

Endpoint behavior of the pion distribution amplitude in QCD sum rules with nonlocal condensates

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Starting from the QCD sum rules with nonlocal condensates for the pion distribution amplitude, we derive another sum rule for its derivative and its “integral derivatives”—defined in this work. We use this new sum rule to analyze the fine details of the pion distribution amplitude in the endpoint region $x \sim 0$. The results for endpoint-suppressed and flattop (or flatlike) pion distribution amplitudes are compared with those we obtained with differential sum rules by employing two different models for the distribution of vacuum-quark virtualities. We determine the range of values of the derivatives of the pion distribution amplitude and show that endpoint-suppressed distribution amplitudes lie within this range, while those with endpoint enhancement—flat-type or Chernyak-Zhitnitsky like—yield values outside this range.

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I. INTRODUCTION

Many of the theoretical and phenomenological analyses of QCD processes rely upon the factorization of the underlying dynamics into a short-distance dominated part, amenable to QCD perturbation theory, and a large-distance part that has to be taken from experiment or be determined by nonperturbative methods. Among such processes, the pion form factors (electromagnetic and transition) play the role of a theoretical laboratory to test various ideas and techniques. The key ingredient in these descriptions is the pion distribution amplitude (DA) $\varphi_\pi(x)$ which represents the pion bound state. At leading twist two it is defined in terms of a nonlocal axial current and reads [1]

$$\begin{aligned} & \langle 0 | \bar{d}(z) \gamma^\mu \gamma_5 [z, 0] u(0) | \pi(P) \rangle \Big|_{z^2=0} \\ &= i f_\pi P^\mu \int_0^1 dx e^{ix(z \cdot P)} \varphi_\pi^{(t=2)}(x, \mu_0^2), \end{aligned} \quad (1.1)$$

where x ($\bar{x} \equiv 1 - x$) is the longitudinal momentum fraction carried by the valence quark (antiquark) in the pion, and the path-ordered exponential, i.e., the lightlike gauge link

$$[z, 0] = \mathcal{P} \exp \left[-ig \int_0^z dy^\mu t^a A_\mu^a(y) \right], \quad (1.2)$$

ensures gauge invariance.

The pion DA has an expansion in the basis of the Gegenbauer polynomials which constitute the eigenfunctions of the one-loop meson evolution equation. At a typical hadronic scale μ_0^2 , which serves as a normalization scale, one then has [2]

$$\begin{aligned} \varphi^{(t=2)}(x; \mu_0^2) &= \varphi^{\text{as}}(x) [1 + a_2(\mu_0^2) C_2^{3/2}(2x - 1) \\ &+ a_4(\mu_0^2) C_4^{3/2}(2x - 1) + \dots], \end{aligned} \quad (1.3)$$

where $\varphi^{\text{as}}(x) = 6x\bar{x}$ is the asymptotic pion DA. By virtue of the leptonic decay $\pi \rightarrow \mu^+ \nu_\mu$, one obtains the normalization $\int_0^1 dx \varphi_\pi^{(t=2)}(x, \mu_0^2) = 1$, which fixes $a_0 = 1$.

Rather than try to derive the pion DA as a whole, one attempts to reconstruct it from its first few moments

$$\langle \xi^N \rangle_\pi \equiv \int_0^1 dx (2x - 1)^N \varphi_\pi(x), \quad (1.4)$$

where $\xi \equiv 2x - 1$. The values of the moments may be determined by means of QCD sum rules (SR)s with local [3] or nonlocal condensates [4–7] or be computed by numerical simulations on the lattice [8–10]. Once they are known, one can use them to reverse engineer the pion DA, with a precision depending upon the influence of the magnitude of the discarded higher-order moments. It was shown in Ref. [11] that, using QCD sum rules with nonlocal condensates, one can de facto resort to the first two Gegenbauer coefficients a_2, a_4 , while a_i with $i = 6, 8, 10$ turn out to be negligible. Once the shape of the pion DA has been determined at some (low) normalization scale μ^2 around 1 GeV², one can evolve the Gegenbauer coefficients to higher values of the momentum scale using the Efremov-Radyushkin-Brodsky-Lepage [2] evolution equation which is determined by means of QCD perturbation theory.

It turns out that another quantity which is intertwined with the form factors of the pion is its inverse moment

$$\langle x^{-1} \rangle_\pi = \int_0^1 dx \frac{1}{x} \varphi_\pi(x). \quad (1.5)$$

This quantity is one of the key ingredients of the pion-photon transition form factor, a process that has attracted

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the continuous attention of theorists [11–27] and experimentalists [28–30]. Actually, the most recent measurement of this observable by the *BABAR* Collaboration [30] has provided controversial results. At moderate values of the momentum transfer, up to 10 GeV², the new high-precision *BABAR* data agree well with the previous CLEO data [29] and can be best described by pion DAs that have their endpoints strongly suppressed [27,31,32]—as the Bakulev-Mikhailov-Stefanis (BMS) model [11] derived from QCD sum rules with nonlocal condensates. By contrast, the high- Q^2 *BABAR* data show an unexpected growth with Q^2 , which cannot be understood on the basis of collinear factorization and calls for pion DAs that have their endpoints strongly enhanced [33,34]. This intriguing behavior provides the basic motivation for the present investigation, though we will not attempt to describe any data.

We shall employ in this work the method of QCD SRs with nonlocal condensates (NLC)s with the aim to estimate the slope of the pion DA in the region $x \sim 0$, trying to understand the fine structure of the pion DA in this region vs the ansatz for the quark-virtuality distribution in the nonperturbative QCD vacuum. Our main interest will be in the behavior of pion DAs with distinct endpoint characteristics. QCD SRs were mainly proposed with the purpose of studying the *integral* characteristics of the pion DA. To overcome this restriction, we shall design an operator for defining *integral derivatives* of the pion DA. These will supplement the results obtained with SRs which employ the *standard derivative* of the pion DA. In this latter case, we will use in our analysis not only a delta-function ansatz for the vacuum-quark-virtuality distribution but also a refined model which describes the large-distance regime more accurately.

The paper is organized as follows. In Sec. II, we briefly discuss the main features of the nonperturbative QCD vacuum in terms of the scalar-quark condensate. We also give the corresponding SRs and recall their main ingredients. Section III is devoted to the calculation of the slope of the pion DA in the endpoint region employing two different techniques: integral SRs and differential SRs. Finally, Sec. IV contains our conclusions, while some important technical details are given in four appendices.

II. NONPERTURBATIVE QCD VACUUM WITH NONLOCAL CONDENSATES

The basic idea underlying the NLC approach is that the vacuum condensates possess a correlation length which endows the vacuum quarks with a nonzero average virtuality $\langle k_q^2 \rangle$ (see, for instance, [35]). To analyze the nonlocality of the vacuum condensate, it is useful to parameterize the lowest one¹ $\langle \bar{q}(0)[0, z]q(z) \rangle \equiv M_S(z^2)$ with the help of the

¹In this work we use the gauge $z^\mu A_\mu = 0$. Therefore, one has $\langle [0, z] = 1$.

vacuum distribution function $f_S(\alpha)$:

$$M_S(z^2) = \langle \bar{q}q \rangle \int_0^\infty f_S(\alpha) e^{\alpha z^2/4} d\alpha \quad (2.1)$$

that describes the distribution of the vacuum-quark virtuality α [4]. Assuming $f_S(\alpha) = \delta(\alpha - \lambda_q^2/2)$ that takes into account only a fixed virtuality λ_q^2 of the vacuum quarks leads to the simplest Gaussian model

$$\langle \bar{q}(0)q(z) \rangle = \langle \bar{q}(0)q(0) \rangle e^{-|z|^2 \lambda_q^2/8} \quad (2.2)$$

for the scalar-quark condensate [4].

The parameter λ_q^2 represents the typical quark momentum in the vacuum and is given by

$$\langle k_q^2 \rangle = \frac{\langle \bar{q}(0)\nabla^2 q(0) \rangle}{\langle \bar{q}(0)q(0) \rangle} \equiv \lambda_q^2. \quad (2.3)$$

In this work, we use the value $\lambda_q^2 = 0.4 \text{ GeV}^2$, which is supported by several analyses, though values within the interval $[0.35 \div 0.45] \text{ GeV}^2$ are still acceptable (see [11,22,36] and references cited therein).

The QCD SRs with nonlocal condensates for the pion DA were first proposed in [4] and were significantly improved in [11] from which we quote

$$\begin{aligned} f_\pi^2 \varphi_\pi(x) + f_{A_1}^2 \varphi_{A_1}(x) e^{-m_{A_1}^2/M^2} + \int_{s_0}^\infty \rho_{\text{pert}}(x) e^{-s/M^2} ds \\ = \int_0^\infty \rho_{\text{pert}}(x) e^{-s/M^2} ds + \Delta\Phi_G(x, M^2) \\ + [\Delta\Phi_S(x, M^2) + \Delta\Phi_V(x, M^2) + \Delta\Phi_T(x, M^2)]_Q. \end{aligned} \quad (2.4)$$

Here, φ_{A_1} is the A_1 -meson DA, whereas f_π and f_{A_1} are, respectively, the decay constants of the A_1 and the π meson. The A_1 -meson state is an effective state that collects the π' and the a_1 meson. The nonperturbative ingredients in the theoretical part of the SR are the gluon-condensate term $\Delta\Phi_G(x, M^2)$ and the quark-condensate contribution $[\dots]_Q$. This latter contribution contains the vector-condensate term (V), the mixed quark-gluon condensate term (T), and the scalar condensate term (S). The explicit expressions for the nonperturbative contributions and the NLO spectral density $\rho_{\text{pert}}^{(\text{NLO})}(x)$ are given in Appendices A and C, respectively. It turns out that in the endpoint region, the first radiative correction in the spectral density, which is of $\mathcal{O}(\alpha_s)$, is too large, thus overshadowing the zeroth order perturbative contribution and the nonperturbative contribution. For that reason, we use in this work the leading-order (LO) approximation $\rho_{\text{pert}}^{(\text{LO})}(x) = 3x\bar{x}/2\pi^2$. In order to include radiative corrections into the spectral density—when analyzing the endpoint region—one would have to resum all radiative corrections, a formidable task outside the scope of the present investigation.

III. SLOPE OF THE PION DA

As already mentioned in the Introduction, the endpoint region of the pion DA turns out to be of particular importance for a variety of pion observables—see [16,37,38] for an in-depth discussion of this issue. For example, in the case of the pion-photon transition form factor, one finds that in order to comply with the CLEO data—and those *BABAR* data close to them—one needs endpoint-suppressed pion DAs, while the high- Q^2 *BABAR* data can only be described with flat-type pion DAs. Therefore, it is crucial to explore the fine details of the pion DA in a region around the origin $x \sim 0$. One way to study the endpoint regime of the pion DA is provided by the inverse moment $\langle x^{-1} \rangle_\pi$ [39]. In Fig. 1, we show the values of

$$\langle x^{-1}(y) \rangle = \frac{1}{\delta} \int_y^{y+\delta} \frac{\phi_\pi(x)}{x} dx \quad (3.1)$$

with $\delta = 0.05$ for some pion DA models with a characteristic behavior in the endpoint region. From this figure one sees that the BMS pion DA (dashed green line), derived with the aid of nonlocal condensates [11], exhibits an evident endpoint suppression, while all other models have more (flattop—dashed-dotted-dotted red line, and CZ—dashed-dotted green line) or less (asymptotic DA) endpoint enhancement. The considered models of the pion DA and their parameters are given in Appendix B. In the present work, we will probe the endpoint region of the pion DA also by another means, namely, the integral derivative which will be defined next.

A. Integral sum rules

We introduce now the integral derivatives in order to discuss the slope of the pion DA. We define these quantities in terms of a set of operators $D^{(n)}$ given by

$$\begin{aligned} [D^{(0)}\varphi](x) &= \varphi'(x), & [D^{(1)}\varphi](x) &= \varphi(x)/x, \\ [D^{(2)}\varphi](x) &= \frac{1}{x} \int_0^x \frac{\varphi(y)}{y} dy. \end{aligned} \quad (3.2)$$

Then, assuming $\varphi(0) = 0$, we get the iterative formula

$$[D^{(n+1)}\varphi](x) = \frac{1}{x} \int_0^x dy [D^{(n)}\varphi](y). \quad (3.3)$$

Thus, each higher derivative within the set of the differential operators $D^{(n)}$ is stronger averaged with respect to x than the previous one. The usefulness of these derivatives derives from the fact that they can be applied to QCD sum rules which in general contain on their right-hand side singular contributions. In Appendix D, we elaborate on $D^{(n)}$ so that here we can focus our attention on the main properties of these derivatives. First, it is obvious that $D^{(n)}$ acts on a linear function as a differentiation operator, i.e., $D^{(n)}ax = a$. Second, assuming that the Taylor expansion of $\varphi(x)$ at $x = 0$ exists, one finds from (3.5)

$$[D^{(\nu+2)}\varphi](x) = \varphi'(0) + \varphi''(0) \frac{x}{2!2^{\nu+1}} + O\left(\frac{x^2}{3^{\nu+1}} \varphi^{(3)}\right), \quad (3.4)$$

which is valid for any *real* ν , as we explain in Appendix D. From the above equation, one can see that the defined operator $D^{(\nu)}$ reproduces at small x and/or large ν the derivative of $\varphi(x)$ at the origin $x = 0$. Strictly speaking, using (3.4), one obtains $\lim_{x \rightarrow 0} [D^{(\nu+2)}\varphi](x) = \varphi'(0)$ (at fixed $\nu \in \mathbb{R}$), and, using (3.5) and (D4), one gets $\lim_{\nu \rightarrow \infty} [D^{(\nu+2)}\varphi](x) = \varphi'(0)$ (at fixed x). For this reason, we may appeal to Eq. (3.3) and call the variation range of the operator $D^{(\nu+2)}$ the integral derivative of φ . Having defined this operator, we can derive the following expression

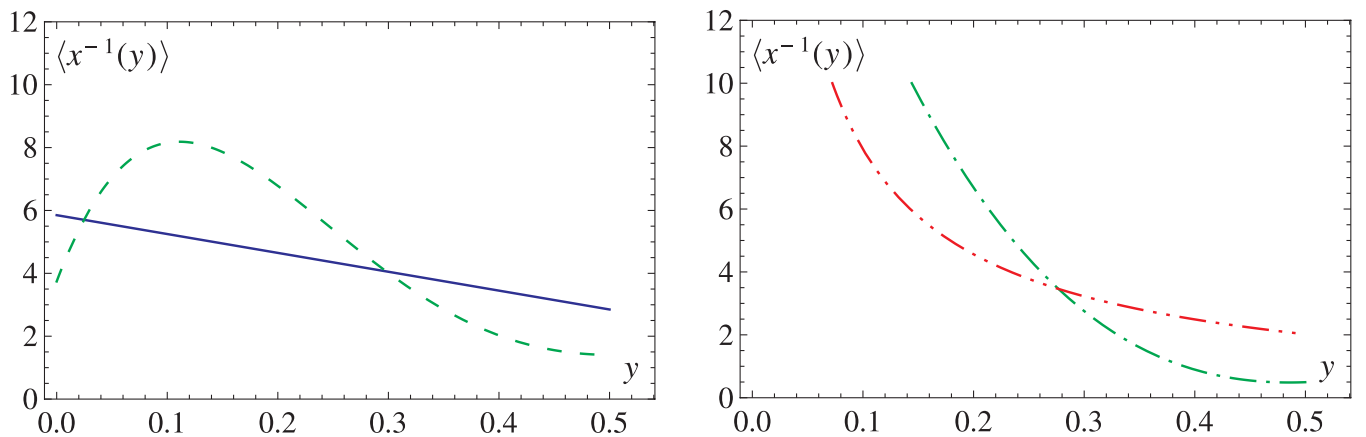


FIG. 1 (color online). Graphical representation of $\langle x^{-1}(y) \rangle$, integrated in the region y to $y + \delta$ for different pion DAs. Left panel: Solid blue line—*asymptotic*; dashed green line—BMS [11]. Right panel: Dashed-dotted green line—CZ [3]; dashed-dotted-dotted red line—flattop DA [Eq. (B1) with $\alpha = 0.1$].

$$[D^{(\nu+2)}\varphi](x) = \frac{1}{x} \int_0^x \varphi(y) f(y, \nu, x) dy, \quad (3.5)$$

where

$$f(y, \nu, x) = \frac{\theta(x-y)}{\Gamma(\nu+1)y} \left(\frac{x}{y}\right)^\nu \quad (3.6)$$

for any real ν (see Appendix D). As it is seen from Eq. (3.5), the function $f(y, \nu, x)$ acts as a ‘‘smooth projector’’ onto the vicinity of the origin of y , as one can appreciate from Fig. 10 in Appendix D.

By applying the operator $[D^{(\nu+2)}]$ on both sides of the QCD SR given by (2.4), we obtain a new SR, viz.,

$$\begin{aligned} & f_\pi^2 [D^{(\nu+2)}\varphi_\pi](x) + f_{A_1}^2 e^{-m_{A_1}^2/M^2} [D^{(\nu+2)}\varphi_{A_1}](x) \\ & + \int_{s_0}^\infty [D^{(\nu+2)}\rho_{\text{pert}}](x) e^{-s/M^2} ds \\ & = \int_0^\infty [D^{(\nu+2)}\rho_{\text{pert}}](x) e^{-s/M^2} ds + [D^{(\nu+2)}\Delta\Phi_G](x, M^2) \\ & + [D^{(\nu+2)}\Delta\Phi_V](x, M^2) + [D^{(\nu+2)}\Delta\Phi_T](x, M^2) \\ & + [D^{(\nu+2)}\Delta\Phi_S](x, M^2). \end{aligned} \quad (3.7)$$

In order to achieve a better stability of this SR, we take into account an effective A_1 -meson state that embodies the π' and the a_1 mesons and has the decay constant $f_{A_1} = 0.227$ GeV and the mass $m_{A_1}^2 = 1.616$ GeV². For the pion-decay constant and the continuum threshold, we use $f_\pi = 0.137$ GeV and $s_0^{\text{NLO}} = 2.25$ GeV², respectively. These values were derived before from the corresponding two-point QCD SRs with nonlocal condensates, see Ref. [11]. However, because in our SR—cf. (3.7)—we have to resort to the LO expression for the spectral density (because the first radiative correction is too large), we adopt a somewhat larger value of the threshold parameter: $s_0 \approx s_0^{\text{NLO}}(1 + \alpha_s/\pi) = 2.61$ GeV². This is done for both the integral as well as the differential SRs, the reason being that we want to preserve the correct normalization of the pion DA.²

As usual, we study the SRs in the fiducial interval of the Borel parameter $M^2 \in [M_{\text{min}}^2, M_{\text{max}}^2]$, where both terms, the continuum contribution and the nonperturbative one, each contributes about 1/3 to the whole SR (2.4). This induces an uncertainty of the order of $(1/3)^2 \rightarrow 10\%$. Moreover, the quantities to be calculated with the SR (the integral derivatives) should not (crucially) depend on the Borel parameter. Therefore, we should take care that this dependence is minimized. To achieve this goal, we attempt to minimize the root-mean-square deviation by varying the $[D^{(\nu+2)}\varphi_{A_1}](x)$ contribution in the fiducial Borel interval. On this account, we can average the M^2 dependence of the pion DA contribution [first term in Eq. (3.7)] in order to get a more reliable form of the SR.

²We thank A. P. Bakulev for useful remarks on this point.

Therefore, we write

$$\begin{aligned} [D^{(\nu+2)}\varphi_\pi^{\text{SR}}](x) & = \langle [D^{(\nu+2)}\varphi_\pi](x, M^2) \rangle \\ & \equiv \frac{1}{M_{\text{max}}^2 - M_{\text{min}}^2} \\ & \times \int_{M_{\text{min}}^2}^{M_{\text{max}}^2} dM^2 [D^{(\nu+2)}\varphi_\pi](x, M^2). \end{aligned} \quad (3.8)$$

In Fig. 2, we show the M^2 dependence of $[D^{(3)}\varphi_\pi](x, M^2)$, obtained from SR (3.7), for different values of $[D^{(3)}\varphi_{A_1}](x)$ and evaluating it for $x = 0.5$. The average value in the fiducial interval is $[D^{(3)}\varphi_\pi^{\text{SR}}](0.5) = 4.8 \pm 0.5$.

We discuss now the integral sum rules and their applications. The main contribution from the singularities in the SR for $[D^{(\nu+2)}\varphi_\pi](x, M^2)$ stems from the x region around $\Delta \equiv \lambda_q^2/(2M^2)$. This is because we used a delta-ansatz model [cf. (A1) and (A2)] for the condensates, which implies that the nonperturbative contributions have delta-function and Heaviside-function behaved terms (A3)–(A8). Therefore, in order to take into account all NLC contributions, we should analyze the region $x \geq 0.4$ and use $M_{\text{min}}^2 \geq 0.6$ GeV², which corresponds to $\Delta \leq 1/3 < x$. Moreover, the image of the operator $D^{(\nu+2)}$ for $\nu \geq 4$ is numerically very close to the result obtained with the differentiation method (see next subsection) for any x . Thus, the integral SR (3.7) becomes close to the differential SR, which we will consider in the next section. For these reasons, we analyze the constructed SR (3.7) for $\nu = 0, 1, 2, 3, 4$, and $x > 0.4$ and present the results in Fig. 3 by the solid line that is inside the light gray strip bounded by the short-dashed lines. For the sake of comparison, the predictions for the asymptotic DA (dashed-dotted line) and the BMS DA bunch—obtained in the NLC SR analysis of Ref. [11]—(shaded band limited by long-dashed lines)

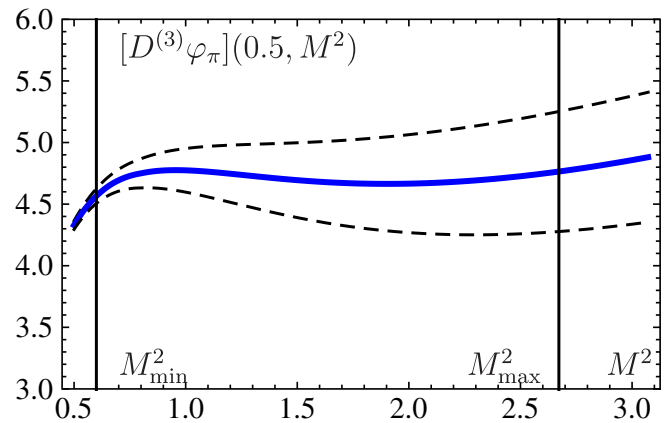


FIG. 2 (color online). Dependence of $[D^{(3)}\varphi_\pi](x, M^2)$ on the auxiliary Borel parameter M^2 for $x = 0.5$. The solid blue line corresponds to $[D^{(3)}\varphi_{A_1}](x) = 6.4$, whereas the dashed lines refer to $[D^{(3)}\varphi_{A_1}](x) = 6.7$ (upper curve) and $[D^{(3)}\varphi_{A_1}](x) = 7.1$ (lower curve), using in all cases $x = 0.5$. The two vertical lines delimit the fidelity region $M^2 \in [M_{\text{min}}^2, M_{\text{max}}^2]$.

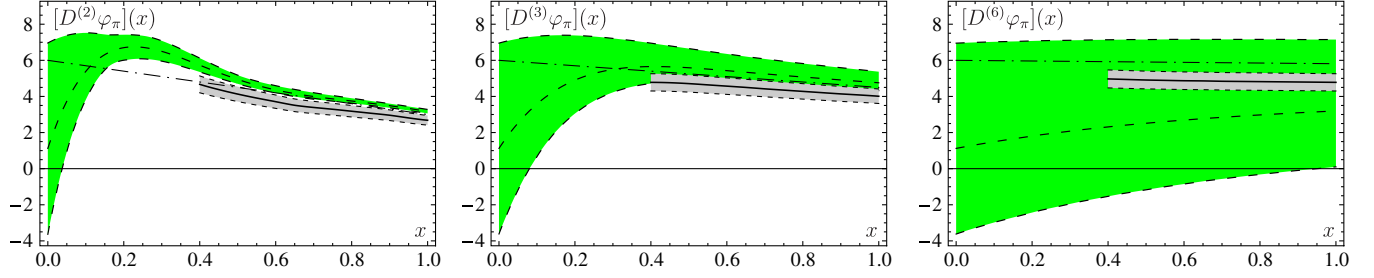


FIG. 3 (color online). We show the x dependence of $[D^{(\nu+2)}\varphi_\pi](x)$ for the BMS bunch of pion DAs [11] (shaded green band within long-dashed lines) in comparison with the SR result (3.8) (narrow gray strip) in all three panels. The left panel shows the predictions for $\nu = 0$, whereas those for $\nu = 1$, and $\nu = 4$ are shown in the middle and the right panel, respectively. The dashed-dotted line represents the asymptotic result $[D^{(\nu+2)}\varphi^{as}](x) = 6 - 3x/2^\nu$.

are also shown. From this figure, we see that our SR estimates for $[D^{(\nu+2)}\varphi_\pi^{\text{SR}}](x)$ agree fairly well with the BMS model—see also Table I. This table shows estimates for the third-order integral derivative of the pion DA for $x = 0.5$, using (i) the sum rule given by Eq. (3.7) and (ii) the pion DA models we discussed above, and also flat-type DAs which we consider below.

We turn now our attention to flat-type DAs. First, we compare the QCD SR result, obtained in (3.8), with what one finds with the flattop model $\varphi_{(B1)}^{\text{flat}}(x) \sim (x(1-x))^\alpha$ given in Appendix B. For this model, one has

$$[D^{(\nu+2)}\varphi_{(B1)}^{\text{flat}}](x) \gg [D^{(\nu+2)}\varphi^{\text{as}}](x) \geq [D^{(\nu+2)}\varphi_\pi^{\text{SR}}](x)$$

for $\alpha < 0.1$

for any real $\nu \in +\mathbb{R}$ and $0 < x < 1$. For the value $\alpha = 0.1$, we find $[D^{(3)}\varphi_{(B1)}^{\text{flat}}](0.5) = 227$, which is much larger and far outside the range of values extracted from our SR.

Second, we consider a particular flat-type pion DA, which is provided by the anti-de Sitter/QCD correspondence in the holographic approach—see, for instance, Refs. [40–43]. In that case, one has $\alpha = 0.5$ yielding $[D^{(3)}\varphi^{\text{hol}}](0.5) = 14$.

Third, we study an alternative flatlike pion DA which results from the Gegenbauer expansion of unity by retain-

ing only the first few harmonics. One obtains

$$\varphi_{(3.9)}^{\text{flat}}(x) = 6x\bar{x} \sum_{n=0}^3 C_{2n}^{3/2}(2x-1) \frac{2(4n+3)}{3(2n+1)(2n+2)} \quad (3.9)$$

with a profile shown in Fig. 4 in comparison with the models already mentioned: BMS—solid line; CZ—long-dashed blue line; flattop DA given by Eq. (B1)—dotted red line; flatlike DA given by Eq. (3.9)—dashed-dotted green line.

It is interesting to notice that using the pion DA given by expression (3.9) to calculate the pion-photon transition form factor according to Radyushkin’s expression (24) in Ref. [33], one actually reproduces the gross features of his results. This proves that the inclusion of the first few Gegenbauer polynomials in (3.9) does not affect the result obtained with $\varphi(x) = 1$ in the momentum range $Q^2 \leq 40 \text{ GeV}^2$ in a crucial way. This type of DA, i.e., (3.9), yields for the integral derivative the value $[D^{(3)}\varphi_{(3.9)}^{\text{flat}}](0.5) = 22.5$, which is much larger than the range of values determined via our SR. This holds true also for the other two flat-type DAs considered above. Recalling that the leading-order QCD sum rules with the minimal Gaussian model for the nonlocal condensates

TABLE I. Results for the third-order integral derivative for $x = 0.5$ and the (usual) derivative of the pion DA, using different SR approaches (first three rows) and pion DA models (six last rows).

	Approach/Model	Integral derivative $[D^{(3)}\varphi_\pi](0.5)$	Derivative $\varphi'_\pi(0)$
1	Integral SR (3.7)	4.7 ± 0.5	5.5 ± 1.5
2	Differential SR (3.14)	...	5.3 ± 0.5
3	SR with smooth NLC (3.18)	...	7.0 ± 0.7
4	BMS bunch [11]	5.7 ± 1.0	1.7 ± 5.3
5	Asymptotic DA	5.25	6
6	CZ DA [45]	15.1	26.2
7	DA from [44]	14	0
8	Flat-type DA, Eq. (3.9)	22.5	72
9	flattop DA [Eq. (B1), $\alpha = 0.1$]	227	$\gg 6$

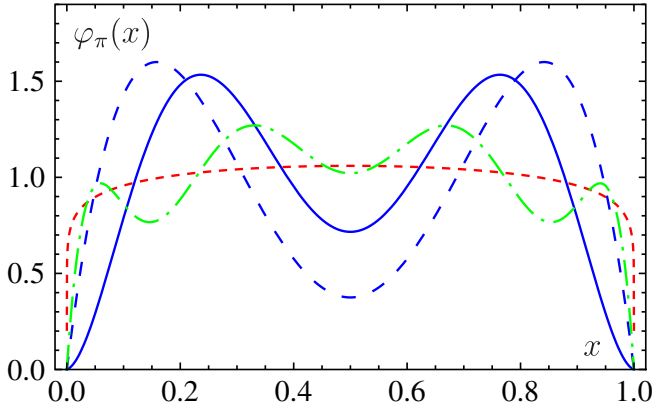


FIG. 4 (color online). Comparison of selected pion DA models. Solid blue line—central line of the BMS bunch [11]; long-dashed blue line—CZ model [3]; dashed-dotted green line—flatlike model given by (3.9), dotted red line—flattop DA from Eq. (B1) with $\alpha = 0.1$. All DAs are normalized at the same scale $\mu_0^2 \simeq 1 \text{ GeV}^2$ with or without evolution [11].

provide much smaller values of the integral derivative of the pion DA, one may conclude that it is very difficult to reconcile flat-type pion DAs with SR (3.7).

On the other hand, the CZ model also yields third-order integral derivatives which are incompatible with the values derived from our SR (3.7)—see Table I. Note that a similar statement also applies to the pion DA proposed in [44], which employs a Brodsky-Huang-Lepage ansatz for the \mathbf{k}_\perp dependence of the pion wave function—see Table I. The upshot of this table is that the SR for the integral derivative of the pion DA is fulfilled by the BMS bunch, whereas flat-type DAs have no overlap with the estimated range of values. The same is true for the CZ model.

We close this subsection by considering the usual derivative $\varphi'_\pi(0)$ of the pion DA, which encapsulates the key characteristics of the pion DA at small x . This quantity can be extracted from Fig. 3, where we have plotted the results for $[D^{(\nu)}\varphi_\pi^{\text{SR}}](x)$ we obtained with the SR (3.7) for different values of ν . To determine $\varphi'_\pi(0)$ one can use expansion (3.4)

$$[D^{(\nu+2)}\varphi_\pi^{\text{SR}}](x) \approx \varphi'_\pi(0) + \varphi''_\pi(0) \frac{x}{2!2^{\nu+1}} \quad (3.10)$$

and subtract the second derivative for which the asymptotic value $\varphi''_\pi(0) = -12(6)$ is used. The involved error ± 6 was estimated by varying (3.10) within the narrow grey strip in Fig. 3. To be more specific, one obtains

$$\begin{aligned} \varphi'_\pi(0) &\approx [D^{(\nu+2)}\varphi_\pi^{\text{SR}}](x) - \varphi''_\pi(0) \frac{x}{2!2^{\nu+1}} \\ &\approx [D^{(\nu+2)}\varphi_\pi^{\text{SR}}](x) + \frac{3x}{2^\nu} = 5.5 \pm 1.5 \end{aligned} \quad (3.11)$$

for any $0.4 < x$ and $0 \leq \nu \leq 4$. The error in (3.11) is a

combination of the uncertainties originating from SR (3.7) and the error in the determination of $\varphi''_\pi(0)$.

The above finding can be compared with what one obtains for the BMS and the CZ model (displayed in Table I)

$$\begin{aligned} \varphi'_\pi(0) &= 6[1 + 6a_2^{\text{BMS}} + 15a_4^{\text{BMS}}] = 1.07^{+5.87}_{-4.68} \\ &= -3.61 \div 6.95 \end{aligned} \quad (3.12)$$

and

$$\varphi'_\pi(0) = 6[1 + 6a_2^{\text{CZ}}] \simeq 26.2, \quad (3.13)$$

respectively, using in both cases the normalization scale $\mu^2 \simeq 1 \text{ GeV}^2$. Quite analogously to the integral derivative, the CZ model yields also for the standard derivative much larger values than those estimated in (3.11). By contrast, the pion DA model proposed in [44]—though it provides a similarly large integral derivative like the CZ DA—has an usual derivative at the origin, which is zero due to the strong exponential suppression of this DA in the vicinity of the origin.

B. Differential sum rules

Another way to study the behavior of the pion DA in the small- x region is provided by the differentiation of the SR (2.4), which yields

$$\begin{aligned} f_\pi^2 \varphi'_\pi(0, M^2) &= \frac{3}{2\pi^2} M^2 (1 - e^{-s_0/M^2}) + 18A_S \Phi' \\ &\quad - f_{A_1}^2 \varphi'_{A_1}(0) e^{-m_{A_1}^2/M^2}. \end{aligned} \quad (3.14)$$

We shall evaluate this SR for the threshold value $s_0 = 2.61 \text{ GeV}^2$, recalling that we employ a LO expression for the spectral density.

Using the simplest delta-ansatz model for the condensates [cf. (A2)], only one nonperturbative term survives, namely, the four-quark condensate

$$\Phi' = \frac{1}{18A_S} \frac{d}{dx} \Delta \Phi_S(x, M^2)|_{x=0}, \quad (3.15)$$

where $\Delta \Phi_S(x, M^2)$ is represented by Eq. (A9)—see Appendix A. The vector-quark condensate (V) and the gluon condensate (G)—recall Eq. (2.4)—give zero contributions in the region $x < \Delta$, where $\Delta = \lambda_q^2/(2M^2) > 0$ with $\lambda_q^2 = 0.4 \text{ GeV}^2$. On the other hand, the antiquark-gluon-quark condensate (T) amounts to a vanishing contribution in the region $x < \min\{\Delta, 1 - 2\Delta\}$. For $M^2 > 0.4 \text{ GeV}^2$ (or equivalently $\Delta < 1/2$) this contribution can also be neglected.

Even if we assume a behavior of the various condensates differing from the delta-ansatz model, using, for instance, a smooth model like (3.18) for the scalar-quark condensate (which implies a decay at large distances not slower than the exponential decay—see further below), the (V), (G), and (T) terms give only a small nonzero contribution in the

small- x region. Therefore, these terms can be neglected in first approximation in both mentioned models. In this section, we shall study the differential SR (3.14) using the mentioned models for the scalar condensate.

The calculation of the nonlocal version of the four-quark-condensate contribution involves only the scalar-quark condensate. The four-quark contribution $\Delta\Phi_S(x, M^2)$ was obtained in [4] and can be used with various types of vacuum-quark distributions $f_S(\alpha)$. Accordingly, the probability of finding vacuum quarks with very large virtualities is very small, as one can see from Fig. 5. Therefore, the distribution function $f_S(\alpha) \simeq 0$ diminishes in the region $\alpha > M^2$, the latter region corresponding to the values $M^2 > M_{\min}^2 \geq 0.6 \text{ GeV}^2$ of the Borel parameter. On the basis of this result, one [46] can use the method of Ref. [4] to obtain the model-independent expression (A9). Then, one finds for the four-quark-condensate contribution to the SR (3.14) the following expression:

$$\Phi' = \int_0^\infty d\alpha \frac{f_S(\alpha)}{\alpha^2} = \langle \bar{q}q \rangle^{-1} \int_0^\infty z^2 M_S(z^2) dz^2. \quad (3.16)$$

We see from this equation that the nonperturbative contribution to the SR is mainly due to the scalar-quark condensate at large and moderate distances $z^2 \sim 4/\langle k_q^2 \rangle$.

For the concrete evaluation of the differential SR, we again employ the same criteria as already used in the integral SR for both the continuum and the nonperturbative terms. Applying these criteria, the low boundary for the Borel parameter turns out to be very small, viz., $M_{\min}^2 = (0.3\text{--}0.4) \text{ GeV}^2$. Though this low value is consistent with the applied criteria, it is not in good agreement with standard QCD sum-rule approaches in which a higher minimum Borel-parameter value is used. Therefore, we use $M_{\min}^2 = 0.6 \text{ GeV}^2$, a value also employed in the QCD SR for the pion DA in Ref. [11] in connection with the moments of the pion DA. Keep also in mind that

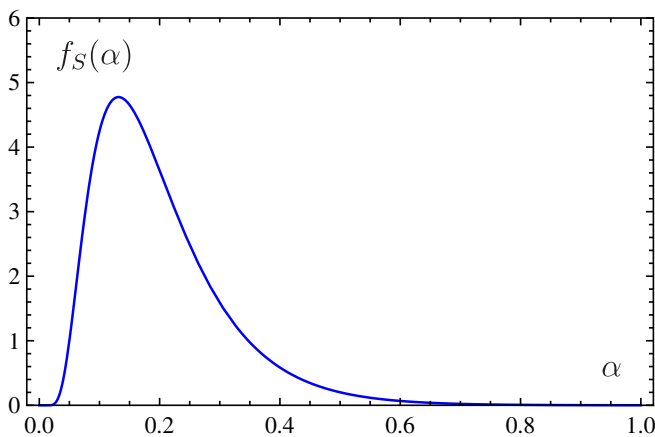


FIG. 5 (color online). The distribution of vacuum-quark virtualities in the smooth model (3.18) for a particular choice of the intrinsic parameters as explained in the text.

using a lower value of the Borel parameter would cause the decrease of the first derivative of the pion DA at the endpoints. To continue, we define the derivative $\varphi'_\pi(0) = \langle \varphi'_\pi(0, M^2) \rangle$ in terms of the mean value in the fiducial interval $M^2 \in [M_{\min}^2, M_{\max}^2]$ following the definition on the RHS of (3.8). This helps minimizing the sensitivity of $\varphi'_\pi(0)$ on the choice of the Borel parameter by means of the variation of the A_1 contribution $\varphi'_{A_1}(0)$.

The delta-ansatz $f_S(\alpha) = \delta(\alpha - \lambda_q^2/2)$ leads to a simple expression for the nonperturbative contribution to SR (3.14), notably,

$$\Phi' = \Phi'_{\text{delta}} = \frac{4}{\lambda_q^4}. \quad (3.17)$$

Note that we defined the A_1 -meson contribution $\varphi'_{A_1}(0) = 6.7$ by means of the minimum of the root-mean-square deviation. On the other hand, the dependence of $\varphi'_\pi(0, M^2)$ on the Borel parameter M^2 for the delta-ansatz model is controlled by Eq. (3.14) and is shown in the left panel of Fig. 6. Thus, the average value of the pion DA derivative in the fiducial Borel interval is $\varphi'_\pi(0) = 5.3(5)$ —see Table I.

We go forward and discuss the consequences of the smooth model for the quark-virtuality distribution in the differential SR.

Though the delta-ansatz model is useful, because of its simplicity, there is an indication from the heavy-quark effective theory [47] that in reality the quark-virtuality distribution f_S should be parameterized in a different way so as to ensure that the scalar condensate decreases exponentially at large distances. Moreover, in order that the vacuum matrix element $\langle \bar{q}(D^2)^N q \rangle$ exists, the quark-virtuality distribution $f_S(\alpha)$ should decrease faster than any power $1/\alpha^{N+1}$ as $\alpha \rightarrow \infty$ [4]. For this reason, the authors of [36,48] suggested a two-tier model for f_S which has a smooth dependence on the quark virtuality α , namely,

$$f_S(\alpha; \Lambda, n, \sigma) = \frac{(\sigma/\Lambda)^n}{2K_n(2\Lambda\sigma)} \alpha^{n-1} e^{-\Lambda^2/\alpha - \alpha\sigma^2}, \quad (3.18)$$

where $K_n(z)$ is the modified Bessel function. This model, the so-called “smooth model”, depends on two parameters Λ and σ that parameterize, respectively, the long- and short-distance behavior of the nonlocal condensates [36]. For large distances $|z| = \sqrt{-z^2}$ this model leads to the asymptotic form

$$\langle \bar{q}(0)q(z) \rangle \xrightarrow{|z| \rightarrow \infty} \langle \bar{q}q \rangle |z|^{-(2n+1)/2} e^{-\Lambda|z|} \frac{2^{(2n-1)/2} \sqrt{\pi} \sigma^n}{\sqrt{\Lambda} K_n(2\Lambda\sigma)}. \quad (3.19)$$

It is instructive to consider a purely exponential decay of the quark-virtuality distribution and study its influence on the quark condensate. This can be realized in the model of [36,48] by choosing $n = 1$, whereas the second parameter $\Lambda = 0.45 \text{ GeV}$ can be taken from the QCD SRs for the

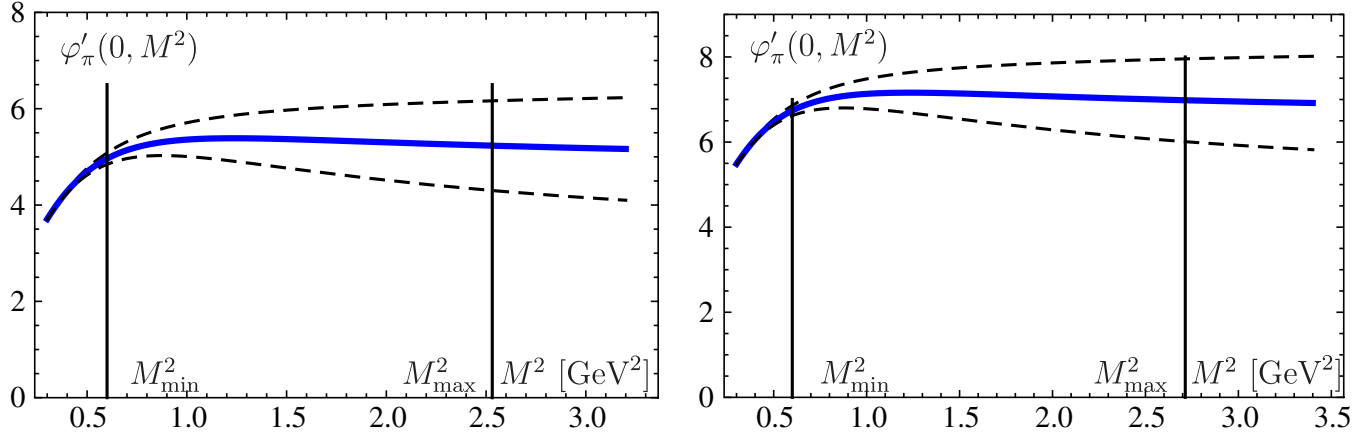


FIG. 6 (color online). Left panel: The solid curve shows the M^2 dependence of $\varphi'_\pi(0, M^2)$ in the SR with the A_1 -meson contribution $\varphi'_{A_1}(0) = 6.7$. The broken lines represent $\varphi'_\pi(0, M^2)$ for $\varphi'_{A_1}(0) = 6.1$ (lower line) and $\varphi'_{A_1}(0) = 7.5$ (upper line). Right panel: The solid curve illustrates the M^2 dependence of $\varphi'_\pi(0, M^2)$ in the SR which employs the smooth quark-condensate model with a A_1 -meson contribution given by $\varphi'_{A_1}(0) = 6.8$. The broken lines denote $\varphi'_\pi(0, M^2)$ at $\varphi'_{A_1}(0) = 6.1$ (lower line) and $\varphi'_{A_1}(0) = 7.5$ (upper line).

heavy-light meson in heavy-quark effective theory—see [47,49]. The two parameters n and Λ are responsible for the large- z behavior of the scalar-quark condensate, cf. Equation (3.19). The third parameter $\sigma^2 = 10 \text{ GeV}^{-2}$ is defined in terms of the parameters n , Λ , and λ_q^2 via the following equation:

$$\int_0^\infty \alpha f_S(\alpha; \Lambda, n, \sigma) d\alpha = \frac{\Lambda}{\sigma} \frac{K_{n+1}(2\Lambda\sigma)}{K_n(2\Lambda\sigma)} = \frac{\lambda_q^2}{2}, \quad (3.20)$$

which we evaluate for the value of the nonlocality parameter $\lambda_q^2 = 0.4 \text{ GeV}^2$. The main effect of using a smooth model for the quark-virtuality distribution relative to the Gaussian form, $f_S(\alpha) = \delta(\alpha - \lambda_q^2/2)$, is the induced increase of the nonperturbative contribution to the SR, so that

$$\begin{aligned} \Phi'_{\text{smooth}} &= \int_0^\infty d\alpha \frac{f_S(\alpha; \Lambda, n, \sigma)}{\alpha^2} \\ &= \frac{\sigma^2}{\Lambda^2} \frac{K_{n-2}(2\Lambda\sigma)}{K_n(2\Lambda\sigma)} > \Phi'_{\text{delta}}. \end{aligned} \quad (3.21)$$

We analyzed the SR (3.14) for this model using a particular choice of its parameters, notably, $f_S(\alpha; \Lambda = 0.45 \text{ GeV}, n = 1, \sigma^2 = 10 \text{ GeV}^{-2})$ and determined the dependence of $\varphi'_\pi(0, M^2)$ on the Borel parameter M^2 . The result is shown graphically in the right panel of Fig. 6. The average value of the derivative $\varphi'_\pi(0, M^2)$ in the fiducial Borel interval is $\varphi'_\pi(0) = 7.0(7)$. Thus, the nonperturbative contribution Φ'_{smooth} , obtained from the smooth model, is approximately 2 times larger than the analogous contribution Φ'_{delta} in the delta ansatz: $\Phi'_{\text{smooth}} \approx 2.3\Phi'_{\text{delta}}$. In addition to this result, marked by a black dot, we show in Fig. 7 the dependence of $\varphi'_\pi(0)$ on the choice parameters n and Λ of the smooth model. From this picture and the relation (3.16), we may come to the conclusion

that, choosing a model for the condensate that has a slower decay at large distances (small n or Λ), may cause an increase of the nonperturbative contribution to the SR (2.4) and entail also an increase of the value $\varphi'_\pi(0)$. On the other hand, choosing a model for the condensate with a faster decay at large distances (large n or Λ), may lead to a decrease of the nonperturbative contribution to the SR (2.4) and therefore to a decrease of the value $\varphi'_\pi(0)$.

In the last column of Table I, we collect the values of the (usual) pion DA derivative at $x \approx 0$, using different SR approaches (first three rows) and pion DA models (last six rows).

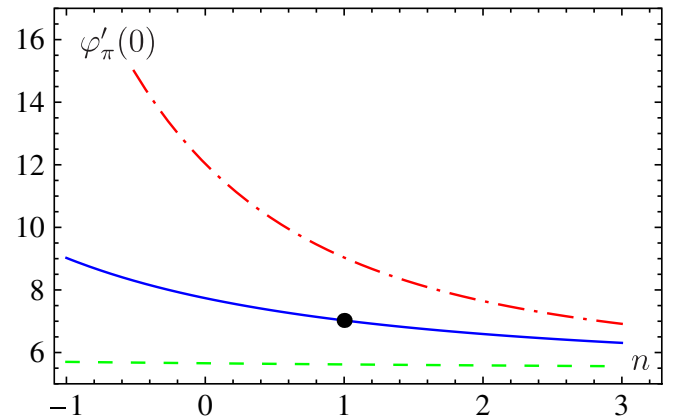


FIG. 7 (color online). Dependence of the derivative $\varphi'_\pi(0)$ on the parameter n of the smooth scalar condensate model (3.18) for $\Lambda = 0.3 \text{ GeV}$ (dashed-dotted red line), $\Lambda = 0.45 \text{ GeV}$ (solid blue line), and $\Lambda = 1 \text{ GeV}$ (dashed green line). Here, we show only the central value of the derivative $\varphi'_\pi(0)$ we obtained from the SR analysis (3.14). The black dot symbol marks the position, which corresponds to the model parameters $n = 1$, $\Lambda = 0.45 \text{ GeV}$, and $\varphi'_\pi(0) = 7.0(7)$, that corresponds to the third row in Table I.

IV. CONCLUSIONS

The current investigation was partly motivated by the recent results obtained by the *BABAR* Collaboration [30] on the pion-photon transition form factor which indicate an unexpected growth of this quantity with Q^2 above $\sim 10 \text{ GeV}^2$ up to the highest momentum value of 40 GeV^2 measured. As it was pointed out in [27] (see also [31,32]), such behavior is impossible for endpoint-suppressed pion DAs and brings into play a flat-type profile for the pion DA, as proposed in Refs. [33,34]. Therefore, it appears to be of crucial importance to have a theoretical tool in our hands able to reveal the particular characteristics of pion DAs precisely in the kinematic endpoint region.

The method mostly used in the past to extract such information employs the inverse moment $\langle x^{-1} \rangle_\pi$ —extensively discussed in [11,22,23,25,37]. In the present work, we proposed another, more direct, way to access the endpoint characteristics of the pion DA, which makes use of its derivatives. The first step was to define the notion of integral derivatives $[D^{(\nu)} \varphi_\pi](x)$ by means of an appropriate operator $D^{(\nu)}$ [cf. (3.5)]. Next, we formulated an integral sum rule, Eq. (3.7), for these quantities by taking into account an effective A_1 -meson contribution. Using this sum rule, we determined the range of values of the integral derivatives and displayed it in the first entry of the second column of Table I showing it also graphically in Fig. 3. We also calculated the integral derivative for different characteristic pion DA models and listed its value in the same Table. The list of pion DAs includes the BMS, the CZ, the asymptotic, and two options for flat-type DA models, one given by Eq. (B1) with $\alpha = 0.1$ —“flattop” model—the other being a flatlike model parameterized in terms of Eq. (3.9). We also analyzed the usual derivative of the pion DA in the vicinity of the origin. This procedure is helpful in revealing the fine details of the pion distribution amplitude in the endpoint region—see Table I for the results.

Our findings can be summarized as follows. First, at the end of Sec. III A, we applied the result for the integral derivatives $[D^{(\nu+2)} \varphi_\pi](x)$ in order to reproduce the usual derivative of $\varphi_\pi(x)$ at the origin $x = 0$. The result $\varphi'_\pi(0) = 5.5 \pm 1.5$ is shown in the third column of Table I in comparison with some characteristic models for the pion DA. Second, in Sec. III B, we studied the derivative of the pion DA $\varphi'_\pi(0)$ in terms of the differentiation of the SR (2.4) that leads to the differential SR (3.14). It turns out that the only nonperturbative content in this SR is mainly defined by the scalar-quark condensate. The nonperturbative contribution is proportional to the second inverse moment (3.16) of the distribution $f_S(\alpha)$ of the vacuum-quark virtualities and is defined by the behavior of the quark condensate at large and moderate distances between the vacuum quarks. The results for the derivative of the pion DA $\varphi'_\pi(0)$ are shown in Table I for the delta-ansatz model

(A1) and also for the two-tier smooth model (3.18) with the model parameters $n = 1$, $\Lambda = 0.45 \text{ GeV}$. The dependence of the derivative of the pion DA $\varphi'_\pi(0)$ on the choice of the model parameters n and Λ is illustrated in Fig. 7.

As we see from Table I, the results from the differential (3.14) and the integral SRs (3.7) agree with each other. The integral derivative of the pion DA, based on a new SR derived in this work, remains smaller than the asymptotic value and overlaps with the range of values determined with the BMS bunch of pion DAs, while there is no agreement with the CZ DA and the flat-type models considered. The same conclusions can be drawn also for the usual derivative of the pion DA, which follows from the differential SR (3.14). It is worth mentioning that, employing the integral and the differential sum rules, we found that the leading-order QCD sum rules (2.4), which employ the minimal Gaussian model for the nonlocal condensates, cannot be satisfied by flat-type pion distribution amplitudes.

Given that an increasing behavior of the scaled pion-photon transition form factor can only be achieved with flat-type pion DAs, the independent experimental confirmation of this effect, e.g., by the BELLE Collaboration, becomes extremely crucial for our theoretical understanding of basic QCD exclusive processes. From a broader perspective, the *BABAR* effect is a further indication that exclusive processes may be inaccessible to perturbative QCD, as pointed out many years ago by Isgur and Llewellyn-Smith [52].

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APPENDIX A: EXPRESSIONS FOR THE NONLOCAL CONTRIBUTIONS TO THE SUM RULES

In Secs II and III, we used the following expressions for the vacuum distribution functions:

$$f_S(\nu) = \delta(\nu - \lambda_q^2/2); \quad f_V(\nu) = \delta'(\nu - \lambda_q^2/2); \quad (\text{A1})$$

$$f_{T_i}(\alpha_1, \alpha_2, \alpha_3) = \delta(\alpha_1 - \lambda_q^2/2)\delta(\alpha_2 - \lambda_q^2/2) \times \delta(\alpha_3 - \lambda_q^2/2). \quad (\text{A2})$$

The meaning of these expressions and their connection to the initial NLCs has been discussed in detail in Refs. [4,7]. The contributions to the QCD SR (2.4), with nonlocal condensates $\Delta\Phi_T(x, M^2)$ associated with these expressions, are shown below (see Fig. 8). Here, and in what follows, we use $\Delta \equiv \lambda_q^2/(2M^2)$, $\bar{\Delta} \equiv 1 - \Delta$.

Then we obtain

$$\Delta\Phi_S(x, M^2) = \frac{A_S}{M^4} \frac{18}{\bar{\Delta}\Delta^2} \{ \theta(\bar{x} > \Delta > x) \bar{x} [x + (\Delta - x) \ln(\bar{x})] + (\bar{x} \rightarrow x) + \theta(1 > \Delta) \theta(\Delta > x > \bar{\Delta}) [\bar{\Delta} + (\Delta - 2\bar{x}) \ln(\Delta)] \}, \quad (\text{A3})$$

$$\Delta\Phi_V(x, M^2) = \frac{A_S}{M^4} (x\delta'(\bar{x} - \Delta) + (\bar{x} \rightarrow x)), \quad (\text{A4})$$

$$\Delta\Phi_T(x, M^2) = \Delta\Phi_{T_1}(x, M^2) + \Delta\Phi_{T_2}(x, M^2) + \Delta\Phi_{T_3}(x, M^2),$$

$$\Delta\Phi_{T_1}(x, M^2) = -\frac{3A_S}{M^4} \left[[\delta(x - 2\Delta) - \delta(x - \Delta)] \left(\frac{1}{\Delta} - 2 \right) \theta(1 > 2\Delta) + \theta(2\Delta > x) \theta(x > \Delta) \theta(x > 3\Delta - 1) \frac{\bar{x}}{\Delta} \times \left[\frac{3x}{\Delta} - 6 - \frac{1 + \bar{x}}{\bar{\Delta}} \right] \right] + (\bar{x} \rightarrow x), \quad (\text{A5})$$

$$\Delta\Phi_{T_2}(x, M^2) = \frac{4A_S}{M^4} \bar{x} \left[\frac{\delta(x - 2\Delta)}{\Delta} \theta(1 > 2\Delta) - \theta(2\Delta > x) \theta(x > \Delta) \theta(x > 3\Delta - 1) \frac{1 + 2x - 4\Delta}{\bar{\Delta}\Delta^2} \right] + (\bar{x} \rightarrow x), \quad (\text{A6})$$

$$\Delta\Phi_{T_3}(x, M^2) = \frac{3A_S \bar{x}}{M^4 \bar{\Delta} \Delta} \left\{ \theta(2\Delta > x) \theta(x > \Delta) \theta(x > 3\Delta - 1) \left[2 - \frac{\bar{x}}{\Delta} - \frac{\Delta}{\bar{\Delta}} \right] \right\} + (\bar{x} \rightarrow x), \quad (\text{A7})$$

$$\Delta\Phi_G(x, M^2) = \frac{\langle \alpha_s GG \rangle}{24\pi M^2} (\delta(x - \Delta) + (\bar{x} \rightarrow x)). \quad (\text{A8})$$

In the above equations, we used the abbreviation $A_S = \frac{8\pi\alpha_s}{81} \langle \bar{q}q \rangle^2$, while for the quark and the gluon condensates the standard estimates [50] $\alpha_s \langle \bar{q}q \rangle^2 = 1.83 \times 10^{-4} \text{ GeV}^6$, $\frac{\langle \alpha_s GG \rangle}{12\pi} = 0.0012 \text{ GeV}^4$, and $\lambda_q^2 = \frac{\langle \bar{q}(ig\sigma_{\mu\nu}G^{\mu\nu})q \rangle}{2\langle \bar{q}q \rangle} = 0.4 \text{ GeV}^2$, normalized at $\mu^2 \approx 1 \text{ GeV}^2$, have been adopted. In Sec. III B, we applied not only the delta ansatz,

discussed above, but also the smooth model, proposed in Refs. [48,51], which is based on an exponential decay of the condensate (3.19). For this reason, we use the model-independent expression for the four-quark contribution [4,46] to obtain Eq. (3.16)

$$\Delta\Phi_S(x, M^2) = \frac{18A_S}{M^4} \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 f_S(\alpha_1) f_S(\alpha_2) \frac{x\theta(\Delta_1 - \bar{x})}{\Delta_1^2 \Delta_2 \bar{\Delta}_1^2} \left[\bar{x} \Delta_2 \bar{\Delta}_1 + \ln \left(\frac{x\Delta_1 \bar{\Delta}_2}{x\Delta_1 - (\Delta_1 - \bar{x})\Delta_2} \right) \Delta_1 (\Delta_1 - \bar{x}) \bar{\Delta}_2 \right] + (\bar{x} \rightarrow x), \quad (\text{A9})$$

where $\Delta_i \equiv \alpha_i/M^2$, $\bar{\Delta}_i \equiv 1 - \Delta_i$, and $\bar{x} \equiv 1 - x$.

APPENDIX B: MODELING THE PION DA

In our work, we compared the results following from the differential SR (3.14) for the usual derivative of the pion DA $\varphi'_\pi(0)$ with those obtained with the help of the integral derivative $[D^{(\nu)}\varphi_\pi](x)$, i.e., (3.7), evaluating them for the BMS DA bunch, the CZ model, and a couple of flat-type pion DAs.

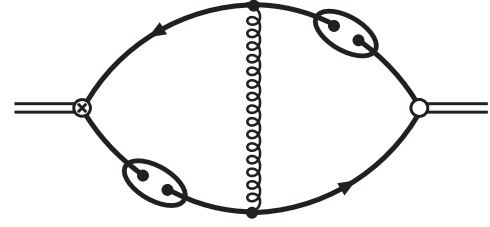


FIG. 8. Typical diagram for the four-quark nonlocal condensate $\Delta\Phi_S(x, M^2)$ encoding nonperturbative input in (2.4).

The BMS bunch of pion DAs was obtained [11] as the sum of the first three terms in the Gegenbauer-polynomial expansion (1.3). Their Gegenbauer coefficients were obtained from an analysis of the SR (2.4) for the $\langle \xi^{2N} \rangle$ moments ($N = 0, 1, 2, 3, 4, 5$) with the nonlocal-condensate contributions being presented in Appendix A. It was found that only the two first Gegenbauer coefficients a_2 and a_4 give sizeable contributions to the $\langle \xi^{2N} \rangle$ moments, whereas the higher Gegenbauer terms give merely tiny contributions. For this reason, the BMS bunch DAs are two-parameter models. The central values of a_2 and a_4 for the whole BMS bunch in the a_2, a_4 space are $a_2^{\text{BMS}} = 0.187$ and $a_4^{\text{BMS}} = -0.129$ using as a normalization scale $\mu^2 \simeq 1 \text{ GeV}^2$ [11].

On the other hand, the CZ pion DA contains only the first nontrivial Gegenbauer polynomial corresponding to the coefficient a_2 [3]. It was derived from QCD SRs for the $\langle \xi^{2N} \rangle$ moments ($N = 0, 1, 2$), using local condensates. For the sake of consistency, we use here a value of $a_2^{\text{CZ}} = 0.56$ obtained after evolution to the normalization scale $\mu^2 \simeq 1 \text{ GeV}^2$, see for more details [22].

One of the two flat-type models considered in this paper is defined by

$$\varphi_{(\text{B1})}^{\text{flat}}(x) = \frac{\Gamma(2(\alpha + 1))}{\Gamma^2(\alpha + 1)} x^\alpha (1 - x)^\alpha. \quad (\text{B1})$$

For this model, we find the following expressions:

$$[D^{(2)} \varphi_{(\text{B1})}^{\text{flat}}](x) = \frac{\Gamma(2(\alpha + 1))}{\Gamma^2(\alpha + 1)} B_x(\alpha, 1 + \alpha),$$

$$[D^{(\nu+2)} \varphi_{(\text{B1})}^{\text{flat}}](x) \approx \alpha^{-\nu} [D^{(2)} \varphi_{(\text{B1})}^{\text{flat}}](x).$$

These expressions can be generalized to any real differentiation index $\nu \in +\mathbb{R}$ for $0 < x < 1$ to get

$$[D^{(\nu+2)} \varphi_{(\text{B1})}^{\text{flat}}](x) \gg [D^{(\nu+2)} \varphi^{\text{as}}](x) \geq [D^{(\nu+2)} \varphi_\pi^{\text{SR}}](x)$$

$$\text{for } \alpha \leq 0.1,$$

$$227 \gg 5.25 \geq 4.7(5)$$

for the particular values $\alpha = 0.1$, $\nu = 1$, and $x = 0.5$;

$$[D^{(\nu+2)} \varphi_{(\text{B1})}^{\text{flat}}](x) > [D^{(\nu+2)} \varphi^{\text{as}}](x) \geq [D^{(\nu+2)} \varphi_\pi^{\text{SR}}](x)$$

$$\text{for } \alpha < 1,$$

$$14 > 5.25 \geq 4.7(5) \quad (\text{B2})$$

for the particular values $\alpha = 0.5$, $\nu = 1$, and $x = 0.5$.

The other flat-type model, given by Eq. (3.9), was already discussed in the text.

APPENDIX C: THE SPECTRAL DENSITY

The spectral density $\rho_{\text{pert}}(x)$ with NLO accuracy was obtained in [4,7] and was found to be

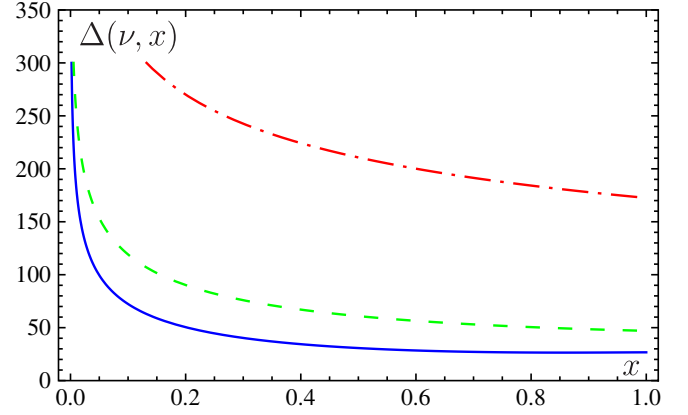


FIG. 9 (color online). x dependence of the function $\Delta(\nu, x)$ for $\nu = 0$ (solid blue line), $\nu = 1$ (dashed green line), and $\nu = 4$ (dashed-dotted red line).

$$\rho_{\text{pert}}(x) = 3x\bar{x} \left[1 + \frac{\alpha_s}{4\pi} C_F \left(5 - \frac{\pi^2}{3} + \ln^2(\bar{x}/x) \right) \right] \frac{1}{2\pi^2}. \quad (\text{C1})$$

For our numerical calculations, which take into account radiative corrections, we use the following function

$$\Delta(\nu, x) = 100 \frac{[D^{(\nu+2)} \rho_{\text{pert}}^{\text{NLO}}](x)}{[D^{(\nu+2)} \rho_{\text{pert}}^{\text{LO}}](x)} - 100. \quad (\text{C2})$$

The x dependence of the function $\Delta(\nu, x)$ is illustrated in Fig. 9 for different values of ν : solid blue line— $\nu = 0$, dashed green line— $\nu = 1$, and dashed-dotted red line— $\nu = 4$.

APPENDIX D: PROPERTIES OF THE INTEGRAL OPERATOR $D^{(\nu+2)}$

In order to ensure a weak dependence of the results on the particular model for the condensates adopted, and in order to include all condensate contributions, one has to construct the SR by integrating the pion DA SR (2.4) over a large enough interval of x . For this reason, we introduced in Sec. III A the integral derivatives $[D^{(n)} \varphi](x)$, with the two lowest-order ones being given in terms of Eq. (3.2). The next higher derivative reads

$$\begin{aligned} [D^{(3)} \varphi](x) &= \frac{1}{x} \int_0^x \frac{dy}{y} \int_0^y \frac{\varphi(t)}{t} dt = \frac{1}{x} \int_0^x \frac{\varphi(t)}{t} dt \int_t^x \frac{dy}{y} \\ &= \frac{1}{x} \int_0^x \frac{\varphi(t)}{t} \ln\left(\frac{x}{t}\right) dt. \end{aligned} \quad (\text{D1})$$

Assuming $\varphi(0) = 0$, we find

$$[D^{(n+1)} \varphi](x) = \frac{1}{x} \int_0^x dy [D^{(n)} \varphi](x). \quad (\text{D2})$$

To obtain an expression for $[D^{(n+2)} \varphi](x)$ for any $n \geq 0$ and $n \in \mathbb{N}$, one has to rearrange the integration order to get

$$\begin{aligned}
[D^{(n+2)}\varphi](x) &= \frac{1}{x} \int_0^x \frac{dy_n}{y_n} \int_0^{y_n} \frac{dy_{n-1}}{y_{n-1}} \dots \int_0^{y_2} \frac{dy_1}{y_1} \\
&\quad \times \int_0^{y_1} \frac{\varphi(t)}{t} dt \\
&= \frac{1}{x} \int_0^x \frac{\varphi(t)}{t} \frac{1}{n!} \left(\int_t^x \frac{dy}{y} \right)^n dt. \quad (D3)
\end{aligned}$$

This can be readily generalized to the expression (3.5) using (3.6) that allows one to establish this transformation for any real ν , $\nu \in \mathbb{R}$. The dependence on y of the function $f(y, \nu, x)$ is shown in Fig. 10 for $x = 0.6$ and $\nu = 0, 1, 4$. As one sees from Eq. (3.5), the function $f(y, \nu, x)$ acts as a smooth projector to the area around the origin of y .

If a Taylor expansion exists for $\varphi(x)$ at $x = 0$, then by applying (3.5) one finds

$$\begin{aligned}
[D^{k+2}\varphi](x) &= \varphi'(0) + \varphi''(0) \frac{x}{2!2^{k+1}} + \sum_{n=2}^{\infty} \varphi^{(n+1)}(0) \\
&\quad \times \frac{x^n}{(n+1)!(n+1)^{k+1}}. \quad (D4)
\end{aligned}$$

Using (3.5) in combination with (D4), one can obtain the following properties of the operator $D^{(\nu)}$:

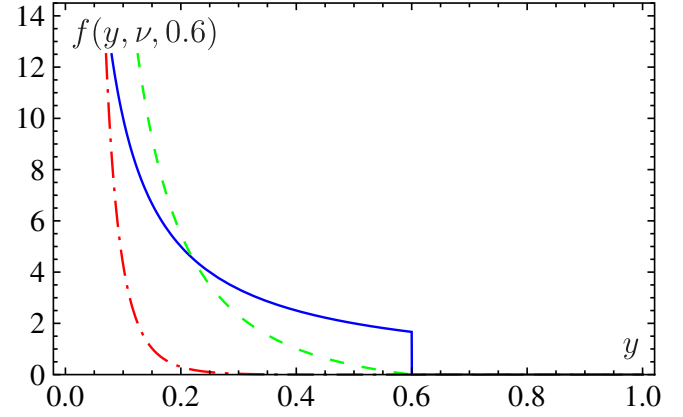


FIG. 10 (color online). y dependence of the function $f(y, \nu, 0.6)$ for $\nu = 0$ (solid blue line), $\nu = 1$ (dashed green line), and $\nu = 4$ (dashed-dotted red line).

$\lim_{x \rightarrow 0} [D^{(\nu)}\varphi](x) = \lim_{\nu \rightarrow \infty} [D^{(\nu)}\varphi](x) = \varphi'(0)$ and $D^{(\nu)}(ax + bx^2) = a + 2^{1-\nu}bx$. Moreover, from (D4) it follows that the introduced operator $[D^{(\nu)}\varphi](x)$ reproduces at small x and large ν the derivative of $\varphi(x)$ at $x = 0$.

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