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Seeing a c-theorem with holography

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Using the anti-de Sitter conformal theory correspondence, we examine holographic renormalization group flows in a framework where the bulk gravity contains higher curvature interactions. This holographic model allows us to distinguish the flow of the different central charges in dual theory. For example, in four dimensions, one finds that the flow of the central charge a is naturally monotonic but that of c is not. Our results agree with Cardy's proposal to extend Zamolodchikov's c-theorem to higher dimensions. We are also led to formulate a novel c-theorem for a universal coefficient appearing in the entanglement entropy of the fixed point conformal theories in any (including an odd) number of dimensions.

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I. INTRODUCTION

Zamolodchikov's c-theorem [1] is a remarkable result for quantum field theories in $d = 2$. A direct outcome of the c-theorem is that in any renormalization group (RG) flow connecting two fixed points,

$$(c)_{\mathrm{UV}} \geq (c)_{\mathrm{IR}}. \tag{1}$$

That is, the central charge of conformal field theory (CFT) describing the ultraviolet fixed point is larger than (or equal to) that at the infrared fixed point. The proof relies only on the Euclidian group of symmetries, the existence of a conserved stress-energy tensor and unitarity in the field theory.

There have been various suggestions on how such a result might extend to quantum field theories in higher d. Cardy [2] conjectured a monotonic flow for the coefficient of the A-type trace anomaly. For even d, the trace anomaly for a CFT is in a curved background is given by [3]

$$\langle T^a{}_a \rangle = \sum B_i I_i - 2(-)^{d/2} A E_d \tag{2}$$

where E_d is the Euler density in d dimensions and I_i are the independent Weyl invariants of weight $-d$ [4]. Cardy's proposal is then that A satisfies a relation like Eq. (1) along RG flows between two fixed points in any even d. Of course, this coincides with Zamolodchikov's result in $d = 2$ where $A = c/12$.

Efforts made towards proving Cardy's conjectured c-theorem have focused $d = 4$ in which case Eq. (2) contains two terms, i.e., there is a single Weyl invariant $I_1 = C_{abcd}C^{abcd}$. The common nomenclature denotes the two central charges as $c = 16\pi^2 B_1$ and $a = A$. Numerous nontrivial examples have been found supporting Cardy's conjecture, including perturbative fixed points [5] and supersymmetric gauge theories [6]. The latter investigations also demonstrated that the $d = 4$ central charge c will not generically satisfy Eq. (1). Further, as we review below, support for a c-theorem in higher dimensions was found [7] with the anti-de Sitter conformal theory (AdS/CFT) correspondence [8].

However, a general proof is lacking and, in fact, a counterexample to Cardy's c-theorem in $d = 4$ was proposed in [9]. If this counterexample survives further scrutiny, there are two obvious possibilities: A "c-theorem" exists in higher d but the quantity satisfying Eq. (1) is not the central charge A. Alternatively, the central charge A satisfies Eq. (1) under RG flows in higher d but only when some more stringent requirements are imposed than in $d = 2$. In the latter case, the challenge would be to identify the precise conditions for which the analog of Eq. (1) is satisfied in higher (even) d.

In the following, we find evidence for the second alternative within the framework of the AdS/CFT correspondence. We examine the c-theorem using a holographic model with a higher curvature gravity theory [10,11] which allows one to distinguish the central charges a and c in the dual CFT. Hence, we are able to discriminate between the behavior of these two central charges in RG flows, and we find that only a has a natural monotonic flow. We are also able to extend our analysis to holographic CFT's in arbitrary higher d. Our results there suggest the following general conjecture:

Placing a d-dimensional CFT on $S^{d-1} \times R$ and calculating the entanglement entropy of the ground state between two halves of the sphere, one finds a universal contribution: $S_{\mathrm{univ}} \propto a_d^*$ [as detailed in Eq. (22)]. Further in RG flows between fixed points, $(a_d^*)_{\mathrm{UV}} \geq (a_d^*)_{\mathrm{IR}}$.

This conjecture then gives us a framework in which to consider the c-theorem for d even or odd. As described below, this conjecture actually coincides to Cardy's proposal for even d. We now discuss the holographic origin of this conjecture.

II. HOLOGRAPHIC C-THEOREM

The AdS/CFT correspondence has emerged as a powerful tool to study the behavior of strongly coupled CFT's in diverse dimensions [8]. Within this framework, [7] considered the c-theorem where one begins with $(d+1)$-dimensional Einstein gravity coupled to various matter fields:

$$I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x\sqrt{-g}(R + \mathcal{L}_{\text{matter}}). \tag{3}$$

The matter theory is assumed to have various stationary points where $\mathcal{L}_{\text{matter}} = d(d-1)\alpha_i^2/L^2$ with some canonical scale L. The fixed points are distinguished by different values of α as indicated by the subscript and at these points, the gravity vacuum is simply AdS_{d+1} with the curvature scale given by $\tilde{L}^2 = L^2/\alpha_i^2$. RG flows between critical points can be described with a metric of the form

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}_{d-1}^2) + dr^2. \tag{4}$$

This metric becomes that for AdS_{d+1} with $A(r) = r/\tilde{L}$ at the stationary points. Now define [7]:

$$a(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)(\ell_P A'(r))^{d-1}}, \tag{5}$$

where "prime" denotes a derivative with respect to r. Then for general solutions of the form (4), one finds

$$\begin{aligned} a'(r) &= -\frac{(d-1)\pi^{d/2}}{\Gamma(d/2)\ell_P^{d-1}A'(r)^d} A''(r) \\ &= -\frac{\pi^{d/2}}{\Gamma(d/2)\ell_P^{d-1}A'(r)^d}(T^t{}_t - T^r{}_r) \geq 0. \end{aligned} \tag{6}$$

Above in the second equality, the Einstein equations are used to eliminate $A''(r)$ in favor of components of the stress tensor. The final inequality assumes that the matter fields obey the null energy condition [12]. Given this monotonic evolution of $a(r)$ with r and the standard connection between r and energy scale in the CFT, $a(r)$ always decreases in flowing from the UV to the IR.

To make better contact with the dual CFT, it is simplest to focus the discussion on $d = 4$ at this point. Then with the holographic trace anomaly [13] for the AdS_5 stationary points, one finds

$$a(r)|_{\text{AdS}} = \pi^2\tilde{L}^3/\ell_P^3 = a. \tag{7}$$

That is, the value of the flow function (5) matches precisely that of the central charge a in the dual CFT at each of the fixed points. Hence with the assumption of the null energy condition, the holographic CFT's dual to the gravity theory (3) satisfy Cardy's c-theorem. Of course, one must add that the two central charges are precisely equal [13], i.e., $a = c$, for the class of $d = 4$ CFT's dual to Einstein gravity. Hence these holographic models do not distinguish between the flow of a and c.

It has long been known that to construct a holographic model where $a \neq c$, the gravity action must include higher curvature interactions [14]. In part, this motivated the construction of quasitopological gravity [10]

$$I = \frac{1}{2\ell_P^3} \int d^5x\sqrt{-g}\left[\frac{12\alpha^2}{L^2} + R + \frac{\lambda}{2}L^2\mathcal{X}_4 + \frac{7\mu}{4}L^4\mathcal{Z}_5\right], \tag{8}$$

where $\mathcal{X}_4$ is the "Gauss-Bonnet" interaction, a combination of curvature-squared terms which corresponds to the Euler density in four dimensions, and $\mathcal{Z}_5$ is a particular interaction involving curvature-cubed terms developed in [10,15]. This holographic model allows one to explore the full three-parameter space of coefficients controlling the two- and three-point functions of the stress tensor in a general four-dimensional CFT [16]. However, one must keep in mind that this action (8) was not derived from string theory. Rather it was constructed to facilitate the exploration of a broader class of holographic CFT's while maintaining control within the gravity calculations. Further, the gravitational couplings in Eq. (8) can be constrained by various consistency requirements of the dual CFT [11]. A generalization of the model (8) to $d = 2$ CFT's and c-theorems was considered in [17].

In comparison to the action presented in [10], we have replaced the cosmological constant term by $12\alpha^2/L^2$ with the study of RG flows in mind. The idea is that as in Eq. (3) the gravity theory is coupled to a standard matter action with various stationary points which yield different values for the parameter α^2. The curvature scale of the AdS_5 vacua are related to the scale L appearing in Eq. (8) by $\tilde{L}^2 = L^2/f_\infty$, where

$$\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^2 = 0. \tag{9}$$

Using standard techniques [13], one determines the central charges of the CFT's dual to these AdS_5 vacua [11]:

$$a = \pi^2\tilde{L}^3/\ell_P^3(1 - 6\lambda f_\infty + 9\mu f_\infty^2), \tag{10}$$

$$c = \pi^2\tilde{L}^3/\ell_P^3(1 - 2\lambda f_\infty - 3\mu f_\infty^2). \tag{11}$$

Considering the metric ansatz (4), it is straightforward to construct two flow functions:

$$a(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3}(1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^4), \tag{12}$$

$$c(r) \equiv \frac{\pi^2}{\ell_P^3 A'(r)^3}(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4). \tag{13}$$

These are chosen as the simplest extensions of Eq. (5) with $d = 4$ which yield the two central charges at the fixed points, i.e., $a(r)|_{\text{AdS}} = a$ and $c(r)|_{\text{AdS}} = c$. Now for a general RG flow solution, one finds

$$a'(r) = -3\frac{A''(r)}{A'(r)}c(r) = -\frac{\pi^2}{\ell_P^3 A'(r)^4}(T^t{}_t - T^r{}_r) \geq 0, \tag{14}$$

where as before we are using the gravitational equations of motion and assuming that the matter sector obeys the null energy condition. We might note that the null energy condition plays a central role in gravitational physics [12] and that violations of this condition are argued to lead to instabilities quite generally [18]. While it is possible that our assumption is unduly conservative in the present context, we regard it as a reasonable restriction on the matter sector in order to produce a holographic framework with a consistent dual CFT. With the latter assumption, $a(r)$ evolves monotonically along the holographic RG flows and we can conclude that the central charge (10) satisfies the analog of Eq. (1). One can also consider the behavior of $c(r)$ along RG flows but there is no clear way to establish that $c'(r)$ has a definite sign. Hence, this holographic model provides another broad class of four-dimensional theories which support Cardy's proposal that the central a (rather than any other central charge) evolves monotonically along RG flows.

Given this result and those above for Einstein gravity, it is straightforward to extend our holographic analysis of quasitopological gravity to an arbitrary spacetime dimension. Beginning with the equations of motion which are proportional to $T^t{}_t - T^r{}_r$, one can engineer the following flow function [19]

$$a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)(\ell_P A'(r))^{d-1}} \times \left(1 - \frac{2(d-1)}{d-3}\lambda L^2 A'(r)^2 - \frac{3(d-1)}{d-5}\mu L^4 A'(r)^4\right). \tag{15}$$

By construction, $a_d(r)$ satisfies the following:

$$a_d'(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)\ell_P^{d-1}A'(r)^d}(T^t{}_t - T^r{}_r) \geq 0, \tag{16}$$

where we again assume the null energy condition. Note that here we must also ensure that $A'(r) > 0$ for odd d—the details of this proof will be given in [20]. If we define the fixed point value as $a_d^* \equiv a_d(r)|_{\mathrm{AdS}}$, then

$$a_d^* = \frac{\pi^{d/2}\tilde{L}^{d-1}}{\Gamma(d/2)\ell_P^{d-1}}\left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty - \frac{3(d-1)}{d-5}\mu f_\infty^2\right), \tag{17}$$

and the result in Eq. (16) guarantees that

$$(a_d^*)_{\mathrm{UV}} \geq (a_d^*)_{\mathrm{IR}}. \tag{18}$$

Having found that a_d^* satisfies a c-theorem, one is left to determine what this quantity corresponds to in the dual CFT. By construction for $d = 4$, this is precisely the central charge a. Motivated by Cardy's general conjecture for even d, it is natural to compare a_d^* to the coefficient A in Eq. (2). In fact, using the approach of [21], one readily confirms that there is a precise match:

$$a_d^* = A \quad \text{for even } d. \tag{19}$$

Hence again, we find support for Cardy's conjecture with this broad class of holographic CFT's. However, we must seek a broader definition of a_d^* to also incorporate odd d.

Examining the results for black hole entropy in quasitopological gravity [10], we observe the following: If the dual CFT is placed on a $(d-1)$-dimensional hyperbolic plane (i.e., the d-dimensional space is $H^{d-1} \times R$), the energy density ρ_E of the ground state is negative. If a temperature is introduced and the system is heated up to the point where $\rho_E = 0$, the entropy density becomes

$$s = (4\pi)^{d/2}\Gamma(d/2)a_d^* T^{d-1} = \frac{2\pi}{\pi^{d/2}}\Gamma(d/2)\frac{a_d^*}{L^{d-1}}, \tag{20}$$

where L denotes the radius curvature of H^{d-1}. Now this leads to an interesting question: The bulk geometry is precisely AdS_{d+1} in "unusual" coordinates, i.e.,

$$ds^2 = \frac{dr^2}{(\frac{r^2}{\tilde{L}^2} - 1)} - \left(\frac{r^2}{\tilde{L}^2} - 1\right)dt^2 + r^2 d\Sigma_2^{d-1}, \tag{21}$$

where $d\Sigma^{d-1}$ denotes the line element on H^{d-1} with unit curvature, and so why should there be any entropy at all? The answer is that the foliation of AdS_{d+1} in Eq. (21) divides the boundary into two halves—see Fig. 1. Hence, the entropy is interpreted as the entanglement entropy between these two halves—related ideas were discussed in [22]. Note that if we integrate over the hyperbolic horizon out to some maximum radius which is then given the conventional interpretation of a short-distance cutoff, i.e., $\delta = L^2/\rho_{\max}$, the leading contribution in Eq. (20) takes the form $S \propto a_d^*(L/\delta)^{d-2}$. Hence despite doing a $(d-1)$-dimensional integral, the leading divergence produced by the hyperbolic geometry has precisely the power

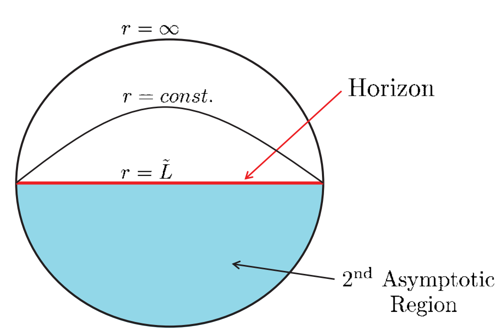

FIG. 1 (color online). A slice of constant t through the AdS_{d+1} metric in Eq. (21). This slice bears some similarity to the Einstein-Rosen bridge in a Schwarzschild black hole [12]. Note that only half of the AdS boundary is reached in the limit $r \to \infty$. The other half is reached from the second asymptotic region "behind the horizon."

expected for the "area law" contribution to the entanglement entropy in a d-dimensional CFT [23]. However, this divergent contribution is not universal. Rather a universal contribution is extracted from the subleading terms. The form of the universal contribution to the entanglement entropy depends on whether d is odd or even:

$$S_{\text{univ}} = \begin{cases} (-)^{(d/2)-1} 4a_d^* \log(L/\delta) & \text{for even } d, \\ (-)^{(d-1)/2} 2\pi a_d^* & \text{for odd } d. \end{cases} \tag{22}$$

With a Weyl transformation, the boundary metric can be brought to that on $S^{d-1} \times R$ and Eq. (22) can then be interpreted as the universal contribution to the entanglement entropy between the two halves of the S^{d-1} for ground state of the CFT. Hence, this framework allows us to interpret the coefficient a_d^* in terms of entanglement entropy in the dual CFT for odd or even d. Further our previous analysis shows that this coefficient obeys a c-theorem in the holographic RG flows.

III. DISCUSSION

Although our holographic calculations here referred to quasitopological gravity for concreteness, our findings hold for a much wider class of higher curvature gravity theories in arbitrary dimensions [20]. In all of these instances, we still assume that the null energy condition holds and that the gravitational couplings are independent of the matter fields. It will be interesting to explore the extent to which these conditions can be relaxed while still requiring the existence of a holographic c-theorem. In any event, having identified this interesting behavior in the RG flows of a broad class of holographic CFT's, it is natural to conjecture that the same c-theorem should apply more broadly for RG flows outside of any holographic framework. Hence, we propose the general conjecture presented above in the introduction. Now one would like to determine what evidence one might find for this general conjecture, again, outside of a holographic framework.

For even dimensions, one can calculate the entanglement entropy for a CFT divided by a smooth boundary making use of the trace anomaly [23,24]. In general, the result will depend on all of the coefficients appearing in Eq. (2). However, our conjecture refers to a very specific background geometry and a specific boundary dividing this space. Applying the approach of [23,24] for this geometry in arbitrary even d, one finds that the coefficient of the universal entanglement entropy (22) is precisely $a_d^* = A$ [20]. While this matches Eq. (19) for our holographic model, our result here is a general statement about S_{univ} on $S^{d-1} \times R$ with any CFT in even d. Hence our conjectured c-theorem coincides precisely with Cardy's proposal for arbitrary even d. Therefore any evidence for Cardy's conjecture also supports the present conjecture. However, the present results also frame Cardy's conjecture in the context of entanglement entropy. Taking into account both of these perspectives should prove useful in better understanding the precise conditions under which A satisfies Eq. (1) for higher (even) dimensions, e.g., in particular, for $d = 4$.

We emphasize that specifying the geometry in which one calculates the entanglement entropy is an important feature of our conjecture. As noted above, our prescription for the geometry was crucial to have $S_{\text{univ}} \propto A$ in even d. We can infer that the entanglement entropy decreases along RG flows for several known examples in $d = 2$ and 3 [25–27]. However, we cannot say that these examples provide direct support of our conjecture primarily because they do not concern the precise geometry specified there. We also anticipate that with further study one can identify other geometries for which the universal entanglement entropy is proportional to a_d^* [20]. For example, $S_{\text{univ}} \propto a$ for a $d = 4$ CFT when calculated for a spherical boundary in flat space [24].

Of course, entanglement entropy has previously been considered in the context of RG flows and c-theorems. In particular, [28] establishes an entropic c-theorem in $d = 2$ based purely on considerations of Lorentz symmetry and the strong subadditivity. This construction is distinct from our conjecture in $d = 2$, where in fact the latter simply coincides with Zamalodchikov's c-theorem [1].

The higher curvature terms play an important role in our holographic analysis as they allow us to unambiguously identify a_d^* with the A-type anomaly in any even d. Similarly we are also able to distinguish a_d^* from $\hat{C}$, the coefficient appearing in the free energy density, i.e., $f = \hat{C}T^d$. This coefficient was also considered in efforts to extend the c-theorem to higher dimensions [29]. In our holographic model, it does not appear that this coefficient always varies monotonically in RG flows [20].

In closing, we note that our calculation of the holographic entanglement entropy did not make reference to the standard proposal conjectured by [23]. We would also note that this is also the first calculation of such a quantity that includes the contributions of higher curvature interactions in the bulk gravity theory. However, in the Einstein gravity limit, i.e., $\mu = 0 = \lambda$, our result (22) coincides with that calculated using the standard proposal. The present calculation may point the way to a more systematic derivation of holographic entanglement entropy.

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