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Gravitational deflection of light in the Schwarzschild–de Sitter space-time

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Recent studies suggest that the cosmological constant affects the gravitational bending of photons, although the orbital equation for light in Schwarzschild–de Sitter space-time is free from a cosmological constant. Here we argue that the very notion of a cosmological constant independent of the photon orbit in the Schwarzschild–de Sitter space-time is not proper. Consequently, the cosmological constant has some clear contributions to the deflection angle of light rays. We stress the importance of the study of photon trajectories from the reference objects in bending calculations, particularly for asymptotically nonflat space-time. When such an aspect is taken into consideration, the contribution of a cosmological constant to the effective bending is found to depend on the distances of the source and the reference objects.

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I. INTRODUCTION

A number of recent cosmological observations indicate the presence of a cosmological constant with a value of $\Lambda \simeq 10^{-52}/;m^{-2}$ (e.g. see [1] and references therein). Consequently, the exterior space-time due to a static spherically symmetric mass distribution is the Schwarzschild–de Sitter (SDS) space-time[2]. Recently, working with the SDS geometry, Rindler and Ishak demonstrated [3] that, contrary to the widely held idea [4,5], there is a small contribution of the cosmological constant (Λ) in the gravitational bending of light that diminishes the deflection angle when Λ is positive, although the orbital equation for light in SDS space-time is free from Λ. In a subsequent work Ishak *et al.* [6] have further shown that the contribution of Λ to the bending of light could be significant (larger than the second order term) for many lens systems, such as a cluster of galaxies.

Some later works [7–9] support the conclusion of Rinder and Ishak. In particular, Sereno [7] showed that the deflection angle in SDS space-time contains a term that describes local coupling between the lens (characterized by mass) and Λ. He provided a general expression for the bending angle and claimed that the results of Rindler and Ishak [3] can be recovered from his expression for a specific radial distance of the observer/source.

Some investigations [10–12], however, questioned the findings of Rindler and Ishak. The criticisms about the Rindler-Ishak method mainly rest on the fact that in the SDS universe the lens, source, and observer are moving relative to each other, which has not been incorporated by Rindler and Ishak in their (original) analysis. Khriplovich and Pomeransky [10] accommodated such a dynamic feature by working with Friedmann-Robertson-Walker (FRW) coordinates, whereas Park [11] analyzed the problem by expanding the null geodesic equations following the McVitte metric to first order in mass in newly defined physical spatial coordinates consistent with the expanding universe. Both the works concluded that the cosmological constant plays no role in gravitational lensing. In order to avoid coordinate dependent artifacts, Simpson *et al.* [12] employed the standard technique of cosmological perturbations, and by working in the Newtonian gauge, they obtained that the potential in the perturbed FRW metric has no explicit dependence on the cosmological constant. Thus they concluded that the Λ dependence of the bending angle obtained from the Kottler metric is a gauge artifact result. In a subsequent work Ishak *et al.* [13] addressed the criticisms and stated that the conclusion of no contribution of Λ to the bending angle is mainly due to the improper dropping of relevant terms in calculating the deflection angle, though in [12] the analytic solution for the potential is also verified numerically with no sign of the Λ contribution. Thus the source of the disagreement between the results of [3,12] remains unclear. On the other hand, Sereno [7] argued that the separate Λ contribution is absent in the bending expression of [10,11] because it is included in the angular diameter distance through which bending is expressed in [10,11].

Here we question the very concept of the Λ independency of the orbital equation of light in the SDS geometry. Our reasoning is that the first order differential equation of the null geodesic in the SDS geometry contains a Λ that drops out at the second order. So the solution of the second order differential equation for the null geodesic must also satisfy the parent first order differential equation, and thereby the orbital solution should include Λ. When one integrates the second order differential equation of the null geodesic, the Λ should reappear in the solution through an integration constant.

More importantly, the measurement of the bending requires a ‘‘straight’’ reference line (except in the special case of the Einstein ring). Usually the path of light rays from a reference source (or the same source but in a different position) having an impact parameter much larger than that for the light rays from the source is considered as the reference line for measuring the bending. But in the SDS geometry the light trajectory of such a reference source cannot be treated as a straight line. This is because, in contrast to the gravitational effect due to the mass that falls off quickly with distance, the influence of Λ increases with distance from the source. Hence it is expected that in the SDS geometry the light rays from the reference source will also be affected, possibly to a higher degree, by Λ.

In the present work we will first compute the bending angle in the SDS space-time, taking the proper Λ involved in the solution of the light trajectory, and then we will show that when the light path from the reference source is taken into consideration, the resultant bending in the SDS geometry will appear to increase rather than decrease due to the Λ effect. The possibility of the detection of the Λ effect from the bending angle measurement will be discussed.

In deriving the deflection angle in the SDS geometry we would assume, as is usually done, that the whole lensing system, consisting of the source, the reference, the lens and the observer, is an isolated one; light rays, while moving from the source/reference to the observer, are not influenced by any other object outside the system. We will restrict our discussion only to the case of local (within the galaxy) scale; we are not going to consider the situation involving large distance scales, such as lensing due to clusters of galaxies, etc. Since the global cosmic expansion is not supposed to affect local structures, we consider that the observer, the lens, and the source are static; no relative motion exists between them. Our sole objective is to estimate the bending angle correctly in the presence of the cosmological constant and also to explore whether the Λ contribution to the bending angle can be detected experimentally, in principle.

The plan of the paper is the following. In the next section we will first obtain the orbit equation for light rays in the SDS geometry, and consequently, we will derive the expression for the gravitational deflection angle. In Sec. III we will estimate the expected deviation in the image position of the source with respect to a reference source. Finally we will conclude with our results in Sec. IV.

II. GRAVITATIONAL DEFLECTION IN THE SDS SPACE-TIME

For the computation of the deflection angle we will follow the procedure described in [14]. We consider the following geometrical configuration for the phenomenon of gravitational bending of light. The light emitted by the distant source (S) is deviated by the gravitational source (lens, L), and reaches the observer (O). The angles are measured with respect to the polar axis which is parallel to the undeflected ray (in the absence of a massive object) and passes through the center of the lens. Such a choice of the polar axis has been justified in [13]. The point L is taken as the origin of the coordinate system. Our first target is to estimate theoretically the deflection angle and, subsequently, the image position in the context of SDS geometry.

The metric for the SDS or Kottler space-time is given by (we are using units such that $G = c = 1$)

$$ds^2 = -f_\Lambda(r)dt^2 + \frac{dr^2}{f_\Lambda(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where

$$f_\Lambda(r) = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right), \quad (2)$$

with m being the mass of the lens object. For this space-time the null geodesic equation involving r and ϕ is given by (see [14])

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{f_\Lambda}{r^2} - \frac{1}{b^2} = 0, \quad (3)$$

where $b \equiv r^2 d\phi/dp$ is the first integral of motion that behaves as the impact parameter at large distances, and p is an affine parameter along the null geodesic. Writing $u = 1/r$ and differentiating the above equation with respect to ϕ, one gets the second order differential equation for the null geodesic, which does not contain Λ, as given hereunder,

$$\frac{d^2u}{d\phi^2} + u = 3mu^2, \quad (4)$$

which is the same as that for the Schwarzschild space-time. The general solution of the above path equation up to first order accuracy in m reads as

$$u = \frac{\sin\phi}{R} + \frac{3m}{2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right). \quad (5)$$

The above solution must be a solution of Eq. (3) also, which implies

$$\frac{1}{R} - \frac{m}{R^2} = \left(\frac{1}{b^2} + \frac{\Lambda}{3}\right)^{1/2}. \quad (6)$$

Equation (5), together with the above relation, implies that the orbit equation of light rays in the SDS geometry does contain Λ.

The coordinate angular velocity is given by

$$\frac{dr}{d\phi} = -\frac{r^2}{R}\cos\phi\left(1 - \frac{2m}{R}\sin\phi\right), \quad (7)$$

which vanishes at the coordinate distance of closest approach (r_o) that occurs when $\phi = \pi/2$. The parameter R is thus related with r_o through the following relation:

$$\frac{1}{r_o} = \frac{1}{R} + \frac{m}{R^2}. \tag{8}$$

Note that R is an integration constant which is related to b and Λ via Eq. (6). Since the parameters b and Λ are involved in the parent geodesic equation [Eq. (3)], one must express the solution in terms of these parameters rather than R. It may be convenient to express the orbit of the photon in terms of R [as in Eq. (5)] but one has to note that this parameter (R) has no independent locus standi. Hence the coordinate distance of closest approach [Eq. (8)] also depends on Λ through Eq. (6).

For asymptotically flat space-times such as the Schwarzschild space-time, the direction of asymptotic light rays is usually evaluated by applying the limit $r \to \infty$ in the orbit equation, and the angle between the two asymptotic directions gives the total deflection angle. However, $r \approx \sqrt{3/\Lambda}$ gives the de Sitter horizon. Hence $r \to \infty$ does not make any sense in SDS space-time. This was one of the main objections raised by Rindler and Ishak [3] against the conventional treatment of calculating bending in SDS space-time. As a solution they proposed to consider the angle that the tangent to the light trajectory made with a coordinate direction at a given point which for the general metric (1) is given by

$$\tan(\psi) = r f(r)^{1/2} |d\phi/dr|. \tag{9}$$

For the null geodesic the above equation reduces to [14]

$$\tan(\psi) = \left[\frac{f(r_o)}{f(r)}\frac{r^2}{r_o^2} - 1\right]^{-1/2}, \tag{10}$$

which to the leading order in m and Λ gives

$$\tan(\psi) = \frac{r_o}{r} + \frac{m}{r} - \frac{m r_o}{r^2} - \frac{\Lambda r_o r}{6} + \frac{\Lambda r_o^3}{6r}. \tag{11}$$

When $r \gg r_o$ the angles ϕ and ψ will be small, and consequently, one may take $\sin(\phi) \to \phi$ and $\tan(\psi) \to \psi$. Using Eq. (5) we get the angle between the tangent to the light trajectory at point (r, ϕ) and the polar axis to the leading order in m, Λ, and r_o/r,

$$|\epsilon| = |\psi - \phi| = \frac{2m}{r_o} - \frac{m r_o}{r^2} - \frac{\Lambda r_o r}{6} + \frac{\Lambda r_o^3}{6r}. \tag{12}$$

At this juncture an important question is, at what coordinate point(s) is the angle ψ to be determined? It appears that the selection of the points on the orbit at which the tangents are to be drawn for estimation of angles remains somewhat arbitrary in the literature. For instance, in their basic work, while obtaining the bending angle for the SDS metric, Rindler and Ishak [3] used the point $\phi = 0$ (the corresponding r follows from the orbit equation) purely on the basis of convention, whereas in a subsequent work [6] the angle was determined at the boundary of the SDS vacuole. While the former choice is not a proper one [the angular position of the observer cannot be taken as zero in the coordinate system considered here; if it is taken forcefully the observer distance will become fixed by the distance of closest approach, as may be seen from Eq. (5) and (8)], the latter choice has limited applicability. Thus a straightforward approach should be to calculate the angles directly at the location of the observer and the source.

The angle of transmission ϵ_s and reception ϵ_o with respect to the polar axis can be straightway computed from Eq. (12) at the location of the source ($d_{\rm LS}$, ϕ_s) and the observer ($d_{\rm OL}$, ϕ_o), respectively, and the total deflection angle thus reads

$$|\epsilon| = \frac{4m}{r_o} - 2m r_o\left(\frac{1}{d_{\rm LS}^2} + \frac{1}{d_{\rm OL}^2}\right) - \frac{\Lambda r_o}{6}(d_{\rm OL} + d_{\rm LS}) + \frac{\Lambda r_o^3}{6}\left(\frac{1}{d_{\rm OL}} + \frac{1}{d_{\rm LS}}\right). \tag{13}$$

The distance of closest approach is a coordinate dependent variable; it is not a measurable one. Identifying the measured radius of an object with coordinate distance of closest approach works tolerably well only up to the first order level for a standard or an isotropic coordinate system, but such an approximation does not work at second or higher order in m [15]. It is proper to express the bending angle in terms of a coordinate independent quantity such as the apparent impact parameter b.

The relation between b and $r_{\rm min}$ may be obtained from Eq. (3),

$$\frac{1}{r_o} - \frac{m}{r_o^2} = \frac{1}{b} - \frac{\Lambda b}{6}. \tag{14}$$

Exploiting the above relation, one finally gets, to the leading order in m and Λ, the total deflection angle in terms of the impact parameter,

$$|\epsilon| = \frac{4m}{b} - 2mb\left(\frac{1}{d_{\rm LS}^2} + \frac{1}{d_{\rm OL}^2}\right) + \frac{2m\Lambda b}{3} - \frac{\Lambda b}{6}(d_{\rm OL} + d_{\rm LS}) + \frac{\Lambda b^3}{6}\left(\frac{1}{d_{\rm OL}} + \frac{1}{d_{\rm LS}}\right). \tag{15}$$

The fourth term on the right-hand side of the above equation [or the third term in Eq. (13)] may appear to be unphysical, as it rapidly increases with lens-source distance, but it just reflects the gravitational potential in the presence of Λ. Basically this term arises due to the asymptotically nonflat nature of the SDS geometry. Both the source and the observer are located at the nonflat region of space-time produced by the lens, and hence it is natural that the final expression of the bending angle will contain terms relating to the gravitational potential at the location of the source and the observer. Also note that the source-lens distance cannot exceed the de Sitter horizon radius in any case. An analogous term in the expression of the bending angle was obtained by Ishak *et al.* [6] for Schwarzschild–de Sitter geometry in the framework of the Einstein-Strauss vacuole model [16].

One may note that the Λ contribution part in the above expression is not the same as the one obtained in [3]. This is mainly due to the use of the modified (proper) orbit equation. Moreover, in [3] the angle ψ has been determined at a different (improper) coordinate point. If we forcefully take $\phi = 0$ in Eq. (5), then $1/r = 2m/R^2$. On substitution of this r, the third term on the right-hand side of Eq. (12) becomes $\Lambda R^3/(12m)$, which is what Rindler and Ishak found in their work [3]. The third term on the right-hand side of Eq. (15) is the one that Sereno qualified as local [7] since it contains m and Λ but not the positional coordinates of the source/observer.

It appears from the above expression that, even in a Solar System observation, the Λ contribution to the bending can be considerable if the source is a large distance away from us. For instance, in the case of a light ray grazing the limb of the Sun (so that $b \simeq R_\odot$) and if the source distance is 10 kpc, which is roughly equal to the distance of the Sun from the Galactic center, the ratio of the Λ contribution to the main general relativistic contribution is $\simeq \frac{\Lambda b^2 d_{\rm LS}}{24m}$, which is about 4×10^{-19}. Note that in the Solar System the influence of the cosmological constant is known to be maximum in the case of the perihelion shift of the mercury orbit, where the Λ contribution is about 10^{-15} of the total shift.

III. DEVIATION OF IMAGE POSITIONS BETWEEN THE SOURCE AND THE REFERENCE OBJECT

The gravitational bending of light trajectories has been measured experimentally with high precision. At the early stages the bending was measured in Solar System experiments by comparing the apparent positions of stars when light trajectories from the stars came close to the solar disc but remained visible (normally during a solar eclipse), with their positions half a year earlier when the stars were on the opposite side of the Earth from the Sun. Thereby, light rays from these sources did not come too close to the Sun on their way to the Earth. In modern high precision measurements of gravitational deflection using the interferometric technique, angular positions of stars are measured as a function of time with respect to other sources having larger impact parameters, treating the latter objects as references. For instance, in an effort to test the gravitational theories, the change in angle between the quasar $3C279$, which is occulted by the Sun each October, and the quasar $3C273$ from their angular separation of about 9.5° has been measured just before and after occultation, and the results are found to be in accordance with the prediction of general relativity to first order accuracy in $M_\odot/R_\odot$.

The deviation of the image position (θ) of the source from its actual (which would have been seen by the observer in the absence of the lens) position (β) can be obtained from the lens equation, which is given by [17]

$$\tan\beta = \frac{d_{\rm OL}}{d_{\rm OS}} \frac{\sin\theta}{\cos(\delta - \theta)} - \frac{d_{\rm OS} - d_{\rm OL}}{d_{\rm OS}} \tan(\delta - \theta), \quad (16)$$

where the angles are with respect to the optic axis (the line joining the observer and the lens) and $d_{\rm OS}/\cos\beta$ is the distance between the observer and the source. Hence in this scenario $\theta = \psi$. For small angles, i.e. when $\theta, \beta, \delta \ll 1$, the lens equation reduces to

$$\beta \simeq \theta - \frac{d_{\rm LS}}{d_{\rm OS}} \delta. \quad (17)$$

As mentioned already, experimentally the effect of the lens on the photon trajectory is obtained by measuring the bending with respect to the photon trajectory from a second source that may be called the reference source. The distance of closest approach for the light path from the reference has to be much larger than that for the photon trajectory from the source.

Thus when angles are small, the angular difference between the images of the source and the reference, as to be revealed to the observer, is

$$\theta^{\rm R} - \theta^{\rm S} = \beta^{\rm R} - \beta^{\rm S} + \left(\frac{d_{\rm LR}\delta^{\rm R}}{d_{\rm OR}} - \frac{d_{\rm LS}\delta^{\rm S}}{d_{\rm OS}}\right), \quad (18)$$

where the superscripts R and S denote the reference and the source, respectively, and $d_{\rm OR}/\cos\beta^{\rm R}$ is the distance between the observer and the reference object. As the observer changes his/her position, both β and θ of the source as well as of the reference will change. The difference between the impact parameter or the closest approach of the light path as the observer changes his/her position from one point to the other is expected to be the same for both the source and the reference. In other words, $b_2^{\rm S} - b_1^{\rm S}$, where the subscripts 1 and 2 refer to parameter b at positions 1 and 2 of the observer, respectively, which should be the same as $b_2^{\rm R} - b_1^{\rm R}$, particularly when the source and the reference are a large distance away from the lens. To the leading order the difference in angle as the observer changes position is finally

$$\begin{aligned} \delta\alpha &= \theta_2^{\rm R} - \theta_2^{\rm S} - (\theta_1^{\rm R} - \theta_1^{\rm S}) \\ &\simeq -4m\left(\frac{d_{\rm LR}}{d_{\rm OR} b_1^{\rm R}} - \frac{d_{\rm LS}}{d_{\rm OS} b_1^{\rm S}}\right) - \frac{2m\Lambda b}{3}\left(\frac{d_{\rm LR}}{d_{\rm OR}} - \frac{d_{\rm LS}}{d_{\rm OS}}\right) \\ &\quad + \frac{\Lambda \delta b d_{\rm OL}}{6}\left(\frac{d_{\rm LR}}{d_{\rm OR}} - \frac{d_{\rm LS}}{d_{\rm OS}}\right) + \frac{\Lambda \delta b d_{\rm OL}}{6}\left(\frac{d_{\rm LR}^2}{d_{\rm OR}} - \frac{d_{\rm LS}^2}{d_{\rm OS}}\right), \end{aligned} \quad (19)$$

where $\Delta b \equiv b_2^i - b_1^i$, and i stands for the source/reference. Here we assume that the impact parameter (or the distance of closest approach) for both the source and the reference is smaller at position 1 than at position 2. When both the reference object and the source are far away from the lens in comparison to the lens-observer distance, one may take $d_{\rm LR}/d_{\rm OR} \simeq d_{\rm LS}/d_{\rm OS} \simeq 1$. In such a case the relative deflection angle becomes

$$\delta\alpha \simeq \frac{4m}{b_1^{\rm S}} - \frac{\Lambda \delta b}{6}(d_{\rm LR} - d_{\rm LS}). \tag{20}$$

So the deflection angle up to the accuracy we considered here still contains a cosmological constant involved term unless $d_{\rm LR} = d_{\rm LS}$.

Referring back to the example cited in the previous section, if $d_{\rm LR} - d_{\rm LS} \sim 10$ kpc and $\delta b\, {\rm sim} R_\odot$, the contribution of the bending angle due to Λ will be about 4×10^{-19} of the total bending angle. The expected angular precision of the planned astrometric missions using optical interferometry is at the level of microarcseconds, at least 12 orders lower than the Λ contribution on the bending angle when the lens system is within the galaxy.

So the natural temptation will be to consider the extragalactic sources and/or lenses for which $d_{\rm OL}$ and $d_{\rm LS}$ will be much higher. But in that case the expression for the bending angle has to be obtained in the frame of a comoving observer. Note that the cosmological expansion is unlikely to affect local structures—local overdensities in the matter distribution to inhibit space from expanding [18]. Accordingly, in this work the whole lensing system, consisting of the source, reference, lens, and observer, is considered as static. This is the usual practice of estimating bending angles for a local lens system in the framework of general relativity without Λ, though cosmic expansion is also a generic feature of Friedmann cosmology without Λ.

In the frame of the comoving observer the photon path may get distorted due to the (apparent) relative motion between the source, the lens, and the observer [12]. Consequently, the aberration effect will come into play [12]. One should also note that the cosmological expansion cannot be described equivalently by the relative motion of the observer and the source (cosmological redshift and Doppler redshift are not the same). Hence for the bending angle expression in the case of the extragalactic lens system, further investigation is required. Besides, one also has to consider that in such a situation the effect of Λ due to cosmological expansion may dominate over the geometric term [19].

IV. CONCLUSION

We conclude the following.

(1) The field equations of general relativity are modified when Λ is introduced in the theory. Consequently, Λ starts affecting not only the cosmological dynamics of the Universe, but also the local gravitational phenomena. The present investigation ascertains the recent claim that the cosmological constant does affect the gravitational deflection phenomenon like many other local gravitational phenomena, such as gravitational time delay, perihelion shift of the orbit of the planets, etc. [4]. For galactic sources and lenses, the contribution of Λ to the bending angle is, however, quite small, not detectable by a near future experiment.

(2) To the leading order there are two terms involving the cosmological constant in the expression of bending; one of them is purely local in the sense that it does not contain any information about the location of the observer/source. Instead, this term describes the coupling between the lens and the cosmological constant, as first pointed out by Sereno [7]. Interestingly, this term has the same signature as that of the classical expression of general relativistic bending ($4m/b$); i.e. this term will cause an increase of the bending angle. The other term, which is the dominating one, involves the radial distances of the source and the observer, and it bears the repulsive characteristics of the positive cosmological constant.

While studying gravitational bending in Schwarzschild–de Sitter geometry or in any asymptotic nonflat space-time, it is also important to study the photon trajectories from reference objects with respect to which the bending will be measured. When such an aspect is taken into consideration, the Λ contribution to the effective bending is found to depend on the distances of the source and the reference objects [Eq. (19) or Eq. (20)]. In principle, the Λ effect can be detected from the bending angle measurement by choosing suitable source and reference objects.

In the instance of the formation of the Einstein ring, however, no reference object is needed. In that particular case the ring radius will be smaller than that of the Schwarzschild geometry due to the Λ contribution, as noted in [7,20].

The effect of Λ is likely to be prominent for sources of large distances, particularly for an extragalactic lens system. However, to address such a situation one has to work in the frame of a coming observer [12]. Moreover, in such situations the effect of Λ due to cosmological expansion may dominate over the geometric term [19]. More investigation is needed in this respect for a definite conclusion. It is also interesting to examine the influence of other matters of the Universe on the system. An investigation has been undertaken in this direction.

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