

at i th order in the expansion. Note that while the difference of the gain and loss terms is linear in the phase-space distributions, the gain term contains quadratic terms. In the definition of $G_{\lambda\lambda'\omega\omega'}^{(i)}$ we use the unexpanded distribution functions in the Bose enhancement factors.

The expanded collision term can now be written in the form

$$C_{ab}[f] = \frac{3}{4} n_e \sigma_T \int_0^\infty dp' p' \int \frac{d\Omega'}{4\pi} \left[c^{(1)} + c_\Delta^{(2)} + c_v^{(2)} + c_{\Delta v}^{(2)} + c_{vv}^{(2)} + c_K^{(2)} \right]_{ab}, \quad (100)$$

where Ω' denotes the solid angle of the scattered photon momentum vector \mathbf{p}' . This expression includes the first-order term

$$c_{ab}^{(1)} = S_{0,\lambda\lambda'\omega\omega'} \left[\delta(p - p') GL^{(1)} + \mathbf{v}_e^{(1)} \cdot (\mathbf{p} - \mathbf{p}') \frac{\partial \delta(p - p')}{\partial p'} GL^{(0)} \right]_{\lambda\lambda'\omega\omega'} \quad (101)$$

(summation over repeated photon polarization indices $\lambda, \lambda', \omega, \omega'$ is understood), and the second-order term split into five contributions according to

$$c_{\Delta,ab}^{(2)} = S_{0,\lambda\lambda'\omega\omega'} \delta(p - p') GL_{\lambda\lambda'\omega\omega'}^{(2)} \quad (102)$$

$$c_{v,ab}^{(2)} = S_{0,\lambda\lambda'\omega\omega'} \mathbf{v}_e^{(2)} \cdot (\mathbf{p} - \mathbf{p}') \frac{\partial \delta(p - p')}{\partial p'} GL_{\lambda\lambda'\omega\omega'}^{(0)} \quad (103)$$

$$c_{\Delta v,ab}^{(2)} = S_{0,\lambda\lambda'\omega\omega'} \mathbf{v}_e^{(1)} \cdot (\mathbf{p} - \mathbf{p}') \frac{\partial \delta(p - p')}{\partial p'} GL_{\lambda\lambda'\omega\omega'}^{(1)} + S_{1,\lambda\lambda'\omega\omega'}^i \delta(p - p') v_e^{(1)i} GL_{\lambda\lambda'\omega\omega'}^{(1)} \quad (104)$$

$$c_{vv,ab}^{(2)} = S_{0,\lambda\lambda'\omega\omega'} \frac{1}{2} [\mathbf{v}_e^{(1)} \cdot (\mathbf{p} - \mathbf{p}')]^2 \frac{\partial^2 \delta(p - p')}{\partial p'^2} GL_{\lambda\lambda'\omega\omega'}^{(0)} + S_{1,\lambda\lambda'\omega\omega'}^i \mathbf{v}_e^{(1)} \cdot (\mathbf{p} - \mathbf{p}') v_e^{(1)i} \frac{\partial \delta(p - p')}{\partial p'} GL_{\lambda\lambda'\omega\omega'}^{(0)} \quad (105)$$

$$c_{K,ab}^{(2)} = S_{0,\lambda\lambda'\omega\omega'} \frac{(\mathbf{p} - \mathbf{p}')^2}{2m_e} \left(\frac{\partial \delta(p - p')}{\partial p'} GL_{\lambda\lambda'\omega\omega'}^{(0)} - 2 \frac{\partial \delta(p - p')}{\partial p'} G_{\lambda\lambda'\omega\omega'}^{(0)} + T_e \frac{\partial^2 \delta(p - p')}{\partial p'^2} GL_{\lambda\lambda'\omega\omega'}^{(0)} \right) + S_{1,\lambda\lambda'\omega\omega'}^i \frac{(p - p')^i}{m_e} \left(-\delta(p - p') G_{\lambda\lambda'\omega\omega'}^{(0)} + T_e \frac{\partial \delta(p - p')}{\partial p'} GL_{\lambda\lambda'\omega\omega'}^{(0)} \right). \quad (106)$$

Because of the delta functions the integral over p' can be performed after a few partial integrations. We also sum over polarizations and integrate over the solid angle, whenever possible. We define the integral operator

$$I[\dots] = \frac{1}{2p} \int_0^\infty dp' p' \int \frac{d\Omega'}{4\pi} [\dots] \quad (107)$$

such that

$$C_{ab}[f] = \frac{3}{2} n_e \sigma_T p \times I[c^{(1)} + c_\Delta^{(2)} + c_v^{(2)} + c_{\Delta v}^{(2)} + c_{vv}^{(2)} + c_K^{(2)}]_{ab}, \quad (108)$$

and work out the six terms separately. The first-order term yields

$$I[c_{ab}^{(1)}] = \frac{1}{2} \int \frac{d\Omega'}{4\pi} S_{0,\lambda\lambda'\omega\omega'} \left[GL_{|p=p'}^{(1)} - \mathbf{v}_e^{(1)} \cdot (\mathbf{n} - 2\mathbf{n}') GL_{|p=p'}^{(0)} - \mathbf{v}_e^{(1)} \cdot (\mathbf{n} - \mathbf{n}') p \frac{\partial}{\partial p'} GL_{|p=p'}^{(0)} \right]_{\lambda\lambda'\omega\omega'} \quad (109)$$

after partial integration. We note that S_0 and S_1^i depend on the direction of \mathbf{p}' but not on its magnitude p' . The integral over the delta function sets \mathbf{p}' to $p\mathbf{n}'$. The subscript “ $p = p'$ ” means that $f_{ab}^{(i)}(p') = f_{ab}^{(i)}(p\mathbf{n}')$ in the expressions (99) for the gain and loss terms. The zeroth-order distribution function does not depend on the momentum direction and is unpolarized, hence

$$\left[GL_{|p=p'}^{(0)} \right]_{\lambda\lambda'\omega\omega'} = 0, \quad \left[p \frac{\partial}{\partial p'} GL_{|p=p'}^{(0)} \right]_{\lambda\lambda'\omega\omega'} = 2\delta_{a\lambda} \delta_{\omega b} \delta_{\lambda'\omega'} p \frac{\partial f_I^{(0)}}{\partial p}. \quad (110)$$

Inserting the expression (96) for $S_{0,\lambda\lambda'\omega\omega'}$ into (109) we next obtain

$$I[c_{ab}^{(1)}] = \frac{1}{2} \delta^{ik} \delta^{jl} \int \frac{d\Omega'}{4\pi} \left\{ 2\epsilon_a^i(\mathbf{n}) \epsilon_b^{*j}(\mathbf{n}) \left[\epsilon_{\lambda'}^{*k}(\mathbf{n}') \epsilon_{\omega'}^l(\mathbf{n}') f_{\lambda'\omega'}^{(1)}(p\mathbf{n}') - \mathbf{v}_e^{(1)} \cdot (\mathbf{n} - \mathbf{n}') \epsilon_{\lambda'}^{*k}(\mathbf{n}') \epsilon_{\omega'}^l(\mathbf{n}') p \frac{\partial f_I^{(0)}}{\partial p} \right] - \epsilon_{\lambda'}^i(\mathbf{n}) \epsilon_{\omega'}^{*j}(\mathbf{n}) \epsilon_{\lambda'}^{*k}(\mathbf{n}') \epsilon_{\omega'}^l(\mathbf{n}') [\delta_{a\lambda} f_{\omega b}^{(1)}(p) + \delta_{\omega b} f_{a\lambda}^{(1)}(p)] \right\}. \quad (111)$$

This can be further simplified using

$$\int \frac{d\Omega'}{4\pi} \epsilon_{\lambda'}^{*k}(\mathbf{n}') \epsilon_{\omega'}^l(\mathbf{n}') = \int \frac{d\Omega'}{4\pi} [\delta^{kl} - n'^k n'^l] = \frac{2}{3} \delta^{kl} \quad (112)$$

and $\epsilon_{\lambda'}^i(\mathbf{n}) \epsilon_{\omega'}^{*i}(\mathbf{n}) = \delta_{\lambda\omega}$ to obtain the final result

$$I[c_{ab}^{(1)}] = -\frac{2}{3} f_{ab}^{(1)}(p) - \frac{2}{3} \delta_{ab} \mathbf{n} \cdot \mathbf{v}_e^{(1)} p \frac{\partial f_I^{(0)}(p)}{\partial p} + \int \frac{d\Omega'}{4\pi} \epsilon_a^i(\mathbf{n}) \epsilon_b^{*j}(\mathbf{n}) \left[\epsilon_{\lambda'}^{*i}(\mathbf{n}') \epsilon_{\omega'}^j(\mathbf{n}') f_{\lambda'\omega'}^{(1)}(p\mathbf{n}') \right]. \quad (113)$$

This expression is equivalent to the standard result for the first-order polarized collision term [29,30].

The second-order terms can be calculated in a similar way without further complications though the algebra gets lengthier, when the matrix element S_1^i is involved. We also note that the gain term alone, which contains quadratic terms in

the phase-space distributions, appears only in $c_{K,ab}^{(2)}$ at second order, but the simpler zeroth-order expression

$$G_{\lambda\lambda'\omega\omega'}^{(0)} = 2\delta_{a\lambda}\delta_{\omega b}\delta_{\lambda'\omega'}f_I^{(0)}(p') \left[1 + f_I^{(0)}(p) \right] \quad (114)$$

is needed there. The result for the integrated second-order terms is

$$I[c_{\Delta,ab}^{(2)}] = -\frac{2}{3}f_{ab}^{(2)}(p) + \int \frac{d\Omega'}{4\pi} \epsilon_a^i(\mathbf{n})\epsilon_b^{*j}(\mathbf{n}) \left[\epsilon_{\lambda'}^{*i}(\mathbf{n}')\epsilon_{\omega'}^j(\mathbf{n}')f_{\lambda'\omega'}^{(2)}(p\mathbf{n}') \right], \quad (115)$$

$$I[c_{v,ab}^{(2)}] = -\frac{2}{3}\delta_{ab}\mathbf{n} \cdot \mathbf{v}_e^{(2)} p \frac{\partial f_I^{(0)}(p)}{\partial p}, \quad (116)$$

$$I[c_{\Delta v,ab}^{(2)}] = \frac{2}{3}\mathbf{n} \cdot \mathbf{v}_e^{(1)} f_{ab}^{(1)}(p) + \int \frac{d\Omega'}{4\pi} \left\{ S_{1,ijkl}^m \mathbf{v}_e^{(1)m} - \delta^{ik}\delta^{jl} \left[\mathbf{v}_e^{(1)} \cdot (\mathbf{n} - 2\mathbf{n}') + \mathbf{v}_e^{(1)} \cdot (\mathbf{n} - \mathbf{n}') p \frac{\partial}{\partial p} \right] \right\} \\ \times \epsilon_a^i(\mathbf{n})\epsilon_b^{*j}(\mathbf{n}) \left[\epsilon_{\lambda'}^{*k}(\mathbf{n}')\epsilon_{\omega'}^l(\mathbf{n}')f_{\lambda'\omega'}^{(1)}(p\mathbf{n}') \right], \quad (117)$$

$$I[c_{vv,ab}^{(2)}] = \delta_{ab} \left[\frac{2}{3}\mathbf{v}_e^{(1)2} + \frac{2}{3}(\mathbf{n} \cdot \mathbf{v}_e^{(1)})^2 \right] p \frac{\partial f_I^{(0)}(p)}{\partial p} + \delta_{ab} \left[\frac{2}{15}\mathbf{v}_e^{(1)2} + \frac{1}{3}(\mathbf{n} \cdot \mathbf{v}_e^{(1)})^2 \right] p^2 \frac{\partial^2 f_I^{(0)}(p)}{\partial p^2}, \\ - \epsilon_a^i(\mathbf{n})\epsilon_b^{*j}(\mathbf{n}) \frac{1}{15} \mathbf{v}_e^{(1)i} \mathbf{v}_e^{(1)j} p^2 \frac{\partial^2 f_I^{(0)}(p)}{\partial p^2}, \quad (118)$$

$$I[c_{K,ab}^{(2)}] = \frac{2}{3}\delta_{ab} \left\{ \frac{4p}{m_e} f_I^{(0)}(p)(1 + f_I^{(0)}(p)) + \left[\frac{4T_e}{m_e} + \frac{p}{m_e}(1 + 2f_I^{(0)}(p)) \right] p \frac{\partial f_I^{(0)}(p)}{\partial p} + \frac{T_e}{m_e} p^2 \frac{\partial^2 f_I^{(0)}(p)}{\partial p^2} \right\}. \quad (119)$$

Here $S_{1,ijkl}^m$ equals $S_{1,\lambda\lambda'\omega\omega'}^i$ with the polarization vectors stripped off, i.e.

$$S_{1,ijkl}^m = \delta^{ik}(\delta^{jm}n^l + \delta^{lm}n^j) + \delta^{jl}(\delta^{im}n^k + \delta^{km}n^i). \quad (120)$$

C. Fourier and multipole transformation

The final step in the derivation of the collision term consists in applying the multipole transformation operator (69), to perform the Fourier transformation, and to convert the equations to the Stokes parameter basis. Taking into account the expansion of the prefactors $1/p^0$ in (17) and the free electron density n_e in (100), the right-hand side of the Boltzmann hierarchy is of the form

$$\frac{\partial}{\partial \eta} f_{X,lm}^{(1)}(\mathbf{k}) + \dots = L \left[\frac{1}{p^0} C_X[f] \right] = \frac{a}{E} L \left[C_X[f]^{(1)} \right] = \frac{3}{2} n_e^{(0)} \sigma_{Ta} \times \hat{I}[c_X^{(1)}](\mathbf{k}) \quad (121)$$

at first order, and

$$\frac{\partial}{\partial \eta} f_{X,lm}^{(2)}(\mathbf{k}) + \dots = L \left[\frac{1}{p^0} C_X[f] \right] = \frac{a}{E} L \left[C_X[f]^{(2)} \right] + \frac{a}{E} A^{(1)} L \left[C_X[f]^{(1)} \right] \\ = \frac{3}{2} n_e^{(0)} \sigma_{Ta} \times \left\{ \hat{I} \left[c_{\Delta}^{(2)} + c_v^{(2)} + c_{\Delta v}^{(2)} + c_{vv}^{(2)} + c_K^{(2)} \right]_X(\mathbf{k}) \right. \\ \left. + \left(A^{(1)} + \left[\frac{\delta \rho_b}{\rho_b} \right]^{(1)} + \left[\frac{\delta x_e}{x_e} \right]^{(1)} \right) (\mathbf{k}_1) \hat{I}[c_X^{(1)}](\mathbf{k}_2) \right\} \quad (122)$$

at second order. In order to arrive at the last equality in each equation we used (108) for the collision term, (92) for the free electron density, and set $E = p$, since the collision term refers explicitly to photons. We also define the operation $\hat{I}[\dots] = L[I[\dots]]$. The Fourier transformation of the collision term is trivial, since it does not contain spatial derivatives. Products of position-dependent functions sim-

ply turn into convolutions as indicated by the momentum argument. In the remainder of this subsection we neglect these arguments to avoid notational complications, but we restore them in the summary of Sec. V. To complete the calculation of the Boltzmann hierarchy, it now remains to work out the multipole transformation of (113) and (115)–(119).

Inserting the multipole expansion (43) for $f_{\lambda'\omega'}^{(1)}(\mathbf{p}\mathbf{n}')$ and applying the L operator (69) the angular integrals can be expressed in terms of the matrices

$$Q_{ab,lm}^{ij} = \frac{1}{\sqrt{4\pi}} \int d\Omega \epsilon_b^{*i}(\mathbf{n}) \epsilon_a^j(\mathbf{n}) Y_{lm}^s(\mathbf{n}), \quad (123)$$

where $s = 0$ is implied for $ab = ++$, $--$ and $s = \pm 2$ for $ab = \mp \pm$. Since the polarization vectors are spin 1 objects, the Q matrices are nonzero only for $l \leq 2$. To transform to the Stokes parameter basis we use the matrices (51) and define

$$Q_{X,lm}^{ij} = U_{X,[ab]} Q_{[ab],lm} \quad (124)$$

in analogy with (50) for the phase-space distributions. In the *IVEB* basis the Q matrices vanish for $X = B$ for any value of l . The nonvanishing Q matrices are given explicitly in (A14). The trace

$$\text{tr}(Q_{ab,lm}^\dagger Q_{cd,l'm'}) = Q_{ab,lm}^{ij*} Q_{cd,l'm'}^{ij} \equiv \frac{1}{3} \omega_{ab;cd}^{(l)} \delta_{ll'} \delta_{mm'} \quad (125)$$

is diagonal in the multipole indices and defines the ω symbols. Similarly in the *IVEB* basis

$$\text{tr}(Q_{X,lm}^\dagger Q_{Y,l'm'}) = Q_{X,lm}^{ij*} Q_{Y,l'm'}^{ij} \equiv \frac{1}{3} \omega_{XY}^{(l)} \delta_{ll'} \delta_{mm'} \quad (126)$$

for

$$\omega_{XY}^{(l)} = U_{X,[ab]}^* U_{Y,[cd]} \omega_{ab;cd}^{(l)} \quad (127)$$

Only very few of the $\omega_{XY}^{(l)}$ are not zero. In particular, $\omega_{IE}^{(2)} = \omega_{IE}^{(2)} = -\sqrt{3/50}$ is the only off-diagonal term that couples to polarization. The other nonvanishing values are summarized in (A15).

After these preliminaries we turn to the explicit calculation beginning with the first-order term. With the definitions (123) and (125) it is straightforward to obtain from (113) the expression

$$\hat{I}[c_{ab}^{(1)}] = \frac{2}{3} \left\{ -f_{ab,lm}^{(1)}(p) - \delta_{ab} \delta_{l1} \mathbf{v}_{e,[m]}^{(1)} p \frac{\partial f_I^{(0)}(p)}{\partial p} + \frac{1}{2} \omega_{ba;\omega'\lambda'}^{(l)} f_{\lambda'\omega',lm}^{(1)}(p) \right\}. \quad (128)$$

Note that the phase-space distributions $f_{ab,lm}^{(1)}(p)$ and helicity components of the electron bulk-velocity field $\mathbf{v}_{e,[m]}^{(1)}$ also depend on the Fourier mode vector \mathbf{k} . This dependence is suppressed in this subsection as mentioned above. The transformation to the Stokes parameter basis requires the calculation of $U_{X,[ab]} \hat{I}[c^{(1)}]_{ab}$. For this purpose we use

$$\begin{aligned} U_{X,[ab]} \omega_{ba;\omega'\lambda'}^{(l)} f_{\lambda'\omega',lm}^{(1)}(p) &= U_{X,[ab]} U_{[ba],X'}^{-1*} \omega_{X'Y'}^{(l)} U_{[\omega'\lambda'];Y'}^{-1} U_{[\lambda'\omega'];Y}^{-1} f_{Y,lm}^{(1)}(p) \\ &\stackrel{(52)}{=} 2 U_{X,[ab]} U_{[ab],X'}^{-1} \omega_{X'Y'}^{(l)} U_{[\lambda'\omega'];Y'}^{-1*} U_{Y,[\lambda'\omega']}^* f_{Y,lm}^{(1)}(p) \\ &= 2 \delta_{XX'} \omega_{X'Y'}^{(l)} \delta_{Y'Y} f_{Y,lm}^{(1)}(p) \\ &= 2 \omega_{XY}^{(l)} f_{Y,lm}^{(1)}(p) \end{aligned} \quad (129)$$

and obtain

$$\begin{aligned} \hat{I}[c_X^{(1)}] &= \frac{2}{3} \left\{ -f_{X,lm}^{(1)}(p) - \delta_{Xl} \delta_{l1} \mathbf{v}_{e,[m]}^{(1)} p \frac{\partial f_I^{(0)}(p)}{\partial p} \right. \\ &\quad \left. + \omega_{XY}^{(l)} f_{Y,lm}^{(1)}(p) \right\}. \end{aligned} \quad (130)$$

When inserted into (121) we reproduce the first-order collision term in the Boltzmann hierarchy for the polarized phase-space distributions in a notation similar to [31]. The last term in brackets describes the generation of the E -polarization quadrupole in Thomson scattering.

The second-order terms can be calculated in a similar way. The first two, $I[c_{\Delta,ab}^{(2)}]$ and $I[c_{v,ab}^{(2)}]$, have the same

structure as the first-order term and can be obtained from (130) without additional work:

$$\hat{I}[c_{\Delta,X}^{(2)}] = \frac{2}{3} \left\{ -f_{X,lm}^{(2)}(p) + \omega_{XY}^{(l)} f_{Y,lm}^{(2)}(p) \right\}, \quad (131)$$

$$\hat{I}[c_{v,X}^{(2)}] = -\frac{2}{3} \delta_{Xl} \delta_{l1} \mathbf{v}_{e,[m]}^{(2)} p \frac{\partial f_I^{(0)}(p)}{\partial p}. \quad (132)$$

By far the most complicated expression to transform to multipole variables is the term $I[c_{\Delta v,ab}^{(2)}]$ in (117). In terms of the Q matrices and ω coefficients introduced before we find

$$\begin{aligned}
\hat{I}[c_{\Delta v, X}^{(2)}] &= \frac{2}{3} \sum_{m_2=-1}^1 \sum_{l_1=|l-1|}^{l+1} \sum_{m_1=-l_1}^{l_1} i^{l-l_1-1} v_{e, [m_2]}^{(1)} \\
&\times \left\{ \begin{pmatrix} l_1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} \left[\begin{pmatrix} l_1 & 1 & l \\ F_X & 0 & F_X \end{pmatrix} \sum_Y H_{XY}^*(l_1+1-l) f_{Y, l_1 m_1}^{(1)}(p) \right. \right. \\
&+ \left. \sum_{Y, Z} \begin{pmatrix} l_1 & 1 & l \\ F_Y & 0 & F_Y \end{pmatrix} \omega_{XZ}^{(l)} H_{ZY}^*(l_1+1-l) \left(2f_{Y, l_1 m_1}^{(1)}(p) + p \frac{\partial}{\partial p} f_{Y, l_1 m_1}^{(1)}(p) \right) \right] \\
&- (-1)^{m_2} \frac{2l+1}{2l_1+1} \begin{pmatrix} l & 1 & l_1 \\ m & -m_2 & m_1 \end{pmatrix} \begin{pmatrix} l & 1 & l_1 \\ F_X & 0 & F_X \end{pmatrix} \\
&\times \left. \sum_{Y, Z} H_{XZ}^*(l+1-l_1) \omega_{ZY}^{(l_1)} \left(f_{Y, l_1 m_1}^{(1)}(p) + p \frac{\partial}{\partial p} f_{Y, l_1 m_1}^{(1)}(p) \right) \right\} \\
&+ \frac{2}{3} \sum_{m_2=-1}^1 \sum_{l_1=|l-1|}^{l+1} \sum_{m_1=-l_1}^{l_1} i^{l-l_1} \sum_{Y, Z} 3f_{Y, l_1 m_1}^{(1)}(p) \\
&\times \left\{ (-1)^{m_2} \sum_{L=|l-1|}^{l+1} \sum_{M=-L}^L \frac{2l+1}{\sqrt{(2l_1+1)(2L+1)}} \begin{pmatrix} l & 1 & L \\ m & -m_2 & M \end{pmatrix} \begin{pmatrix} l & 1 & L \\ F_X & 0 & F_X \end{pmatrix} \right. \\
&\times H_{XZ}^*(l+1-L) [v_e^{(1)j} \xi_{m_2}^k Q_{Z, LM}^{ij*} Q_{Y, l_1 m_1}^{ik} + v_e^{(1)i} \xi_{m_2}^k Q_{Z, LM}^{ij*} Q_{Y, l_1 m_1}^{kj}] \\
&+ \sum_{L=|l-1|}^{l+1} \sum_{M=-L}^L \sqrt{\frac{2l+1}{2L+1}} \begin{pmatrix} l_1 & 1 & L \\ m_1 & m_2 & M \end{pmatrix} \begin{pmatrix} l_1 & 1 & L \\ F_Y & 0 & F_Y \end{pmatrix} \\
&\times \left. [v_e^{(1)k} \xi_{m_2}^j Q_{X, lm}^{ij*} Q_{Z, LM}^{ik} + v_e^{(1)k} \xi_{m_2}^i Q_{X, lm}^{ij*} Q_{Z, LM}^{kj}] H_{ZY}^*(l_1+1-L) \right\}. \tag{133}
\end{aligned}$$

We have made the sums over Y, Z explicit here. While providing a closed expression for any $X = I, V, E, B$, this result is not very transparent. Recalling that the Q matrices and ω coefficients vanish for $l, l_1, L > 2$ and noting that the Clebsch-Gordan coefficients are nonzero only if the angular momenta differ by no more than one, we see that $\hat{I}[c_{\Delta v, X}^{(2)}]$ vanishes when $l > 3$. For any particular X the sums can be worked out explicitly at the expense of introducing explicit values of the Clebsch-Gordan coefficients.

We give the corresponding simpler expressions in our summary of the Boltzmann hierarchy in Sec. V. The last two terms to be converted to the multipole representation, $I[c_{vv, ab}^{(2)}]$ and $I[c_{K, ab}^{(2)}]$, are relatively simple, since they depend only on the unperturbed phase-space densities. In particular, $I[c_{K, ab}^{(2)}]$ does not contain any perturbation variables, hence angular dependence, since it arises from the nonrelativistic expansion. The result is

$$\begin{aligned}
\hat{I}[c_{vv, X}^{(2)}] &= \frac{2}{3} \left\{ \delta_{XI} p \frac{\partial f_I^{(0)}(p)}{\partial p} \left[\delta_{l0} \delta_{m0} v_e^{(1)2} - \sum_{m_1, m_2=-1}^1 v_{e, [m_1]}^{(1)} v_{e, [m_2]}^{(1)} i^l \begin{pmatrix} 1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} 1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \right] \right. \\
&+ \delta_{XI} p^2 \frac{\partial^2 f_I^{(0)}(p)}{\partial p^2} \left[\delta_{l0} \delta_{m0} \frac{1}{5} v_e^{(1)2} - \sum_{m_1, m_2=-1}^1 \frac{1}{2} v_{e, [m_1]}^{(1)} v_{e, [m_2]}^{(1)} i^l \begin{pmatrix} 1 & 1 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} 1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \right] \\
&\left. - i^l \sqrt{2l+1} \frac{1}{10} v_e^{(1)i} v_e^{(1)j} Q_{X, lm}^{ij*} p^2 \frac{\partial^2 f_I^{(0)}(p)}{\partial p^2} \right\}, \tag{134}
\end{aligned}$$

$$\hat{I}[c_{K, X}^{(2)}] = \frac{2}{3} \delta_{XI} \delta_{l0} \delta_{m0} \left\{ \frac{4p}{m_e} f_I^{(0)}(p) \left(1 + f_I^{(0)}(p) \right) + \left[\frac{4T_e}{m_e} + \frac{p}{m_e} \left(1 + 2f_I^{(0)}(p) \right) \right] p \frac{\partial f_I^{(0)}(p)}{\partial p} + \frac{T_e}{m_e} p^2 \frac{\partial^2 f_I^{(0)}(p)}{\partial p^2} \right\}. \tag{135}$$

Here and above in (133) we expressed the result directly in the Stokes parameter basis. The result in the circular polarization basis is obtained by omitting the sums over Y, Z and replacing $H_{PQ}^*(\dots) \rightarrow \delta_{PQ}$ (any P, Q); by substituting $\delta_{XI} \rightarrow \delta_{ab}$, $X \rightarrow ab$, $Y \rightarrow \lambda' \omega'$, $F_X \rightarrow -s$, $F_Y \rightarrow -s'$ (with s chosen according to the value of ab and s' according to $\lambda' \omega'$), as well as $\omega_{XY}^{(l)} \rightarrow \omega_{ba, \omega' \lambda'}^{(l)}/2$, $Q_{X, lm}^{ij*} \rightarrow Q_{ba, lm}^{ij*}$, $Q_{Y, lm}^{ij} \rightarrow Q_{\omega' \lambda', lm}^{ij}$ (any i, j, l, m) in (130)–(135).

V. BOLTZMANN HIERARCHY AT SECOND ORDER—SUMMARY OF EQUATIONS

At this point we can return to our original notation and express the phase-space densities in terms of the comoving momentum $q = ap$, using $p \partial f_X(p) / \partial p = q \partial f_X / \partial q$. In the following we leave away the photon momentum argument q on the phase-space densities but restore the Fourier mode momentum writing $f_{X,lm}(\mathbf{k}_i)$. As mentioned above, by taking the four values of X separately, we can evaluate the angular momentum sums over l, l' , etc., and obtain a more explicit form of the Boltzmann hierarchy. We summarize the second-order equations in this section. For convenience we recall the first-order equations in the absence of first-order vector and tensor modes in the present notation:

$$\begin{aligned} \frac{\partial}{\partial \eta} f_{l,lm}^{(1)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{l,(\pm 1)m_1}^{(1)}(\mathbf{k}) k^{[m_2]} C_{m_1 m}^{\pm, l} - \delta_{l0} q \frac{\partial f_I^{(0)}}{\partial q} \dot{D}^{(1)}(\mathbf{k}) - i \delta_{l1} q \frac{\partial f_I^{(0)}}{\partial q} k^{[m]} A^{(1)}(\mathbf{k}) \\ = |\dot{\kappa}| \left\{ -f_{l,lm}^{(1)}(\mathbf{k}) + \delta_{l0} f_{l,00}^{(1)}(\mathbf{k}) - \delta_{l1} q \frac{\partial f_I^{(0)}}{\partial q} v_{e,[m]}^{(1)}(\mathbf{k}) + \delta_{l2} \frac{1}{10} \left(f_{l,2m}^{(1)}(\mathbf{k}) - \sqrt{6} f_{E,2m}^{(1)}(\mathbf{k}) \right) \right\}, \end{aligned} \quad (136)$$

$$\frac{\partial}{\partial \eta} f_{V,lm}^{(1)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{V,(\pm 1)m_1}^{(1)}(\mathbf{k}) k^{[m_2]} C_{m_1 m}^{\pm, l} = |\dot{\kappa}| \left\{ -f_{V,lm}^{(1)}(\mathbf{k}) + \delta_{l1} \frac{1}{2} f_{V,1m}^{(1)}(\mathbf{k}) \right\}, \quad (137)$$

$$\begin{aligned} \frac{\partial}{\partial \eta} f_{E,lm}^{(1)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{E,(\pm 1)m_1}^{(1)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{\pm, l} - i f_{B,lm_1}^{(1)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{0,l} \\ = |\dot{\kappa}| \left\{ -f_{E,lm}^{(1)}(\mathbf{k}) - \delta_{l2} \frac{\sqrt{6}}{10} \left(f_{l,2m}^{(1)}(\mathbf{k}) - \sqrt{6} f_{E,2m}^{(1)}(\mathbf{k}) \right) \right\}, \end{aligned} \quad (138)$$

$$\frac{\partial}{\partial \eta} f_{B,lm}^{(1)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{B,(\pm 1)m_1}^{(1)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{\pm, l} + i f_{E,lm_1}^{(1)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{0,l} = |\dot{\kappa}| \left\{ -f_{B,lm}^{(1)}(\mathbf{k}) \right\}. \quad (139)$$

Here we introduced the abbreviation

$$\dot{\kappa} = -n_e^{(0)} \sigma_T a < 0 \quad (140)$$

for the collision rate. Furthermore, here and below a summation over $m_2 = 0, \pm 1$ is implicitly understood in terms containing the index m_2 , and m_1 is equal to $m - m_2$. We also introduce the coupling coefficients

$$\begin{aligned} C_{m_1 m}^{+,l} = -\frac{\sqrt{(l+1 \pm m)(l+2 \pm m)}}{\sqrt{2}(2l+3)} \quad C_{m_1 m}^{+,l} = \frac{\sqrt{(l+1)^2 - m^2}}{2l+3} \quad C_{m_1 m}^{-,l} = \frac{\sqrt{(l-1 \mp m)(l \mp m)}}{\sqrt{2}(2l-1)} \quad C_{m_1 m}^{-,l} = \frac{\sqrt{l^2 - m^2}}{2l-1} \\ D_{m_1 m}^{+,l} = \frac{\sqrt{(l-1)(l+3)}}{l+1} C_{m_1 m}^{+,l} \quad D_{m_1 m}^{-,l} = \frac{\sqrt{l^2 - 4}}{l} C_{m_1 m}^{-,l} \quad D_{m_1 m}^{0,l} = \mp \frac{\sqrt{2(l+1 \pm m)(l \mp m)}}{l(l+1)} \quad D_{m_1 m}^{0,l} = -\frac{2m}{l(l+1)}, \end{aligned} \quad (141)$$

as well as

$$\begin{aligned} R_{m_1 m}^{+,l} = -(l+2) C_{m_1 m}^{+,l}, \quad R_{m_1 m}^{-,l} = (l-1) C_{m_1 m}^{-,l}, \\ K_{m_1 m}^{+,l} = -(l+2) D_{m_1 m}^{+,l}, \quad K_{m_1 m}^{-,l} = (l-1) D_{m_1 m}^{-,l}, \\ K_{m_1 m}^{0,l} = -D_{m_1 m}^{0,l}. \end{aligned} \quad (142)$$

Note that we may choose \mathbf{k} such that it points into the three-direction, in which case $k_{[\pm 1]} = 0$ and the first-order equations become particularly simple.

We now present our main result, the Boltzmann hierarchy for the second-order perturbations to the polarized

phase-space densities. Recall that the equations are given in conformal Newtonian gauge for a comoving and aligned observer ($U_i = 0, \theta_i = 0$) under the assumptions of vanishing first-order vector and tensor modes ($B_i^{(1)} = E_{ij}^{(1)} = 0$). In the equations given below we keep terms involving the first-order perturbations $f_{B,lm}^{(1)}$ of the B -polarization density to display their structure. Of course, under the above assumptions there is no B polarization in first order, so $f_{B,lm}^{(1)}$ vanishes, and the corresponding terms can be neglected in numerical evaluations. The equations read

$$\begin{aligned}
& \frac{\partial}{\partial \eta} f_{I,lm}^{(2)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{I,(l\pm 1)m_1}^{(2)}(\mathbf{k}) k^{[m_2]} C_{m_1 m}^{\pm, l} - \delta_{l0} q \frac{\partial f_I^{(0)}}{\partial q} \dot{D}^{(2)}(\mathbf{k}) \\
& + \delta_{l1} q \frac{\partial f_I^{(0)}}{\partial q} \left(-i k_{[m]} A^{(2)}(\mathbf{k}) + \dot{B}_{[m]}^{(2)}(\mathbf{k}) \right) - \delta_{l2} q \frac{\partial f_I^{(0)}}{\partial q} \alpha_m \dot{E}_{[m]}^{(2)}(\mathbf{k}) \\
& - \dot{D}^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{I,lm}^{(1)}(\mathbf{k}_2) + \sum_{\pm} (\mp i) \left\{ k_1^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) f_{I,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) R_{m_1 m}^{\pm, l} \right. \\
& + \left. \left(k_2^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) - k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} \right) f_{I,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{\pm, l} \right\} \\
& + 2 \delta_{l0} q \frac{\partial f_I^{(0)}}{\partial q} \dot{D}^{(1)}(\mathbf{k}_1) D^{(1)}(\mathbf{k}_2) + \delta_{l1} q \frac{\partial f_I^{(0)}}{\partial q} \left(i k_1^{[m]} A^{(1)}(\mathbf{k}_1) \left(A^{(1)} + D^{(1)} \right) (\mathbf{k}_2) \right) \\
= & |\dot{\mathbf{k}}| \left\{ -f_{I,lm}^{(2)}(\mathbf{k}) + \delta_{l0} f_{I,00}^{(2)}(\mathbf{k}) - \delta_{l1} q \frac{\partial f_I^{(0)}}{\partial q} \mathbf{v}_{e,[m]}^{(2)}(\mathbf{k}) + \delta_{l2} \frac{1}{10} \left(f_{I,2m}^{(2)}(\mathbf{k}) - \sqrt{6} f_{E,2m}^{(2)}(\mathbf{k}) \right) \right. \\
& + \left(A^{(1)} + \left[\frac{\delta \rho_b}{\rho_b} \right]^{(1)} + \left[\frac{\delta x_e}{x_e} \right]^{(1)} \right) (\mathbf{k}_1) \left(-f_{I,lm}^{(1)}(\mathbf{k}_2) + \delta_{l0} f_{I,00}^{(1)}(\mathbf{k}_2) - \delta_{l1} q \frac{\partial f_I^{(0)}}{\partial q} \mathbf{v}_{e,[m]}^{(1)}(\mathbf{k}_2) \right. \\
& + \left. \delta_{l2} \frac{1}{10} \left(f_{I,2m}^{(1)} - \sqrt{6} f_{E,2m}^{(1)} \right) (\mathbf{k}_2) \right) \\
& + \sum_{\pm} (\mp 1) \mathbf{v}_{e,[m_2]}^{(1)}(\mathbf{k}_1) f_{I,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{\pm, l} \\
& + \delta_{l0} \left\{ -\mathbf{v}_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left(q \frac{\partial}{\partial q} f_{I,1m_1}^{(1)} + 2 f_{I,1m_1}^{(1)} \right) (\mathbf{k}_2) C_{m_1 m}^{+, 0} \right. \\
& - \left. \mathbf{v}_{e,[m_1]}^{(1)}(\mathbf{k}_1) \mathbf{v}_{e,[m_2]}^{(1)}(\mathbf{k}_2) \left(4q \frac{\partial f_I^{(0)}}{\partial q} + q^2 \frac{\partial^2 f_I^{(0)}}{\partial q^2} \right) C_{m_1 m}^{+, 0} \right. \\
& + \left. \frac{4q}{m_e} f_I^{(0)} (1 + f_I^{(0)}) + \left[\frac{4T_e}{m_e} + \frac{q}{m_e} (1 + 2f_I^{(0)}) \right] q \frac{\partial f_I^{(0)}}{\partial q} + \frac{T_e}{m_e} q^2 \frac{\partial^2 f_I^{(0)}}{\partial q^2} \right\} \\
& + \delta_{l1} \mathbf{v}_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left\{ - \left(q \frac{\partial}{\partial q} + 1 \right) f_{I,0m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{-, 1} \right. \\
& + \left. \frac{1}{10} \left(q \frac{\partial}{\partial q} + 4 \right) \left(f_{I,2m_1}^{(1)} - \sqrt{6} f_{E,2m_1}^{(1)} \right) (\mathbf{k}_2) C_{m_1 m}^{+, 1} \right\} \\
& + \delta_{l2} \left\{ \frac{1}{10} \mathbf{v}_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left[\left(q \frac{\partial}{\partial q} - 1 \right) f_{I,1m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{-, 2} \right. \right. \\
& - \left. \left(q \frac{\partial}{\partial q} + 4 \right) \left(f_{I,3m_1}^{(1)} - \sqrt{\frac{10}{3}} f_{E,3m_1}^{(1)} \right) (\mathbf{k}_2) C_{m_1 m}^{+, 2} + \sqrt{6} \left(q \frac{\partial}{\partial q} + 1 \right) f_{B,2m_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{0, 2} \right] \\
& + \left. \mathbf{v}_{e,[m_1]}^{(1)}(\mathbf{k}_1) \mathbf{v}_{e,[m_2]}^{(1)}(\mathbf{k}_2) \left(q \frac{\partial f_I^{(0)}}{\partial q} + \frac{11}{20} q^2 \frac{\partial^2 f_I^{(0)}}{\partial q^2} \right) C_{m_1 m}^{-, 2} \right\} \\
& + \delta_{l3} \frac{1}{10} \mathbf{v}_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left(-q \frac{\partial}{\partial q} + 1 \right) \left(f_{I,2m_1}^{(1)} - \sqrt{6} f_{E,2m_1}^{(1)} \right) (\mathbf{k}_2) C_{m_1 m}^{-, 3} \left. \right\}, \tag{143}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial \eta} f_{V,lm}^{(2)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{V,(l\pm 1)m_1}^{(2)}(\mathbf{k}) k^{[m_2]} C_{m_1 m}^{\pm, l} \\
& - \dot{D}^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{V,lm}^{(1)}(\mathbf{k}_2) + \sum_{\pm} (\mp i) \left\{ k_1^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) f_{V,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) R_{m_1 m}^{\pm, l} \right. \\
& + \left. \left(k_2^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) - k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} \right) f_{V,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{\pm, l} \right\} \\
& = |\dot{\kappa}| \left\{ -f_{V,lm}^{(2)}(\mathbf{k}) + \delta_{l1} \frac{1}{2} f_{V,1m}^{(2)}(\mathbf{k}) \right. \\
& + \left(A^{(1)} + \left[\frac{\delta \rho_b}{\rho_b} \right]^{(1)} + \left[\frac{\delta x_e}{x_e} \right]^{(1)} \right) (\mathbf{k}_1) \left(-f_{V,lm}^{(1)}(\mathbf{k}_2) + \delta_{l1} \frac{1}{2} f_{V,1m}^{(1)}(\mathbf{k}_2) \right) \\
& + \sum_{\pm} (\mp 1) v_{e,[m_2]}^{(1)}(\mathbf{k}_1) f_{V,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{\pm, l} \\
& + \delta_{l0} \frac{1}{2} v_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left(q \frac{\partial}{\partial q} + 3 \right) f_{V,1m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{+, 0} \\
& + \delta_{l1} \frac{1}{2} v_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left(q \frac{\partial}{\partial q} f_{V,0m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{-, 1} - \left(q \frac{\partial}{\partial q} + 3 \right) f_{V,2m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{+, 1} \right) \\
& \left. - \delta_{l2} \frac{1}{2} v_{e,[m_2]}^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{V,1m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{-, 2} \right\}, \tag{144}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial \eta} f_{E,lm}^{(2)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{E,(l\pm 1)m_1}^{(2)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{\pm, l} - i f_{B,lm_1}^{(2)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{0, l} \\
& - \dot{D}^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{E,lm}^{(1)}(\mathbf{k}_2) + \sum_{\pm} (\mp i) \left\{ k_1^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) f_{E,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) K_{m_1 m}^{\pm, l} \right. \\
& + \left. \left(k_2^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) - k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} \right) f_{E,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{\pm, l} \right\} \\
& - i k_1^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) f_{B,lm_1}^{(1)}(\mathbf{k}_2) K_{m_1 m}^{0, l} \\
& - i \left(k_2^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) - k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} \right) f_{B,lm_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{0, l} \\
& = |\dot{\kappa}| \left\{ -f_{E,lm}^{(2)}(\mathbf{k}) - \delta_{l2} \frac{\sqrt{6}}{10} \left(f_{I,2m}^{(2)}(\mathbf{k}) - \sqrt{6} f_{E,2m}^{(2)}(\mathbf{k}) \right) \right. \\
& + \left(A^{(1)} + \left[\frac{\delta \rho_b}{\rho_b} \right]^{(1)} + \left[\frac{\delta x_e}{x_e} \right]^{(1)} \right) (\mathbf{k}_1) \left(-f_{E,lm}^{(1)}(\mathbf{k}_2) - \delta_{l2} \frac{\sqrt{6}}{10} \left(f_{I,2m}^{(1)}(\mathbf{k}_2) - \sqrt{6} f_{E,2m}^{(1)}(\mathbf{k}_2) \right) \right) \\
& + \sum_{\pm} (\mp 1) v_{e,[m_2]}^{(1)}(\mathbf{k}_1) f_{E,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{\pm, l} - v_{e,[m_2]}^{(1)}(\mathbf{k}_1) f_{B,lm_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{0, l} \\
& + \delta_{l2} \frac{\sqrt{6}}{10} \left\{ v_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left[\left(-q \frac{\partial}{\partial q} + 1 \right) f_{I,1m_1}^{(1)}(\mathbf{k}_2) C_{m_1 m}^{-, 2} \right. \right. \\
& + \left. \left. \left(q \frac{\partial}{\partial q} + 4 \right) \left(f_{I,3m_1}^{(1)} - \sqrt{\frac{10}{3}} f_{E,3m_1}^{(1)} \right) (\mathbf{k}_2) C_{m_1 m}^{+, 2} - \sqrt{6} \left(q \frac{\partial}{\partial q} + 1 \right) f_{B,2m_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{0, 2} \right] \right. \\
& \left. - \frac{1}{2} v_{e,[m_1]}^{(1)}(\mathbf{k}_1) v_{e,[m_2]}^{(1)}(\mathbf{k}_2) q^2 \frac{\partial^2 f_I^{(0)}}{\partial q^2} C_{m_1 m}^{-, 2} \right\} \\
& + \delta_{l3} \frac{\sqrt{6}}{10} v_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left(q \frac{\partial}{\partial q} - 1 \right) \left(f_{I,2m_1}^{(1)} - \sqrt{6} f_{E,2m_1}^{(1)} \right) (\mathbf{k}_2) D_{m_1 m}^{-, 3} \left. \right\} \tag{145}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial \eta} f_{B,lm}^{(2)}(\mathbf{k}) + \sum_{\pm} (\mp i) f_{B,(l\pm 1)m_1}^{(2)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{\pm, l} + i f_{E,lm_1}^{(2)}(\mathbf{k}) k^{[m_2]} D_{m_1 m}^{0, l} \\
& - \dot{D}^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} f_{B,lm}^{(1)}(\mathbf{k}_2) + \sum_{\pm} (\mp i) \left\{ k_1^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) f_{B,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) K_{m_1 m}^{\pm, l} \right. \\
& + \left. \left(k_2^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) - k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} \right) f_{B,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{\pm, l} \right\} \\
& + i k_1^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) f_{E,lm_1}^{(1)}(\mathbf{k}_2) K_{m_1 m}^{0, l} \\
& + i \left(k_2^{[m_2]} \left(A^{(1)} - D^{(1)} \right) (\mathbf{k}_1) - k_1^{[m_2]} A^{(1)}(\mathbf{k}_1) q \frac{\partial}{\partial q} \right) f_{E,lm_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{0, l} \\
& = |\dot{\kappa}| \left\{ -f_{B,lm}^{(2)}(\mathbf{k}) - \left(A^{(1)} + \left[\frac{\delta \rho_b}{\rho_b} \right]^{(1)} + \left[\frac{\delta x_e}{x_e} \right]^{(1)} \right) (\mathbf{k}_1) f_{B,lm}^{(1)}(\mathbf{k}_2) \right. \\
& + \sum_{\pm} (\mp 1) v_{e,[m_2]}^{(1)}(\mathbf{k}_1) f_{B,(l\pm 1)m_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{\pm, l} + v_{e,[m_2]}^{(1)}(\mathbf{k}_1) f_{E,lm_1}^{(1)}(\mathbf{k}_2) D_{m_1 m}^{0, l} \\
& \left. + \delta_{l2} \frac{\sqrt{6}}{10} v_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left(q \frac{\partial}{\partial q} + 2 \right) \left(f_{l,2m_1}^{(1)} - \sqrt{6} f_{E,2m_1}^{(1)} \right) (\mathbf{k}_2) D_{m_1 m}^{0, 2} \right\}. \tag{146}
\end{aligned}$$

These are the dynamical equations for the second-order photon variables. The source terms depend on products of first-order perturbations as well as on the second-order perturbations $A^{(2)}$, $D^{(2)}$, $B_{[m]2}^{(2)}$, $E_{[m]}^{(2)}$ to the metric and to the bulk electron velocity $v_{e,[m]}^{(1)}$. To close the system of equations, these quantities must be determined from the second-order Einstein and fluid equations.

At this point it seems appropriate to compare our results to those given in [18]. We already mentioned that the collision term in [18] takes a different form before expansion of the phase-space distributions around the equilibrium distributions, but that these structural differences drop out at second order, at least for the frequency-integrated equations. The derivation of the expanded equations in [18] follows a different method from the one employed in the present paper by first considering the collision term in the electron rest frame, and then performing the boost to the frame, in which the electron fluid moves with bulk velocity \mathbf{v}_e . In contrast, we work directly in this frame adopting the Maxwell-Boltzmann distribution (91) for the electrons. Both methods should give the same results, since the Lorentz noncovariance of the shifted Maxwell-Boltzmann distribution is a higher-order effect. For a detailed comparison we note that only the frequency-integrated equations for the quantities $\Delta_{X,lm}^{(n)}(\eta, \mathbf{k})$ defined in (148) below are given explicitly in [18] and that the contribution from c_K in (135) is neglected. The integrated equations can be obtained from the above by applying the substitution rules (149). After doing this we find that the structure of the equations is in complete agreement but we observe differences in the following terms: the octupole collision source term for E -mode polarization [the δ_{l3} term in our (145)] has different numerical coefficients (this is corrected in the arXiv version of [18]); in the B -mode equation [our (146)] the coupling coefficient ${}^{\pm 1}\lambda_l^m$ differs

from our corresponding $D_{m\pm 1,m}^{0,l}$ in the collision term and second-order Liouville operator, and the terms corresponding to the last line before the equality sign in (146) are missing [32].

VI. DISCUSSION

While a numerical or even qualitative evaluation of the second-order Boltzmann hierarchy is beyond the scope of the present paper, we briefly discuss the sources of B polarization contained in the equations, and the tight-coupling limit. Before proceeding to the discussion of the collision term we note the different l dependences in the weak-lensing and gravitational time-delay terms [33], which we identify as the product terms of $A^{(1)}$, $D^{(1)}$, a mode momentum k_1 or k_2 , and $f_{X,lm}^{(1)}$ on the left-hand side of the Boltzmann equations (143), (44), (145), and (146). While lensing of $X = I, V, E, B$ on itself is proportional to l for large l , since $R^{\pm, l}, K^{\pm, l} \propto l$ (for large l), the corresponding time-delay effect is only of order 1, since $C^{\pm, l}, D^{\pm, l} \propto 1$. In contrast, for conversion of E into B polarization and vice versa, weak lensing and time delay are effects of the same order, and both coefficients involved, $K^{0,l}$ and $D^{0,l}$, are only of order $1/l$ for large l .

A. B -mode polarization from scattering

There are two sources of B -mode polarization in the photon propagation terms on the left-hand side of the Boltzmann equations. A well-known mechanism is the generation of B polarization when polarized radiation propagates through an inhomogeneous universe, usually referred to as the weak-lensing effect. It appears first at second-order and is contained in the terms involving the product of the metric perturbation $A^{(1)}$ or $D^{(1)}$ with the first-

order E -mode distribution $f_{E,lm}^{(1)}$ in the last two lines before the equality sign in (146).

B -mode polarization is further generated in the presence of vector or tensor metric perturbations. Around photon decoupling Thomson scattering generates the vector and tensor components of the E -polarization quadrupole which is subsequently partially converted to B polarization through free-streaming. In the present scenario we assume that there are no first-order vector or tensor metric perturbations. In the absence of any primordial vector or tensor perturbations, they will still be generated at second-order, however. B -mode polarization induced by these second-order perturbations through free-streaming has been estimated in [11]. The effect turns out to be relatively small, though comparable to the weak-lensing effect in the small l region of the BB anisotropy spectrum.

The full second-order Boltzmann equations exhibit further sources for B polarization through the collision term, which are absent in the first-order equation (139), which contains only the damping term $-f_{B,lm}^{(1)}(\mathbf{k})$ on the right-hand side. The second-order collision term in (146) contains products of the electron velocity and first-order intensity and E -mode perturbations. Of particular interest is the term

$$\delta_{l2} \frac{\sqrt{6}}{10} v_{e,[m_2]}^{(1)}(\mathbf{k}_1) \left(q \frac{\partial}{\partial q} + 2 \right) f_{l,2m_1}^{(1)}(\mathbf{k}_2) D_{m_1 m_2}^{0,2}, \quad (147)$$

which can generate a B -mode quadrupole directly from the intensity quadrupole rather than indirectly through E polarization. A numerical analysis of the B polarization generated from this term will be presented in [14].

B. Tight-coupling limit

We now examine the second-order equations in the regime where the electrons and photons are strongly coupled by Thomson scattering. For the following discussion, we are not interested in the frequency dependence of the photon distribution functions and integrate over q . We define the frequency-integrated multipoles

$$\Delta_{X,lm}^{(n)}(\eta, \mathbf{k}) = \frac{\int dq q^3 f_{X,lm}^{(n)}(\eta, \mathbf{k}, q)}{\int dq q^3 f_I^{(0)}(q)}. \quad (148)$$

In the fluid description of photon radiation $\Delta_{l,00}^{(n)}$ equals the fractional perturbations of the photon number density, and $\Delta_{l,1m}^{(n)} = 4v_{\gamma,[m]}^{(n)}$ is related to the bulk velocity of the photon fluid.

Using partial integration, derivatives on photon distributions can be eliminated, resulting in the following substitution rules in the Boltzmann equations in Sec. V:

$$\begin{aligned} f_I^{(0)} &\rightarrow 1 \\ q \frac{\partial f_I^{(0)}}{\partial q} &\rightarrow -4 \\ q^2 \frac{\partial^2 f_I^{(0)}}{\partial q^2} &\rightarrow 20 \\ f_{X,lm}^{(n)} &\rightarrow \Delta_{X,lm}^{(n)} \\ q \frac{\partial f_{X,lm}^{(n)}}{\partial q} &\rightarrow -4 \Delta_{X,lm}^{(n)}. \end{aligned} \quad (149)$$

The only term to which these rules cannot be applied is the c_K contribution from (135) to the collision term for $f_{l,00}^{(2)}$, which contains nonlinear terms in the photon distribution. Inserting the Bose-Einstein distribution for the zeroth-order $f_I^{(0)}$ to calculate (148) for this term, we find

$$\begin{aligned} \frac{4q}{m_e} f_I^{(0)} \left(1 + f_I^{(0)} \right) + \left[\frac{4T_e}{m_e} + \frac{q}{m_e} (1 + 2f_I^{(0)}) \right] q \frac{\partial f_I^{(0)}}{\partial q} \\ + \frac{T_e}{m_e} q^2 \frac{\partial^2 f_I^{(0)}}{\partial q^2} \rightarrow \frac{4(T_e - T)}{m_e}. \end{aligned} \quad (150)$$

But in the strongly coupled electron-photon plasma the electron and photon temperatures coincide, so this term makes no contribution to the frequency-integrated Boltzmann equations.

In the tight-coupling regime the collision rate $|\dot{\kappa}|$ is larger than any other scale of interest. The collision term drives the system to equilibrium, which makes the left-hand sides of the Boltzmann equations small. Thus the Boltzmann equations can be satisfied only if the coefficients of $|\dot{\kappa}|$ in the collision term on the right-hand side nearly vanish. At leading order in the expansion in $1/|\dot{\kappa}|$ this enforces a number of relations among the perturbation variables.

Looking at the first-order equations in Sec. V we immediately find $\Delta_{V,lm}^{(1)} = \Delta_{B,lm}^{(1)} = 0$ for all lm . The intensity equation (136) has no collision term for $l = 0$, so the intensity monopole is unconstrained in the tight-coupling limit. For the dipole $l = 1$, we obtain the familiar relation

$$\Delta_{l,1m}^{(1)}(\mathbf{k}) = 4v_{e,[m]}^{(1)}(\mathbf{k}), \quad (151)$$

which implies that the bulk velocities of the photon and electron plasma are equal. (The continuity equation for the electron fluid yields the same relation.) Continuing with the quadrupoles, we obtain from (136) and (138), the equations

$$\frac{9}{10} \Delta_{l,2m}^{(1)} = -\frac{\sqrt{6}}{10} \Delta_{E,2m}^{(1)} \quad \frac{2}{5} \Delta_{E,2m}^{(1)} = -\frac{\sqrt{6}}{10} \Delta_{l,2m}^{(1)}, \quad (152)$$

which imply $\Delta_{l,2m}^{(1)} = \Delta_{E,2m}^{(1)} = 0$. Likewise, all higher multipoles vanish. It follows that there is no polarization in the tight-coupling limit, as expected, and only the intensity monopole and dipole are unsuppressed.

We now consider the second-order equations in the tight-coupling regime. It is straightforward to see that as in first order, circular and B polarization vanish, $\Delta_{v,lm}^{(2)} = \Delta_{B,lm}^{(2)} = 0$, as well as the multipoles higher than $l = 2$ for I and E . The collision term for the intensity monopole is no longer zero, but vanishes at leading order in the tight-coupling expansion after inserting the relation (151), so the monopole is again unconstrained. Setting $l = 1$ in (143) we find the tight-coupling relation

$$\Delta_{I,1m}^{(2)}(\mathbf{k}) = 4\left(v_{e,[m]}^{(2)}(\mathbf{k}) + v_{e,[m]}^{(1)}(\mathbf{k}_1)\Delta_{I,00}^{(1)}(\mathbf{k}_2)\right), \quad (153)$$

which is similar to (151) but contains a term quadratic in the first-order perturbations. Finally, we examine the quadrupoles. For $l = 2$, we can write (143) and (145) in the form

$$\begin{aligned} \dot{\Delta}_{I,2m}^{(2)} + \dots &= -|\dot{\kappa}| \left[\Delta_{I,2m}^{(2)} + P_I^{(m)} \right], \\ \dot{\Delta}_{E,2m}^{(2)} + \dots &= -|\dot{\kappa}| \left[\Delta_{E,2m}^{(2)} + \sqrt{6}P_E^{(m)} \right], \end{aligned} \quad (154)$$

with

$$\begin{aligned} P_I^{(m)}(\mathbf{k}) &= -\frac{1}{10} \left[\Delta_{I,2m}^{(2)}(\mathbf{k}) - \sqrt{6}\Delta_{E,2m}^{(2)}(\mathbf{k}) \right] \\ &\quad - 9v_{e,[m_1]}(\mathbf{k}_1)v_{e,[m_2]}(\mathbf{k}_2)C_{m_1 m_2}^{-,2} \\ P_E^{(m)}(\mathbf{k}) &= \frac{1}{10} \left[\Delta_{I,2m}^{(2)}(\mathbf{k}) - \sqrt{6}\Delta_{E,2m}^{(2)}(\mathbf{k}) \right] \\ &\quad - 10v_{e,[m_1]}(\mathbf{k}_1)v_{e,[m_2]}(\mathbf{k}_2)C_{m_1 m_2}^{-,2}. \end{aligned} \quad (155)$$

Setting the right-hand sides of (154) to zero yields the tight-coupling relations

$$\begin{aligned} \Delta_{I,2m}^{(2)}(\mathbf{k}) &= 10v_{e,[m_1]}(\mathbf{k}_1)v_{e,[m_2]}(\mathbf{k}_2)C_{m_1 m_2}^{-,2}, \\ \Delta_{E,2m}^{(2)}(\mathbf{k}) &= 0. \end{aligned} \quad (156)$$

We therefore find that there is no polarization in tight-coupling at second order. However, contrary to the first order, there exists a nonvanishing intensity quadrupole quadratic in the electron bulk velocity, as expected. In Cartesian components and before Fourier transformation, Eq. (156) corresponds to

$$\Delta_I^{ij,(2)}(\mathbf{x}) = \frac{4}{3} \left(v_e^i(\mathbf{x})v_e^j(\mathbf{x}) - \frac{1}{3}\delta^{ij}v_e(\mathbf{x})^2 \right) \quad (157)$$

in agreement with [19], where this result has been obtained from the unpolarized Boltzmann hierarchy. (A factor of 2 difference arises due to the different convention for expanding quantities X to second order.) It follows from the above that the size of the quadrupole is not modified when the full polarized set of equations is employed, since the E -polarization quadrupole vanishes in tight coupling.

Our results are at variance, however, with [20], where it has been found that the tight-coupling intensity quadrupole provides a large source for B -mode polarization. The argument is based on an incomplete expression for the

E -polarization source term (155). Since the authors of [20] did not have the Boltzmann equations for E and B polarization at second order available, the source term without the product of first-order perturbations was used,

$$P_E^{(m)} \rightarrow \frac{1}{10} \left[\Delta_{I,2m}^{(2)} - \sqrt{6}\Delta_{E,2m}^{(2)} \right] \rightarrow \frac{\Delta_{I,2m}^{(2)}}{4}. \quad (158)$$

The source term was further simplified using the second of the relations (152) for the second-order modes as done after the second arrow above, which implies the assumption that there exists E polarization in tight-coupling in contradiction with (156). We conclude that in this case it is clearly important that the full second-order polarized equations are used. Then it follows from (155) that $P_E^{(m)} = 0$, and thus there is no source term in tight coupling that would yield E and therefore B polarization from the line-of-sight solutions of (154). The large effect reported in [20] is therefore absent. Polarization is only generated, also at second-order, once the scattering rate drops sufficiently so that corrections to tight coupling become relevant.

VII. CONCLUSION

In this paper we derived the complete Boltzmann hierarchy for the polarized photon phase-space distributions at second order in conformal Newtonian gauge and in the local observer rest frame under the assumption that vector and tensor perturbations are formally of second order. This assumption is well-motivated by the fact that our primary aim is to study the B -mode polarization and non-Gaussianity induced at second order, when the primordial sources are small. A first analysis shows that the B -mode collision term contains new sources that involve the intensity of the perturbation rather than its E polarization. In tight-coupling we obtain the intensity quadrupole found earlier from the unpolarized Boltzmann hierarchy but no E -mode polarization. The equations presented here set the stage for their numerical evaluation, which we plan to present in a subsequent paper.

ACKNOWLEDGMENTS

We thank P. Fischer for collaboration during an early stage of this work, C. Pitrou for correspondence on Ref. [18] and helpful suggestions, and K. Klingmüller and Y. Y. Y. Wong for comments on the manuscript. M. B. wishes to thank the theory groups at the University of Zürich and CERN for their hospitality during 2008 and 2009, when part of this work was performed. This work is supported in part by the Gottfried Wilhelm Leibniz program of the Deutsche Forschungsgemeinschaft.

APPENDIX A: SUMMARY OF DEFINITIONS

1. Tetrad components

The tetrad components before specifying conformal Newtonian gauge are given to second order by

$$\begin{aligned}
 [e_0]^0 &= \frac{1}{a} \left(1 - A + \frac{3}{2} A^{(1)2} - B_i^{(1)} U_i^{(1)} + \frac{1}{2} U_i^{(1)} U_i^{(1)} \right) \\
 [e_0]^i &= \frac{U_i}{a} \\
 [e_k]^0 &= \frac{1}{a} \left(U_k - B_k + (D^{(1)} - A^{(1)}) U_k^{(1)} + (D^{(1)} \right. \\
 &\quad \left. + 2A^{(1)} B_k^{(1)} + E_{kj}^{(1)} (U_j^{(1)} + B_j^{(1)}) \right) \\
 [e_k]^i &= \frac{1}{a} \left(\delta_{ik} \left(1 - D + \frac{3}{2} D^{(1)2} \right) - E_{ik} + \frac{1}{2} B_i^{(1)} B_k^{(1)} \right. \\
 &\quad \left. - \frac{1}{2} U_i^{(1)} U_k^{(1)} - 3D^{(1)} E_{ik}^{(1)} - \frac{3}{2} E_{ij}^{(1)} E_{jk}^{(1)} \right).
 \end{aligned} \tag{A1}$$

Quantities without superscript are expanded according to $X = X^{(1)} + X^{(2)} + \dots$. The one simplification that has been made is that we set to zero the angles θ_k , which defines the orientation of the local inertial coordinate axes relative to those of x^i . The expressions in conformal Newtonian gauge adopted in this paper are given in (2).

2. Spin-weighted spherical harmonics

The spin-weighted spherical harmonics are defined for $l \geq |s|$ and $|m| \leq l$ by

$$\begin{aligned}
 Y_{lm}^s(\theta, \varphi) &= \left(\frac{2l+1}{4\pi} \frac{(l+m)!(l-m)!}{(l+s)!(l-s)!} \right)^{1/2} \sin^{2l} \frac{\theta}{2} \\
 &\quad \times \sum_r \binom{l-s}{r} \binom{l+s}{r+s-m} (-1)^{l-r-s+m} e^{im\varphi} \cot^{2r+s-m} \frac{\theta}{2},
 \end{aligned} \tag{A2}$$

such that for $s = 0$ the standard spherical harmonics are recovered. Y_{lm}^s carries spin s , since under a rotation of the coordinate system with angle $\Delta\Psi$ is transforms as

$$Y_{lm}^{s'} = e^{is\Delta\Psi} Y_{lm}^s. \tag{A3}$$

For any given s the spin-weighted spherical harmonics define a complete set of functions on the sphere obeying

the orthogonality relations

$$\int d\Omega Y_{lm}^{s*} Y_{l'm'}^s = \delta_{ll'} \delta_{mm'}. \tag{A4}$$

Under complex conjugation $Y_{lm}^{s*} = (-1)^{m+s} Y_{l-m}^{-s}$. A product of two spin-weighted spherical harmonics can be combined to a single one using

$$Y_{l_1 m_1}^{s_1} Y_{l_2 m_2}^{s_2} = \sum_{l, m, s} \frac{\sqrt{(2l_1+1)(2l_2+1)}}{\sqrt{4\pi(2l+1)}} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ -s_1 & -s_2 & -s \end{pmatrix} Y_{lm}^s. \tag{A5}$$

The summation ranges are restricted by the triangular equation for the Clebsch-Gordan coefficients

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \neq 0 \quad \text{if } |l_2 - l_1| \leq l \leq l_1 + l_2, \tag{A6}$$

and

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = 0 \quad \text{if } m \neq m_1 + m_2. \tag{A7}$$

This implies in particular $s = s_1 + s_2$ in (A5). Furthermore, the Clebsch-Gordan coefficients satisfy the following relation:

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{l_1+l_2-l} \begin{pmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m \end{pmatrix}. \tag{A8}$$

The spin-raising and -lowering operators are defined by

$$\begin{aligned}
 \bar{\delta}_s &= -\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta} \frac{\partial}{\partial\varphi} + s \cot\theta, \\
 \delta_s &= -\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial\varphi} - s \cot\theta.
 \end{aligned} \tag{A9}$$

We then have

$$\bar{\delta}_s Y_{lm}^s = l_s^+ Y_{lm}^{s+1}, \quad \delta_s Y_{lm}^s = -l_s^- Y_{lm}^{s-1}, \tag{A10}$$

where

$$l_s^\pm = \sqrt{(l \mp s)(l \pm s + 1)}. \tag{A11}$$

3. Unit vector in the spherical basis

The coefficients ξ_m^i and χ_{2m}^{ij} defined in (66) which express n^i and $n^i n^j$ in terms of spherical harmonics are given explicitly by

$$\xi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \xi_{\pm 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp 1 \\ i \\ 0 \end{pmatrix}, \quad (\text{A12})$$

$$\chi_{20} = \frac{1}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \chi_{2,\pm 1} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & i \\ \mp 1 & i & 0 \end{pmatrix} \quad \chi_{2,\pm 2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \mp i & 0 \\ \mp i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A13})$$

4. Q matrices and ω coefficients

The nonvanishing Q matrices introduced in (124) read in the *IVEB* basis

$$\begin{aligned} Q_{I,00}^{ij} &= \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ Q_{I,20}^{ij} &= \frac{1}{6\sqrt{5}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\ Q_{I,21}^{ij} &= \frac{1}{2\sqrt{30}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix} \quad Q_{I,2-1}^{ij} = \frac{1}{2\sqrt{30}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & i & 0 \end{pmatrix} \\ Q_{I,22}^{ij} &= \frac{1}{2\sqrt{30}} \begin{pmatrix} -1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_{I,2-2}^{ij} = \frac{1}{2\sqrt{30}} \begin{pmatrix} -1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ Q_{V,10}^{ij} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ Q_{V,11}^{ij} &= \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix} \quad Q_{V,1-1}^{ij} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix} \\ Q_{E,20}^{ij} &= \frac{1}{\sqrt{30}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ Q_{E,21}^{ij} &= \frac{1}{2\sqrt{5}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ -1 & -i & 0 \end{pmatrix} \quad Q_{E,2-1}^{ij} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix} \\ Q_{E,22}^{ij} &= \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_{E,2-2}^{ij} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A14})$$

The nonzero traces (126) are

$$\omega_{II}^{(0)} = 1, \quad \omega_{VV}^{(1)} = \frac{1}{2}, \quad \omega_{II}^{(2)} = \frac{1}{10}, \quad \omega_{EE}^{(2)} = \frac{3}{5}, \quad \omega_{IE}^{(2)} = \omega_{EI}^{(2)} = -\sqrt{\frac{3}{50}}. \quad (\text{A15})$$

- [1] D. Larson *et al.*, [arXiv:1001.4635](#).
- [2] G. F. Smoot *et al.*, *Astrophys. J.* **396**, L1 (1992).
- [3] J. Kovac *et al.*, *Nature (London)* **420**, 772 (2002).
- [4] C. L. Bennett *et al.* (WMAP Collaboration), *Astrophys. J. Suppl. Ser.* **148**, 1 (2003).
- [5] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, *Phys. Rep.* **402**, 103 (2004).
- [6] M. Liguori, E. Sefusatti, J. R. Fergusson, and E. P. S. Shellard, [arXiv:1001.4707](#).
- [7] D. Baumann *et al.* (CMBPol Study Team Collaboration), *AIP Conf. Proc.* **1141**, 10 (2009).
- [8] A. Lewis and A. Challinor, *Phys. Rep.* **429**, 1 (2006).
- [9] M. Zaldarriaga and U. Seljak, *Phys. Rev. D* **58**, 023003 (1998).
- [10] W. Hu and A. Cooray, *Phys. Rev. D* **63**, 023504 (2000).
- [11] S. Mollerach, D. Harari, and S. Matarrese, *Phys. Rev. D* **69**, 063002 (2004).
- [12] S. Dodelson and J. M. Jubas, *Astrophys. J.* **439**, 503 (1995).
- [13] N. Bartolo, S. Matarrese, and A. Riotto, *J. Cosmol. Astropart. Phys.* 06 (2006) 024.
- [14] M. Beneke, C. Fidler, and K. Klingmüller (unpublished).
- [15] N. Bartolo, S. Matarrese, and A. Riotto, [arXiv:1001.3957](#).
- [16] C. Pitrou, J. P. Uzan, and F. Bernardeau, [arXiv:1003.0481](#).
- [17] C. Fidler, Diplom thesis, RWTH Aachen University, 2007.
- [18] C. Pitrou, *Classical Quantum Gravity* **26**, 065006 (2009).
- [19] N. Bartolo, S. Matarrese, and A. Riotto, *J. Cosmol. Astropart. Phys.* 01 (2007) 019.
- [20] N. Bartolo, S. Matarrese, S. Mollerach, and A. Riotto, [arXiv:astro-ph/0703386](#).
- [21] A. Challinor, *Phys. Rev. D* **62**, 043004 (2000).
- [22] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [23] G. Dautcourt and K. Rose, *Astron. Nachr.* **299**, 13 (1978).
- [24] G. Sigl and G. Raffelt, *Nucl. Phys.* **B406**, 423 (1993).
- [25] A. Kosowsky, *Ann. Phys. (N.Y.)* **246**, 49 (1996).
- [26] D. I. Nagirner and J. Poutanen, *Astron. Astrophys.* **379**, 664 (2001).
- [27] J. Portsmouth and E. Bertschinger, [arXiv:astro-ph/0412094](#).
- [28] L. Senatore, S. Tassev, and M. Zaldarriaga, *J. Cosmol. Astropart. Phys.* 08 (2009) 031.
- [29] J. R. Bond and G. Efstathiou, *Astrophys. J.* **285**, L45 (1984).
- [30] W. Hu, U. Seljak, M. J. White, and M. Zaldarriaga, *Phys. Rev. D* **57**, 3290 (1998).
- [31] U. Seljak, (unpublished).
- [32] C. Pitrou, the author of [18], agrees with these corrections (private communication).
- [33] We thank C. Pitrou for drawing our attention to this point.