$\mu - \tau$ symmetry and charged lepton mass hierarchy in a supersymmetric D_4 model

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In this paper we discuss a supersymmetric $D_4 \times Z_5$ model which leads to vanishing reactor mixing angle $\theta_{13} = 0$ and maximal atmospheric mixing $\theta_{23} = \pi/4$ in the lepton sector at leading order, due to the preservation of nontrivial distinct D_4 subgroups in the charged lepton and neutrino sectors, respectively. The solar mixing angle θ_{12} remains undetermined and is expected to be of order one. Since right-handed charged leptons transform as singlets under D_4, the charged lepton mass hierarchy can be naturally accounted for. The model predicts inverted mass hierarchy for neutrinos. Additionally, we show that, unlike in most of the other models of this type, all vacuum expectation values of gauge singlets (flavons) can be determined through mass parameters of the superpotential. Next-to-leading order corrections to lepton masses and mixings are calculated and shown to be under control; in particular, the corrections to $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ are of the order of the generic expansion parameter $\epsilon \approx 0.04$ and arise dominantly from the charged lepton sector.

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I. INTRODUCTION

The observed mixing pattern in the lepton sector [1],

$$\begin{aligned} \sin^2\theta_{12} &= 0.318^{+0.042}_{-0.028}, \\ \sin^2\theta_{23} &= 0.50^{+0.13}_{-0.11}, \\ \text{and} \quad \sin^2\theta_{13} &= 0.013^{+0.026}_{-0.013}\ (2\sigma), \end{aligned} \tag{1}$$

is well compatible with special mixing patterns in which θ_{13} vanishes and θ_{23} is maximal,

$$\sin^2\theta_{23} = \tfrac{1}{2} \quad \text{and} \quad \sin\theta_{13} = 0. \tag{2}$$

Equation (2) is derived from a $\mu - \tau$ symmetric neutrino mass matrix [2], in the charged lepton mass basis. At the same time, the solar mixing angle θ_{12} is not predicted. It is natural to assume a symmetry to be the origin of such a mixing pattern.

It has been shown [3] that in the case of a nontrivial breaking of a dihedral flavor symmetry in the lepton (quark) sector, one element $U_{\alpha i}$ of the lepton (quark) mixing matrix is given in terms of group theoretical quantities only,[1]

$$|U_{\alpha i}| = \left|\cos\left(\frac{\pi(k_1 - k_2)\mathrm{j}}{n}\right)\right|, \tag{3}$$

where n refers to the group index of the dihedral group D_n, j to the index of the two-dimensional irreducible representation $\underline{\mathbf{2}}_{\mathbf{j}}$ under which two of the three left-handed fields transform, and $k_{1,2}$ are associated with the generating elements BA^{k_1} and BA^{k_2} of the Z_2 subgroups preserved in the charged lepton (down quark) and neutrino (up quark) sectors, respectively. (A and B are the generators of the original dihedral group D_n.) Note that k_1 and k_2 have to be distinct to get a nontrivial value for $|U_{\alpha i}|$. Considering the lepton sector, one sees that choosing $n = 4$ makes it possible to achieve e.g. $|U_{\mu 3}| = \frac{1}{\sqrt{2}}$. Using a specific set of flavons to break D_4 to the subgroup Z_2 generated by BA^{k_1} in the charged lepton sector, $|U_{\tau 3}| = \frac{1}{\sqrt{2}}$ and $U_{e3} = 0$ can also be enforced, thus leading to $\mu - \tau$ symmetric lepton mixings.

In the model presented here D_4 is accompanied by the cyclic symmetry Z_5. Both symmetries are spontaneously broken by flavon vacuum expectation values (VEVs). We use as a framework the minimal supersymmetric standard model (MSSM). It is not the first time that $\mu - \tau$ symmetry is deduced from the group D_4 [4,6,10,11]. However, in these models, producing the mass hierarchy among charged leptons is usually nontrivial. In contrast, the observed charged lepton mass hierarchy

$$m_e\colon m_\mu\colon m_\tau \approx \epsilon^2\colon \epsilon\colon 1 \quad \text{with} \quad \epsilon \approx \lambda^2 \approx 0.04, \tag{4}$$

with λ being the Wolfenstein parameter [12], is a natural outcome of our model. In order to achieve this, it is essential that the right-handed charged lepton fields transform as (different) singlets under the flavor symmetry D_4, instead of being in the representations $\underline{\mathbf{1}} + \underline{\mathbf{2}}$. Notice also that the introduction of a $U(1)$ symmetry is not necessary for ending up with Eq. (4); see [13–15] as well. $\mu - \tau$ symmetric mixing is not the only pattern which can be achieved in a D_4 model. In [16] it is shown that soft breaking of D_4 still allows $\theta_{13} = 0$, whereas θ_{23} is no longer maximal, with its deviation from maximal mixing being proportional to the soft breaking of D_4. The D_4 models discussed in [17] lead to a scaling behavior of the

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[1]For specific realizations see [4–9].

elements of the (light) neutrino mass matrix. Similar to the results of [16], θ_{13} vanishes and θ_{23} is, in general, expected to be nonmaximal. A possible origin of D_4 in the context of certain string theory models is examined in [18], and a D_4 model in the framework of deconstructed extra dimensions can be found in [19].

At LO, our model leads to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ with no constraints on the solar mixing angle θ_{12}, apart from predicting its value generically to be of order one. In the limit of preserved subgroups, m_μ and m_τ are generated of the correct order of magnitude. Since m_τ arises from a nonrenormalizable operator, small and moderate values of $\tan\beta$ are preferred. Neutrinos exhibit inverted mass hierarchy, and the lightest neutrino mass m_3 fulfills $m_3 \gtrsim 0.015$ eV. We find at LO a strong correlation among m_3 and the measure of neutrinoless double beta decay $|m_{ee}|$, a rather weak one among m_3 and $\tan\theta_{12}$, as well as a restricted range for Majorana phases. These results are very similar to those found in [4]. We study the model to next-to-leading order (NLO) and show that corrections to the predictions $\theta_{13} = 0$ and θ_{23} being maximal are of order ϵ and dominated by the charged lepton sector. Also θ_{12} undergoes corrections; but these are not particularly interesting, because its value is not a prediction of our model anyway. The mass of the electron is generated by NLO corrections. The masses of the other charged leptons and neutrinos are slightly corrected as well.

Furthermore, an appropriate construction of the flavon superpotential allows us to determine all flavon VEVs through couplings with positive mass dimension. By choosing the latter to be of order $\epsilon\Lambda$, where Λ is the generic cutoff scale of our theory, all flavon VEVs are of order $\epsilon\Lambda$. Determining all flavon VEVs by mass parameters is usually not the case in such types of models, which thus encounter free parameters among the flavon VEVs [equivalent to flat directions in the flavon (super)potential]; see e.g. [4,5,13,20].

The paper is structured as follows: in Sec. II we briefly introduce the group D_4 and discuss its subgroups. The model is outlined and the LO results for lepton masses and mixings are presented in Sec. III. Section IV contains the discussion of the flavon superpotential at LO and NLO. In Sec. V NLO corrections to lepton masses and mixings are shown to be well under control. We conclude in Sec. VI.

II. D_4 GROUP THEORY

The dihedral group D_4 has eight (distinct) elements and five irreducible representations denoted here as $\underline{\mathbf{1}}_{\mathbf{i}}$, i = 1, …, 4 and $\underline{\mathbf{2}}$. All representations are real, and only $\underline{\mathbf{2}}$ is faithful. D_4 is generated by A and B, which can be chosen as [21]

$$\mathrm{A} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

for $\underline{\mathbf{2}}$. Note that we have chosen A to be a complex matrix, although $\underline{\mathbf{2}}$ is a real representation. For $(a_1, a_2)^t \sim \underline{\mathbf{2}}$ we thus find that $(a_2^*, a_1^*)^t$ transforms as $\underline{\mathbf{2}}$ under D_4. The generators A and B of the one-dimensional representations are

$$\underline{\mathbf{1}}_{\mathbf{1}}\colon \mathrm{A} = 1, \quad \mathrm{B} = 1, \tag{6}$$

$$\underline{\mathbf{1}}_{\mathbf{2}}\colon \mathrm{A} = 1, \quad \mathrm{B} = -1, \tag{7}$$

$$\underline{\mathbf{1}}_{\mathbf{3}}\colon \mathrm{A} = -1, \quad \mathrm{B} = 1, \tag{8}$$

$$\underline{\mathbf{1}}_{\mathbf{4}}\colon \mathrm{A} = -1, \quad \mathrm{B} = -1. \tag{9}$$

The character table can be found in e.g. [4]. A and B fulfill the relations

$$\mathrm{A}^4 = 1, \qquad \mathrm{B}^2 = 1, \quad \text{and} \quad \mathrm{ABA} = \mathrm{B}. \tag{10}$$

The Kronecker products involving $\underline{\mathbf{1}}_{\mathbf{i}}$ are the following:

$$\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{i}} = \underline{\mathbf{1}}_{\mathbf{1}}, \qquad \underline{\mathbf{1}}_{\mathbf{1}} \times \underline{\mathbf{1}}_{\mathbf{i}} = \underline{\mathbf{1}}_{\mathbf{i}} \quad \text{for i} = 1, \ldots, 4,$$
$$\underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{1}}_{\mathbf{3}} = \underline{\mathbf{1}}_{\mathbf{4}}, \qquad \underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{1}}_{\mathbf{4}} = \underline{\mathbf{1}}_{\mathbf{3}}, \quad \text{and} \quad \underline{\mathbf{1}}_{\mathbf{3}} \times \underline{\mathbf{1}}_{\mathbf{4}} = \underline{\mathbf{1}}_{\mathbf{2}}.$$

For $s_i \sim \underline{\mathbf{1}}_{\mathbf{i}}$ and $(a_1, a_2)^t \sim \underline{\mathbf{2}}$ we find

$$\begin{pmatrix} s_1 a_1 \\ s_1 a_2 \end{pmatrix} \sim \underline{\mathbf{2}}, \qquad \begin{pmatrix} s_2 a_1 \\ -s_2 a_2 \end{pmatrix} \sim \underline{\mathbf{2}},$$
$$\begin{pmatrix} s_3 a_2 \\ s_3 a_1 \end{pmatrix} \sim \underline{\mathbf{2}}, \quad \text{and} \quad \begin{pmatrix} s_4 a_2 \\ -s_4 a_1 \end{pmatrix} \sim \underline{\mathbf{2}}.$$

The four one-dimensional representations contained in $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$ read, for $(a_1, a_2)^t$, $(b_1, b_2)^t \sim \underline{\mathbf{2}}$,

$$a_1 b_2 + a_2 b_1 \sim \underline{\mathbf{1}}_{\mathbf{1}}, \qquad a_1 b_2 - a_2 b_1 \sim \underline{\mathbf{1}}_{\mathbf{2}},$$
$$a_1 b_1 + a_2 b_2 \sim \underline{\mathbf{1}}_{\mathbf{3}}, \quad \text{and} \quad a_1 b_1 - a_2 b_2 \sim \underline{\mathbf{1}}_{\mathbf{4}}.$$

These formulas are special cases of the expressions, given for dihedral symmetries D_n with a general index n, which can be found e.g. in [3,22].

In order to understand how maximal atmospheric mixing and vanishing θ_{13} arise in our model, it is relevant to discuss the subgroups of D_4. All its subgroups are Abelian: $Z_2 \cong D_1$, Z_4 and $D_2 \cong Z_2 \times Z_2$. In the following we are only interested in Z_2 subgroups generated by BA^k, with $k = 0, \ldots, 3$, because these are preserved in the charged lepton and neutrino sectors at LO. Noting

$$(\mathrm{BA}^k)^2 = \mathrm{BA}^k\mathrm{BA}^k = \mathrm{BA}^{k-1}\mathrm{BA}^{k-1} = \ldots = \mathrm{B}^2 = 1$$

shows that BA^k indeed generates a Z_2 symmetry. Since k can take integer values between 0 and 3, we find four possible Z_2 subgroups of this type. Apart from VEVs of fields transforming trivially under D_4, a Z_2 group given through BA^k is left unbroken by a nonvanishing VEV of a singlet transforming as $\underline{\mathbf{1}}_{\mathbf{3}}$ if $k = 0$, 2 holds, and of one singlet transforming as $\underline{\mathbf{1}}_{\mathbf{4}}$ for $k = 1$, 3. Additionally, it is left intact by fields $\psi_{1,2}$ forming a doublet, if their VEVs have the following structure:

$$\begin{pmatrix} \langle \psi_1 \rangle \\ \langle \psi_2 \rangle \end{pmatrix} \propto \begin{pmatrix} \mathrm{e}^{-(\pi i k/2)} \\ 1 \end{pmatrix}. \tag{11}$$

As mentioned in the Introduction, in order to get nontrivial mixing, the Z_2 subgroups preserved in the charged lepton and the neutrino sectors have to have different indices k_l and k_ν. To achieve maximal atmospheric mixing, we need e.g. k_l to be odd, whereas k_ν has to be even. This is analogous to the constraints found for the indices $k_{l,\nu}$ in [4]. In the following section we show that it is phenomenologically irrelevant whether k_l is 1 or 3, as well as whether k_ν is 0 or 2; i.e. the result for the lepton mixing angles is, in all cases, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. In Sec. IV a simple flavon superpotential is constructed which naturally leads to an odd index k_l and an even index k_ν.

III. OUTLINE OF THE MODEL AND RESULTS AT LO

In this section the transformation properties of all fields (apart from the driving fields responsible for the vacuum alignment) under the flavor symmetry $D_4 \times Z_5$ are presented. Similar to the model discussed in [4], the framework is the MSSM and the left-handed lepton doublets L_i transform as $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}$ under D_4. In order to accommodate the charged lepton mass hierarchy, we assign the right-handed charged leptons e_i^c to the three singlets $\underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{1}}_{\mathbf{3}} + \underline{\mathbf{1}}_{\mathbf{4}}$ and not to $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}$ as done in [4]. In contrast to the model given in [4], we assign the Z_5 charges to the matter superfields and $h_{u,d}$ such that $h_{u,d}$ are uncharged under $D_4 \times Z_5$ and, more importantly, we allow the generations of one type, L_i and e_i^c, to transform differently under Z_5. In this way the additional cyclic symmetry Z_5 not only plays the role of a symmetry which separates the charged lepton and neutrino sectors (at LO), but it also plays the role of a Froggatt-Nielsen symmetry [23]. As one can see, we use for the five different multiplets L_1, $L_D = (L_2, L_3)^t$, e_i^c all five different possible Z_5 charges. As in [4] the light neutrino masses arise from the effective operator $L_i L_j h_u^2/\Lambda$. The flavons relevant for achieving $\mu - \tau$ symmetry in the lepton sector at LO are ψ_e and ψ_ν, which form doublets under D_4. The VEV of ψ_e is aligned through the flavon superpotential in such a way that either the Z_2 subgroup generated by BA or by BA^3 is preserved in the charged lepton sector, whereas the special form of $\langle \psi_{\nu;1,2} \rangle$ preserves either a Z_2 subgroup arising from B or from BA^2. As we will show, it is not relevant for phenomenology which of the two subgroups in each sector is chosen. The relevant aspect is the fact that different subgroups are preserved. The two additional flavons η_1 and η_3 are Froggatt-Nielsen–type fields. Since they do not transform under D_4, they do not play a role in the preservation of different subgroups of D_4. They allow for nonzero (11) and (23), (32) entries in the neutrino mass matrix at LO so that we arrive at a neutrino mass matrix containing three independent parameters, related to the nonzero VEVs of the flavons η_1, η_3, and ψ_ν. These parameters are fixed by the two mass squared differences and the solar mixing angle θ_{12}. Furthermore, the field η_1 is relevant for generating the appropriate hierarchy among the charged lepton masses, because m_τ arises dominantly from $L_D e_3^c h_d \psi_e/\Lambda$, while the mass of the muon is generated through the operator $L_D e_2^c h_d \psi_e \eta_1/\Lambda^2$. The mass of the electron originates mainly from one subleading operator involving three flavons, $L_1 e_1^c h_d \psi_e \psi_\nu \eta_1/\Lambda^3$. Since the tau lepton mass stems from a nonrenormalizable operator, $v_d = \langle h_d \rangle$ is expected to be of the order of the electroweak scale, and thus small and moderate values of $\tan\beta = \langle h_u \rangle/\langle h_d \rangle = v_u/v_d$ are preferred in this model. The right order of magnitude of m_τ can be achieved for $\langle \psi_e \rangle/\Lambda \approx \epsilon \approx 0.04$. The ratio $m_\mu : m_\tau \sim \epsilon : 1$ also requires that $\langle \eta_1 \rangle/\Lambda \approx \epsilon$. As we will see in the next section all flavon VEVs are expressed in terms of two mass parameters M_1 and M_2. Choosing the latter to be of order $\epsilon\Lambda$ leads, unless accidental cancellations occur, to all flavon VEVs being of that order as well. As a consequence, the electron mass m_e is expected to fulfill $m_e : m_\tau \sim \epsilon^2 : 1$ due to its origin from a three flavon operator. Apart from generating the electron mass, operators involving several flavons (as well as shifts in the flavon VEVs) lead to small deviations from the LO result that θ_{23} is maximal and θ_{13} vanishes. As we show below, these deviations are dominated by corrections associated with the charged lepton sector, while all subleading effects in the neutrino sector are of relative order ϵ^2. All fields appearing in the Yukawa couplings and their transformation properties under $D_4 \times Z_5$ are captured in Table I.

In order to demonstrate how maximal atmospheric mixing and the vanishing reactor mixing angle arise, we first take into account only operators which are suppressed by at maximum $1/\Lambda^2$ and in which the flavon doublet ψ_ν only couples to neutrinos, whereas ψ_e couples to charged leptons. In this way the Z_2 subgroups in charged lepton and neutrino sectors remain exactly preserved. We show subsequently that the inclusion of further operators, compatible with the symmetries of the model, and shifts in the flavon VEVs only slightly correct the results achieved at this level. With the above restrictions the allowed operators in the superpotential are[2]

[2]We do not list the operator $L_1 e_2^c h_d \psi_e^2/\Lambda^2$, because plugging in the vacuum given in Eq. (13) renders this possible contribution to the charged lepton mass matrix zero. However, we discuss the operator in Sec. V, since it leads to a nonvanishing contribution, once the shifts in the flavon VEVs are taken into account.

TABLE I. Particle content of the model. L_i denotes the three left-handed lepton $SU(2)_L$ doublets, e_i^c the right-handed charged leptons, and $h_{u,d}$ the MSSM Higgs doublets. The flavons $\psi_{e;1,2}$ and $\psi_{\nu;1,2}$ only transform under $D_4 \times Z_5$. Also, the Froggatt-Nielsen–type fields η_1 and η_3 are gauge singlets and only carry a nonzero Z_5 charge. $\omega = \mathrm{e}^{(2\pi i)/5}$ is the generating element of Z_5.

Field	L_1	L_D	e_1^c	e_2^c	e_3^c	h_u	h_d	$\psi_{e;1,2}$	$\psi_{\nu;1,2}$	η_1	η_3
D_4	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_2$	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$
Z_5	ω^2	ω	1	ω^3	ω^4	1	1	1	ω^2	ω	ω^3

$$\begin{aligned} W_l = {} & \frac{y_1^e}{\Lambda}(L_2 e_3^c h_d \psi_{e;1} - L_3 e_3^c h_d \psi_{e;2}) \\ & + \frac{y_2^e}{\Lambda^2}(L_2 e_2^c h_d \psi_{e;1} + L_3 e_2^c h_d \psi_{e;2})\eta_1 \\ & + \frac{y_1^\nu}{\Lambda^2} L_1 L_1 h_u^2 \eta_1 + \frac{y_2^\nu}{\Lambda^2}(L_2 L_3 + L_3 L_2) h_u^2 \eta_3 \\ & + \frac{y_3^\nu}{\Lambda^2} L_1 (L_2 h_u^2 \psi_{\nu;2} + L_3 h_u^2 \psi_{\nu;1}) \\ & + \frac{y_3^\nu}{\Lambda^2}(L_2 \psi_{\nu;2} + L_3 \psi_{\nu;1}) h_u^2 L_1. \end{aligned} \tag{12}$$

As shown in Sec. IV, the vacuum structure is

$$\begin{aligned} \langle \psi_{e;2} \rangle &= -i\rho_e \langle \psi_{e;1} \rangle = w_e, \\ \langle \psi_{\nu;2} \rangle &= \rho_\nu \langle \psi_{\nu;1} \rangle = w_\nu, \\ \langle \eta_1 \rangle &= w_1, \quad \text{and} \quad \langle \eta_3 \rangle = w_3 \end{aligned} \tag{13}$$

with $\rho_{e,\nu} = \pm 1$. The different choices of $\rho_{e,\nu}$ correspond to the different possible values of the subgroup indices k_l and k_ν, respectively. Since they are not uniquely fixed by the flavon superpotential, we keep $\rho_{e,\nu} = \pm 1$ as parameters and check that all possibilities lead to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. We arrive at fermion mass matrices of the form (M_l is given in the left-right convention)

$$\begin{aligned} M_l &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\rho_e y_2^e w_1/\Lambda & i\rho_e y_1^e \\ 0 & y_2^e w_1/\Lambda & -y_1^e \end{pmatrix} \frac{w_e}{\Lambda} v_d \quad \text{and} \\ M_\nu &= \begin{pmatrix} y_1^\nu w_1 & y_3^\nu w_\nu & \rho_\nu y_3^\nu w_\nu \\ y_3^\nu w_\nu & 0 & y_2^\nu w_3 \\ \rho_\nu y_3^\nu w_\nu & y_2^\nu w_3 & 0 \end{pmatrix} \frac{v_u^2}{\Lambda^2}. \end{aligned} \tag{14}$$

For

$$w_e, w_\nu, w_1, w_3 \approx \epsilon \Lambda \tag{15}$$

with $\epsilon \approx 0.04$, muon and tau lepton masses read

$$\begin{aligned} m_\mu &= \sqrt{2}|y_2^e| w_1 w_e v_d/\Lambda^2 \approx \epsilon^2 v_d \quad \text{and} \\ m_\tau &= \sqrt{2}|y_1^e| w_e v_d/\Lambda \approx \epsilon v_d, \end{aligned} \tag{16}$$

while the electron remains massless at this stage and acquires a mass from operators with three flavon insertions; see Eq. (34). The charged lepton mass matrix can be diagonalized through the usual biunitary transformation (U_l and V_l) so that

$$U_l^\dagger M_l V_l = \mathrm{diag}(m_e, m_\mu, m_\tau), \tag{17}$$

with U_l given by

$$U_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i\rho_e/\sqrt{2} & -i\rho_e/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \tag{18}$$

and V_l is a diagonal matrix, ensuring that the entries of M_l in the charged lepton mass basis are real and positive. The unitary matrix U_ν diagonalizing $M_\nu M_\nu^\dagger$ can be written in the form

$$U_\nu = \begin{pmatrix} \cos\phi_\nu & \sin\phi_\nu \mathrm{e}^{-i\gamma_\nu} & 0 \\ -\rho_\nu \sin\phi_\nu \mathrm{e}^{i\gamma_\nu}/\sqrt{2} & \rho_\nu \cos\phi_\nu/\sqrt{2} & -\rho_\nu/\sqrt{2} \\ -\sin\phi_\nu \mathrm{e}^{i\gamma_\nu}/\sqrt{2} & \cos\phi_\nu/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \tag{19}$$

with ϕ_ν and γ_ν being real functions of the entries of M_ν. For the absolute values of the lepton mixing matrix we then find

$$|U_{\mathrm{MNS}}| = \begin{pmatrix} |\cos\phi_\nu| & |\sin\phi_\nu| & 0 \\ |\sin\phi_\nu|/\sqrt{2} & |\cos\phi_\nu|/\sqrt{2} & 1/\sqrt{2} \\ |\sin\phi_\nu|/\sqrt{2} & |\cos\phi_\nu|/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \tag{20}$$

showing that $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ hold, while the solar mixing angle θ_{12}, related to ϕ_ν, is not fixed in our model. As one can see, the form of $|U_{\mathrm{MNS}}|$ and thus the result for the mixing angles do not depend on a particular choice of ρ_e and ρ_ν.

In the charged lepton mass basis the neutrino mass matrix $M_\nu' = U_l^\dagger M_\nu U_l^*$ reveals a texture zero in its (23) entry. According to the findings in [24] a consequence of the presence of this texture zero together with $\theta_{13} = 0$ is that the neutrino mass spectrum has to have inverted hierarchy and, furthermore, the lightest neutrino mass m_3 cannot vanish. In our model this is not a generic result attributed to the flavor symmetry $D_4 \times Z_5$ and its breaking pattern, but rather, it is due to the fact that a flavon coupling to the neutrino sector and transforming as $(\underline{\mathbf{1}}_3, \omega^3)$ under (D_4, Z_5) is absent. A numerical analysis similar to the one found in [4] can also be performed for this model. The results of such an analysis are quantitatively very similar to those found in [4] for the case of a nonzero (23) entry of the neutrino mass matrix M_ν' in a spontaneously CP violating framework. These include the relation $|m_{ee}| \approx m_3$, a weak correlation between $\tan\theta_{12}$ and m_3, and rather strong correlations of the two Majorana phases and the mass m_3, as well as a lower bound on m_3 around 0.015 eV. Plots showing these results can be found in [4] in Figs. 1–4. All results are found to be independent of the different choices of ρ_ν and ρ_e. Noting that the third neutrino mass m_3 is given by $|y_2^\nu| w_3 v_u^2/\Lambda^2$, we find that in order to correctly reproduce the light neutrino mass scale of around 0.1 eV, a value of the cutoff scale $\Lambda \approx 4 \times 10^{12}$ GeV for $v_u \approx 100$ GeV is necessary.

Comparing our results to those found in [4], we see that the difference between the two models lies in the fact that in [4] the subgroup in the charged lepton sector is D_2, while it is only a Z_2 subgroup in our model. As a consequence, the model in [4] leads to $\mu - \tau$ symmetric mixing through preserving certain D_4 subgroups only, whereas in the model here the absence of a flavon transforming as $\underline{\mathbf{1}}_4$ under D_4 from the charged lepton sector is relevant for achieving $\theta_{23} = \pi/4$ and $\theta_{13} = 0$; for details see Appendix B in [4].

IV. FLAVON SUPERPOTENTIAL

In the following we align the VEVs of the fields ψ_e and ψ_ν correctly and determine all VEVs through the two mass parameters M_1 and M_2 of the superpotential, thus avoiding flat directions in the potential associated with VEVs which remain undetermined. We consider the F-terms of a new set of fields, the driving fields, as the source of the alignment conditions. These fields are, like the flavons, gauge singlets and transform, in general, under $D_4 \times Z_5$. We assume the presence of a continuous R-symmetry $U(1)_R$ under which all matter superfields have charge $+1$, flavons and the superfields $h_{u,d}$ charge 0, and driving fields charge $+2$, so that the superpotential, responsible for the alignment of the flavon VEVs, will be linear in the driving fields. This $U(1)_R$ symmetry contains R parity as a subgroup. As motivation for its presence, we take the fact that such symmetries typically play an important role in models of supersymmetry (SUSY) breaking [25].

A. Renormalizable level

In order to align the vacuum of ψ_ν we introduce one driving field, σ_ν^0, transforming as $\underline{\mathbf{1}}_4$ under D_4, similar to [4], and as ω under Z_5. The alignment of the vacuum of ψ_e is achieved in a similar way by coupling it to a field σ_e^0 transforming as $\underline{\mathbf{1}}_3$ and being invariant under Z_5. We add a field χ_e^0, which is neutral under $D_4 \times Z_5$, to allow for a coupling of mass dimension two, which fixes the size of the VEV of ψ_e. The actual size of this mass parameter, and thus also of the VEV ψ_e, is chosen by hand to be of order $\epsilon\Lambda$. The VEVs of ψ_ν and of the Froggatt-Nielsen fields η_1 and η_3 are deduced from the F-terms of three further fields, χ_ν^0, η_3^0, and η_4^0 transforming as shown in Table II. Note that the field η_4^0 also allows for a coupling with positive mass dimension in the flavon superpotential. Again, this mass parameter is chosen to be of order $\epsilon\Lambda$ so that the VEVs of ψ_ν, η_1, and η_3 are of that order as well and especially of the same order as the VEV of ψ_e whose size is given by the other mass parameter of the superpotential. Apart from σ_e^0 and σ_ν^0, which are responsible for the alignment of $\langle \psi_{e;1,2} \rangle$ and $\langle \psi_{\nu;1,2} \rangle$, all other driving fields transform trivially under D_4. The flavon superpotential $W_{\rm fl}$ is given, at the renormalizable level, as[3]

TABLE II. Driving fields of the model necessary to align the VEVs of ψ_e and ψ_ν and to determine the flavon VEVs through mass parameters of the superpotential. All these fields have charge $+2$ under $U(1)_R$.

Field	σ_e^0	σ_ν^0	χ_e^0	χ_ν^0	η_3^0	η_4^0
D_4	$\underline{\mathbf{1}}_3$	$\underline{\mathbf{1}}_4$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$
Z_5	1	ω	1	ω	ω^3	ω^4

$$\begin{aligned} W_{\rm fl} = {} & a_e \sigma_e^0 (\psi_{e;1}^2 + \psi_{e;2}^2) + M_1^2 \chi_e^0 + b_e \chi_e^0 \psi_{e;1} \psi_{e;2} \\ & + a_\nu \sigma_\nu^0 (\psi_{\nu;1}^2 - \psi_{\nu;2}^2) + b_\nu \chi_\nu^0 \psi_{\nu;1} \psi_{\nu;2} \\ & + c_\nu \chi_\nu^0 \eta_1 \eta_3 + d \eta_3^0 (\psi_{e;1} \psi_{\nu;2} + \psi_{e;2} \psi_{\nu;1}) \\ & + f \eta_3^0 \eta_1^2 + M_2 \eta_4^0 \eta_1 + g \eta_4^0 \eta_3^2, \end{aligned} \quad (21)$$

with a_e, b_e, a_ν, b_ν, c_ν, d, f, and g being complex numbers with absolute values of order one. We note that the two mass parameters M_1 and M_2 can be chosen as real and positive without loss of generality. Assuming that the flavons acquire their VEVs in the SUSY limit, i.e. (soft) SUSY breaking effects can be safely neglected, we can use the F-terms of the driving fields to determine the vacuum structure of the flavons. The first two equations read

$$\begin{aligned} \frac{\partial W_{\rm fl}}{\partial \sigma_e^0} &= a_e (\psi_{e;1}^2 + \psi_{e;2}^2) = 0 \quad \text{and} \\ \frac{\partial W_{\rm fl}}{\partial \sigma_\nu^0} &= a_\nu (\psi_{\nu;1}^2 - \psi_{\nu;2}^2) = 0. \end{aligned} \quad (22)$$

We find as solutions

$$\begin{aligned} \langle \psi_{e;2} \rangle &= -i \rho_e \langle \psi_{e;1} \rangle = w_e \quad \text{and} \\ \langle \psi_{\nu;2} \rangle &= \rho_\nu \langle \psi_{\nu;1} \rangle = w_\nu, \end{aligned} \quad (23)$$

with $\rho_e = \pm 1$ and $\rho_\nu = \pm 1$. We see that we preserve a Z_2 subgroup of D_4 generated by either BA ($k_l = 1$) or BA^3 ($k_l = 3$) in the charged lepton sector, whereas in the neutrino sector the subgroup is either generated by B ($k_\nu = 0$) or by BA^2 ($k_\nu = 2$). Note further that the VEV alignment, i.e. the relations between the VEVs of the upper and lower components of the D_4 doublets ψ_e and ψ_ν [see Eq. (23)], is independent of the choice of parameters, like the mass parameters M_1 and M_2. From

$$\frac{\partial W_{\rm fl}}{\partial \chi_e^0} = M_1^2 + b_e \psi_{e;1} \psi_{e;2} = 0 \quad (24)$$

it follows that the VEV of ψ_e is fixed through the mass scale M_1 to be

[3] We can safely neglect the term $\chi_e^0 h_u h_d$, because the VEVs of all flavons are much larger than the electroweak scale.

$$w_e^2 = i\rho_e \frac{M_1^2}{b_e}. \tag{25}$$

The last three F-term equations,

$$\frac{\partial W_{\rm fl}}{\partial \chi_\nu^0} = b_\nu \psi_{\nu;1}\psi_{\nu;2} + c_\nu \eta_1 \eta_3 = 0, \tag{26a}$$

$$\frac{\partial W_{\rm fl}}{\partial \eta_3^0} = d(\psi_{e;1}\psi_{\nu;2} + \psi_{e;2}\psi_{\nu;1}) + f\eta_1^2 = 0, \tag{26b}$$

$$\frac{\partial W_{\rm fl}}{\partial \eta_4^0} = M_2 \eta_1 + g\eta_3^2 = 0, \tag{26c}$$

allow us to determine the VEVs of ψ_ν, η_1, and η_3 as functions of the two mass parameters M_1 and M_2. For M_1 and M_2 being of the order $\epsilon\Lambda$, we find that all flavon VEVs are of that order as well. Since we do not have a model of dynamical flavor symmetry breaking, M_1 and M_2 being of order $\epsilon\Lambda$ is a parameter choice and not a prediction of our model. $\langle \psi_e \rangle \neq 0$ is a necessary consequence of Eq. (24), showing that D_4 is always spontaneously broken. The other flavon VEVs could, in principle, vanish. However, if we assume that the VEV of one of the fields ψ_ν, η_1, and η_3 does not vanish, then the nonvanishing of the other two follows. Because of the fact that all terms in $W_{\rm fl}$ are, by construction, linear in the driving fields, the F-terms of the flavons vanish in any case for vanishing VEVs of all driving fields. Then the allowed term $\chi_e^0 h_u h_d$ does not give rise to a μ-term either.

B. Corrections from nonrenormalizable terms

Including nonrenormalizable terms with three flavons leads to corrections of the alignment achieved at LO. We find nine terms that contribute at this level to the flavon superpotential (again, all possible terms involving the superfields h_u and h_d are neglected),

$$\begin{aligned}\Delta W_{\rm fl} = {}& x_1 \sigma_e^0 \eta_1 (\psi_{\nu;1}^2 + \psi_{\nu;2}^2)/\Lambda + x_2 \sigma_e^0 \eta_3 (\psi_{e;1}\psi_{\nu;1} + \psi_{e;2}\psi_{\nu;2})/\Lambda + x_3 \chi_e^0 \eta_1 \psi_{\nu;1}\psi_{\nu;2}/\Lambda \\ & + x_4 \chi_e^0 \eta_3 (\psi_{e;1}\psi_{\nu;2} + \psi_{e;2}\psi_{\nu;1})/\Lambda + x_5 \chi_e^0 \eta_1^2 \eta_3/\Lambda + x_6 \chi_\nu^0 \eta_3^3/\Lambda + x_7 \eta_3^0 \eta_3 \psi_{\nu;1}\psi_{\nu;2}/\Lambda + x_8 \eta_3^0 \eta_1 \eta_3^2/\Lambda \\ & + x_9 \eta_4^0 \eta_1 \psi_{e;1}\psi_{e;2}/\Lambda. \end{aligned} \tag{27}$$

Note that there is no correction at this level involving the driving field σ_ν^0. We can parametrize the VEVs of ψ_e and ψ_ν as

$$\langle \psi_e \rangle = \begin{pmatrix} i\rho_e (w_e + \delta w_{e;1}) \\ w_e + \delta w_{e;2} \end{pmatrix} \quad \text{and} \quad \langle \psi_\nu \rangle = \begin{pmatrix} \rho_\nu (w_\nu + \delta w_{\nu;1}) \\ w_\nu + \delta w_{\nu;2} \end{pmatrix} \tag{28}$$

and find for the shifts in linear expansion that

$$\frac{\delta w_{e;i}}{w_e}, \qquad \frac{\delta w_{\nu;i}}{w_\nu} \sim \epsilon, \qquad \delta w_{e;1} \neq \delta w_{e;2}, \quad \text{and} \quad \delta w_{\nu;1} = \delta w_{\nu;2}. \tag{29}$$

Since the shifts of $\langle \psi_{\nu;i} \rangle$ are the same, the vacuum alignment is preserved up to this level, and we can absorb the shifts $\delta w_{\nu;i}$ into a redefinition of w_ν. The equality of $\delta w_{\nu;i}$ is due to the fact that at the first nonrenormalizable level the superpotential $\Delta W_{\rm fl}$ does not contain terms involving σ_ν^0. The shifts of the VEVs of the singlets η_1 and η_3 are of the same order of magnitude as $\delta w_{e;i}$ and $\delta w_{\nu;i}$. However, we do not mention them explicitly, because their effect on lepton masses and mixings can always be absorbed into a redefinition of Yukawa couplings or VEVs w_1 and w_3.

V. LEPTON MASSES AND MIXINGS AT NLO

In general, NLO corrections arise from two sources: (i) shifts in the flavon VEVs and (ii) operators with two and more flavon insertions, evaluated by plugging in the LO form of the VEVs, which have not been considered in Sec. III. As we show in the following, all such additional contributions change the LO results only slightly. However, their discussion is relevant, because such terms generate the electron mass and govern the deviations from the LO results $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. We discuss all such terms which give rise to contributions up to and including ϵ^3 (in units of v_d and v_u^2/Λ, respectively), and in the case of the (13) element of the charged lepton mass matrix, we also discuss corrections of order $\epsilon^4 v_d$.

Considering the charged lepton sector, we find the following additional operators involving two and more flavons which contribute to the (23) and (33) elements of M_l (order one coefficients are omitted and operators are not written in D_4 components),

$$\frac{1}{\Lambda^2} L_D e_3^c h_d \psi_\nu \eta_3 + \frac{1}{\Lambda^3} L_D e_3^c h_d \psi_e^3, \tag{30}$$

where the last term actually gives rise to two independent contributions; i.e. there are two different possibilities to arrive at a D_4-invariant coupling, involving the fields L_D, e_3^c, h_d and three powers of the flavon ψ_e, each of which is accompanied by a complex order one coefficient.[4] This

[4]As an example of the two independent contractions, we give the pair of terms

$$\psi_{e;1}\psi_{e;2}(L_2 e_3^c h_d \psi_{e;1} - L_3 e_3^c h_d \psi_{e;2})/\Lambda^3,$$
$$(L_2 e_3^c h_d \psi_{e;2}^3 - L_3 e_3^c h_d \psi_{e;1}^3)/\Lambda^3.$$

holds similarly for several of the operators mentioned in the following. The (22) and (32) elements get corrected through

$$\frac{1}{\Lambda^3} L_D e_2^c h_d \psi_\nu^3 + \frac{1}{\Lambda^3} L_D e_2^c h_d \psi_e \eta_3^2 + \frac{1}{\Lambda^3} L_D e_2^c h_d \psi_\nu \eta_1 \eta_3, \tag{31}$$

with the first term being responsible for two independent contributions. At the same time, these entries also receive contributions from the shifts in the VEV of ψ_e, which we indicate by[5]

$$\frac{1}{\Lambda} L_D e_3^c h_d \delta\psi_e + \frac{1}{\Lambda^2} L_D e_2^c h_d \delta\psi_e \eta_1. \tag{32}$$

The (21) and (31) elements are generated only through three flavon insertions of the form

$$\frac{1}{\Lambda^3} L_D e_1^c h_d \psi_e \psi_\nu^2 + \frac{1}{\Lambda^3} L_D e_1^c h_d \psi_\nu \eta_1^2 + \frac{1}{\Lambda^3} L_D e_1^c h_d \psi_e \eta_1 \eta_3. \tag{33}$$

Note that the first operator gives rise to three independent terms. Similarly, the (11) element arises from

$$\frac{1}{\Lambda^3} L_1 e_1^c h_d \psi_e \psi_\nu \eta_1. \tag{34}$$

The generation of the (12) and (13) elements is somewhat special, because the lowest order operators which could give rise to nonzero (12) and (13) elements are

$$\frac{1}{\Lambda^2} L_1 e_2^c h_d \psi_e^2 \quad \text{and} \quad \frac{1}{\Lambda^2} L_1 e_3^c h_d \psi_\nu^2. \tag{35}$$

However, plugging in the LO result for the VEVs of ψ_e and ψ_ν, we see that these operators give no contribution. Taking into account the shifts arising at a relative order ϵ, we then find that the (12) element is generated, whereas the (13) element still vanishes. Operators with three flavons also contribute to the (12) element at this level,

$$\frac{1}{\Lambda^3} L_1 e_2^c h_d \psi_\nu^2 \eta_1 + \frac{1}{\Lambda^3} L_1 e_2^c h_d \psi_e \psi_\nu \eta_3. \tag{36}$$

The (13) element only originates from operators involving four flavons,

$$\frac{1}{\Lambda^4} L_1 e_3^c h_d \psi_e^2 \eta_1 \eta_3 + \frac{1}{\Lambda^4} L_1 e_3^c h_d \psi_e \psi_\nu \eta_1^2 + \frac{1}{\Lambda^4} L_1 e_3^c h_d \psi_e^2 \psi_\nu^2. \tag{37}$$

Again, the last operator allows for two independent contractions. At this level we expect that the (13) element also receives a contribution from subleading shifts in the VEVs of ψ_ν, which, however, we do not calculate, because the specific form of the (13) element is not relevant for the analysis of lepton masses and mixings. Thus, we can parametrize the charged lepton mass matrix including NLO corrections as

$$M_l = \begin{pmatrix} \beta_1^e \epsilon^2 & \beta_2^e \epsilon^2 & \beta_3^e \epsilon^3 \\ \beta_4^e \epsilon^2 & i\rho_e \alpha_2^e \epsilon + \beta_6^e \epsilon^2 & i\rho_e \alpha_1^e + \beta_7^e \epsilon \\ \beta_5^e \epsilon^2 & \alpha_2^e \epsilon & -\alpha_1^e \end{pmatrix} \epsilon v_d, \tag{38}$$

taking into account the sizes of flavon VEVs and of their shifts as given in Eqs. (15) and (29). Through rephasing of right-handed fields we can make α_1^e, α_2^e, and β_5^e real and positive. The other parameters, apart from ϵ, are, in general, complex numbers with absolute values of order one. Note further that the parameters $\alpha_{1,2}^e$ are, up to corrections of order ϵ, determined by the LO operators given in Eq. (12).

We find as a result for the charged lepton masses

$$m_e = (|\beta_1^e|\epsilon^3 + \mathcal{O}(\epsilon^4))v_d, \qquad m_\mu = (\sqrt{2}\alpha_2^e \epsilon^2 + \mathcal{O}(\epsilon^3))v_d, \quad \text{and} \quad m_\tau = (\sqrt{2}\alpha_1^e \epsilon + \mathcal{O}(\epsilon^2))v_d \tag{39}$$

and thus confirm earlier statements about the size and origin of the electron mass in our model. The unitary transformation which has to be applied to the left-handed charged leptons to diagonalize M_l is, up to the first correction in ϵ in each matrix element, of the form

$$U_l \approx \begin{pmatrix} 1 - \left(\frac{|\beta_2^e|}{2\alpha_2^e}\right)^2 \epsilon^2 & \frac{\beta_2^e}{\sqrt{2}\alpha_2^e}\epsilon & -\frac{\beta_3^e}{\sqrt{2}\alpha_1^e}\epsilon^3 \\ -i\rho_e \frac{\beta_2^{e*}}{2\alpha_2^e}\epsilon & \frac{1}{\sqrt{2}}\left(i\rho_e + \frac{\beta_7^{e*}}{2\alpha_1^e}\epsilon\right) & -\frac{1}{\sqrt{2}}\left(i\rho_e + \frac{\beta_7^e}{2\alpha_1^e}\epsilon\right) \\ -\frac{\beta_2^{e*}}{2\alpha_2^e}\epsilon & \frac{1}{\sqrt{2}}\left(1 + i\rho_e \frac{\beta_7^{e*}}{2\alpha_1^e}\epsilon\right) & \frac{1}{\sqrt{2}}\left(1 + i\rho_e \frac{\beta_7^e}{2\alpha_1^e}\epsilon\right) \end{pmatrix}. \tag{40}$$

[5]As mentioned in Sec. IV B the shift in the VEV of the field η_1 can be absorbed into the Yukawa couplings or the LO VEV itself, and thus its contribution is not displayed. The same holds for η_3.

We analyze the corrections to the neutrino mass matrix in a similar way and find that the following operators contribute to the (11) element:

$$\frac{1}{\Lambda^3} L_1 L_1 h_u^2 \eta_3^2 + \frac{1}{\Lambda^4} L_1 L_1 h_u^2 \psi_e^2 \eta_1, \tag{41}$$

and that corrections to the (23) element come from

$$\frac{1}{\Lambda^4} L_D L_D h_u^2 \eta_1^3 + \frac{1}{\Lambda^4} (L_D L_D)_1 h_u^2 (\psi_e^2 \eta_3)_1 + \frac{1}{\Lambda^4} (L_D L_D)_1 h_u^2 (\psi_e \psi_\nu \eta_1)_1. \tag{42}$$

Here we denote by $(\cdots)_1$ the contraction to a trivial singlet $\underline{\mathbf{1}}_{\mathbf{1}}$ of D_4. All these contributions can be absorbed into the LO result. The relation between the (12) and (13) elements is disturbed at a relative order ϵ^2 through the terms

$$\frac{1}{\Lambda^4} L_1 L_D h_u^2 \psi_e^2 \psi_\nu + \frac{1}{\Lambda^4} L_1 L_D h_u^2 \psi_e \eta_1^2. \tag{43}$$

Note that the first operator generates three independent contributions. Since the shift of the VEVs of ψ_ν is, at lowest order, aligned with the LO result [see Eq. (29)], we do not encounter any correction to the LO relation between the (12) and (13) elements at the relative level of ϵ. Among the corrections of order ϵ^3 (in units of v_u^2/Λ) we also expect corrections due to possible deviations from $\langle \psi_{\nu;1} \rangle = \langle \psi_{\nu;2} \rangle$ at the level of $\epsilon^3 \Lambda$. The (22) and (33) elements of M_ν receive two types of corrections. The first one still preserves the $\mu - \tau$ symmetric structure of the LO result and generates entries of order $\epsilon^3 v_u^2/\Lambda$,

$$\frac{1}{\Lambda^4} (L_D L_D)_3 h_u^2 (\psi_e^2 \eta_3)_3 + \frac{1}{\Lambda^4} (L_D L_D)_3 h_u^2 (\psi_e \psi_\nu \eta_1)_3. \tag{44}$$

The second type breaks $\mu - \tau$ symmetry and stems from very similar operators,

$$\frac{1}{\Lambda^4} (L_D L_D)_4 h_u^2 (\psi_e^2 \eta_3)_4 + \frac{1}{\Lambda^4} (L_D L_D)_4 h_u^2 (\psi_e \psi_\nu \eta_1)_4. \tag{45}$$

As one can see, the only difference is the contraction to another D_4 singlet; in the first case we contract to a $\underline{\mathbf{1}}_{\mathbf{3}}$, indicated by $(\cdots)_3$, whereas in the second case the contraction is to $\underline{\mathbf{1}}_{\mathbf{4}}$, denoted by $(\cdots)_4$. Also these $\mu - \tau$ symmetry breaking contributions arise at the level $\epsilon^3 v_u^2/\Lambda$. Thus, at NLO M_ν takes the form

$$M_\nu = \begin{pmatrix} \alpha_1^\nu & \alpha_3^\nu + \beta_1^\nu \epsilon^2 & \rho_\nu \alpha_3^\nu \\ \alpha_3^\nu + \beta_1^\nu \epsilon^2 & \beta_2^\nu \epsilon^2 & \alpha_2^\nu \\ \rho_\nu \alpha_3^\nu & \alpha_2^\nu & \beta_3^\nu \epsilon^2 \end{pmatrix} \epsilon \frac{v_u^2}{\Lambda}. \tag{46}$$

Note that the parameters $\alpha_{1,2,3}^\nu$ and $\beta_{1,2,3}^\nu$ are complex numbers with absolute values of order one. Similar to the charged lepton sector, the parameters $\alpha_{1,2,3}^\nu$ coincide at LO in the expansion in ϵ with those given in Eqs. (12) and (14).

Recomputing the mass spectrum of the neutrinos, we find that all masses are corrected by terms of relative order ϵ^2. Relevant for calculating the deviations from maximal atmospheric mixing and $\theta_{13} = 0$ is the form of the eigenvector associated with the third neutrino mass m_3. We find that its LO form [see Eq. (19)] receives corrections only at order ϵ^2. Thus, the results for the mixing angles θ_{13} and θ_{23} at NLO are dominated by corrections coming from the charged lepton sector [see Eq. (40)], and we get

$$|U_{e3}| = \sin\theta_{13} \approx \frac{|\beta_2^e|}{2\alpha_2^e} \epsilon, \tag{47}$$

$$\sin^2\theta_{23} \approx \frac{1}{2}\left(1 - \rho_\nu \frac{\mathrm{Re}(\beta_7^e)}{\alpha_1^e} \epsilon\right). \tag{48}$$

The solar mixing angle θ_{12}, whose exact value is not predicted in our model, but only given in terms of the parameters of the neutrino mass matrix at LO, also undergoes corrections of order ϵ. However, these are not of particular interest.

VI. CONCLUSIONS

We have presented a SUSY D_4 model which leads to maximal atmospheric mixing and vanishing θ_{13}, while keeping the solar mixing angle θ_{12} undetermined, though it is expected to be large, in general. These predictions originate from the nontrivial breaking of D_4 to distinct Z_2 subgroups generated by BA^k, $k = 0, 1, 2, 3$, in the charged lepton and neutrino sectors. As we have shown, it does not matter to which of these Z_2 subgroups D_4 is actually broken. It is enough to achieve that the index k_l in the charged lepton sector is odd, whereas k_ν, the index in the neutrino sector, is even. At the same time, the hierarchy among charged leptons can be naturally produced. Apart from D_4, which is responsible for the $\mu - \tau$ symmetric mixing pattern, we employ a cyclic symmetry Z_5 in order to separate the flavons coupling to neutrino and charged lepton sectors at LO, as well as to achieve appropriately the mass hierarchy $m_e \ll m_\mu \ll m_\tau$. Because of a restricted choice of flavon fields we find that neutrinos have inverted mass hierarchy with the lightest neutrino mass $m_3 \gtrsim 0.015$ eV. These properties together with nontrivial relations among m_3, $|m_{ee}|$, $\tan\theta_{12}$, and the Majorana phases are shared with a similar SUSY D_4 model, found in [4]. In the latter, however, the hierarchy among charged leptons could not be reproduced without fine-tuning. This is due to the fact that right-handed charged leptons also transform as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ under D_4. This problem is solved in the present model by assigning them to three (distinct) one-dimensional representations of D_4. The result of $\mu - \tau$ symmetric lepton mixings remains untouched.

A particular feature of our model is that the flavon superpotential not only leads to the vacuum alignment which is needed in order to achieve the predictions

$\theta_{23} = \pi/4$ and $\theta_{13} = 0$, but also determines all flavon VEVs through two mass parameters, M_1 and M_2. We choose both of them to be of order $\epsilon\Lambda$, so that all flavon VEVs are also of that particular order. Furthermore, flat directions in the flavon (super)potential, associated with free parameters among the VEVs, are avoided. In contrast, in (almost) all models of this type these are generically present. The existence of terms with couplings of positive mass dimension also ensures the spontaneous breaking of the flavor symmetry D_4 and so destabilizes the trivial vacuum in which all flavon VEVs vanish.

We have carefully studied NLO corrections arising from higher-dimensional operators involving several flavons in the Yukawa sector as well as in the flavon superpotential. The latter induce, in general, shifts in the flavon VEVs. In the particular case we discussed here, these operators do not disturb, at NLO level, the LO vacuum structure of the flavons coupling dominantly to the neutrino sector. We eventually find that all corrections in the neutrino sector are of relative order ϵ^2. In contrast to this, the results in the charged lepton sector get corrected at a relative order ϵ. For this reason, the deviations from $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ of order ϵ stem from the charged lepton sector only. This is a further feature which distinguishes the present model from the one of [4], in which deviations from $\mu - \tau$ symmetric lepton mixing are governed by corrections from the neutrino sector and which at NLO lead to $\theta_{23} - \pi/4$ being much smaller than θ_{13}. In the present model NLO corrections are furthermore relevant to generate the electron mass of order $\epsilon^3 v_d$. As frequently happens in such models, the mass of the tau lepton is generated through a nonrenormalizable operator so that small and moderate values of $\tan\beta$ are preferred.

Several models leading to $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ have been considered before in the literature. The model most closely related to the one illustrated is discussed in [4]. However, as mentioned, this model needs some fine-tuning to get the correct charged lepton mass hierarchy. The model in [6], on which the one in [4] is actually based, is a non-SUSY model and thus incorporates no solution to the hierarchy problem. Nevertheless, in this model it has been shown [6,26] that $m_\mu \ll m_\tau$ can arise from a softly broken (additional) symmetry, whereas m_e being small can be attributed to a small VEV of one of the Higgs doublets present in the model. A model constructed in the same spirit as [6] can be found in [7], having as a flavor group $S_3 \times Z_2$ (recall that S_3 is isomorphic to D_3). Its results are very similar to those of the model in [6], apart from fewer constraints on the neutrino mass spectrum. The D_4 model presented in [10] is a variant of the model in [6] in which the flavor symmetry is only broken by flavons, and as a consequence, the model only contains one Higgs doublet. Thus, strong bounds coming from flavor changing neutral current processes and lepton flavor violation are avoided. However, due to the fact that the model is, like the one in [6], non-SUSY, the desired alignment of the flavon VEVs can only be achieved by fine-tuning one quartic coupling in the flavon potential to be zero. In order to produce correctly the charged lepton mass hierarchy, a tuning is necessary, similar to that present in the original model [6]. A SUSY version of [10] is constructed in [11]. Yet, this model cannot be regarded as satisfactory, because any discussion on the vacuum alignment, which is crucial in order to achieve $\mu - \tau$ symmetric lepton mixings, is missing. A neat example of a minimalistic SUSY model leading to θ_{13} being zero and $\theta_{23} = \pi/4$ is given in [13]. The flavor group is $S_3 \times Z_3$.[6] The structure of the model is very similar to ours, and in the neutrino sector a Z_2 subgroup of the same type as given here is preserved by appropriate flavon VEVs. However, in the charged lepton sector an alignment is employed which completely breaks S_3, but efficiently generates the charged fermion mass hierarchy. The alignment is studied in detail in [13], but, as usual, two parameters among the flavon VEVs remain undetermined, thus giving rise to flat directions in the flavon (super)potential.

In summary, we have constructed a SUSY D_4 model, which predicts $\mu - \tau$ symmetric lepton mixing through breaking D_4 to distinct Z_2 subgroups in the charged lepton and neutrino sectors, respectively, and at the same time naturally accommodates the mass hierarchy $m_e \ll m_\mu \ll m_\tau$. Furthermore, all flavon VEVs are determined through the two mass parameters of the superpotential. Thus, undetermined free parameters among the flavon VEVs are avoided, which are a problem, often encountered in models of such type. On the basis of this model it might be very interesting to consider an extension to the quark sector. As has been shown in [5], the Cabibbo angle might also arise from a nontrivial breaking of a dihedral group to distinct Z_2 subgroups in up and down quark sectors, and so its size might have a similar origin as the prediction of $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ in the lepton sector.

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[6]In [13] an extension of the model to the quark sector is also discussed, which entails adding another cyclic symmetry Z_3' to reproduce the quark mass spectra. However, it requires some enhancement in order to generate a large enough Cabibbo angle.

[1] T. Schwetz, M. A. Tortola, and J. W. F. Valle, New J. Phys. **10**, 113011 (2008).

[2] T. Fukuyama and H. Nishiura, arXiv:hep-ph/9702253; R. N. Mohapatra and S. Nussinov, Phys. Rev. D **60**, 013002 (1999); E. Ma and M. Raidal, Phys. Rev. Lett. **87**, 011802 (2001); **87**, 159901(E) (2001); C. S. Lam, Phys. Lett. B **507**, 214 (2001); P. F. Harrison and W. G. Scott, Phys. Lett. B **547**, 219 (2002); T. Kitabayashi and M. Yasue, Phys. Rev. D **67**, 015006 (2003).

[3] A. Blum, C. Hagedorn, and M. Lindner, Phys. Rev. D **77**, 076004 (2008).

[4] A. Adulpravitchai, A. Blum, and C. Hagedorn, J. High Energy Phys. 03 (2009) 046.

[5] A. Blum, C. Hagedorn, and A. Hohenegger, J. High Energy Phys. 03 (2008) 070; A. Blum and C. Hagedorn, Nucl. Phys. **B821**, 327 (2009).

[6] W. Grimus and L. Lavoura, Phys. Lett. B **572**, 189 (2003).

[7] W. Grimus and L. Lavoura, J. High Energy Phys. 08 (2005) 013.

[8] C. S. Lam, Phys. Rev. D **74**, 113004 (2006); Phys. Lett. B **656**, 193 (2007).

[9] A. Adulpravitchai, A. Blum, and W. Rodejohann, New J. Phys. **11**, 063026 (2009).

[10] H. Ishimori, T. Kobayashi, H. Ohki, Y. Omura, R. Takahashi, and M. Tanimoto, Phys. Lett. B **662**, 178 (2008).

[11] H. Ishimori, T. Kobayashi, H. Ohki, Y. Omura, R. Takahashi, and M. Tanimoto, Phys. Rev. D **77**, 115005 (2008).

[12] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).

[13] F. Feruglio and Y. Lin, Nucl. Phys. **B800**, 77 (2008).

[14] Y. Lin, Nucl. Phys. **B813**, 91 (2009).

[15] G. Altarelli and D. Meloni, J. Phys. G **36**, 085005 (2009).

[16] W. Grimus, A. S. Joshipura, S. Kaneko, L. Lavoura, and M. Tanimoto, J. High Energy Phys. 07 (2004) 078.

[17] R. N. Mohapatra and W. Rodejohann, Phys. Lett. B **644**, 59 (2007); A. Blum, R. N. Mohapatra, and W. Rodejohann, Phys. Rev. D **76**, 053003 (2007).

[18] T. Kobayashi, S. Raby, and R. J. Zhang, Nucl. Phys. **B704**, 3 (2005); T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby, and M. Ratz, Nucl. Phys. **B768**, 135 (2007); P. Ko, T. Kobayashi, J. h. Park, and S. Raby, Phys. Rev. D **76**, 035005 (2007); **76**, 059901(E) (2007).

[19] G. Seidl, arXiv:hep-ph/0301044.

[20] G. Altarelli and F. Feruglio, Nucl. Phys. **B741**, 215 (2006).

[21] J. S. Lomont, *Applications of Finite Groups* (Academic Press, New York, 1959), p. 346; P. E. Desmier and R. T. Sharp, J. Math. Phys. (N.Y.) **20**, 74 (1979); J. Patera, R. T. Sharp, and P. Winternitz, J. Math. Phys. (N.Y.) **19**, 2362 (1978); A. D. Thomas and G. V. Wood, *Group Tables* (Shiva Publishing Limited, Orpington, 1980).

[22] P. H. Frampton and T. W. Kephart, Int. J. Mod. Phys. A **10**, 4689 (1995).

[23] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. **B147**, 277 (1979).

[24] A. Merle and W. Rodejohann, Phys. Rev. D **73**, 073012 (2006).

[25] K. A. Intriligator and N. Seiberg, Classical Quantum Gravity **24**, S741 (2007).

[26] W. Grimus and L. Lavoura, J. Phys. G **30**, 73 (2004).