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Nuclei as near BPS Skyrmions

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We study a generalization of the Skyrme model with the inclusion of a sixth-order term and a generalized mass term. We first analyze the model in a regime where the nonlinear σ and Skyrme terms are switched to zero, which leads to well-behaved analytical Bogomol'nyi-Prasad-Sommerfeld–type solutions. Adding contributions from the rotational energy, we reproduce the mass of the most abundant isotopes to rather good accuracy. These BPS-type solutions are then used to compute the contributions from the nonlinear sigma and Skyrme terms when these are switched on. We then adjust the four parameters of the model using two different procedures and find that the additional terms only represent small perturbations to the system. We finally calculate the binding energy per nucleon and compare our results with the experimental values.

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I. INTRODUCTION

The Skyrme model [1] is nowadays one of the strongest candidates for a description of the low-energy regime of QCD. Developed in the beginning of the 60's by T.H.R. Skyrme, it consists of a nonlinear theory of mesons where its main feature is the presence of topological solitons as solutions. Each of these solutions is associated with a conserved topological charge, the winding number, which Skyrme interpreted as the baryon number, thus leading him to state that the solitons are baryons emerging from a meson field. The $\frac{1}{N_C}$ expansion of QCD introduced by t'Hooft [2] in the mid-70's and the later connection from Witten [3] with the model developed by Skyrme brought some support to this interpretation.

Since its original formulation, the Skyrme model has been able to predict the properties of the nucleon within a precision of 30%. Several modifications to the model have been considered to improve these predictions, from the generalization of the mass term [4–6] to the explicit addition of vector mesons [7,8], aside from higher order terms in derivatives of the pion field [4]. Unfortunately, the analysis of these models has been hampered by their nonlinear nature and the absence of analytical solutions. Indeed, all the solutions rely on numerical computation at some point, whether one uses the rational map ansatz [9], which turns out to be a rather good approximation of the angular dependence, or a full fledge numerical algorithm, like simulated annealing [10,11], to find an exact solution of the energy functional. Clearly, even a prototype model with analytical solutions would allow going deeper in the investigation of the properties and perhaps identifying novel features of the Skyrmions.

In a recent study, Adam, Sanchez-Guillen, and Wereszczynski (ASW) [12] obtained an analytical solution by considering a model consisting only of a term of order six in derivatives and a potential, which correspond to the customary mass term for pions in the Skyrme model [13]. Their calculations lead to a compacton-type solution with size growing as $n^{1/3}$, where n the winding number is identified with the baryon number, a result in general agreement with experimental observations. Another important remark on their study is that their solutions are of BPS-type, i.e. they saturate a Bogomol'nyi's bound. Even though physical nuclei do not saturate such a bound, the small value of the binding energy may be one of the motivations for solution of this type. Let us also mention that recently Sutcliffe [14] found that BPS-type Skyrmions emerge from models when a large number of vector mesons are added to the Skyrme model. However, the analysis of ASW neglects rotational or isorotational energies of nuclei, and perhaps the oddest feature of the model is that it does not contain any of the terms that Skyrme originally introduced in his model, the nonlinear σ and so-called Skyrme terms which are of order 2 and 4 in derivatives, respectively. Being an effective theory of QCD, there is nevertheless no reason to omit such contributions. In their work, ASW further suggest that their analytical solutions found could be used to compute the contributions from the terms of the original Skyrme lagrangian assuming they are small and do not affect significantly the overall solutions. Unfortunately the nature of the solution leads to singularities in the computation of the energies related to the nonlinear σ and Skyrme terms.

In this work, we find analytical BPS-type solutions for a Lagrangian similar to the one in [12], which allows considering contributions from the original Skyrme Lagrangian as small perturbations. The analysis also includes contributions for (iso)rotational energies providing a more realistic description of nuclei. The paper is divided as follows: in Sec. II, we introduce the general form of this generalized Skyrme model and find expressions for the static energies. Next, we quantify semiclassically the zero modes of the Skyrmions, which will allow computing rotational contributions to the total energy coming from the spin as well as the isospin of the nuclei. In Sec. IV, we choose an adequate potential (or mass term) and switch off the nonlinear σ and Skyrme terms. We then find a simple analytical form of the BPS-type solutions for the remaining Lagrangian. It turns out that all the properties of the nuclei can be calculated analytically. In Sec. V, we use the solution to compute the properties of the full Lagrangian. Fitting the different parameters of the model with nuclear mass data [15], we verify that the contributions from the nonlinear σ and Skyrme terms remain small and that the analytical solution is a good approximation.

II. LAGRANGIAN OF THE SKYRME MODEL

The model proposed by ASW is based on the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_6 - \mu^2 V \\ &= -\frac{3}{2}\frac{\lambda^2}{16^2}\,\mathrm{Tr}([L_\mu, L_\nu][L^\nu, L^\lambda][L_\lambda, L_\mu]) \\ &\quad -\frac{1}{2}\mu^2\,\mathrm{Tr}[1-U],\end{aligned} \tag{1}$$

where $U = \phi_0 + i\tau_i\phi_i$ is the $SU(2)$ matrix representing the meson fields, and $L_\mu = U^\dagger\partial_\mu U$ is the left-handed current. The model leads to BPS-type solitons. The constants λ and μ are the only free parameters of the model, with units MeV^{-1} and MeV^2, respectively. Using scaling arguments, one can show that the term of order 6 in field derivatives $\mathcal{L}_6$ prevents the soliton from shrinking to zero size while the second term, often called the mass term, stabilizes the solution against arbitrary expansion.

On the other hand, the original Skyrme model consists of the two completely different terms

$$\mathcal{L} = \mathcal{L}_{nl\sigma} + \mathcal{L}_{\mathrm{Sk}} \tag{2}$$

with

$$\mathcal{L}_{nl\sigma} = \alpha\,\mathrm{Tr}[L_\mu L^\mu], \qquad \mathcal{L}_{\mathrm{Sk}} = \beta\,\mathrm{Tr}([L_\mu, L_\nu]^2), \tag{3}$$

the nonlinear σ and so-called Skyrme terms, which are of order 2 and 4 in derivatives, respectively. Here $[\alpha] = \mathrm{MeV}^2$ and β is a dimensionless constant.

We shall consider here a model containing the four terms i.e. an extension of the Skyrme model with a sixth order term in derivatives and generalized mass term. The Lagrangian density reads

$$\begin{aligned}\mathcal{L} = &-\mu^2 V(U) - \alpha\,\mathrm{Tr}[L_\mu L^\mu] + \beta\,\mathrm{Tr}([L_\mu, L_\nu]^2) \\ &-\frac{3}{2}\frac{\lambda^2}{16^2}\,\mathrm{Tr}([L_\mu, L_\nu][L^\nu, L^\lambda][L_\lambda, L_\mu]).\end{aligned} \tag{4}$$

We are interested in the regime where α and β are small, so that $\mathcal{L}_{nl\sigma}$ and $\mathcal{L}_{\mathrm{Sk}}$ can be considered as small perturbations to (1). Usually, the potential $V(U)$ is chosen such that it reproduces the mass term for pions when small fluctuations of the fields are considered

$$U = e^{2i\tau_i(\pi_i/F_\pi)} \sim 1 + 2i\tau_i\frac{\pi_i}{F_\pi}, \tag{5}$$

where $F_\pi = 4\sqrt{\alpha}$ is interpreted as the pion decay constant. Since U is a $SU(2)$ matrix, the meson fields obey the condition

$$\phi_0^2 + \phi_i^2 = 1, \tag{6}$$

which limits the number of degrees of freedom to three. The boundary condition at infinity must be $U(r\to\infty) =$ constant. Any constant matrix could be used since they are all related by a global isorotation. We chose by convenience

$$U(r\to\infty) = I_{2\times 2} \tag{7}$$

with $I_{2\times 2}$ the two-dimensional unit matrix. This ensures that each solution for the Skyrme field falls into a topological sector characterized by a conserved topological charge

$$B = -\frac{\epsilon^{ijk}}{48\pi^2}\int d^3x\,\mathrm{Tr}(L_i[L_j, L_k]). \tag{8}$$

The static energy can then be calculated using

$$E_{\mathrm{stat}} = -\int d^3x\,\mathcal{L}_{\mathrm{stat}}. \tag{9}$$

We may conveniently write a general solution as

$$U = e^{i\mathbf{n}\cdot\tau F} = \cos F + i\mathbf{n}\cdot\tau\sin F, \tag{10}$$

where $\hat{\mathbf{n}}$ is the unit vector

$$\hat{\mathbf{n}} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$$

or

$$\phi_a = (\cos F, \sin F\sin\Theta\cos\Phi, \sin F\sin\Theta\sin\Phi, \sin F\cos\Theta).$$

Following ASW [12], we consider solutions that saturate the Bogomol'nyi's bound for (1) using the form

$$F = F(r), \qquad \Theta = \theta, \qquad \Phi = n\phi, \tag{11}$$

where n is an integer. The static energy (4) becomes

$$\begin{aligned}E_{\mathrm{stat}} &= -\int d^3x\,\mathcal{L}_{\mathrm{stat}} \\ &= 4\pi\int r^2 dr\bigg(\mu^2 V + \frac{9\lambda^2}{16}n^2 F'^2\frac{\sin^4 F}{r^4} \\ &\quad + 2\alpha\bigg[F'^2 + (n^2+1)\frac{\sin^2 F}{r^2}\bigg] \\ &\quad + 16\beta\frac{\sin^2 F}{r^2}\bigg[(n^2+1)F'^2 + n^2\frac{\sin^2 F}{r^2}\bigg]\bigg),\end{aligned} \tag{12}$$

where $F' = \partial F/\partial r$, and the topological charge is simply $B = n$.

In order to represent physical nuclei, we have to quantize the solitons using a semiclassical method described in the next section. Using the appropriate spin and isospin numbers, we then calculate the total energy for each nucleus.

III. QUANTIZATION

Because the topological solitons occupy a spatial volume that is nonzero, usual quantization procedures are no longer available. We therefore have to use a semiclassical quantization method by adding explicit time dependence to the zero modes of the Skyrmion. Performing time-dependent (iso)rotations on the Skyrme field by $SU(2)$ matrix $A(t)$ and $B(t)$ yield

$$\tilde{U}(\mathbf{r}, t) = A(t)U(R(B(t))\mathbf{r})A(t), \tag{13}$$

where $R_{ij}(B(t)) = \frac{1}{2}\,\mathrm{Tr}[\tau_i B \tau_j B^\dagger]$ is the associated $SO(3)$ rotation matrix. Upon insertion of this ansatz in the time-dependent part of (4), we write the rotational Lagrangian as

$$\mathcal{L}_{\mathrm{rot}} = \frac{1}{2}a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j, \tag{14}$$

with U_{ij}, V_{ij}, and W_{ij} the inertia tensors

$$U_{ij} = -\int d^3x \Big\{ 2\alpha\,\mathrm{Tr}(T_i T_j) + 4\beta\,\mathrm{Tr}([L_k, T_i][L_k, T_j]) + \frac{9\lambda^2}{16^2}\mathrm{Tr}([T_i, L_k][L_k, L_n][L_n, T_j]) \Big\}, \tag{15}$$

$$V_{ij} = -\epsilon_{ikl}\epsilon_{jmn}\int d^3x x_k x_m \Big\{ 2\alpha\,\mathrm{Tr}(L_l L_n) + 4\beta\,\mathrm{Tr}([L_p, L_l][L_p, L_n]) + \frac{9\lambda^2}{16^2}\mathrm{Tr}([L_l, L_p] \times [L_p, L_q][L_q, L_n]) \Big\}, \tag{16}$$

$$W_{ij} = \epsilon_{jkl}\int d^3x x_k \Big\{ 2\alpha\,\mathrm{Tr}(T_i L_l) + 4\beta\,\mathrm{Tr}([L_p, T_j] \times [L_p, L_n]) + \frac{9\lambda^2}{16^2}\mathrm{Tr}([T_i, L_m][L_m, L_n][L_n, L_l]) \Big\}, \tag{17}$$

and $T_i = iU^\dagger[\frac{\tau_i}{2}, U]$. Assuming a solution of the form (10), all inertia tensors become diagonal. Furthermore, one can show that $U_{11} = U_{22} \neq U_{33}$ and that similar identities hold for the V_{ij} and W_{ij} tensors. Finally the general expressions for the moments of inertia coming from each pieces of the Lagrangian read

$$U_{11} = \frac{4\pi}{3}\int r^2 dr \sin^2 F \Big(8\alpha + 16\beta\Big[4F'^2 + (3n^2+1) \times \frac{\sin^2 F}{r^2}\Big] + \frac{9\lambda^2}{4}\frac{(3n^2+1)}{4}F'^2\frac{\sin^2 F}{r^2}\Big), \tag{18}$$

$$V_{11} = \frac{4\pi}{3}\int r^2 dr \sin^2 F \Big(2(n^2+3)\alpha + 16\beta\Big[(n^2+3)F'^2 + 4n^2\frac{\sin^2 F}{r^2}\Big] + \frac{9\lambda^2}{4}n^2 F'^2 \frac{\sin^2 F}{r^2}\Big), \tag{19}$$

and the expression for U_{33} can be obtained by setting $n = 1$ in the integrand of (18). It turns out that expressions (15)–(17) lead to $W_{11} = W_{22} = 0$ for $|n| \geq 2$ and $n^2 U_{33} = nW_{33} = V_{33}$. Otherwise, for $|n| = 1$, where the solution has spherical symmetry, we get

$$W_{11} = \frac{4\pi}{3}\int r^2 dr \sin^2 F \Big(8\alpha + 16\beta\Big(4\frac{\sin^2 F}{r^2} + 4F'\Big) + \frac{9\lambda^2}{4}F'^2\frac{\sin^2 F}{r^2}\Big). \tag{20}$$

Following Houghton and Magee [11], we now write the rotational Hamiltonian as

$$H_{\mathrm{rot}} = \frac{1}{2}\Big[\frac{(L_1 + W_{11}\frac{K_1}{U_{11}})^2}{V_{11} - \frac{W_{11}^2}{U_{11}}} + \frac{(L_2 + W_{22}\frac{K_2}{U_{22}})^2}{V_{22} - \frac{W_{22}^2}{U_{22}}} + \frac{(L_3 + W_{33}\frac{K_3}{U_{33}})^2}{V_{33} - \frac{W_{33}^2}{U_{33}}} + \frac{K_1^2}{U_{11}} + \frac{K_2^2}{U_{22}} + \frac{K_3^2}{U_{33}} \Big], \tag{21}$$

with (K_i) L_i the body-fixed (iso)rotation momentum canonically conjugate to a_i and b_i, respectively. The expression for the rotational energy of the nucleon has been obtained in [11] and reads, for a spherical symmetry,

$$E_{\mathrm{rot}}^N = \frac{3}{8U_{11}}. \tag{22}$$

For the deuteron, the rotational energy has been calculated assuming an axial symmetric solution [16]

$$E_{\mathrm{rot}}^D = \frac{1}{2V_{11}} + \frac{1}{2V_{22}}, \tag{23}$$

which reduces to

$$E_{\mathrm{rot}}^D = \frac{1}{V_{11}} \tag{24}$$

for the axial ansatz (10). It is also easy to calculate the rotational energies for nuclei with winding number $|n| \geq 3$. The axial symmetry of the solution imposes the constraint $L_3 + nK_3 = 0$, which is simply the statement that a spatial rotation by an angle θ about the axis of symmetry can be compensated by an isorotation of $-n\theta$ about the τ_3 axis. It also implies that $n^2 U_{33} = nW_{33} = V_{33}$. Then, recalling that $W_{11} = W_{22} = 0$ for these values of n, the rotational Hamiltonian reduces to

$$H_{\mathrm{rot}} = \frac{1}{2}\Big[\frac{\mathbf{L}^2}{V_{11}} + \frac{\mathbf{K}^2}{U_{11}} + \Big(\frac{1}{U_{33}} - \frac{1}{U_{11}} - \frac{n^2}{V_{11}} \Big) K_3^2 \Big]. \tag{25}$$

These momenta are related to the usual space-fixed isospin $\mathbf{I}$ and spin $\mathbf{J}$ by the orthogonal transformations

$$I_i = -R(A_1)_{ij} K_j, \tag{26}$$

$$J_i = -R(A_2)^T_{ij} L_j. \tag{27}$$

According to (26) and (27), we see that the Casimir invariants satisfy $\mathbf{K}^2 = \mathbf{I}^2 = I(I+1)$ and $\mathbf{L}^2 = \mathbf{J}^2 = J(J+1)$, so the rotational Hamiltonian is given by

$$H_{\rm rot} = \frac{1}{2}\left[\frac{J(J+1)}{V_{11}} + \frac{I(I+1)}{U_{11}} + \left(\frac{1}{U_{33}} - \frac{1}{U_{11}} - \frac{n^2}{V_{11}}\right)K_3^2\right]. \tag{28}$$

IV. BPS-TYPE SOLUTIONS

Let us consider a model similar to [12] composed of the term of order six in derivatives plus a potential by setting $\alpha, \beta = 0$,

$$\mathcal{L} = -\frac{3}{2}\frac{\lambda^2}{16^2}\,\mathrm{Tr}([L_\mu, L_\nu][L^\nu, L^\lambda][L_\lambda, L_\mu]) - \mu^2 V(U). \tag{29}$$

Using the results of Sec. II, the static energy is

$$E_{\rm stat} = 4\pi \int dr\left(\frac{9\lambda^2 n^2}{4}\frac{\sin^4 F}{4r^2}F'^2 + \mu^2 V(U)\right). \tag{30}$$

The minimization of the static energy of the soliton, leads to the differential equation for F

$$\frac{9\lambda^2 n^2}{4}\frac{\sin^2 F}{2r^2}\partial_r\left(\frac{\sin^2 F}{r^2}F'\right) - \mu^2 V_F = 0. \tag{31}$$

A change of variable $z = \frac{2\sqrt{2}\mu r^3}{9n\lambda}$ allows (31) to be written in a simple form

$$\sin^2 F\partial_z[\sin^2 F(\partial_z F)] - \mu^2\frac{\partial V}{\partial F} = 0. \tag{32}$$

This last equation can be integrated

$$\frac{1}{2}\sin^4 F(F_z)^2 = V \tag{33}$$

and, inserting the expression for z, provides an expression which amounts to a statement of equipartition of the energy, i.e. the term of order 6 in derivatives and the potential contribute equally to the total energy. ASW has shown that a solution of (33) saturates the Bogomol'nyi's bound [12]. From (33), we obtain the following useful relation between the function F and the potential:

$$\int dF\frac{\sin^2 F}{\sqrt{2V}} = \pm(z - z_0) \tag{34}$$

with z_0 an integration constant.

Now comes the time to choose a specific potential. The choice for mass term of the Skyrme is not unique and indeed has been the object of several discussions [4,6,13]. The usual mass term $V = 1 - \cos F$ was considered in [12]. Solving (34) for F led to

$$F(r) = \begin{cases} 2\arccos(\nu r^3) & \text{for } r \in [0, \nu^{-(1/3)}] \\ 0 & \text{for } r \geq \nu^{-(1/3)} \end{cases}, \tag{35}$$

where $\nu = \frac{\mu}{18n\lambda}$ is a constant depending on the parameters λ, μ, and n. Note that F' diverges as $r \to \nu^{-(1/3)}$. Since this solution saturates the Bogomol'nyi's bound, the static energy is proportional to the baryon number $B = n$.

A question arises as to how would the nonlinear σ and Skyrme term affect the energy of such Skyrmions. Switching them on slowly by moving α and β away from zero could give an estimate of their contributions. Unfortunately, it turns out that simply substituting the solution (35) in the expression for energy associated with the full Lagrangian leads to divergences. So, however small the parameters α and β are, these BPS solutions cannot be considered as appropriate approximations of the solutions for (4).

Yet, it could be interesting to analyze the full Lagrangian (4) in a regime close to a BPS Skyrmion. For this purpose, we propose to write the potential in the form of the generalized mass term introduced by [4]

$$\begin{aligned} V &= -\sum_{k=1}^{\infty} C_k\,\mathrm{Tr}[U^k + U^{\dagger k} - 2] \\ &= -4\pi\sum_{k=1}^{\infty}\int r^2 dr 8C_k \sin^2\left(\frac{k\xi}{2}\right). \end{aligned} \tag{36}$$

The main motivation for this choice is that the potential can be written in a simple form in terms of pion fields. Furthermore, this particular framework insures that one recovers the chiral symmetry breaking pion mass term $-\frac{1}{2}m_\pi^2\pi\cdot\pi$ in the limit of small pion field fluctuations provided

$$\sum_{k=1}^{\infty} k^2 C_k = -\frac{m_\pi^2 F_\pi^2}{16}. \tag{37}$$

For practical purposes, one requires (i) an expression of the potential that is simple enough to allow the analytical integration of the left-hand side of Eq. (34), (ii) that the results lead to an invertible function to be able to write the chiral profile F as a function of r, and finally (iii) that $F(r)$ is well behaved. A most convenient choice is

$$V = \sin^2\left(\frac{F}{2}\right)\cos^6\left(\frac{F}{2}\right). \tag{38}$$

Expanding the expression (38), the coefficients C_k are

$$\begin{aligned} &C_1 = -\frac{\mu^2}{128}, \qquad C_2 = \frac{\mu^2}{128}, \qquad C_3 = \frac{\mu^2}{128}, \\ &C_4 = \frac{\mu^2}{512}, \qquad C_{k>4} = 0. \end{aligned} \tag{39}$$

Integrating (34), we get the general solution

$$F(r) = 2\arccos(e^{\pm\nu(r^3 - r_0^3)}) \tag{40}$$

with $\nu = \frac{\mu}{18n\lambda}$. In order that the baryon number corresponds to $|B| = n$, one must require that

$$F(\infty) - F(0) = \mp\pi$$

for B positive or negative, respectively. Accordingly, we choose the boundary conditions $F(0) = 0$ and $F(\infty) = \mp\pi$, which sets the integration constant r_0 to zero and leads to

$$F(r) = \mp 2|\arccos(e^{-\nu r^3})|, \tag{41}$$

where we use the absolute value to dispose of the sign ambiguity of the arccos function. Note that here, contrary to [12], we do not get a compacton-type solution but a well-behaved function, with a continuous first derivative. All calculations regarding energy can be performed analytically, i.e. static energy and the moments. For example, the baryon density is given by the radial function

$$B(r) = \frac{2\mu}{3\pi^2\lambda} e^{-(\mu r^3/6n\lambda)}(1 - e^{-(\mu r^3/9n\lambda)}),$$

which upon integration leads to baryon number $B = n$. Experimentally, the size of the nucleus is known to behave as

$$R = R_0 B^{1/3} = (1.25\ \text{fm})B^{1/3},$$

where $R_0 = 1.25$ fm. It is interesting to note that the baryon number distribution is zero at $r = 0$ but has maximum value $\frac{\sqrt{3}\mu}{8\pi^2\lambda}$ independent of n, which is positioned at

$$r_{\max} = \left(\frac{9\lambda}{\mu}\ln\left(\frac{4}{3}\right)\right)^{1/3} B^{1/3}\,m, \tag{42}$$

where $r_{\max}$ here is in units of MeV^{-1}. Accordingly the size of the nucleus $r_{\max}$ is proportional to $B^{1/3}$ with R_0 depending only on the ratio λ/μ. Similarly, expressions can be obtained for energy and moment of inertia densities. Using (40), they yield

$$\begin{aligned} E &= 2n\pi\mu\lambda \\ V_{11} &= n^2 U_{33} = \frac{4n^2}{(3n^2+1)} U_{11} \\ &= 2\pi\left(\frac{\lambda n}{3\mu}\right)^{5/3}\mu^2\Gamma\left(\frac{2}{3}\right)(16\cdot 3^{1/3} - 9\cdot 2^{2/3}), \end{aligned} \tag{43}$$

where $\Gamma(x)$ is the gamma function. Combining these results in (28)

$$H_{\text{rot}} = \frac{1}{2U_{11}}\left[J(J+1)\frac{(3n^2+1)}{4n^2} + I(I+1) - K_3^2\right]. \tag{44}$$

Note that this last result only holds for $\alpha = \beta = 0$ and the solution (40). The last term in H_{rot} is either zero or negative. Depending on the dimension of the spin and isospin representation, the diagonalization of this Hamiltonian will lead to a number of possible eigenstates. We are interested in the lowest eigenvalue of H_{rot}, which points towards the eigenstate $|i, i_3, k_3\rangle|j, j_3, l_3\rangle$ with the largest possible eigenvalue k_3. Since $\mathbf{K}^2 = \mathbf{I}^2$ and $\mathbf{L}^2 = \mathbf{J}^2$, the state with highest weight is characterized by $k_3 = i$ and $l_3 = j$ and, since nuclei are built out of B fermions, we must have $j \leq B/2$. On the other hand, the axial symmetry of the solutions implies that $k_3 = -l_3/n$. We recall that these solutions are approximations. Then for even B nuclei, the integer part of $|l_3/n|$ is

$$0 \leq |k_3| = \left[\left|\frac{l_3}{n}\right|\right] \leq \left[\left|\frac{j}{n}\right|\right] \leq \left[\left|\frac{B}{2n}\right|\right] = 0,$$

so it leads to $|k_3| = 0$. Similarly, for half-integer spin nuclei,

$$\frac{1}{2} \leq |k_3| \leq \left|\frac{j}{n}\right| \leq \left|\frac{B}{2n}\right| = \frac{1}{2}.$$

So we shall assume for simplicity that the largest possible eigenvalue k_3 is

$$\kappa = \max(|k_3|) = \begin{cases} 0 & \text{for } B = \text{even} \\ \frac{1}{2} & \text{for } B = \text{odd} \end{cases}.$$

Then, lowest possible rotational energy is given by

$$E_{\text{rot}} = \frac{1}{2U_{11}}\left[j(j+1)\frac{(3n^2+1)}{4n^2} + i(i+1) - \kappa^2\right]. \tag{45}$$

It remains to fix the values of the parameters λ and μ. In order to do so, we choose as input parameters the experimental mass of the nucleon and, for simplicity, a nucleus X with zero (iso)rotational energy (i.e. a nucleus with zero spin and isospin). The total energy of these two states are

$$\begin{aligned} E_N &= \left(E + \frac{3}{8U_{11}}\right)\Big|_{n=1} \\ &= 2\pi\mu\lambda + \frac{1}{\mu^2}\left(\frac{3\mu}{\lambda}\right)^{5/3}\frac{3}{16\pi\Gamma(\frac{2}{3})(16\cdot 3^{1/3} - 9\cdot 2^{2/3})} \end{aligned} \tag{46}$$

$$E_X = E|_{n=B} = 2B\pi\mu\lambda. \tag{47}$$

Solving for λ and μ we get

$$\begin{aligned} \lambda &= \frac{3\cdot 3^{1/4}}{(E_X)^{1/4}\sqrt{\pi}((E_X - nE_N)(9\cdot 2^{2/3} - 16\cdot 3^{1/3})\Gamma(\frac{5}{3}))^{3/4}}\mu \\ &= \frac{(E_X)^{5/4}((E_X - nE_N)(9\cdot 2^{2/3} - 16\cdot 3^{1/3})\Gamma(\frac{5}{3}))^{3/4}}{24\cdot 3^{1/4}\sqrt{\pi}}. \end{aligned} \tag{48}$$

As an example, we choose the nucleus X to be the helium-4, the first doubly magic number nucleus. The mass of the helium-4 nucleus has no (iso)rotational parts, since it has zero spin and isospin. Setting the mass of the nucleon as the average mass of the proton and neutron i.e. $E_N = 938.919$ MeV and the mass of the helium nucleus to $E_{\text{He}} = 3727.38$ MeV, we obtain the numerical values $\lambda = 0.00491505\ \text{MeV}^{-1}$ and $\mu = 30174.2\ \text{MeV}^2$. We shall refer to this set of parameters as Set Ia.

Experimentally, the size of the nucleus is known to behave as

$$R = R_0 B^{1/3}$$

with $R_0 = 1.25$ fm. We get a similar behavior for $r_{\max}$ in (42)

$$r_{\max} = (1.4798\ \text{fm})B^{1/3}. \tag{49}$$

Combining (48) with (12) and (45), the mass of any nucleus can be expressed as an analytical function of the input parameters E_N and E_{He}. In general, it depends on the baryon number as well as the spin and the isospin of the isotope. The spin of the most abundant isotopes are known. The isospins are not so well known, so we resort to the usual assumption that the most abundant isotopes correspond to states with lowest isorotational energy, i.e. states where the isospin I has the lowest value that the conservation of the third component of isospin I_3 allows. Accordingly,

$$I = |I_3| = \frac{1}{2}|\#\text{ of proton} - \#\text{ of neutron}| = \left|\frac{A}{2} - Z\right|. \tag{50}$$

Table I shows the relative deviation of the predictions with regard to experimental values of nuclear masses of a few isotopes. The predictions are accurate to 0.4% or better even for heavier nuclei. Part of this accuracy is probably due to the fact that the static energy of a BPS-type solution is proportional to B, so if it dominates, the nuclear masses should follow approximately the same pattern. However, the predictions remain surprising good, even though our calculations include rotational energy, and the model involves only to two free parameters λ and μ.

The computations were repeated using as input parameter $X = {}^{16}\text{O}$ and ${}^{40}\text{Ca}$, two other doubly magic nuclei (also shown in Table I, Set Ib and Set Ic, respectively). These set the parameters to $\lambda = 0.00449295\ \text{MeV}^{-1}$ and $\mu = 32977.0\ \text{MeV}^2$ and to $\lambda = 0.00426504\ \text{MeV}^{-1}$ and $\mu = 34717.8\ \text{MeV}^2$, respectively. Using these heavier elements as input parameters changes slightly the overall predicting accuracy. Whereas, the best overall accuracy is achieved using ${}^{16}\text{O}$ parametrization in Set Ib, the lightest isotopes are best described by choosing ${}^{4}\text{He}$ as input (Set Ia). Note that the lightest nuclei have lower moments of inertia and get relatively large rotational contribution to their mass. Consequently, their masses are expected to be more sensitive to the parameters affecting rotational energy. Likewise, since the ratio λ/μ decreases for $X = {}^{16}\text{O}$ and ${}^{40}\text{Ca}$, the size of the nucleus also decreases with $R_0 = 1.3943$ fm and 1.3470 fm, respectively.

Given this relative accuracy, one may wonder how switching on the nonlinear σ and Skyrme terms can improve or affect these results. Indeed, the last results suggest that these contributions need not be very large. This aspect is analyzed in the next section.

TABLE I. Relative deviation from experimental nuclear masses.

B	Nucleus	$\frac{E_X - E_{\text{exp}}}{E_{\text{exp}}}$ Set Ia	Set Ib	Set Ic	E_{exp} (MeV)
1	nucleon	input	input	input	938.919
2	${}^{2}\text{H}$	−0.0032	−0.0037	−0.0041	1875.61
3	${}^{3}\text{H}$	−0.0040	−0.0047	−0.0052	2808.92
4	${}^{4}\text{He}$	input	−0.0010	−0.0016	3727.38
6	${}^{6}\text{Li}$	−0.0017	−0.0026	−0.0032	5601.52
7	${}^{7}\text{Li}$	−0.0014	−0.0023	−0.0029	6533.83
9	${}^{9}\text{Li}$	−0.0006	−0.0015	−0.0022	8392.75
10	${}^{10}\text{B}$	−0.0004	−0.0013	−0.0019	9324.44
16	${}^{16}\text{O}$	0.0010	input	−0.0006	14895.1
20	${}^{20}\text{Ne}$	0.0010	0.0001	−0.0006	18617.7
40	${}^{40}\text{Ca}$	0.0016	0.0006	input	37214.7
56	${}^{56}\text{Fe}$	0.0018	0.0008	0.0002	52089.8
238	${}^{238}\text{U}$	0.0004	−0.0006	−0.0012	221696

V. NONLINEAR σ AND SKYRME TERMS

Let us now consider the full Lagrangian in (4), assuming that the contribution the nonlinear σ and Skyrme terms can be set arbitrarily small so that (40) represents a good approximation to the exact solution. Inserting the solution in (12) and in the expression for the various moments of inertia, one get additional contributions proportional to α and β

$$\begin{aligned} E_{\text{stat}} = {} & 2n\pi\mu\lambda + 16\pi\alpha\left(\frac{n\lambda}{3\mu}\right)^{1/3}\Gamma\left(\frac{1}{3}\right) \\ & \times\left((2 - 2^{2/3})(n^2+1) + 2\zeta\left(\frac{7}{3}\right)\right) \\ & + \frac{128\pi\beta}{3}\left(\frac{3\mu}{n\lambda}\right)^{1/3}\Gamma \\ & \times\left(\frac{2}{3}\right)((8(2\cdot 3^{1/3} - 2^{2/3}) - 7\cdot 2^{1/3})n^2 + 2^{1/3}) \end{aligned} \tag{51}$$

and

$$E_{\text{rot}} = \frac{1}{2}\left[\frac{j(j+1)}{V_{11}} + \frac{i(i+1)}{U_{11}} + \left(\frac{1}{U_{33}} - \frac{1}{U_{11}} - \frac{n^2}{V_{11}}\right)\kappa^2\right], \tag{52}$$

with $\kappa = 0$ or $\frac{1}{2}$ for even and odd B, respectively and

$$\begin{aligned} U_{11} = {} & 64\pi\alpha\left(\frac{n\lambda}{\mu}\right) + \frac{512\pi\beta}{9}\left(\frac{3n\lambda}{\mu}\right)^{1/3}\Gamma \\ & \times\left(\frac{1}{3}\right)(12^{1/3} + (3n^2+1)(-4 + 6^{1/3}(1 + 2^{1/3}))) \\ & + 2\pi\left(\frac{\lambda n}{3\mu}\right)^{5/3}\mu^2\Gamma\left(\frac{2}{3}\right)(16\cdot 3^{1/3} - 9\cdot 2^{2/3})\frac{(3n^2+1)}{4n^2} \end{aligned} \tag{53}$$

$$U_{33} = 64\pi\alpha\left(\frac{n\lambda}{\mu}\right) + \frac{512\pi\beta}{9}\left(\frac{3n\lambda}{\mu}\right)^{1/3}\Gamma \times\left(\frac{1}{3}\right)(12^{1/3} + 4(-4 + 6^{1/3}(1 + 2^{1/3}))) + 2\pi\left(\frac{\lambda n}{3\mu}\right)^{5/3}\mu^2\Gamma\left(\frac{2}{3}\right)(16\cdot 3^{1/3} - 9\cdot 2^{2/3})\frac{1}{n^2} \quad (54)$$

$$V_{11} = 64\pi\alpha\left(\frac{n\lambda}{\mu}\right)\frac{(n^2+3)}{4} + \frac{128\pi\beta}{9}\left(\frac{3n\lambda}{\mu}\right)^{1/3}\Gamma \times\left(\frac{1}{3}\right)((n^2+3)12^{1/3} + 16n^2(-4 + 6^{1/3}(1 + 2^{1/3}))) + 2\pi\left(\frac{\lambda n}{3\mu}\right)^{5/3}\mu^2\Gamma\left(\frac{2}{3}\right)(16\cdot 3^{1/3} - 9\cdot 2^{2/3}) \quad (55)$$

and as above $W_{11} = \delta_{n,1}U_{11}$ otherwise $W_{11} = W_{22} = 0$ for $|n| \geq 2$. Again due to the axial symmetry of the ansatz, $U_{11} = U_{22} \neq U_{33}$, while nondiagonal elements of U_{ij} are zero. Similar identities also hold for the V_{ij} and W_{ij} tensors. Furthermore, we have $n^2U_{33} = nW_{33} = V_{33}$. Relations (51)–(55) bring a clear understanding of the dependence of the masses of the nuclei on the various parameters $B = n$, μ, α, β, and λ, as long as α and β remain relatively small.

In order to estimate the magnitude of the parameter α and β in a real physical case, we perform two more fits: Set II optimizes the four parameters μ, α, β, and λ to reproduce the best fit for the masses of the nuclei, and Set III is done with respect to the ratio of the binding energy (B.E.) over atomic number, B.E.$/A$. More precisely, we use only a subset of table of nuclei [15] composed of the most abundant 144 isotopes (see Fig. 1). This is compared to Set I which was determined in the previous section using the masses of the nucleon and ^{4}He and assuming $\alpha = \beta = 0$. The results are presented in Fig. 1 in the form of B.E.$/A$, as a function of the baryon number for Sets Ia, II, III, and experimental values. The optimal values of the parameters are presented in Table II.

As suspected, the new sets of parameters are very close to Set Ia. The nonlinear σ and Skyrme parameters α and β are very small, but in order to compare, it is best to rescale the static energy with the change of variable $u = (4\mu/3\lambda)^{1/3}r$, such that the relative weight of each term is more apparent. Then the static energy takes the form

$$E_{\rm stat} = 4\pi\left(\frac{3\lambda\mu}{4}\right)\int u^2du\left(V + 2\alpha\left(\frac{4}{3\lambda\mu^2}\right)^{2/3} \times\left[F'^2 + (n^2+1)\frac{\sin^2F}{u^2}\right] + 16\beta\left(\frac{16}{9\lambda^2\mu}\right)^{2/3} \times\frac{\sin^2F}{u^2}\left[(n^2+1)F'^2 + n^2\frac{\sin^2F}{r^2}\right] + n^2F'^2\frac{\sin^4F}{u^4}\right), \quad (56)$$

where $F' = \partial F/\partial u$ and the energy can be expressed in units of $\frac{3\lambda\mu}{4}$. For example for Set II (Set III), the nonlinear σ term is proportional to $\alpha(\frac{4}{3\lambda\mu^2})^{2/3} = 3.73524\times10^{-7}$ (1.43418×10^{-6}) and the Skyrme term to $\beta(\frac{16}{9\lambda^2\mu})^{2/3} = -1.3263\times10^{-6}$ (-9.2355×10^{-6}), while the remaining terms are of order one. Furthermore, the overall factor $\frac{3\lambda\mu}{4}$ remains approximately the same for all the sets. Looking at the numerical results, we observe nonetheless that these two terms are responsible for corrections of the order of 0.01%. Clearly, the small magnitude of these contributions provides support to the assumption that (40) is a good approximation to the exact solution.

Comparing Set II to the original Skyrme Model with a pion mass term, we may identify

$$F_\pi = 4\sqrt{\alpha} = 0.364474 \text{ MeV}$$

$$(\text{Experiment: } F_\pi = 186 \text{ MeV}) \qquad e^2 = \frac{1}{32\beta} = -57019$$

$$(e = 4.84 \text{ for massive pion Skyrme Model})$$

$$m_\pi = \frac{2\sqrt{\alpha}}{\mu} = 231591 \text{ MeV}$$

$$(\text{Experiment: } m_\pi = 138 \text{ MeV}).$$

Set III leads to similar values for F_π, e^2, and m_π, which are orders of magnitude away for the usual values obtained for the Skyrme model. This was to be expected due to the nature of our approximation since we assumed α and β were relatively small, so that the nonlinear σ and Skyrme terms could be treated as perturbations. Of course here these perturbative terms do not play a significant role in the stabilization of the soliton as they do in the Skyrme Model. Indeed the Skyrme term even has the wrong sign, so it would destabilize the soliton against shrinking if it was not for the contribution of order six in derivatives. The size of the soliton is instead determined by the relative magnitude

TABLE II. Value of parameters for different fits.

Nucleus	Set Ia	Set II	Set III
μ (MeV^2)	30174.2	29841.2	29475.7
α (MeV^2)	0	0.00830341	0.0316869
β (dimensionless)	0	-5.48285×10^{-7}	-4.01085×10^{-7}
λ (MeV^{-1})	0.00491505	0.00496265	0.00503994

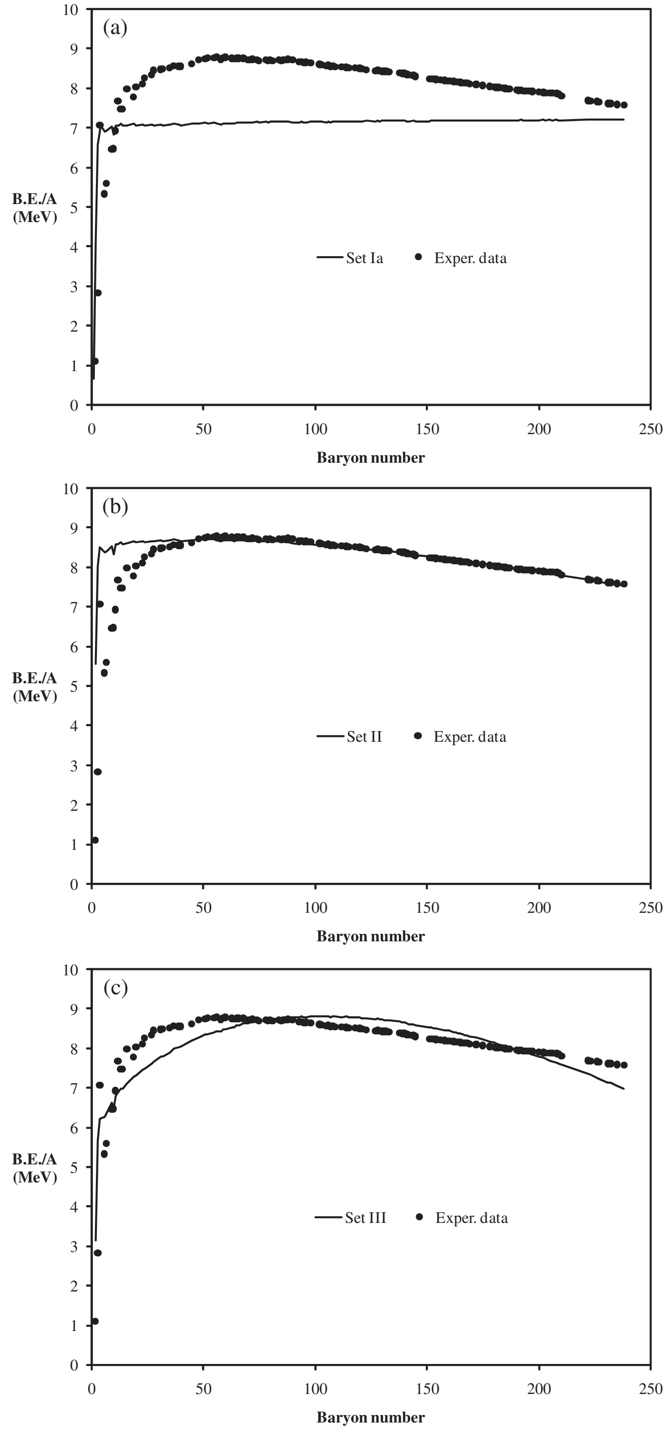

FIG. 1. Ratio of the binding energy (B.E.) over the atomic number A (or baryon number) as a function of A. The experimental data (black circles) are shown along with predicted values (lines) for parametization of (a) Set Ia, (b) Set II and (c) Set III, respectively.

of μ and λ, so there is no need for F_π and m_π to be close to the nucleon mass scale as for the original Skyrme Model. Perhaps the explanation for such a departure is that the parameters of the model are merely bare parameters, and they could differ significantly from their renormalized physical values. Yet, the original Skyrme Model established a link between pion physics with realistic values for F_π and m_π and baryon masses. On the other hand, the model in (4) (in the regime where α and β are small) improved prediction of nuclear masses, but the link to pion physics is more obscure.

We note also that the $B > 1$ solutions of the Skyrme Model display a totally different structure compared to the BPS-type solution analyzed here. It is well known that the lowest-energy $B = 2, 3, 4, \ldots$ solutions of the Skyrme Model exhibit, respectively, toroidal, tetrahedral, cubic, ... baryon density configurations. Such solutions are conveniently represented by an ansatz based on rational maps [9]. The model at hand here leads to spherically symmetric baryon density, at least in the regime of small α and β, where solution (40) can apply. So it seems that the regime dominated by the μ and λ terms leads to spherical configurations, whereas the regime dominated by the nonlinear σ and Skyrme terms shows totally different baryon density distributions. In the absence of a complete analysis, we can only conjecture that the change in configuration is related to which of the four terms are responsible for the stabilization of the soliton and at some critical values of the parameters there is a transition between configurations.

Let us now look more closely at the numerical results presented in Fig. 1. These are in the form of the ratio of the binding energy (B.E.) over the atomic number A as a function of A, which corresponds to the baryon number. The experimental data (black circles) are shown along with predicted value (lines) for parametization of (a) Set Ia, (b) Set II, and (c) Set III, respectively. Clearly Set Ia is less accurate when it comes to reproduce the full set of experimental data but is somewhat successful for the lightest nuclei. This to be expected since the fit relies on the masses of the nucleon and ^{4}He. Yet, all predicted nuclear masses are found to be within a 0.4% precision. In fact, the ratio B.E.$/A$ is rather sensitive to small variation of the nuclear masses so the results in general are surprisingly accurate. On the other hand, Set II, based on the nuclear masses, overestimates the binding energies of the lightest nuclei, while it reproduces almost exactly the remaining experimental values. In Fig. 1(b), the predicted values for large B coincide almost exactly with experimental data. The least square fit based on B.E.$/A$, Set III, is the best fit overall, but in order to better represent the features of the lightest nuclei, it abdicates some of the accuracy found in Set II for $B > 30$.

This apparent dichotomy between the description of the two regions $B \leq 30$ and $B > 30$ may find an explanation in the (iso)rotational contribution to the mass. Indeed light nuclei have smaller sizes and moments of inertia so that their rotational energy contributes to a larger fraction of the total mass as the spins and isospins remain relatively small. On the other hand, the size of heavy nuclei grows as $B^{1/3}$ and their moments of inertia increase accordingly. The spin of the most abundant isotopes are relatively small, while isospin can have relatively large values due to the growing disequilibrium between the number of protons and the number of neutrons in heavy nuclei [see Eq. (50)]. Despite these behaviors, our numerical results show that the rotational energy is less than 1 MeV for $B > 10$ for any of the Sets considered, and its contribution to B.E.$/A$ decreases rapidly as B increases. On the contrary, for $B < 10$, the rotational energy is responsible for large part of the binding energy, which means that B.E.$/A$ should be very sensitive to the way the rotational energy is computed. In our case, we approximated the nucleus as a rigid rotator. One may argue that if rotational deformations due to centrifugal effects were to be considered, it would lead to larger moments of inertia and lower rotational energies. This would predominantly affect the binding energy of the lightest nuclei since this is where rotational energy is most significant. Allowing for such deformation would in general require the full numerical computation of the solution. An easier way to check for deformation is by allowing the ratio of the parameters $\sigma = \mu/\lambda$ in the solution (40) to vary independently from the μ and λ in the model (4) and by repeating the fit with respect to five parameters instead of the four previous ones. This procedure allows for a further adjustment of the size of the soliton in terms of σ with respect to a given choice of model parameters μ, α, β, and λ and would lead to partial deformation of the solution. Such a parametrization is expected to increase both the size and the moments of inertia of the soliton and decrease the total mass of the lightest nuclei, which would be an improvement over the four parameters fit. We evaluated such correction for the nucleon whose relative contribution to mass from rotation is the largest using the parameters of Set II, and we obtained a modest decrease of the mass of the order of 0.16%. Since the rotational energy accounts for much less than 1% of the total energy in most of the nuclei, deformations are not generally expected to be very significant.

VI. CONCLUSION

We have proposed a 4-terms model as a generalization of the Skyrme Model. In the regime where two of the terms are negligible, i.e. $\alpha = \beta = 0$, we find well-behaved analytical solutions for the static solitons. These saturate the Bogomol'nyi's bound with consequence that the static energy is directly proportional to the baryon number B. They differ from those obtained by ASW in an important way: their model leads to compactons at the boundary of which the gradient of the solution is infinite, and so the solution could not be used to approximate the energies in the regime where $\alpha, \beta \neq 0$. Furthermore, one of the major features of our model is that the form of the solutions allows to compute analytically the static and rotational energy and expresses them as a function of the model parameters and B. Fixing the remaining parameters of the model μ and λ leads to rather accurate predictions for the mass of the nuclei.

We then used these BPS-type solutions to compute the mass of the nuclei in the regime where α and β are small but not zero. Indeed, fitting the model parameters to provide the best description of the nuclear mass data leads to that particular regime where the values of α and β turn out to be very small. Yet, we find a noticeable improvement in the size and B.E.$/A$ predictions with respect to those for the $\alpha = \beta = 0$ regime. Even though our 4-term model can be considered a simple extension of the massive pion Skyrme Model (different mass term and an additional term with six derivatives in pion fields) the solution leads to spherically symmetric baryon densities, as opposed to more complex configurations for $B > 1$ standard Skyrmions (e.g. toroidal, tetrahedral, cubic, ...). These results suggest that nuclei could be considered as near BPS Skyrmions.

On the other hand, our results are somewhat puzzling: Considering the nonlinear σ and Skyrme terms as perturbations constrained the values of pion decay constant F_π and pion mass m_π to values which differ by 2 orders of magnitude from that of the Skyrme Model or their experimental values. As we improved predictions for nuclear masses, we seem to have lost the link established by the original Skyrme Model between pion physics and baryons with realistic values for F_π and m_π and baryon masses. It remains an open question as to which set of parameters in our model (or more generally which extension of the Skyrme Model) would give the best description of both pion physics and nuclear properties.

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