Higgs boson self-coupling from two-loop analysis

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The scale invariant of the effective potential of the standard model at two loop is used as a boundary condition under the assumption that the two-loop effective potential approximates the full effective potential. This condition leads with the help of the renormalization-group functions of the model at two loop to an algebraic equation of the scalar self-coupling with coefficients that depend on the gauge and the top quark couplings. It admits only two real positive solutions. One of them, in the absence of the gauge and top quark couplings, corresponds to the nonperturbative ultraviolet fixed point of the scalar renormalization-group function and the other corresponds to the perturbative infrared fixed point. The dependence of the scalar coupling on the top quark and the strong couplings at two-loop radiative corrections is analyzed.

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I. INTRODUCTION

The standard model requires, through Higgs mechanism [1], the existence of a neutral scalar Higgs boson that survives after spontaneous breakdown of the electroweak symmetry. In this mechanism the weak gauge bosons and the charged fermions as well as the scalar Higgs itself acquire their masses from the asymmetry of the vacuum [2]. At tree level the scalar Higgs boson self-coupling λ, and thus the Higgs boson mass (m_H), is a free parameter of the model and its determination self-consistently poses a challenging problem. Many efforts, theoretically and experimentally, have been devoted to it.

One of the primary goals of the LHC is to search for the standard model Higgs boson. Recent high energy experiments excluded the region $m_H \leq 114.4$ GeV [3] and electroweak precession data excluded the region $160\text{ GeV} < m_H < 170\text{ GeV}$ [4]. The region $114.4\text{ GeV} < m_H < 160\text{ GeV}$ is a favored one, while the region $m_H > 180$ GeV, though disfavored, is not ruled out. The current fits lead to $130\text{ GeV} < m_H < 260\text{ GeV}$ [5]. Recently the global fit obtained by the Gfitter Group showed that the 95% C.L. allowed range for the complete fit (including the direct searches) is [114, 153] GeV, and above this range only the values between 180 and 224 GeV are not yet excluded at 3 standard deviations or more [6]. Several papers have presented upper bounds based on avoiding triviality of pure φ^4 theory [7], lower bounds on the bases of vacuum stability [8], and constraints based on some theoretical assumptions. The requirement, for example, that the coefficient of the quadratic divergent term of the square of the scalar Higgs boson mass at one loop is equal to zero leads to the well-known Veltman condition [9], the assumption that the one-loop effective potential effectively represents the full effective potential leads to an algebraic equation that fixes, in the absence of fermion contributions, the scalar self-coupling [10], and from the assumption that the ratio of the scalar coupling to the square of the top quark coupling is scale invariant allows one to fix the Higgs boson mass at one and two loop [11].

In the standard model for the potential ($m^2 < 0$, $\lambda > 0$)

$$V^{(0)} = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4, \tag{1}$$

the two-loop renormalization-group function β_λ for pure φ^4 theory is given by [12]

$$\beta_\lambda = \beta_\lambda^{(1)} + \beta_\lambda^{(2)} = \frac{1}{16\pi^2}(4\lambda^2) + \frac{1}{(16\pi^2)^2}\left(-\frac{26}{3}\lambda^3\right). \tag{2}$$

This equation admits two constant solutions: the perturbative infrared fixed point $\lambda = 0$, and the nonperturbative ultraviolet fixed point $\lambda_{\rm UV} = 72.9$. It is expected, similar to the one-loop case [13], that the perturbative fixed point is removed in the presence of interactions with gauges and fermions while the nonperturbative one persists [14,15]. We show that this is the case, under the assumption that the two-loop effective potential approximates the full effective potential. This assumption leads to an algebraic equation that fixes the scalar self-coupling in the perturbative region. The ultraviolet fixed point, where perturbation expansion breaks down [14,15], is shown to occur in the region where all other couplings, but the scalar coupling, are neglected in comparison. The full effective potential $V(\varphi(t))$, which is constructed from the one-particle-irreducible Green's functions at zero external momenta [16], satisfies, in the mass-independent scheme of Weinberg 't Hooft [17], the renormalization-group equation (RGE) $dV(\varphi(t))/dt = DV(\varphi) = 0$, where D is a first order partial differential operator that depends on the renormalization-group functions of the model and t is an arbitrary running parameter related to the renormalization scale $\mu(t = \ln(\mu/\mu(0))$. This equation is just a statement that V is scale invariant

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$V(\varphi(t)) = V(\varphi(0))$ [16]. When perturbative expansion to a given order is employed to V one may take this scale invariance as a boundary condition for the solution of the first order partial differential equation. In the present work we use this boundary condition at two loop to obtain a relation that fixes the scalar coupling in terms of the gauge and top quark couplings. This relation expresses the validity of the scale invariant of the two-loop effective potential. This paper is organized as follows: in the following section we calculate the effective potential at two-loop approximation using the renormalization-group equation in the mass-independent subtraction scheme. In Sec. III we impose the scale invariance on the two-loop effective potential that allows one to obtain a condition that expresses the validity of the scale invariant at two loop, which depends on the renormalization-group functions at two loop. This condition leads to an algebraic equation of degree 5 in λ, which has only one perturbative real positive solution, and one nonperturbative real positive solution. In Sec. IV the perturbative solution is used to estimate the mass of the Higgs boson at one- and two-loop approximation, the dependence of the Higgs boson mass on the strong interaction coupling α_S, and the top quark mass m_t is discussed, and in this section we apply the matching condition to estimate the physical mass of the Higgs boson. Finally in Sec. V we give our conclusion.

II. CALCULATION OF THE EFFECTIVE POTENTIAL AT TWO LOOP

In the mass-independent subtraction scheme of Weinberg-'t Hooft, the scale invariance of the full effective potential of a renormalizable quantum field theory leads, under infinitesimal transformation, to the RGE of the effective potential [17]:

$$\left(\mu\frac{\partial}{\partial\mu} + \delta_p\beta_p\frac{\partial}{\partial\lambda_p} + \gamma\varphi_c\frac{\partial}{\partial\varphi_c}\right)V_{\mathrm{eff}}(\varphi_c) = 0, \qquad (3)$$

where μ is a renormalization mass parameter and γ is the anomalous dimension, $\lambda_p = (m^2, \lambda, g, g\prime, g_3, g_t)$, $g, g\prime, g_3$ are the SU(2), the U(1), and the SU(3) gauge couplings, respectively, g_t is the top quark coupling and δ_p equals m^2 for mass coupling and 1 otherwise, and β_p is the renormalization-group function:

$$\beta_p = \frac{d\lambda_p}{dt}, \quad \text{with} \quad t = \ln\frac{\mu}{\mu(0)}.$$

In the loop expansion, the effective potential is given by [14]

$$V_{\mathrm{eff}} = V^{(0)} + \Sigma_{n=1}\hbar^{(n)}V^{(n)}. \qquad (4)$$

Following the procedure of Ref. [18], we can write the following recursion equation for the effective potential up to order n:

$$\mu\frac{\partial}{\partial\mu}V^{(n)} + D_nV^{(0)} + D_{n-1}V^{(1)} + \cdots + D_1V^{(n-1)} = 0, \qquad (5)$$

where the differential operator D_n is given by

$$D_n = \delta_p\beta_p^{(n)}\frac{\partial}{\partial\lambda_p} + \gamma^{(n)}\varphi_c\frac{\partial}{\partial\varphi_c}, \qquad (6)$$

$n = 1, 2, 3, \ldots$ and $V^{(n)}$ is the n-loop contribution to V, and $\beta_p^{(n)}$ is the n-loop contribution to the renormalization-group function:

$$\beta_p = \Sigma_{n=1}\hbar^{(n)}\beta_p^{(n)}. \qquad (7)$$

To find the one-loop effective potential we use Eqs. (5)–(7) and the zero-loop (the tree level) potential of Eq. (1), which give

$$V^{(1)} = \frac{1}{48}a_1\varphi^4\ln\frac{\varphi^2}{\mu^2}, \quad \text{where } a_1 = \beta_\lambda^{(1)} + 4\gamma^{(1)}\lambda, \qquad (8)$$

and for the two loop we substitute Eqs. (1) and (8) into Eq. (5) to find

$$V^{(2)} = \frac{1}{48}a\varphi^4\ln\frac{\varphi^2}{\mu^2} + \frac{1}{192}b\varphi^4\left(\ln\frac{\varphi^2}{\mu^2}\right)^2,$$

where

$$a = a_1\gamma^{(1)} + a_2, \quad \text{with} \quad a_2 = \beta_\lambda^{(2)} + 4\gamma^{(2)}\lambda \qquad (9)$$

and

$$b = \beta_\lambda^{(1)}\frac{\partial\beta_\lambda^{(1)}}{\partial\lambda} + 8\beta_\lambda^{(1)}\gamma^{(1)} + 16\lambda(\gamma^{(1)})^2 + G, \qquad (10)$$

with

$$G = \beta_{g\prime}^{(1)}\frac{\partial\beta_\lambda^{(1)}}{\partial g\prime} + 4\lambda\beta_{g\prime}^{(1)}\frac{\partial\gamma^{(1)}}{\partial g\prime} + \beta_g^{(1)}\frac{\partial\beta_\lambda^{(1)}}{\partial g} + 4\lambda\beta_g^{(1)}\frac{\partial\gamma^{(1)}}{\partial g} + \beta_{g_t}^{(1)}\frac{\partial\beta_\lambda^{(1)}}{\partial g_t} + 4\lambda\beta_{g_t}^{(1)}\frac{\partial\gamma^{(1)}}{\partial g_t}. \qquad (11)$$

Therefore, the full effective potential approximated to two-loop effective potential becomes

$$V = V^{(0)} + V^{(1)} + V^{(2)} = A_0 + A_1\varphi^4\ln\frac{\varphi^2}{\mu^2} + A_2\varphi^4\left(\ln\frac{\varphi^2}{\mu^2}\right)^2, \qquad (12)$$

where

$$A_0 = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4, \quad A_1 = \frac{1}{48}(a_1(1+\gamma^{(1)}) + a_2), \qquad (13)$$

$$A_2 = \frac{1}{192}\left(\frac{8}{k}\lambda\beta_\lambda^{(1)} + 4a_1\gamma^{(1)} + G\right). \qquad (14)$$

The renormalization-group functions used in this work are [12]

$$\gamma^{(1)} = \frac{1}{4k}(3g'^2 + 9g^2 - 12g_t^2), \qquad \beta_\lambda^{(1)} = \frac{1}{k}\left(4\lambda^2 - \lambda(3g'^2 + 9g^2 - 12g_t^2) + \frac{9}{4}g'^4 + \frac{9}{2}g'^2g^2 + \frac{27}{4}g^4 - 36g_t^4\right),$$

$$\beta_{g_t}^{(1)} = \frac{1}{k}\left(\frac{9}{2}g_t^3 - 8g_3^2g_t - \frac{9}{4}g^2g_t - \frac{17}{12}g'^2g_t\right), \qquad \beta_{g'}^{(1)} = \frac{41}{6k}g'^3, \qquad \beta_g^{(1)} = \frac{-19}{6k}g^3, \qquad \beta_{g_3}^{(1)} = \frac{-7}{k}g_3^3,$$

$$\gamma^{(2)} = \frac{-1}{k^2}\left(\frac{\lambda^2}{6} + \frac{431}{96}g'^4 + \frac{9}{16}g'^2g^2 - \frac{271}{32}g^4 + \frac{85}{24}g'^2g_t^2 + \frac{45}{8}g^2g_t^2 + 20g_3^2g_t^2 - \frac{27}{4}g_t^4\right),$$

$$\beta_\lambda^{(2)} = \frac{1}{k^2}\left(\frac{-26}{3}\lambda^3 + \lambda^2(6g'^2 + 18g^2 - 24g_t^2) + \lambda\left(\frac{629}{24}g'^4 + \frac{39}{4}g'^2g^2 + \frac{85}{6}g'^2g_t^2 - \frac{73}{8}g^4 + \frac{45}{2}g^2g_t^2 + 80g_3^2g_t^2 - 3g_t^4\right)\right.$$
$$\left. + 180g_t^6 - 192g_3^2g_t^4 - 16g'^2g^4 - \frac{27}{2}g^4g_t^2 + 63g'^2g^2g_t^2 - \frac{57}{2}g'^4g_t^2, - \frac{379}{8}g'^6 - \frac{559}{8}g'^4g^2 - \frac{289}{8}g'^2g^4 + \frac{915}{8}g^6\right),$$

$$\beta_{g_t}^{(2)} = \frac{1}{k^2}g_t\left(-12g_t^4 + g_t^2\left(\frac{131}{16}g'^2 + \frac{225}{16}g^2 + 36g_3^2 - 2\lambda\right) + \frac{1187}{216}g'^4 - \frac{3}{4}g'^2g^2 + \frac{19}{9}g'^2g_3^2, - \frac{23}{4}g^4\right.$$
$$\left. + 9g^2g_3^2 - 108g_3^4 + \frac{1}{6}\lambda^2\right),$$

$$\beta_g^{(2)} = \frac{1}{k^2}g^3\left(\frac{3}{2}g'^2 + \frac{35}{6}g^2 + 12g_3^2 - \frac{3}{2}g_t^2\right), \qquad \beta_{g'}^{(2)} = \frac{1}{k^2}g'^3\left(\frac{199}{18}g'^2 + \frac{9}{2}g^2 + \frac{44}{3}g_3^2 - \frac{17}{6}g_t^2\right),$$

$$\beta_{g_3}^{(2)} = \frac{1}{k^2}g_3^3\left(\frac{11}{6}g'^2 + \frac{9}{2}g^2 - 26g_3^2 - 2g_t^2\right), \tag{15}$$

and

$$k = 16\pi^2.$$

III. A CONDITION FROM THE SCALE INVARIANCE AT TWO LOOP

As stated in the Introduction, when a perturbative expansion of the effective potential to a given order is employed, the boundary condition

$$V(\varphi(t), \lambda_p(t), \mu(t)) = V(\varphi, \lambda_p, \mu), \tag{16}$$

where $\varphi(t) = z\varphi$, $\mu(t) = e^t\mu$ leads to a restriction on the coupling constants that expresses the validity of this scale invariant to this order. Employing these transformations in the two-loop effective potential (12), we obtain the following conditions:

$$A_1(t)z^4 + 4A_2(t)z^4 \ln\left(\frac{z}{e^t}\right) = A_1(0), \tag{17}$$

and

$$A_2(t)z^4 = A_2(0). \tag{18}$$

Differentiating Eq. (17), using Eq. (18) and the definition of γ:

$$\gamma = \frac{1}{z}\frac{dz}{dt}, \tag{19}$$

we obtain the following relation:

$$\frac{dA_1}{dt} + 4A_1\gamma + 4A_2(\gamma - 1) = 0, \quad \text{at } t = 0. \tag{20}$$

Substituting A_1 and A_2 from Eqs. (13) and (14) and using Eqs. (9) and (10) we finally obtain

$$(-1 + \gamma^{(1)} + \gamma^{(2)})\left(\frac{1}{2\pi^2}\beta_\lambda^{(1)} + 4\gamma^{(1)}(\beta_\lambda^{(1)} + 4\gamma^{(1)}\lambda) + G\right)$$
$$+ 4(\gamma^{(1)} + \gamma^{(2)})((1 + \gamma^{(1)})(\beta_\lambda^{(1)} + 4\gamma^{(1)}\lambda)$$
$$+ (\beta_\lambda^{(2)} + 4\gamma^{(2)}\lambda)) + \frac{d}{dt}((1 + \gamma^{(1)})(\beta_\lambda^{(1)} + 4\gamma^{(1)}\lambda)$$
$$+ (\beta_\lambda^{(2)} + 4\gamma^{(2)}\lambda)) = 0, \tag{21}$$

where G is as given in Eq. (11). Substituting Eq. (11) and the one- and two-loop renormalization-group functions from Eq. (15) into Eq. (21), and after simplifying, we obtain an equation of degree 5 in λ:

$$\lambda^5 + C_1\lambda^4 + C_2\lambda^3 + C_3\lambda^2 + C_4\lambda + C_5 = 0, \tag{22}$$

where C_n ($n = 1, 2, \ldots, 5$) are functions in the couplings g, g', g_3, g_t and the explicit expressions of them are given in the Appendix. This equation allows one to obtain the scalar Higgs coupling in terms of the gauge couplings and the top quark coupling at the renormalization scale $\mu = M_Z$, where M_Z is the mass of the weak neutral gauge boson. In the following section we use the matching condition to relate the scalar coupling calculated from Eq. (22) to the physical Higgs boson mass (“the pole mass”) which is the measurable quantity.

IV. THE HIGGS BOSON MASS

In the absence of the couplings g, g', g_3 and g_t, Eq. (22) has only two solutions, one of them corresponds to the ultraviolet fixed point solution $\lambda_{UV} = 72.9$, in which the perturbative expansion breaks down, and the other is the infrared fixed point solution $\lambda = 0$. This result can be taken as a check of the validity of the scale invariant of the effective potential at two-loop corrections. In the presence of these couplings the nonperturbative solution continues and we are left with only one real positive perturbative solution which we use to analyze the dependence of the

TABLE I. Values of the scalar coupling λ obtained from Eq. (22), for different values of the top quark mass m_t (GeV) and the strong coupling α_S; column 4 gives the Higgs boson mass at tree level [Eq. (26)], column 5 at 1 loop [Eq. (27)], column 6 at 2 loop [Eq. (28)], and column 7 the physical Higgs boson mass from Eq. (29).

α_S	m_t (GeV)	λ	$m_{H\text{-tree}}$ (GeV)	$m_{H\text{-1 loop}}$ (GeV)	$m_{H\text{-2 loop}}$ (GeV)	$M_{H\text{-1 loop}}$ (GeV)
0.1073	160	2.451 7	224.56	223.93	226.56	227.16
	162	2.457 64	224.83	223.60	226.3	226.71
	164	2.463 05	225.08	223.21	225.98	226.20
	166	2.467 93	225.30	222.77	225.62	225.64
	168	2.472 24	225.50	222.27	225.2	225.02
	170	2.475 97	225.67	221.70	224.72	224.33
	172	2.479 1	225.81	221.08	224.18	223.59
	174	2.481 59	225.92	220.38	223.58	222.77
	176	2.483 43	226.00	219.62	222.92	221.90
0.1127	160	2.545 99	228.83	229.14	231.88	232.71
	162	2.553 08	229.15	228.88	231.7	232.32
	164	2.559 66	229.45	228.56	231.46	231.87
	166	2.565 71	229.72	228.20	231.18	231.38
	168	2.571 21	229.96	227.77	230.84	230.82
	170	2.576 15	230.19	227.29	230.45	230.21
	172	2.580 52	230.38	226.75	230.00	229.54
	174	2.584 28	230.55	226.15	229.49	228.81
	176	2.587 41	230.69	225.48	228.92	228.01
0.1161	160	2.604 18	231.43	232.31	235.13	236.10
	162	2.611 98	231.78	232.09	234.99	235.75
	164	2.619 27	232.10	231.82	234.81	235.34
	166	2.626 04	232.40	231.50	234.57	234.88
	168	2.632 27	232.68	231.13	234.28	234.38
	170	2.637 95	232.93	230.69	233.93	233.80
	172	2.643 06	233.16	230.20	233.53	233.17
	174	2.647 58	233.35	229.65	233.08	232.49
	176	2.651 49	233.53	229.03	232.57	231.74
0.118	160	2.636 34	232.86	234.05	236.92	237.97
	162	2.644 53	233.22	233.85	236.80	237.63
	164	2.652 22	233.56	233.61	236.64	237.25
	166	2.659 38	233.87	233.31	236.43	236.81
	168	2.666 01	234.17	232.96	236.16	236.32
	170	2.672 09	234.43	232.56	235.85	235.78
	172	2.677 61	234.67	232.09	235.48	235.17
	174	2.682 54	234.89	231.57	235.05	234.51
	176	2.686 88	235.08	230.98	234.56	233.79
0.1202	160	2.673 2	234.48	236.03	238.95	240.10
	162	2.681 84	234.86	235.86	238.87	239.79
	164	2.689 97	235.22	235.65	238.73	239.43
	166	2.697 59	235.55	235.38	238.55	239.02
	168	2.704 67	235.86	235.06	238.31	238.56
	170	2.711 22	236.14	234.68	238.03	238.03
	172	2.717 2	236.40	234.24	237.69	237.45
	174	2.722 61	236.64	233.75	237.29	236.82
	176	2.727 43	236.85	233.20	236.84	236.13

scalar coupling for a range of values of g_3 and g_t. Our input parameters are obtained from the relations:

$$g = \frac{2M_W}{v}, \qquad g\prime = \sqrt{\frac{4M_Z^2}{v^2} - g^2},$$

$$g_t = \sqrt{2}\frac{m_t}{v}, \quad \text{with } v = 246.2 \text{ GeV}.$$

$M_Z = 91.19$ GeV, $M_W = 80$ GeV, $g\prime(M_Z) = 0.355\,54$, and $g(M_Z) = 0.649\,88$. We take the mass of the top quark in the range $160 \text{ GeV} \leq m_t \leq 176$ GeV, and the strong coupling α_S in the range $0.1073 \leq \alpha_S \leq 0.1202$ [19]. In Table I, we present our results for the scalar Higgs coupling in the above ranges of m_t and α_S. The Higgs boson mass, at zero external momentum, is defined by [16]

$$m_H^2 = \frac{d^2V}{d\varphi^2}\bigg|_{\varphi=v}, \tag{23}$$

where v is the vacuum expectation value which has the expansion up to two loop:

$$v = v_0 + \hbar v^{(1)} + \hbar^2 v^{(2)}, \tag{24}$$

and satisfies

$$\frac{dV}{d\varphi}\bigg|_{\varphi=v} = 0, \tag{25}$$

The Higgs boson mass at tree level is

$$m_H^2 = \tfrac{1}{3}\lambda v_0^2, \quad \text{where } v_0 = 246.2 \text{ GeV}, \tag{26}$$

at one loop, using Eq. (8), is

$$m_{H\text{-}1\text{ loop}}^2 = \left(\frac{1}{3}\lambda + \frac{1}{4}a_1 + \frac{1}{3}a_1 \ln\frac{v_1}{\mu}\right)v_1^2, \tag{27}$$

where

$$v_1 = v_0 + \hbar v^{(1)},$$

and at two loop, using Eq. (12), is

$$m_{H\text{-}2\text{ loop}}^2 = \left(\frac{1}{3}\lambda + 12A_1 + 8A_2 + 16(A_1 + 3A_2)\ln\frac{v_2}{\mu} + 32A_2\left(\ln\frac{v_2}{\mu}\right)^2\right)v_2^2, \tag{28}$$

where

$$v_2 = v_1 + \hbar^2 v^{(2)}.$$

A comparison between the tree-level potential and the two-loop effective potential for $\alpha_S = 0.1161$ and $m_t = 171$ GeV is shown in Fig. 1, while for $\alpha_S = 0.118$ and $m_t = 171$ GeV is shown in Fig. 2. Here it should be noted that all other figures with different values of α_S and m_t show similar trends. As can be inferred from these figures the two-loop contribution to the effective potential almost cancels the one-loop contribution in the region in between the positions of the two minima of the potential and as a result the position of the vacuum is slightly shifted (by less than 1%) [20]. For each of the obtained values of the scalar coupling the positivity of the effective potential at two loop is checked from the condition $V(\varphi) > 0$, for $|\varphi| > \varphi^*$, where φ^* is the nontrivial positive solution of $V(\varphi^*) = 0$. When the perturbative real positive solution of Eq. (22) is substituted in the tree-level Eq. (26), one-loop Eq. (27) and the two-loop Eq. (28) we find the mass of the Higgs boson at M_Z to be

$$224.56 \text{ GeV} \leq m_{H\text{-tree}} \leq 236.85 \text{ GeV},$$

$$219.62 \text{ GeV} \leq m_{H\text{-}1\text{ loop}} \leq 236.03 \text{ GeV},$$

$$222.92 \text{ GeV} \leq m_{H\text{-}2\text{ loop}} \leq 238.95 \text{ GeV}.$$

The dependence of $m_{H\text{-}1\text{ loop}}$ and $m_{H\text{-}2\text{ loop}}$ on the top quark mass and the strong coupling α_S is shown in Fig. 3, which is in qualitative agreement with Ref. [14] (see Fig. 3a there) and Ref. [21] (see Fig. 3 there) in the region considered.

Using the matching conditions, the initial value $m_H(M_Z)$ is related to the physical Higgs mass M_H to one-loop order at the scale M_Z [19,21,22] by

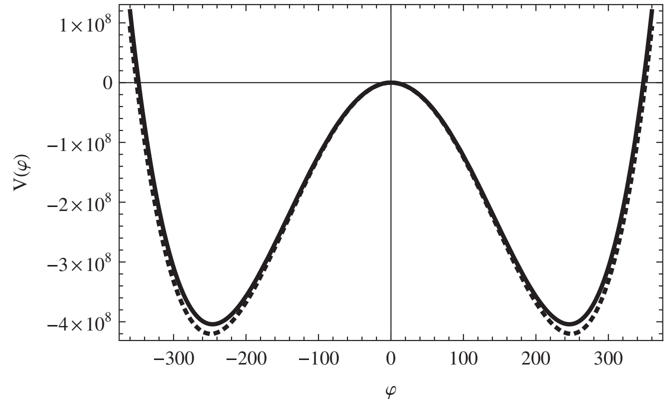

FIG. 1. A comparison between the tree-level (the solid line) and the two-loop (the dashed line) effective potentials for $\alpha_S = 0.1161$ and $m_t = 171$ GeV.

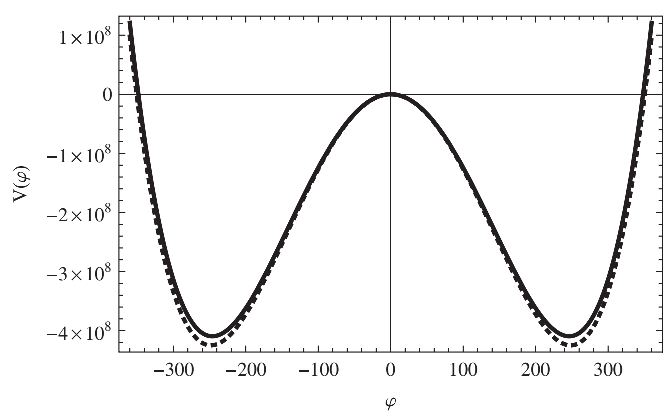

FIG. 2. A comparison between the tree-level (the solid line) and the two-loop (the dashed line) effective potentials for $\alpha_S = 0.118$ and $m_t = 171$ GeV.

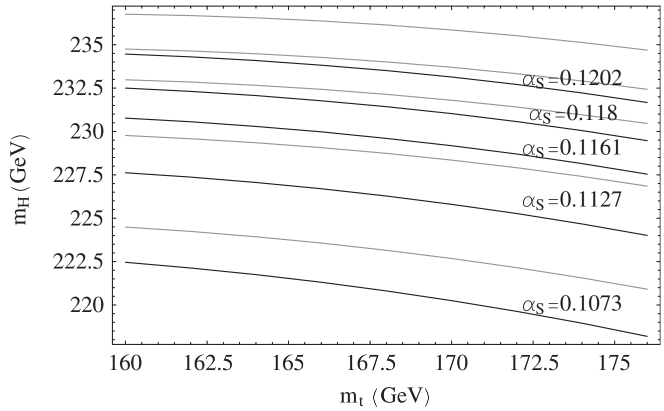

FIG. 3. The Higgs boson mass m_H (GeV) verses the top quark mass m_t (GeV) for different values of the strong coupling α_S. The black lines correspond to the 1-loop case and the gray lines correspond to the 2-loop case.

$$m_H^2 = M_H^2(1 + \delta(M)), \tag{29}$$

where

$$\delta(M) = \frac{1}{16\pi^2}\frac{M_Z^2}{v^2}[\xi f_1(\xi, M) + f_0(\xi, M) + \xi^{-1} f_{-1}(\xi, M)], \tag{30}$$

where $\xi \equiv M_H^2/M_Z^2$ and $f_1(\xi, M)$, $f_0(\xi, M)$, and $f_{-1}(\xi, M)$ are given in the Appendix. Inserting the input parameters, we find that the physical mass of the Higgs boson is 221.9 GeV $\leq M_H \leq$ 240.1 GeV. This result is in the expected range of the recent global fit to all precision electroweak data [5,6]. Now we compare our result with some results of other authors employing different approaches. In Ref. [23] a value of about 218 GeV for the Higgs boson mass (m_H) is obtained from using renormalization-group methods to include all leading-logarithm contributions to the effective potential, and in Ref. [24] a value of $m_H =$ 221 GeV from consideration of scalar-field theory projection of the effective potential. In Ref. [25] using the renormalization-group improved effective potential and results from Ref. [26] they obtained an upper bound of m_H around 196 GeV. In Ref. [27] from their estimate of the top quark mass using the Goldberger-Teller relation and the Veltman condition they found m_H around 317 GeV. The assumption made in Ref. [28] that the scalar coupling equals the top quark coupling led to a value of m_H around 348 GeV. Finally, an estimated value of $m_H =$ 700 GeV has been obtained in Ref. [29] from the assumption that the ratio of the Higgs boson mass to the vacuum expectation value is a cutoff independent.

If the Higgs boson mass is experimentally discovered in the favored region (114–160 GeV), then the calculation of the effective potential shows that the main contribution comes from one loop; in this case, due to the mass of the top quark, the scalar coupling becomes negative, which is a possible indication of physics beyond the standard model.

It is desirable at this point to discuss the stability of our perturbative solution beyond the two-loop order. Unfortunately not all the renormalization-group functions of the standard model are available at present. However under the assumption that the main effect to three-loop order comes from pure φ^4 interaction [30], our finding is that the Higgs boson mass is shifted by an amount of order 1 GeV for the range of values of the top quark mass and the strong gauge coupling considered in this work.

V. CONCLUSION

In this paper we have used the scale invariant of the effective potential at two loop as a boundary condition. This condition which expresses the validity of the scale invariant of effective potential at two loop results in an algebraic equation of degree 5. In the physical ranges of the top quark mass and the strong gauge coupling it admits only two real positive solutions. In the absence of the gauge and the top quark couplings one of these two solutions corresponds to the ultraviolet fixed point of the scalar renormalization-group function, where perturbation theory breaks down, and the other one lies in the perturbative regime. Our findings support, in general, several other works. Of these, the two-loop contribution almost cancels the contribution of the one-loop contribution in the region between the minima of the potential, and thus the vacuum expectation value of the scalar Higgs boson is slightly shifted by renormalization at one-loop and two-loop quantum corrections. The other finding is that the variation of the Higgs boson mass as a function of the top quark mass initially decreases in the region considered. Finally we have made a comparison of our result with some results of other authors employing different approaches.

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APPENDIX

$$C_1 = \frac{1}{420\,864}(-476\,928 g'^2 - 143\,078\,4 g^2 + 189\,388\,8 g_t^2 - 317\,030\,4\pi^2),$$

$$C_2 = \frac{1}{420\,864}(-124\,051\,2 g'^4 + 148\,608 g'^2 g^2 + 106\,848\,0 g^4 - 164\,812\,8 g'^2 g_t^2 - 402\,278\,4 g^2 g_t^2 - 368\,640\,0 g_3^2 g_t^2 + 196\,300\,8 g_t^4 + 420\,249\,6 g'^2\pi^2 + 126\,074\,88 g^2\pi^2 - 168\,099\,84 g_t^2\pi^2),$$

$$C_3 = \frac{1}{420\,864}(299\,721\,6g'^6 + 510\,249\,6g'^4g^2 + 304\,444\,8g'^2g^4 - 405\,734\,4g^6 + 304\,128g'^4g_3^2$$
$$+ 746\,496g^4g_3^2 - 727\,968g'^4g_t^2 + 116\,640g^4g_t^2 + 654\,336g'^2g_3^2g_t^2 + 174\,182g^2g_3^2g_t^2$$
$$+ 895\,795\,2g_3^4g_t^2 - 224\,64g'^2g_t^4 - 606\,528g^2g_t^4 + 265\,420\,8g_3^2g_t^4 - 987\,033\,6g_t^6 - 451\,584g'^4\pi^2$$
$$- 125\,521\,92g^4\pi^2 + 921\,600\,0g'^2g_t^2\pi^2 + 258\,785\,28g^2g_t^2\pi^2 + 707\,788\,8g_3^2g_t^2\pi^2 + 928\,972\,8g_t^4\pi^2$$
$$- 155\,52g'^2g^2(215g_t^2 + 576\pi^2)),$$

$$C_4 = \frac{1}{420\,864}(-373\,968g'^8 - 393\,580\,8g'^6g^2 - 432\,799\,2g'^4g^4 + 164\,808\,0g'^2g^6 + 996\,170\,4g^8 + 836\,352g'^6g_3^2$$
$$+ 380\,160g'^4g^2g_3^2 + 311\,040g'^2g^4g_3^2 + 205\,286\,4g^6g_3^2 + 306\,676\,8g'^6g_t^2 + 486\,043\,2g'^4g^2g_t^2$$
$$+ 591\,624\,0g'^2g^4g_t^2 - 111\,0801\,6g^6g_t^2 - 622\,080g'^4g_3^2g_t^2 - 186\,624\,0g^4g_3^2g_t^2 + 626\,928\,0g'^4g_t^4$$
$$- 127\,267\,2g^4g_t^4 - 102\,988\,80g'^4g^2\pi^2 + 121\,236\,48g'^2g^4\pi^2 + 476\,928\,0g^6\pi^2 - 597\,196\,8g'^4g_t^2\pi^2$$
$$- 179\,159\,04g^4g_t^2\pi^2 - 276\,480\,00g'^2g_t^4\pi^2 - 776\,355\,84g^2g_t^4\pi^2 - 212\,336\,64g_3^2g_t^4\pi^2 + 107\,495\,424g_t^6\pi^2$$
$$- 155\,52g'^2g^2g_t^2(80g_3^2 + 351g_t^2 + 768\pi^2)),$$

$$C_5 = \frac{1}{420\,864}(-521\,739\,9g'^{10} - 788\,114\,1g'^8g^2 - 846\,237\,0g'^6g^4 - 101\,520\,36g'^4g^6 - 973\,827g'^2g^8 + 190\,308\,69g^{10}$$
$$- 720\,403\,2g'^8g_3^2 - 708\,364\,8g'^6g^2g_3^2 - 472\,896\,0g'^4g^4g_3^2 - 299\,635\,2g'^2g^6g_3^2 + 142\,300\,80g^8g_3^2 + 270\,348g'^8g_t^2$$
$$+ 576\,835\,2g'^6g^2g_t^2 + 242\,942\,4g'^4g^4g_t^2 + 260\,474\,4g'^2g^6g_t^2 - 105\,899\,40g^8g_t^2 + 859\,968g'^6g_3^2g_t^2$$
$$+ 796\,780\,8g'^4g^2g_3^2g_t^2 + 625\,363\,2g'^2g^4g_3^2g_t^2 - 185\,950\,08g^6g_3^2g_t^2 + 106\,375\,68g'^4g_3^4g_t^2 + 503\,884\,8g^4g_3^4g_t^2$$
$$- 337\,795\,6g'^6g_t^4 - 190\,674\,0g'^4g^2g_t^4 - 669\,060g'^2g^4g_t^4 + 734\,151\,6g^6g_t^4 - 117\,934\,08g'^4g_3^2g_t^4$$
$$+ 108\,241\,92g^4g_3^2g_t^4 + 792\,576\,0g'^2g_3^4g_t^4 - 149\,299\,20g^2g_3^4g_t^4 + 160\,579\,584g_3^6g_t^4 + 331\,704\,0g'^4g_t^6$$
$$+ 112\,946\,40g^4g_t^6 - 106\,306\,56g'^2g_3^2g_t^6 + 559\,872\,0g^2g_3^2g_t^6 - 261\,273\,600g_3^4g_t^6 + 932\,860\,8g'^2g_t^8$$
$$+ 133\,669\,44g^2g_t^8 + 472\,780\,80g_3^2g_t^8 - 401\,241\,6g_t^{10} - 547\,680\,96g'^8\pi^2 - 575\,539\,20g'^6g^2\pi^2 - 509\,529\,6g'^4g^4\pi^2$$
$$+ 226\,356\,48g'^2g^6\pi^2 - 264\,202\,56g^8\pi^2 - 187\,453\,44g'^6g_t^2\pi^2 + 210\,677\,76g'^4g^2g_t^2\pi^2 - 887\,500\,8g'^2g^4g_t^2\pi^2$$
$$- 425\,502\,72g^6g_t^2\pi^2 + 106\,168\,32g'^4g_3^2g_t^2\pi^2 + 691\,200\,0g'^4g_t^4\pi^2 - 251\,320\,32g^4g_t^4\pi^2 + 760\,872\,96g'^2g_3^2g_t^4\pi^2$$
$$+ 143\,327\,232g^2g_3^2g_t^4\pi^2 + 244\,187\,136g_3^4g_t^4\pi^2 - 492\,687\,36g'^2g_t^6\pi^2 - 403\,107\,84g^2g_t^6\pi^2 - 429\,981\,696g_3^2g_t^6\pi^2$$
$$+ 865\,935\,36g_t^8\pi^2 - 7776g'^2g^2g_t^2(3024g_3^4 - 369g_t^4 + 64(64g_3^2 - 27g_t^2)\pi^2)),$$

$$f_1(\xi, M) = 6\ln\frac{M^2}{M_H^2} + \frac{3}{2}\ln\xi - \frac{1}{2}Z\left(\frac{1}{\xi}\right) - Z\left(\frac{c^2}{\xi}\right) - \ln c^2 + \frac{9}{2}\left(\frac{25}{9} - \frac{\pi}{\sqrt{3}}\right),$$

$$f_0(\xi, M) = -6\ln\frac{M^2}{M_Z^2}\left[1 + 2c^2 - 2\frac{M_t^2}{M_Z^2}\right] + \frac{3c^2\xi}{\xi - c^2}\ln\frac{\xi}{c^2} + 2Z\left(\frac{1}{\xi}\right) + 4c^2Z\left(\frac{c^2}{\xi}\right) + \frac{3c^2\ln c^2}{s^2} + 12c^2\ln c^2 - \frac{15}{2}(1 + 2c^2)$$
$$- 3\frac{M_t^2}{M_Z^2}\left[2Z\left(\frac{M_t^2}{M_Z^2\xi}\right) + 4\ln\frac{M_t^2}{M_Z^2} - 5\right],$$

$$f_{-1}(\xi, M) = 6\ln\frac{M^2}{M_Z^2}\left[1 + 2c^4 - 4\frac{M_t^4}{M_Z^4}\right] - 6Z\left(\frac{1}{\xi}\right) - 12c^4 Z\left(\frac{c^2}{\xi}\right) - 12c^4\ln c^2 + 8(1 + 2c^4) + 24\frac{M_t^4}{M_Z^4}\left[\ln\frac{M_t^2}{M_Z^2} - 2 + Z\left(\frac{M_t^2}{M_Z^2\xi}\right)\right],$$

$$c^2 = \cos^2\theta_W, \qquad s^2 = \sin^2\theta_W,$$

and

$$Z(z) = 2A\arctan\frac{1}{A} \quad \left(z > \frac{1}{4}\right) = A\ln\frac{1 + A}{1 - A} \quad \left(z < \frac{1}{4}\right),$$

with

$$A = \sqrt{|1 - 4z|}.$$

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