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Arbitrary *p*-form Galileons

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We show that scalar, 0-form, Galileon actions—models whose field equations contain only second derivatives—can be generalized to arbitrary even p-forms. More generally, they need not even depend on a single form, but may involve mixed p combinations, including equal p multiplets, where odd p fields are also permitted: We construct, for given dimension D, general actions depending on scalars, vectors, and higher p-form field strengths, whose field equations are of exactly second derivative order. We also discuss and illustrate their curved-space generalizations, especially the delicate nonminimal couplings required to maintain this order. Concrete examples of pure and mixed actions, field equations, and their curved-space extensions are presented.

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I. INTRODUCTION

The geometric ancestors of Galileons [1–5] are the Gauss-Bonnet-Lovelock (GBL) actions

$$I = \int d^D x\, \varepsilon^{\mu\nu\cdots}\varepsilon^{\alpha\beta\cdots} R_{\mu\nu\alpha\beta}\ldots R_{\ldots} e_{\rho\gamma}\ldots e_{\ldots}, \qquad (1)$$

powers of the curvature R whose field equations are nevertheless independent of higher than second metric derivatives. This is achieved by virtue of the Bianchi identities, due to which the R variations do not contribute; only the explicit vielbeins' do, as is especially clear in vielbein/spin connection formalism. Here R is the "field strength" of the (non-Abelian) spin connection $\omega_{\mu\alpha\beta}(e)$; the Levi-Civita symbol $\varepsilon^{\mu\nu\cdots}$ is a tensor density, while $\varepsilon^{\alpha\beta\cdots}$ is a world scalar; $(\mu, \nu, \ldots)$ and $(\alpha, \beta, \ldots)$ are world and local Lorentz indices, respectively. These actions are dimension dependent, yielding vanishing field equations below a certain D, such as $D = 5$ for R^2 and $D = 7$ for R^3. More explicitly, for $D = 5$ say, one $e_{\mu\alpha}$ is required to contract the two leftover indices in $(\varepsilon\varepsilon RR)^{\mu\alpha}$, while there is no $e_{\mu\alpha}$, hence no field equation, in $D = 4$. The mechanism is simple, and as we shall see below, universal: First note that $\delta R_{\mu\nu\alpha\beta} = \mathcal{D}_{[\mu}\delta\omega_{\nu]\alpha\beta}$, where $\mathcal{D}$ is the usual covariant derivative with respect to the spin connection (acting also on local indices), and $\delta\omega$ is a world vector. Therefore, integrating $\mathcal{D}$ by parts (freely past all vielbeins of course) onto the remaining Riemann tensor(s) gives 0 by the cyclic Bianchi identities. So the GBL field equations, $(\varepsilon\varepsilon R\ldots R)^{\mu\alpha} = 0$, just result from removing (any) $e_{\mu\alpha}$ in (1) and are manifestly independent of higher than second vielbein derivatives.

Galileons are scalars whose field equations depend only on second derivatives, hence are invariant under constant shifts of the fields ("positions") and their gradients ("velocities"), recalling old-fashioned Galilean invariance. (Note that this invariance is only meaningful in flat space, since there are no constant vectors or tensors in curved backgrounds.) Their actions bear a formal resemblance to the GBL systems, when expressed as [4]

$$I = \int d^D x\, \varepsilon^{\mu\nu\cdots}\varepsilon^{\alpha\beta\cdots}\partial_\mu\pi\partial_\alpha\pi(\partial_\nu\partial_\beta\pi)\ldots(\partial\partial\pi). \qquad (2)$$

Again, the variations only leave second-order equations $(\partial\partial\pi)\ldots(\partial\partial\pi) = 0$, and for sufficiently low D, where the $\partial\pi\partial\pi \sim$ "e" are absent (for a given power of $\partial\partial\pi \sim$ "R"), Eq. (2) has vanishing variation. This (slightly imperfect) similarity led us to conjecture that (2) could be obtained from a GBL-like action using a metric suitably parametrized in terms of $\partial\pi$; it was indeed elegantly confirmed recently [5] for the R^2 case (and in a suitable limit), as a byproduct of a brane analysis.

The purpose of this paper is to generalize the above models by noting that the properties of 0-forms underlying

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(2) are actually shared by arbitrary even p-forms, and extend to any (dimensionally allowed) admixtures of various p-level fields. Surprisingly, we found a fundamental divide between even (scalars, ...) and odd (vectors, ...) models. The latter turn out, despite initial appearances, to have empty flat-space actions[1] (except of course the standard Maxwell-like $\mathcal{L} = F^2$), i.e., devoid of field equations for any (D, $p = 2n + 1$). However, as we discuss below, they may appear in mixed form, or in multiplets of single p-form, models.

We will work primarily in flat space in order to focus on our main results. As for scalars, the key ingredient here is that the forms' "field strengths" $\omega_{p+1} = dA_p$, are curls which do not become covariant; only the explicit ∇ in $\nabla\omega$ does. Using these "gauge-invariant" field strengths rather than ordinary gradients is both essential to the Galileon aspect and excludes their possible ghost, lower spin gauge components.

Retaining second order upon extension to curved backgrounds is nontrivial; even for scalars, the minimal coupling extension of (2) gave rise to third derivative terms in its stress tensor and hence in the associated gravitational field equations, as well as to third metric derivative terms $\propto \nabla R$ in the π equations. A delicate set of additional, nonminimal, couplings, involving the full curvature tensor was required [3,4] to remove these. This program, though correspondingly more complicated, can in fact be carried out for our present generalized framework in a fashion similar to that of [4] for scalars. Instead of detailing here the straightforward but still rather lengthy derivation for the most general case, we will display some examples of successful covariantization in Sec. IV.

II. p-FORM ACTIONS

We start, to emphasize the pitfalls in this problem, with the obviously simplest—but actually empty—generalization from a scalar (2) to a one-form A_μ with field strength $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$:

$$I = \int d^D x\, \varepsilon^{\mu\nu\cdots}\varepsilon^{\alpha\beta\cdots} F_{\mu\nu}F_{\alpha\beta}(\partial_\rho F_{\gamma\delta}\ldots)(\partial_\epsilon F_{\sigma\tau}\ldots), \tag{3}$$

where the parentheses contain products of ∂F and indices are connected as follows: In the first parenthesis, the index of the derivative ∂ is contracted with the first $\varepsilon^{\mu\nu\cdots}$ whereas those of F are contracted with the second $\varepsilon^{\alpha\beta\cdots}$, and inversely in the second parenthesis. The integrand of (3) is a total divergence which we may write as $\frac{1}{2}\partial_\epsilon[\varepsilon^{\mu\nu\ldots\varphi\chi\cdots}\varepsilon^{\alpha\beta\ldots\epsilon\zeta\cdots}F_{\mu\nu}F_{\sigma\tau}F_{\alpha\beta}(\partial_\rho F_{\gamma\delta}\ldots)(\partial_\zeta F_{\varphi\chi}\ldots)]$. This equality follows by noting that $\frac{1}{2}\partial_\epsilon$ manifestly annihilates all its operands but $F_{\mu\nu}F_{\sigma\tau}$ (on each of which it acts identically), where its actions reproduce the original Lagrangian. Since this conclusion is due to the evenness of $F_{\mu\nu}F_{\sigma\tau}$ upon exchange of their indices $\mu\nu \leftrightarrow \sigma\tau$, the difficulty obviously persists for all (D, $p = 2n + 1$). However, as we shall see in the next section, odd p models can be revived if they are allowed to depend on more than one A_μ.

Fortunately, the direct extension of (2) does exist for even p; it formally resembles (3) where now $\omega_{\lambda\mu\nu\ldots} = \partial_{[\lambda}A_{\mu\nu\ldots]}$. In detail, the general (now nonvanishing) action is

$$I = \int d^D x\, \varepsilon^{\mu\nu\cdots}\varepsilon^{\alpha\beta\cdots}\omega_{\mu\nu\ldots}\omega_{\alpha\beta\ldots}(\partial_\rho\omega_{\gamma\delta\ldots}\ldots) \times (\partial_\epsilon\omega_{\sigma\tau\ldots}\ldots). \tag{4}$$

As in Eq. (3), when the derivative ∂ of a gradient $\partial\omega$ is contracted with one of the two ε, the indices of ω_{p+1} must be contracted with the other ε, otherwise the action would vanish by virtue of the Bianchi identities (i.e., $[d, d] = 0$). The two parentheses of (4) must contain the same number of terms, not greater than $(D - p - 1)/(p + 2)$. [In the lowest, $p = 0$, case, an odd total number of $\partial\omega$ is also permitted, cf. (2).] For fewer terms, there remain extra open indices on each ε that must be contracted between them (using vielbeins in curved space, of course); this yields (up to an overall factor) the same Lagrangian as in the lowest possible dimension. There is actually a formal resemblance between (4) and (2) that emerges from considering the $(p + 1)$ indices of the form ω as a multi-index M replacing the single index of $\pi_{,\mu}$. Then, just as each of the two indices of $\partial_\alpha\partial_\mu\pi$ must be contracted with different ε, here α and M of $\partial_\alpha\omega_M$ must belong to different ε.

It should be clear from our notation and the Bianchi identities that the field equations depend homogeneously on $\partial\omega$ and not at all on ω, hence they enjoy the corresponding Galilean invariance, this time under a shift of $(A_{\mu\nu\ldots}, \omega_{\mu\nu\rho\ldots})$ by constants antisymmetric tensors $(c_{[\mu\nu\ldots]}, k_{[\mu\nu\rho\ldots]})$. For completeness, let us run through the argument, entirely akin to those for gravity and scalars: First, if either pure ω is varied, the explicit curl on its A can only land on the other ω, since Bianchi annihilates any $\partial_\mu\partial_{[\alpha}\omega_{\beta\gamma\ldots]}$ (or $\partial_\alpha\partial_{[\mu}\omega_{\nu\rho\ldots]}$). Likewise, varying any $\partial\omega \sim \partial_\mu\partial_\alpha A$ factor forces each of those two ∂ to land on one of the two pure ω; all other landings vanish, again by Bianchi. The simplest version of Eq. (4), valid for $D \geq p + 1$, does not contain any derivative $\partial\omega$, and is thus the standard kinetic term ω^2 (valid for all p of course, though dynamically nontrivial only when $D > p + 1$). The first novel p-form Galileon action, involving just two $\partial\omega$ factors, requires $D \geq 2p + 3$. For $p = 2$, it reads explicitly

[1]Nevertheless, covariantized versions of trivial flat-space actions can produce nontrivial field equations, proportional to the curvature, as will be illustrated in Sec. IV.

$$I = \int d^7x\, \varepsilon^{\mu\nu\rho\sigma\tau\varphi\chi}\varepsilon^{\alpha\beta\gamma\delta\epsilon\zeta\eta}\omega_{\mu\nu\rho}\omega_{\alpha\beta\gamma}\partial_\sigma\omega_{\delta\epsilon\zeta}\partial_\eta\omega_{\tau\varphi\chi}$$
$$= 36\int d^7x[-9(\omega^\mu{}_{\nu\rho,\sigma}\omega^{\sigma\tau\varphi}\omega_{\tau\varphi\mu,\chi}\omega^{\chi\nu\rho}) - 18(\omega^\mu{}_\nu{}^\rho\omega_{\mu\sigma}{}^\tau\omega^{\varphi\chi\nu,\sigma}\omega_{\varphi\chi\tau,\rho}) - 36(\omega^{\mu\nu\rho}\omega_{\rho\sigma\tau}\omega_{\mu\nu\varphi}{}^{,\sigma}\omega^{\tau\varphi\chi}{}_{,\chi})$$
$$+ 6(\omega_{\mu\nu\rho}\omega^{\mu\nu\rho,\sigma}\omega_{\sigma\varphi\chi}\omega^{\varphi\chi\tau}{}_{,\tau}) + 18(\omega_{\mu\nu}{}^\rho\omega^{\mu\nu\sigma}\omega_{\varphi\chi\rho,\sigma}\omega^{\varphi\chi\tau}{}_{,\tau}) - 3(\omega^{\mu\nu\lambda}\omega_{\rho\sigma\tau,\lambda})^2 - 9(\omega^{\mu\nu\rho}\omega_{\rho\sigma\tau,\lambda})^2$$
$$+ 18(\omega_{\mu\nu\rho}\omega^{\rho\sigma\tau}{}_{,\tau})^2 + 9(\omega^{\mu\nu\rho}\omega_{\mu\nu\sigma,\tau})^2 - 9(\omega_{\mu\nu\rho}\omega^{\mu\nu\sigma}{}_{,\sigma})^2 - (\omega^{\mu\nu\rho}\omega_{\mu\nu\rho,\sigma})^2$$
$$+ (\omega^2)(\omega_{\mu\nu\rho,\sigma})^2 - 3(\omega^2)(\omega^{\mu\nu\rho}{}_{,\rho})^2]. \quad (5)$$

Its field equation

$$\varepsilon^{\mu\nu\rho\sigma\tau\varphi\chi}\varepsilon^{\alpha\beta\gamma\delta\epsilon\zeta\eta}\partial_\rho\omega_{\alpha\beta\gamma}\partial_\sigma\omega_{\delta\epsilon\zeta}\partial_\eta\omega_{\tau\varphi\chi} = 0 \quad (6)$$

is obviously of pure second order; we do not display its, 23 term, expansion.

We conclude this section by discussing another amusing (if somewhat tangential) parallel between tensors and forms, which evokes the well-known conversion [6] of pure divergence $D = 4$ GB into general relativity (GR), with or without cosmological term: upon adding $\pm\Lambda e_{\mu\alpha}e_{\nu\beta}$ to each $R_{\mu\nu\alpha\beta}$ in the topological invariant (1), it becomes proportional to the GR action, since the cross term in $\varepsilon\varepsilon(R \pm \Lambda ee)^2$ is the scalar curvature, while the Λ^2 term is the volume density, cosmological, term. Subtracting $(R + \Lambda ee)^2$ and $(R - \Lambda ee)^2$ actions removes the latter. For scalars, we similarly add $\pm m^2\pi\eta_{\mu\alpha}$ to each of the two $\partial_\mu\partial_\alpha\pi$ in the pure GB-like $I = \int \varepsilon\varepsilon\partial\partial\pi\partial\partial\pi$: this leads to the (massive or massless by subtraction) Klein-Gordon action. These extensions can be made for all forms: thus for the vector Proca/Maxwell actions, add $\sim \pm m^2\eta_{\mu[\alpha}A_{\beta]}$ to each $\partial_\mu F_{\alpha\beta}$ in the otherwise vacuous action $I = \int \varepsilon\varepsilon\partial F\partial F$, etc. Note that our mass construction is valid in all D [$\geq (p + 2)$ of course]; that of GR, for all $D \geq 4$. That the former has a curved-space extension is also obvious.

III. MIXED FORM ACTIONS

Our actions (4) can be further generalized by including several species, i.e., mixtures of various unequal p-forms compatible with a desired D. Labelling these species by $(a, b, \ldots)$, the action takes the formal expression

$$I = \int d^Dx\, \varepsilon^{\mu\nu\ldots}\varepsilon^{\alpha\beta\ldots}\omega^a_{\mu\nu\ldots}\omega^b_{\alpha\beta\ldots}(\partial_\rho\omega^c_{\gamma\delta\ldots}\ldots) \times (\partial_\epsilon\omega^d_{\sigma\tau\ldots}\ldots). \quad (7)$$

The number of indices contracted with the first and second ε must be the same and not greater than D, but the two parentheses may now involve different species and therefore a different number of terms. Here, Bianchi again ensures (exactly as for the single species version) that only $\partial\omega$ appears in the field equations. Hence (flat space) Galilean invariance under translation of all (A_p, ω_{p+1}) by constant antisymmetric tensors (c_p, k_{p+1}) is preserved. Note that odd forms are also allowed in (7), subject to various symmetry constraints. For example, no more than two ∂F factors can be present, otherwise (7) would involve at least one product of the form $\partial_\mu F_{\alpha\beta}\partial_\nu F_{\gamma\delta}$, where the two ∂ and the two F are, respectively, contracted with the same ε tensors. Hence their indices can be interchanged by three permutations, $\mu \leftrightarrow \nu$, $\alpha\beta \leftrightarrow \gamma\delta$, so they vanish identically. This single $\partial F\partial F$ ceiling obviously also applies to higher odd p-forms; instead, the $p = 2n$ models, being even under such permutations, may contain arbitrary powers of $\partial\omega$ consistent with a given D (but conversely, see below for limitations on even p actions).

Let us quote two simple, mixed 0- and 1-form, nontrivial examples; the first Lagrangian is defined in any $D \geq 3$:

$$\mathcal{L} = \varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}F_{\mu\nu}F_{\alpha\beta}\partial_\rho\partial_\gamma\pi = 4F^{\mu\rho}F^\nu{}_\rho\pi_{,\mu\nu} - 2F^2\Box\pi. \quad (8)$$

Both its π and A_λ field equations are obviously of pure second order; explicitly,

$$(F_{\mu\nu,\rho})^2 - 2(F^{\mu\nu}{}_{,\nu})^2 = 0, \quad (9)$$

$$F^{\lambda\mu,\nu}\pi_{,\mu\nu} + F^{\mu\nu}{}_{,\nu}\pi^{,\lambda}{}_\mu - F^{\lambda\mu}{}_{,\mu}\Box\pi = 0. \quad (10)$$

Similarly, in $D \geq 4$, the mixed model

$$\mathcal{L} = \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\partial_\mu\pi\partial_\alpha\pi\partial_\nu F_{\beta\gamma}\partial_\delta F_{\rho\sigma}$$
$$= -8(\pi_{,\mu}F^{\rho\mu,\nu}F_{\rho\sigma}{}^{,\sigma}\pi_{,\nu}) + 4(\pi^{,\mu}F_{\mu\nu,\rho})^2 + 2(\pi^{,\mu}F_{\nu\rho,\mu})^2 - 4(\pi_{,\mu}F^{\mu\nu}{}_{,\nu})^2 - 2(\pi_{,\mu})^2(F_{\nu\rho,\sigma})^2 + 4(\pi_{,\mu})^2(F^{\nu\rho}{}_{,\rho})^2 \quad (11)$$

also yields pure second-order π and A_λ field equations:

$$4(\pi_{,\mu\nu}F^{\rho\mu,\nu}F_{\rho\sigma}{}^{,\sigma}) - 2(F^{\mu\rho,\sigma}\pi_{,\mu\nu}F^\nu{}_{\rho,\sigma}) + 2(F^{\mu\rho}{}_{,\rho}\pi_{,\mu\nu}F^{\nu\sigma}{}_{,\sigma}) - (F_{\rho\sigma,\mu}\pi^{,\mu\nu}F^{\rho\sigma}{}_{,\nu}) + (\Box\pi)(F_{\mu\nu,\rho})^2 - 2(\Box\pi)(F^{\mu\nu}{}_{,\nu})^2 = 0, \quad (12)$$

$$2(\pi_{,\mu\rho}F^{\lambda\mu}{}_{,\nu}\pi^{,\nu\rho}) + 2(\pi^{,\lambda\mu}F_{\mu\nu,\rho}\pi^{,\nu\rho}) + 2(\pi^{,\lambda\rho}\pi_{,\rho\mu}F^{\mu\nu}{}_{,\nu}) - (\pi_{,\mu\nu})^2(F^{\lambda\rho}{}_{,\rho}) - 2(\Box\pi)(\pi_{,\mu\nu}F^{\lambda\mu,\nu}) - 2(\Box\pi)(\pi^{,\lambda}{}_\mu F^{\mu\nu}{}_{,\nu}) + (\Box\pi)^2(F^{\lambda\mu}{}_{,\mu}) = 0. \quad (13)$$

An even simpler class of mixed actions involves a single p-order species, but now as a "multiplet" $A^a_{\mu\nu\ldots}$, for instance pure scalars but with different π^a replacing the single one in (2). This extension even resuscitates odd-p actions: For instance, the simplest bi-vector Lagrangian of the type (3), $\mathcal{L} = \varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{\alpha\beta\gamma\delta\epsilon}F^a_{\mu\nu}F^a_{\alpha\beta}\partial_\rho F^b_{\gamma\delta}\partial_\epsilon F^b_{\sigma\tau}$, is obviously no longer a total divergence. Our reasoning below Eq. (4), showing that the field equations do not involve higher order derivatives, may also be generalized to non-Abelian gauge bosons A^a_μ and their field strengths $F = dA + A \wedge A$, although both the invariances under constant shifts, $A^a_\mu \to A^a_\mu + c^a_\mu$ and $F^a_{\mu\nu} \to F^a_{\mu\nu} + k^a_{[\mu\nu]}$, would then be lost. Indeed, if $\mathcal{D}$ denotes the covariant derivative with respect to the internal space [like the $\mathcal{D}$ below (1) with respect to tangent space], then the Bianchi identities $\mathcal{D}_{[\mu}F^a_{\nu\rho]} = 0$ still hold, therefore Lagrangians of the form $\mathcal{L} = \varepsilon^{\mu\nu\dots}\varepsilon^{\alpha\beta\dots}F^a_{\mu\nu}F^b_{\alpha\beta}(\mathcal{D}_\rho F^c_{\gamma\delta}\dots)(\mathcal{D}_\epsilon F^d_{\sigma\tau}\dots)$ define nonlinear extensions of Yang-Mills theory, while keeping field equations of second (and lower) order.

It is worth noting that one may also add undifferentiated powers of ω beyond the two in the generalized models (7), provided all indices of any one ω (whatever its p order) are contracted with those of a single ε tensor, but not "across" both. Also, no more than two undifferentiated even p-field strengths ω^a_{p+1} are allowed for the same species a, otherwise the action would vanish by oddness, while any number of odd p field strengths may be present. The same reasoning as above indeed shows that no higher derivative than $\partial\omega$ is generated in the field equations, i.e., that they depend at most on second derivatives of the p-forms A. On the other hand, these field equations now involve some pure ω in addition to the usual $\partial\omega$ factors, because at most two derivatives are generated by varying the $\partial\omega$ terms of the action, so that they can act on at most two of the undifferentiated ω. Therefore, this generalization with more than two pure ω results in a loss of the "velocity" invariance $\omega \to \omega + k$. A simple $D \geq 4$ example of this type is

$$\begin{aligned}\mathcal{L} &= \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\partial_\mu\pi\partial_\alpha\pi F_{\nu\rho}F_{\beta\gamma}\partial_\sigma\partial_\delta\pi \\ &= 4(\pi_{,\mu}F^{\mu\nu}\pi_{,\nu\rho}F^{\rho\sigma}\pi_{,\sigma}) + 8(\pi_{,\mu}F^{\mu\nu}F_{\nu\rho}\pi^{,\rho\sigma}\pi_{,\sigma}) \\ &\quad + 2(F^2)(\pi_{,\mu}\pi^{,\mu\nu}\pi_{,\nu}) + 4(\pi_{,\mu})^2(F^\nu{}_\sigma F^{\rho\sigma}\pi_{,\nu\rho}) \\ &\quad - 4(\pi_{,\mu}F^{\mu\nu}F_{\nu\rho}\pi^{,\rho})(\Box\pi) - 2(\pi_{,\mu})^2(F^2)(\Box\pi). \quad (14)\end{aligned}$$

As is clearest from the first expression, its variations involve both first and second (but no higher) derivatives of π and A_μ.

IV. GRAVITATIONAL COUPLING

As stated in the Introduction, second-order preserving extension of even the scalar flat-space actions to curved backgrounds was a rather complicated process, one that becomes more combinatorially involved for higher forms. We content ourselves here with giving the explicit nonminimal extensions for four of our cases (5), (8), (11), and (14), that avoid higher derivatives in both the matter and gravitational (that is, through $T_{\mu\nu}$) field equations. These terms are constructed as for scalars in [4]: All possible pairs of gradients, $\partial\omega^a\partial\omega^b$, must be replaced by suitable contractions of the undifferentiated $\omega^a\omega^b$ with the Riemann tensor, and added to the minimally covariantized flat-space action with suitable coefficients; somewhat more involved counting shows that they require factors $\propto (p_a + 1)(p_b + 1)$, where $p_{a,b}$ denote the orders of the forms $A^{a,b}$. One other difference in the $p > 0$ construction is that $\nabla_\mu\omega_{\alpha\beta\dots}$ are to be distinguished from $\nabla_\alpha\omega_{\mu\nu\dots}$, essentially because of their different ε-index contractions, a distinction irrelevant to the original scalar, $\pi_{;\mu\alpha} = \pi_{;\alpha\mu}$, case. One common feature is that flat-space Galilean invariance is also not restorable by consistent covariantization (nor should it be expected, absent constant vectors or tensors in curved space): the equations now necessarily depend on both second and first derivatives of the fields. For (5), the added terms are

$$\begin{aligned}\Delta I &= -\frac{9}{4}\int d^7x\, \varepsilon^{\mu\nu\rho\sigma\tau\varphi\chi}\varepsilon^{\alpha\beta\gamma\delta\epsilon\zeta\eta}\omega_{\mu\nu\rho}\omega_{\alpha\beta\gamma}\omega_{\lambda\sigma\tau}\omega^\lambda{}_{\delta\epsilon}R_{\varphi\chi\zeta\eta} \\ &= 54\int d^7x\sqrt{-g}\,[24(\omega_{\mu\nu\lambda}\omega^{\lambda\tau\rho}\omega_{\tau\varphi\chi}\omega^{\varphi\chi\sigma}R^{\mu\nu}{}_{\rho\sigma}) + 12(\omega_{\lambda\tau\mu}\omega^{\lambda\tau\nu}\omega_{\varphi\chi\rho}\omega^{\varphi\chi\sigma}R^{\mu\rho}{}_{\nu\sigma}) \\ &\quad + 4(\omega^2)(\omega_{\lambda\mu\nu}\omega^{\lambda\rho\sigma}R^{\mu\nu}{}_{\rho\sigma}) + 12(\omega_{\mu\nu\varphi}\omega^{\mu\nu\chi}\omega^{\tau\varphi\rho}\omega_{\tau\chi\sigma}R^\sigma_\rho) + 18(\omega_{\mu\nu\tau}\omega^{\mu\nu\rho}\omega^{\varphi\chi\tau}\omega_{\varphi\chi\sigma}R^\sigma_\rho) \\ &\quad - 10(\omega^2)(\omega^{\mu\nu\rho}\omega_{\mu\nu\sigma}R^\sigma_\rho) - 3(\omega_{\mu\nu\rho}\omega^{\mu\nu\sigma})^2R + (\omega^2)^2R]. \quad (15)\end{aligned}$$

Similarly, the mixed $D \geq 4$ example (11) acquires the terms

$$\begin{aligned}\Delta\mathcal{L} &= \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\partial_\mu\pi\partial_\alpha\pi F_{\lambda\nu}F^\lambda{}_\beta R_{\rho\sigma\gamma\delta} \\ &= 2\sqrt{-g}\,[-2(\pi_{,\mu}\pi_{,\nu}F_{\rho\lambda}F_\sigma{}^\lambda R^{\mu\rho\nu\sigma}) + 4(\pi_{,\mu}F^{\mu\nu}F_{\nu\rho}R^{\rho\sigma}\pi_{,\sigma}) + 2(\pi_{,\mu})^2(F_{\nu\sigma}F_\rho{}^\sigma R^{\nu\rho}) \\ &\quad + 2(F^2)(\pi_{,\mu}R^{\mu\nu}\pi_{,\nu}) + (\pi_{,\mu}F^{\mu\rho})^2R - (\pi_{,\mu})^2(F^2)R]. \quad (16)\end{aligned}$$

In contrast to the above models, the mixed $D \geq 3$ example (8) and the "non-Galileon" Lagrangian (14) actually require *no* additional terms to preserve second order, since they only contain a single vulnerable—because of second order—$\partial\partial\pi$ factor. It is clear by inspection of (8) and (14) that all (covariant) third derivatives arising from variations here always have the form of a commutator $[\nabla, \nabla]$ acting on a ∂A, that is a—harmless—curvature times first derivatives of fields.

Our final model illustrates the observation made in our footnote that actions trivial in flat space can have nontrivial, dynamical, curvature-dependent extensions: Consider the vector models (3), or more generally, actions (4) for any odd p, which are vacuous in flat space. Their minimal covariantizations are clearly both nonvanishing and of third order. However, one may also add appropriate nonminimal terms that both remove the offending higher derivatives and remain nontrivial. Indeed, the simplest case is the lowest Galileon $D = 5$ vector action,

$$I = \int d^5x\, \varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{\alpha\beta\gamma\delta\epsilon}F_{\mu\nu}F_{\alpha\beta}\nabla_\rho F_{\gamma\delta}\nabla_\epsilon F_{\sigma\tau} = -\frac{1}{2}\int d^5x\, \varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{\alpha\beta\gamma\delta\epsilon}F_{\mu\nu}F_{\alpha\beta}F^{\lambda}{}_{\rho}F_{\delta\gamma}R_{\sigma\tau\lambda\epsilon}. \quad (17)$$

The last equality in (17) exhibits the model's curvature-dependence, and is obtained from the first expression by parts integration. [The metric variation of the curvature in the second expression (17) yields a non-vanishing $T^{\mu\nu} = \partial_\alpha\partial_\beta H^{[\mu\alpha][\nu\beta]}$ even in flat space, despite the model's triviality there; no paradox ensues since this pure superpotential form has vanishing Lorentz generators.] The third derivatives in the resulting field equations can be removed by adding the counterterm

$$\Delta I = \int d^5x\, \varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{\alpha\beta\gamma\delta\epsilon}F_{\mu\nu}F_{\alpha\beta}F^{\lambda}{}_{\rho}F_{\lambda\gamma}R_{\sigma\tau\delta\epsilon}. \quad (18)$$

It differs from the action (17) itself simply by an overall factor and the index change $\delta \leftrightarrow \lambda$ in the last two terms. Their sum,

$$I + \Delta I = -8\int d^5x\sqrt{-g}\,[4(F^{\mu\nu}F^{\rho\lambda}F_{\lambda\tau}F^{\tau\sigma}C_{\mu\nu\rho\sigma}) + 4(F^{\mu}{}_{\lambda}F^{\lambda\nu}F^{\rho}{}_{\tau}F^{\tau\sigma}C_{\mu\rho\nu\sigma}) + (F^2)(F^{\mu\nu}F^{\rho\sigma}C_{\mu\nu\rho\sigma})], \quad (19)$$

depends only on the Weyl tensor $C_{\mu\nu\rho\sigma}$ [for no obvious $D = 5$ reason, though (19) is manifestly conformal invariant in $D = 10$]; as per design, both its $T_{\mu\nu}$ and field equations depend on at most second derivatives.

Details of our models' constructions, of their general nonminimal compensating gravitational extensions, applications for instance in the spirit of [7], and other open questions, e.g., possible supersymmetrization, may be presented elsewhere.

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