PHYSICAL REVIEW D **82,** 045028 (2010)

Quantum radiation reaction force on a one-dimensional cavity with two relativistic moving mirrors

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(Received 23 April 2010; published 27 August 2010)

We consider a real massless scalar field inside a cavity with two moving mirrors in a two-dimensional spacetime, satisfying the Dirichlet boundary condition at the instantaneous position of the boundaries, for arbitrary and relativistic laws of motion. Considering vacuum as the initial field state, we obtain formulas for the exact value of the energy density of the field and the quantum force acting on the boundaries, which extend results found in previous papers [D. T. Alves, E. R. Granhen, H. O. Silva, and M. G. Lima, Phys. Rev. D **81**, 025016 (2010); L. Li and B.-Z. Li, Phys. Lett. A **300**, 27 (2002); L. Li and B.-Z. Li, Chin. Phys. Lett. **19**, 1061 (2002); L. Li and B.-Z. Li, Acta Phys. Sin. **52**, 2762 (2003); C. K. Cole and W. C. Schieve, Phys. Rev. A **64**, 023813 (2001)]. For the particular cases of a cavity with just one moving boundary, nonrelativistic velocities, or in the limit of infinity length of the cavity (a single mirror), our results coincide with those found in the literature.

DOI: 10.1103/PhysRevD.82.045028 PACS numbers: 03.70.+k, 11.10.Wx, 42.50.Lc

The dynamical Casimir effect has been investigated since the 1970s [1–3], and has attracted growing attention. It is related to problems like particle creation in cosmological models and radiation emitted by collapsing black holes [2,4], decoherence [5], entanglement [6], and the Unruh effect [7], among others. Models of a single mirror have been investigated and also cavities with one moving boundary have been studied in many papers (for a review see Ref. [8]). In contrast, the problem of a cavity with two moving boundaries has been investigated recently and relatively few papers on this subject are found in the literature (for instance, Refs. [9–14]). A cavity with two oscillating mirrors can exhibit situations of constructive and destructive interference in the number of created particles, depending on the relation among the phase difference of each boundary, the amplitudes, and frequencies of oscillation [9,10,13]. Ji, Jung, and Soh [9], considering the expansion of the quantizing field over a instantaneous basis and a perturbative approach, investigated the problem of interference in the particle creation for a one-dimensional cavity with two boundaries moving according to prescribed, nonrelativistic, and oscillatory (small amplitudes) laws of motion. Dalvit and Mazzitelli [10] extended the field solution obtained by Moore [1] for the case of a one-dimensional cavity with two moving boundaries, deriving a set of generalized Moore's equations, also obtaining the expected energy-momentum tensor for this model, generalizing the corresponding formula obtained by Fulling and Davies [2]. In Ref. [10] the set of generalized Moore's equations was solved for the case of a resonant oscillatory movement with small amplitude, via the renormalization-group procedure. Li and Li [15] applied the geometrical method, proposed by Cole and Schieve [16], to solve exactly the generalized Moore equations obtained by Dalvit and Mazzitelli [10]. Lambrecht, Jaekel, and Reynaud investigated the problem of the radiation emitted outside of a cavity with two moving mirrors (with transmission amplitudes different from zero) moving in vacuum [14]. They deduced, for arbitrary laws of motion, an exact formula for the field outside the cavity, using a method analogous to the one described in Ref. [16], and also they used the field solution to get an exact formula for the energy density radiated to the right of the cavity, applying their results to a specific class of harmonic motions as well as calculating the energy stored inside the cavity for this case [14]. The method developed in Ref. [14], if applied to find the general solution of the field inside a cavity with two moving mirrors, transmission coefficient equal to zero, and arbitrary trajectories, naturally must result in a solution for the field equivalent to the recursive exact solution found in Ref. [15]. This field solution, inserted in the renormalized formula for the energy-momentum tensor found in Ref. [10], enables the achievement of the exact value of the energy density inside the cavity. However, this procedure requires, in general, numerical and recursive calculations to get the first, second, and third derivatives of each function found in the set of generalized Moore's equations proposed in Ref. [10]. In this context, Li and Li obtained the exact behavior of the energy density in a cavity for particular sinusoidal laws of motion, with small amplitude [17].

In the present paper, we deduce formulas that give directly the exact values of the quantum force and energy density inside a nonstatic cavity, for arbitrary laws of motion for the moving boundaries. We explore the geometrical approach already used in Refs. [14–17], but, as done in Ref. [18], we focus straight on the reflections process of the energy density itself. Considering vacuum as the initial field state, we show that the energy density in a given point of the spacetime can be obtained by tracing back a sequence of null lines, connecting the value of the energy density at the given spacetime point to a certain known value of the energy at a point in the "static zone," where the initial field modes are not affected by the disturbance caused by the movement of the boundaries. An advantage of the formula obtained in the present paper is that the energy density exhibits a clear structure, with a factor corresponding to the initial vacuum energy density, and others depending only on the laws of motion. Our formulas generalize those found in literature [18], where this problem is approached for a cavity with only one moving mirror. For the particular cases of a cavity with just one moving boundary, nonrelativistic velocities, or in the limit of large length of the cavity (a single mirror), our results coincide with those found in the literature [18–20].

We consider a real massless scalar field satisfying the Klein-Gordon equation (we assume throughout this paper $\hbar = c = 1$): $(\partial_t^2 - \partial_x^2)\phi(t,x) = 0$, and obeying Dirichlet conditions imposed at the left boundary located at $x = L(t)$, and also at the right boundary located at $x = R(t)$, where $L(t)$ and $R(t)$ are arbitrary prescribed laws of motion, with $R(t<0) = L_0$ and $L(t<0) = 0$, where L_0 is the length of the cavity in the static situation. Let us start considering the field operator, solution of the wave equation, given by [10]

$$\hat{\phi}(t,x) = \sum_{k=1}^{\infty}[\hat{a}_k \psi_k(t,x) + \hat{a}_k^\dagger \psi_k^*(t,x)],$$

where the field modes are

$$\psi_k(t,x) = \frac{i}{\sqrt{4\pi k}}[e^{-ik\pi G(v)} - e^{-ik\pi F(u)}], \tag{1}$$

with $v = t + x$, $u = t - x$, and

$$G[t + L(t)] - F[t - L(t)] = 0 \tag{2a}$$

$$G[t + R(t)] - F[t - R(t)] = 2. \tag{2b}$$

The set of equations (2), obtained by Dalvit and Mazzitelli exploiting the conformal invariance of the model [10], is a generalization of the Moore equation [1], which can be recovered doing $L(t) = 0$ in these equations. The renormalized energy density in the cavity is given by [10]

$$\langle T_{00}(t,x)\rangle = -f_G(v) - f_F(u), \tag{3}$$

where

$$f_G(z) = \frac{1}{24\pi}\left\{\frac{G'''(z)}{G'(z)} - \frac{3}{2}\left[\frac{G''(z)}{G'(z)}\right]^2 + \frac{\pi^2}{2}[G'(z)]^2\right\}, \tag{4a}$$

$$f_F(z) = \frac{1}{24\pi}\left\{\frac{F'''(z)}{F'(z)} - \frac{3}{2}\left[\frac{F''(z)}{F'(z)}\right]^2 + \frac{\pi^2}{2}[F'(z)]^2\right\}. \tag{4b}$$

Li and Li [15] solved exactly (2a) and (2b), applying the geometrical method proposed by Cole and Schieve [16]. The explicit formulas for F and G obtained in Ref. [15] can be used to calculate the energy density (3). However, this procedure requires, in general, numerical and recursive calculations to get each one of the functions F', F'', F''', G', G'', and G'''. In the present paper, instead of solving (2), we use Eqs. (2a), (2b), (4a), and (4b) to obtain the following set of equations for the functions f_G and f_F:

$$f_G[t + R(t)] = f_F[t - R(t)]A_R(t) + B_R(t), \tag{5a}$$

$$f_G[t + L(t)] = f_F[t - L(t)]A_L(t) + B_L(t), \tag{5b}$$

with

$$A_q(t) = \left[\frac{1 - q'(t)}{1 + q'(t)}\right]^2, \tag{6}$$

$$B_q(t) = -\frac{1}{12\pi}\frac{q'''(t)}{[1 + q'(t)]^3[1 - q'(t)]} - \frac{1}{4\pi}\frac{q''^2(t)q'(t)}{[1 + q'(t)]^4[1 - q'(t)]^2}, \tag{7}$$

where, hereafter, q can represent R or L. Equations (5a) and (5b) are an extension of the corresponding equation for f, valid for a cavity with just one moving mirror, found in Ref. [19]. If we consider the particular case of $L(t) = 0$ in Eq. (5), we recover the corresponding result found in Ref. [19]. For $(t < 0)$ we have $f_G(v) = f_F(u) = f^{(s)} = \pi/(48L_0^2)$, and $\langle T_{00}\rangle = -\pi/(24L_0^2)$, which is the Casimir energy density for this model. Our aim is to solve the Eqs. (5a) and (5b) recursively, using a geometrical point of view.

Let us first examine the cavity in the nonstatic situation $(t > 0)$. The field modes in Eq. (1) are formed by left- and right-propagating parts. As causality requires, the field in region I ($v < L_0$ and $u < 0$) (see Fig. 1) is not affected by

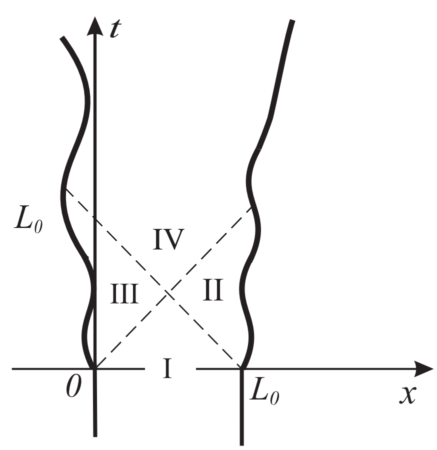

FIG. 1. Boundaries trajectories (solid lines). The dashed lines are null lines separating region I from II and III, and these from region IV.

the boundaries motion, so that, in this sense, this region is considered as a "static zone." In region II ($v > L_0$ and $u < 0$), the right-propagating parts of the field modes remain unaffected by the boundaries motion, so that region II is also a static zone for these modes. On the other hand, the left-propagating parts in region II are, in general, affected by the boundary movement. Similarly, in region III ($u > 0$ and $v < L_0$), the left-propagating parts of the field modes are not affected by the boundaries motion, but the right-propagating parts are. In region IV ($v > L_0$ and $u > 0$), both the left- and right-propagating parts are affected. In summary, the functions corresponding to the left- and right-propagating parts of the field modes are considered in the static zone if their arguments are, respectively $v < L_0$ and $u < 0$. Then, we have $f_G(v < L_0) = f^{(s)}$ and $f_G(u < 0) = f^{(s)}$.

For a certain spacetime point $(\tilde{t}, \tilde{x})$, the energy tensor $\langle T_{00}(\tilde{t}, \tilde{x})\rangle$ is known if its left- and right-propagating parts, taken over, respectively, the null lines $v = z_1$ and $u = z_2$ (where $z_1 = \tilde{t} + \tilde{x}$ and $z_2 = \tilde{t} - \tilde{x}$), are known; or, in other words, $\langle T_{00}(\tilde{t}, \tilde{x})\rangle$ is known if $f_G(v)|_{v=z_1}$ and $f_F(u)|_{u=z_2}$ are known. Li and Li [15] used a recursive method [16] to obtain the functions G and F for general laws of motion of the boundaries, tracing back a sequence of null lines until a null line gets into the static zone where the G or F functions are known. Here, we adopt this method to obtain f_G and f_F, extending the work done in Ref. [15]. Let us assume that $(\tilde{t}, \tilde{x})$ belongs to region IV, and that the null line $v = z_1$ intersects the moving mirror trajectory at the point $[t_1, R(t_1)]$ [see Fig. 2(a)], so that $\tilde{t} + \tilde{x} = t_1 + R(t_1)$. We have $f_G(v)|_{v=z_1} = f_G[t_1 + R(t_1)]$. Using Eq. (5a), we get $f_G[t_1 + R(t_1)] = f_F[t_1 - R(t_1)]A_R(t_1) + B_R(t_1)$. If $t_1 - R(t_1) < 0$, then the null line $u = t_1 - R(t_1)$ is already in the static zone [Fig. 2(a)], so that we can write $f_F[t_1 - R(t_1)] = f^{(s)}$, and also $f_G[t_1 + R(t_1)] = f^{(s)}A_R(t_1) + B_R(t_1)$, and we can say that the number of reflections n_G to get into the static zone is, in this case, $n_G = 1$. On the other hand, if $t_1 - R(t_1) > 0$ [case shown in Fig. 2(b)] we can draw another null line $v = t_2 + L(t_2)$ intersecting the world line of the left boundary at the point $[t_2, L(t_2)]$, with $t_1 - R(t_1) = t_2 - L(t_2)$. In this case we have, using (5b), $f_G[t_1 + R(t_1)] = \{f_G[t_2 + L(t_2)] - B_L(t_2)\}A_R(t_1)/A_L(t_2) + B_R(t_1)$. If $t_2 + L(t_2) < L_0$ [see Fig. 2(b)], then $f_G[t_2 + L(t_2)] = f^{(s)}$, $f_G[t_1 + R(t_1)] = \{f^{(s)} - B_L(t_2)\}A_R(t_1)/A_L(t_2) + B_R(t_1)$, and $n_G = 2$. If $t_2 + L(t_2) > L_0$, we assume that the null line $v = t_2 + L(t_2)$ intersects the right boundary at the point $[t_3, R(t_3)]$, then $t_2 + L(t_2) = t_3 + R(t_3)$ and we get $u = t_3 - R(t_3)$. We repeat this procedure until a null line gets into a static zone, where the function f_F or f_G is known. In summary, we obtain for f_G:

$$f_G(z) = f^{(s)}\tilde{A}_G(z) + \tilde{B}_G(z), \tag{8}$$

where, for $n_G(z)$ even, we have

$$\tilde{A}_G(z) = \prod_{k=0}^{(n_G(z))/2}\left[(1-\delta_{k,0})\frac{A_R[t_{2k-1}(z)]}{A_L[t_{2k}(z)]} + \delta_{k,0}\right], \tag{9a}$$

$$\tilde{B}_G(z) = \sum_{k=0}^{(n_G(z))/2}\left\{(1-\delta_{k,0})\left[\frac{B_R[t_{2k-1}(z)]A_L[t_{2k}(z)]}{A_R[t_{2k-1}(z)]} - B_L[t_{2k}(z)]\right]\prod_{j=0}^{k}\left[(1-\delta_{j,0})\frac{A_R[t_{2j-1}(z)]}{A_L[t_{2j}(z)]} + \delta_{j,0}\right]\right\}, \tag{9b}$$

with δ symbolizing Kronecker's delta function. For $n_G(z)$ odd we have

$$\tilde{A}_G(z) = \prod_{k=0}^{(n_G(z)-1)/2}\left[\frac{A_R[t_{2k+1}(z)]}{(1-\delta_{k,0})A_L[t_{2k}(z)] + \delta_{k,0}}\right], \tag{10a}$$

$$\tilde{B}_G(z) = \sum_{k=0}^{(n_G(z)-1)/2}\left\{[B_R[t_{2k+1}(z)] - (1-\delta_{k,0}) \times B_L[t_{2k}(z)]]\prod_{j=0}^{k}\left[(1-\delta_{j,0})\frac{A_R[t_{2j-1}(z)]}{A_L[t_{2j}(z)]} + \delta_{j,0}\right]\right\}. \tag{10b}$$

Note that the number n_G of reflections and the sequence of instants $t_1, \ldots, t_{n_G}$ depend on the argument z. The set of instants mentioned in Eqs. (9) and (10) are calculated via [15]

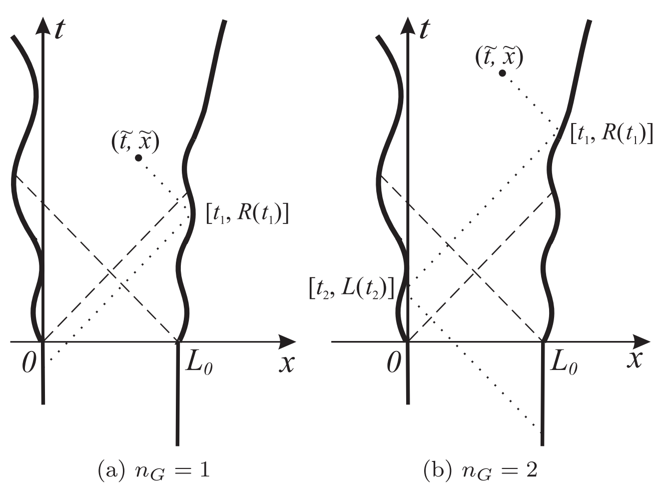

FIG. 2. Sequence of null lines (dotted lines) connecting a point $(\tilde{t}, \tilde{x})$ to a static zone. The dashed lines are null lines separating region I from II and III, and these from region IV, as presented in Fig. 1. In Fig. 2(a), we see the case of one reflection ($n_G = 1$), whereas in Fig. 2(b) we see the case $n_G = 2$.

$$z = t_1 + R(t_1), \qquad t_{2l+1} - R(t_{2l+1}) = t_{2l+2} - L(t_{2l+2}),$$
$$t_{2l+2} + L(t_{2l+2}) = t_{2l+3} + R(t_{2l+3}), \qquad l = 0, 1, 2 \ldots. \tag{11}$$

To solve recursively the set of equations (5) for f_F, we start assuming that the null line $u = \tilde{t} - \tilde{x}$ intersects the world line of the left mirror at the point $[\bar{t}_1, L(\bar{t}_1)]$, so that $\tilde{t} - \tilde{x} = \bar{t}_1 - L(\bar{t}_1)$. Thus we have $f_F(u)|_{u=z_2} = f_F[\bar{t}_1 - L(\bar{t}_1)]$. Using Eq. (5b), we get $f_F[\bar{t}_1 - L(\bar{t}_1)] = \{f_G[\bar{t}_1 + L(\bar{t}_1)] - B_L(\bar{t}_1)\}/A_L(\bar{t}_1)$. If $\bar{t}_1 + L(\bar{t}_1) < L_0$, then the null line $v = \bar{t}_1 + L(\bar{t}_1)$ is already in the static zone, so that we can write $f_G[\bar{t}_1 + L(\bar{t}_1)] = f^{(s)}$, and also $f_F[\bar{t}_1 - L(\bar{t}_1)] = \{f^{(s)} - B_L(\bar{t}_1)\}/A_L(\bar{t}_1)$, and we can say that the number of reflections n_F to get into the static zone is, in this case, $n_F = 1$ [see Fig. 3(a)]. On the other hand, if $\bar{t}_1 + L(\bar{t}_1) > L_0$ [as shown in Fig. 3(b)] we need to find $f_G[\bar{t}_1 + L(\bar{t}_1)]$ recursively via Eq. (8). In general, we get

$$f_F(z) = f^{(s)}\tilde{A}_F(z) + \tilde{B}_F(z), \tag{12}$$

where

$$\tilde{A}_F(z) = \frac{\tilde{A}_G\{\bar{t}_1(z) + L[\bar{t}_1(z)]\}}{A_L[\bar{t}_1(z)]}, \tag{13a}$$

$$\tilde{B}_F(z) = \frac{\tilde{B}_G\{\bar{t}_1(z) + L[\bar{t}_1(z)]\} - B_L[\bar{t}_1(z)]}{A_L[\bar{t}_1(z)]}, \tag{13b}$$

with the function $\bar{t}_1(z)$ calculated via

$$z = \bar{t}_1 - L(\bar{t}_1). \tag{14}$$

The formulas (9), (10), and (13) generalize those for $\tilde{A}$ and $\tilde{B}$ found in Ref. [18], which are valid for a cavity with just the right boundary in movement.

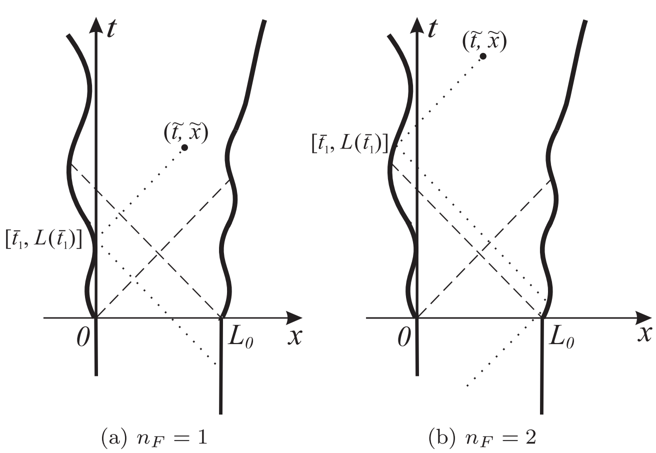

FIG. 3. Sequence of null lines (dotted lines) connecting a point $(\tilde{t}, \tilde{x})$ to a static zone. The dashed lines are null lines separating region I from II and III, and these from region IV, as presented in Fig. 1. In Fig. 3(a), we see the case of one reflection ($n_F = 1$), whereas in Fig. 3(b) we see the case $n_F = 2$.

From Eqs. (3), (8), and (12), we get the exact formula for the renormalized energy density as

$$\langle T_{00}(t, x)\rangle = -f^{(s)}[\tilde{A}_G(v) + \tilde{A}_F(u)] - \tilde{B}_G(v) - \tilde{B}_F(u). \tag{15}$$

Equation (15) gives directly the exact values for the energy density in a nonstatic cavity for arbitrary laws of motion $R(t)$ and $L(t)$. Since $T_{00} = T_{11}$ in this model, we have the following exact formulas for the renormalized quantum forces $\mathcal{F}_R = \langle T_{00}[t, R(t)]\rangle$ and $\mathcal{F}_L = -\langle T_{00}[t, L(t)]\rangle$ (see Refs. [20,21]) acting, respectively, on the right and left boundaries:

$$\mathcal{F}_R(t) = -f^{(s)}\{\tilde{A}_G[t + R(t)] + \tilde{A}_F[t - R(t)]\} - \tilde{B}_G[t + R(t)] - \tilde{B}_F[t - R(t)], \tag{16}$$

$$\mathcal{F}_L(t) = f^{(s)}\{\tilde{A}_G[t + L(t)] + \tilde{A}_F[t - L(t)]\} + \tilde{B}_G[t + L(t)] + \tilde{B}_F[t - L(t)]. \tag{17}$$

Next we examine the behavior of these forces in each region pointed out in Fig. 1.

In region I (Fig. 1), we have $n_G = n_F = 0$. Then, Eqs. (9) and (13) give $\tilde{A}_G(z) = \tilde{A}_F(z) = 1$ and $\tilde{B}_G(z) = \tilde{B}_F(z) = 0$. This results, as expected, in the static Casimir force

$$\mathcal{F}_R^{(\mathrm{Cas})} = -\mathcal{F}_L^{(\mathrm{Cas})} = -\pi/(24L_0^2),$$

acting on the boundaries.

In region II, we have $n_G = 1$ and $n_F = 0$. For this case, Eq. (13) gives $\tilde{A}_F(u) = 1$ and $\tilde{B}_F(u) = 0$, whereas from Eq. (10) we have $\tilde{A}_G(v) = A_R[t_1(v)]$ and $\tilde{B}_G(v) = B_R[t_1(v)]$. To calculate the force $\mathcal{F}_R(t)$ in Eq. (16) we do $v \to t + R(t)$, and obtain $t_1(v)$ as already discussed: $t + R(t) = t_1 + R(t_1) \Rightarrow t_1 = t$. Then we get $\tilde{A}_G[t + R(t)] = A_R(t)$ and $\tilde{B}_G[t + R(t)] = B_R(t)$. The force $\mathcal{F}_R(t)$ on the right boundary in region II, now relabeled as $\mathcal{F}_R^{(\mathrm{II})}(t)$, is

$$\mathcal{F}_R^{(\mathrm{II})}(t) = -f^{(s)}[1 + A_R(t)] - B_R(t). \tag{18}$$

From this formula, we can obtain an analytical result for an arbitrary law of motion $R(t)$. Note that in Eq. (18) the subscript L is not found, since the quantum force for the world line in region II has no influence of the movement of the left boundary. Considering the limit $L_0 \to \infty$ we recover the quantum radiation force $\mathcal{F}_q^{(-\mathrm{u})}$ corresponding to the unbounded field, acting on the left side of a single mirror: $\lim_{L_0\to\infty}\mathcal{F}_R^{(\mathrm{II})}(t) = \mathcal{F}_R^{(-\mathrm{u})}(t)$, where

$$\mathcal{F}_q^{(-\mathrm{u})}(t) = -B_q(t). \tag{19}$$

In the nonrelativistic limit, from (18) we get $\mathcal{F}_R^{(\mathrm{II})}(t) \approx \mathcal{F}_R^{(\mathrm{Cas})} + \dddot{R}/(12\pi)$, and adding the limit $L_0 \to \infty$

we recover the approximate quantum radiation force $\mathcal{F}_R^{(\mathrm{II})}(t) \approx \dddot{R}/(12\pi)$, which acts on the left side of a single mirror [22].

In region III, we have $n_G = 0$ and $n_F = 1$. For this case, Eqs. (9) and (13) give $\tilde{A}_F(u) = 1/A_L[\bar{t}_1(u)]$; $\tilde{B}_F(u) = -B_L[\bar{t}_1(u)]/A_L[\bar{t}_1(u)]$; $\tilde{A}_G(v) = 1$; $\tilde{B}_G(v) = 0$. Considering $u \to t - L(t)$ and $t - L(t) = \bar{t}_1 - L(\bar{t}_1) \Rightarrow \bar{t}_1 = t$, the force $\mathcal{F}_L(t)$ on the left boundary in this region, now relabeled as $\mathcal{F}_L^{(\mathrm{III})}(t)$, is

$$\mathcal{F}_L^{(\mathrm{III})}(t) = f^{(s)}\left\{1 + \frac{1}{A_L(t)}\right\} - \frac{B_L(t)}{A_L(t)}. \tag{20}$$

Considering the limit $L_0 \to \infty$ we recover the quantum radiation force $\mathcal{F}_q^{(+\mathrm{u})}$ corresponding to the unbounded field, acting on the right side of a single mirror: $\lim_{L_0\to\infty}\mathcal{F}_L^{(\mathrm{III})}(t) = \mathcal{F}_L^{(+\mathrm{u})}(t)$, where

$$\mathcal{F}_q^{(+\mathrm{u})}(t) = -\frac{B_q(t)}{A_q(t)}. \tag{21}$$

From Eqs. (19) and (21) we recover the total quantum force $\mathcal{F}_q^{(\mathrm{u})}(t)$ acting on a single mirror at vacuum, with a prescribed trajectory $x = q(t)$:

$$\begin{aligned}\mathcal{F}_q^{(\mathrm{u})}(t) &= \mathcal{F}_q^{(-\mathrm{u})}(t) + \mathcal{F}_q^{(+\mathrm{u})}(t)\\ &= (1+\dot{q}^2)\{[\ddot{q}^2\dot{q}/(2\pi)]/(1-\dot{q}^2)^4\\ &\quad + [\dddot{q}/(6\pi)]/(1-\dot{q}^2)^3\},\end{aligned}$$

which is in agreement with that found in the literature (see Ref. [20]). In the nonrelativistic limit, we reobtain the approximate quantum radiation force $\mathcal{F}_q^{(\mathrm{u})}(t) \approx \dddot{q}/(6\pi)$ [22].

To compute the total forces $\mathcal{F}_R^{(\mathrm{tot})}$ and $\mathcal{F}_L^{(\mathrm{tot})}$ acting on, respectively, the right and left boundaries, for any of the regions II, III, or IV shown in Fig. 1, we need, in addition to Eqs. (16) and (17), to take into account the remaining dynamical Casimir forces corresponding to the vacuum field outside the cavity, which are given by Eqs. (19) and (21). We write

$$\mathcal{F}_R^{(\mathrm{tot})} = \mathcal{F}_R(t) + \mathcal{F}_R^{(+\mathrm{u})}(t), \tag{22}$$

$$\mathcal{F}_L^{(\mathrm{tot})} = \mathcal{F}_L(t) + \mathcal{F}_L^{(-\mathrm{u})}(t). \tag{23}$$

Equations (22) and (23) enable us to calculate directly and analytically the total quantum forces acting on both mirrors for arbitrary laws of motion $R(t)$ and $L(t)$, in regions II or III, because for these regions Eqs. (16) and (17) are replaced by their particular cases given by Eqs. (18) and (20).

In region IV (see Fig. 1), in general it is difficult to obtain exact analytical results for the quantum forces (16) and (17), for arbitrary trajectories $R(t)$ and $L(t)$. The difficulty is in solving equations like $t_1 - R(t_1) = t_2 - L(t_2)$ [see Eq. (11)], which arise after a second reflection ($n_G \geq 2$ or/and $n_F \geq 2$). Trajectories can be constructed to give analytical solutions to these equations, but a large class of relevant laws of motion do not result in exact analytical solutions. However, our results enable us to obtain exact numerical results for the quantum force acting on the moving boundaries of a cavity for an arbitrary law of movement, including nonoscillating movements with large amplitudes, which are out of reach of the perturbative approaches found in the literature, as we will examine next. In this context, let us apply our formulas to the following particular nontrivial trajectory, which is based on the one proposed by Haro in Ref. [23]:

$$L(t) = \kappa_L \ln[\cosh(t)], \tag{24a}$$

$$R(t) = L_0 + \kappa_R \ln[\cosh(t)]. \tag{24b}$$

Considering, for instance, $\kappa_R = -\kappa_L = 0.1$ [Fig. 4(a)], we have an expanding cavity with large amplitude and relativistic velocities. If we consider $\kappa_R = \kappa_L = 0.1$ [Fig. 4(b)], we have the mirrors in movement with relativistic velocities, but keeping constant the cavity length.

In Figs. 5 and 6, using our formulas (9)–(13) and (22), we plot the time evolution of the quantum force $\mathcal{F}_R^{(\mathrm{tot})}(t)$ and $\mathcal{F}_R(t)$ for, respectively, the cases $\kappa_R = -\kappa_L = 0.1$ [see Fig. 4(a)], and $\kappa_R = \kappa_L = 0.1$ [see Fig. 4(b)]. We can see discontinuities of the derivatives for $\mathcal{F}_R^{(\mathrm{tot})}$ and $\mathcal{F}_R(t)$. These discontinuities always occur when the front of the wave in the energy density meets the right boundary. In the case, for instance, shown in Fig. 5, when $t = 0$ the left boundary starts to move and generate a wave in the energy density, propagating rightward and meeting the right boundary at the instant $t = \tau_1 \approx 1.05$, calculated via equation $\tau_1 - R(\tau_1) = 0$, and which corresponds to the first discontinuity of the derivative shown in Fig. 5. At $t = 0$, another front of wave is generated by the right boundary, propagating leftward and meeting the left

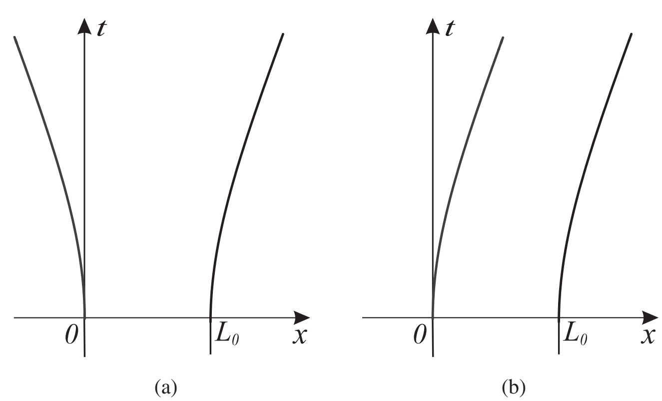

FIG. 4. The solid lines show the boundaries trajectories described in Eq. (24). Part (a) describes the case $\kappa_R = -\kappa_L = 0.1$, whereas (b) describes the case $\kappa_R = \kappa_L = 0.1$.

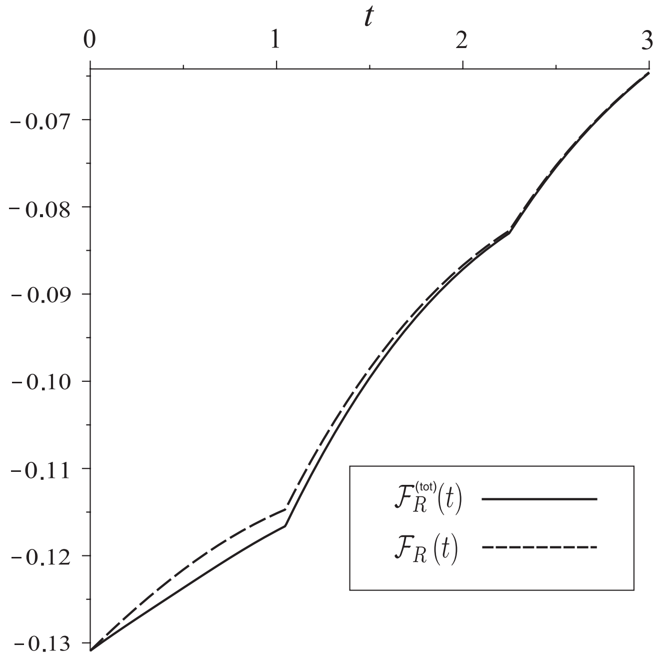

FIG. 5. The solid line shows the total force $\mathcal{F}_R^{(\mathrm{tot})}(t)$, whereas the dashed line shows the force $\mathcal{F}_R(t)$, both for the law of movement (24), with $\kappa_R = -\kappa_L = 0.1$ and $L_0 = 1$.

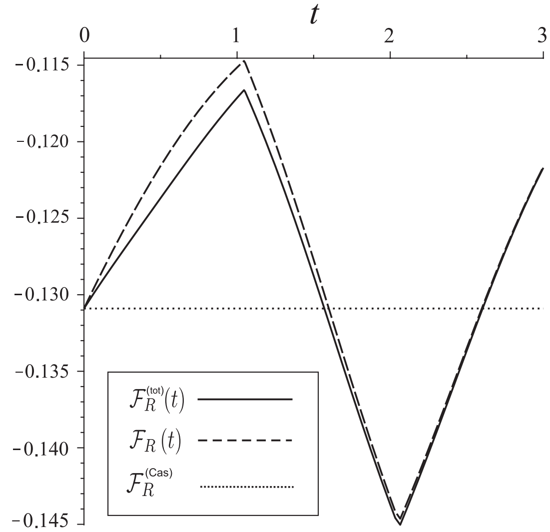

FIG. 6. The solid line shows the total force $\mathcal{F}_R^{(\mathrm{tot})}(t)$, the dashed line shows the force $\mathcal{F}_R(t)$, both for the law of movement (24), with $\kappa_R = \kappa_L = 0.1$ and $L_0 = 1$. The dotted line shows the static Casimir force.

boundary at the instant $\tau_1 \approx 1.05$, and then reflected back and meeting the right boundary at the instant $\tau_2 \approx 2.25$, calculated from the equation $\tau_2 - R(\tau_2) = \tau_1 - L(\tau_1)$. This instant corresponds to the second discontinuity of the derivative shown in Fig. 5. Since the length of the cavity remains the same in the case shown in Fig. 4(b), the quantum force $\mathcal{F}_R^{(\mathrm{tot})}(t)$ oscillates around the static Casimir force (Fig. 6), whereas it goes to zero in the case shown in Fig. 5, where the boundaries go to an asymptotic behavior of infinity length and constant velocity.

Summarizing our results, the formulas obtained in the present paper enable us to get directly exact values for the energy density of the field and the quantum force acting on the boundaries of a nonstatic cavity, for arbitrary laws of motion for the moving boundaries and vacuum as the initial state of the field. Equations (5a) and (5b) are an extension of the corresponding equation for a cavity with just one moving boundary found in Ref. [19], and the achievement of f_G and f_F recursively, tracing back a sequence of null lines, can be viewed as an extension of the work done in Ref. [15]. Formulas (9), (10), and (13) generalize those found in Ref. [18].

From the formulas obtained here, the required calculations to obtain $\langle T_{00}(t, x)\rangle$ consist of solving the set of equations (11) for $z = t + x$, and directly using the results in Eqs. (8), (9), (10a), and (10b), as well as solving (11) for $z = \bar{t}_1 + L(\bar{t}_1)$, where $t - x = \bar{t}_1 - L(\bar{t}_1)$, and directly using the results in Eqs. (12), (13a), and (13b). In contrast, the usual way found in the literature requires the solution of (11) for $t + x$, and also of the set of equations including the first, second, and third derivatives of (11); the use of these results to calculate G', G'', and G''', according to the recursive formula for G found in Ref. [15]; to get the solution of (11) for $z = \bar{t}_1 + L(\bar{t}_1)$, where $t - x = \bar{t}_1 - L(\bar{t}_1)$, and again it is necessary to solve the set of equations formed by the first, second and third derivatives of (11); finally, the use of these results to calculate F', F'', and F''', from the formula for F also given in Ref. [15]. Then, the method presented here is substantially more direct and can be straightforward inserted into a computer routine to provide numerical results for the energy density. Moreover, Eq. (15) has the advantage of exhibiting a clear structure for the energy density: the information about the vacuum energy density (given by the boundary condition) is stored in $f^{(s)}$, whereas all information about the motion of the mirrors is stored in the functions $\tilde{A}$ and $\tilde{B}$.

For the particular cases of a cavity with just one moving boundary, nonrelativistic velocities, or in the limit of infinity length of the cavity (a single mirror), our results are in agreement with those found in the literature [2,18–20,22]. The present results enable investigation of several problems (usually treated by perturbative approaches in the literature) with an exact approach and also out of the regime of small amplitudes. For instance, those related to the inertial forces in the Casimir effect with two moving mirrors [12], or the interference phenomena in the photon production [9]. These issues are under investigation and will be discussed in future papers.

We acknowledge A. L. C. Rego, C. Farina, and P. A. Maia Neto for valuable discussions. We are grateful to C. Farina, A. L. C. Rego, and H. O. Silva for careful reading of this paper. This work was supported by CNPq and CAPES–Brazil.

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