Gravitationally induced zero modes of the Faddeev-Popov operator in the Coulomb gauge for Abelian gauge theories

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It is shown that on curved backgrounds, the Coulomb gauge Faddeev-Popov operator can have zero modes even in the Abelian case. These zero modes cannot be eliminated by restricting the path integral over a certain region in the space of gauge potentials. The conditions for the existence of these zero modes are studied for static spherically symmetric spacetimes in arbitrary dimensions. For this class of metrics, the general analytic expression of the metric components in terms of the zero modes is constructed. Such expression allows one to find the asymptotic behavior of background metrics, which induce zero modes in the Coulomb gauge, an interesting example being the three-dimensional anti–de Sitter spacetime. Some of the implications for quantum field theory on curved spacetimes are discussed.

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I. INTRODUCTION

Black hole physics as well as the description of the early Universe often require the definition of a quantum field theory on a curved background. This can be particularly subtle when the theory is of the Yang-Mills type. Indeed, in order to define the classical and quantum dynamics of a gauge theory, it is necessary to perform a gauge fixing. This is already known to be a nontrivial issue in flat spacetime for non-Abelian gauge theories. The Gribov ambiguity [1], whose appearance is closely related to the nontrivial topology of the space of non-Abelian gauge connections, prevents one from achieving a global gauge fixing in linear derivative gauges like Lorenz, Coulomb, and Landau gauges.[1] These copies appear when the gauge potential is large enough, in a sense that is explained in the following section. Another way to see this problem is to notice that whenever copies appear, the Faddeev-Popov (FP) operator acquires zero modes, so that the FP determinant vanishes, and this prevents one from defining properly the path integral. Since the presence of the copies in QCD is a nonperturbative phenomenon one can safely compute Feynman diagrams in linear derivative gauges for the perturbative regime. Indeed, it was pointed out for the first time in [1], that the Gribov ambiguity could also provide a natural explanation for confinement in QCD, once the path integral is restricted to a region in which the FP operator is positive definite and therefore free of zero modes (see e.g. [2]; other reviews containing more recent results are [3–5]). These ambiguities in the global fixing, of a gauge which is linear in the derivatives, may also arise in the context of Abelian gauge theories [6], since they can also get a nontrivial structure of the fiber bundle due, for example, to finite temperature effects [7,8].

All the cited examples have in common that at least for small enough vector potentials, the Coulomb or the Euclidean Lorenz gauge fixings are well defined. The aim of this work is to point out the existence of an infinite number of zero modes for the FP operator for the Coulomb gauge, on certain curved spacetimes, independently of whether the gauge connection is large or not.[2] This is due to the basic fact that on a curved background, the gauge fixings which are linear in the derivative are defined in terms of the covariant derivative instead of a simple partial derivative. Therefore the equation for the zero modes of the Faddeev-Popov operator depends explicitly on the metric tensor of the background, through the Christoffel symbols. In particular in the Coulomb gauge, the equation for the zero modes of the FP operator is the curved Laplace equation, $\nabla_i \nabla^i U = 0$, where i is an index on the spacelike section, and in a curved spacetime, nonsingular solutions to this equation may exist. These solutions may fulfill strong enough falloff conditions in order to define a proper (normalizable) gauge transformation. When this occurs, the restriction of the path integral to some region of the functional space of vector potential does not help in eliminating the zero modes, since these appear due to the properties of the background.

Here the appearance of these "gravitationally" induced zero modes is studied for static spherically symmetric backgrounds in arbitrary dimensions. We construct a static spherically symmetric background, which generates zero modes of the Coulomb gauge FP operator. We probe also

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[1]The mentioned gauge conditions, being linear in the derivative of the gauge field, define suitable gauge fixings in the Euclidean path integral framework.

[2]There are many other results that also show the subtleties involved in defining a proper quantization of gauge theories on curved spacetimes, as for example [9]; the classic review of these effects being [10].

that the three-dimensional anti–de Sitter (AdS_3) spacetime belongs to this family. This is of special importance for the AdS/CFT correspondence [11].

The structure of the paper is the following: in Sec. II, an introduction to the subtleties of gauge fixing in Yang-Mills theory is given. In Sec. III, we discuss a set of necessary conditions that a spherically symmetric spacetime should satisfy in order to induce zero modes of the FP operator in the Coulomb gauge. In Sec. IV, a number of simple examples are considered, including AdS_3, the three-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole [12], and a four-dimensional wormhole [13]. Some discussion is given and possible solutions to the problem of gauge fixing are finally drawn in the last section.

II. GAUGE-FIXING PROBLEMS IN NON-ABELIAN YANG-MILLS THEORY

In this section a short discussion of the well-known gauge-fixing problems in non-Abelian Yang-Mills theory will be presented. These results will suggest later a very natural and simple solution (yet, the physical consequences of such a solution are highly nontrivial) to the gravitationally induced gauge-fixing problems analyzed in the next sections.

The appearance of zero modes in the FP operator in non-Abelian Yang-Mills on flat backgrounds is related to the well-known Gribov problem. In theories with local symmetries the path integral measure is ill-defined: the action is invariant under a very large class of local symmetries acting on the fields, so that path integrating over all the field configurations gives rise to an overcounting, since many configurations are gauge equivalent. The idea is then to choose only one field configuration in each equivalence class, which is achieved by a gauge-fixing term. The gauge fixing is in general not well defined, since, when the field amplitude is large enough, the gauge-fixing functional may have the same value when evaluated in gauge-equivalent configurations, violating then the assumption that only one configuration in each class will fulfill the gauge-fixing expression. In gauge theories such a problem can be avoided, for instance, by using the axial gauge or the temporal,[3] which possesses its own subtleties (see, for instance, [14]).

The Yang-Mills Lagrangian depends on a Lie algebra valued differential one-form A_μ^a (the most studied case being the one in which the vector potential is in the adjoint representation of the gauge group) as follows:

$$L = -\frac{1}{4g^2}\,\mathrm{tr}F_{\mu\nu}F^{\mu\nu},$$
$$(F_{\mu\nu})^a = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^a.$$

[3]Other methods to deal with this issue can be found in the original reference [1] and, for instance, in the review [4].

Such a Lagrangian is invariant under the following (finite) gauge transformations:

$$A_\mu \rightarrow A'_\mu = U^\dagger A_\mu U + U^\dagger \partial_\mu U, \qquad U^\dagger = U^{-1}, \quad (1)$$

where U is a group-valued scalar.[4]

Because of the above mentioned invariance, the Lagrangian contains nonphysical degrees of freedom which would imply an overcounting of configurations in the Euclidean path integral, and for this reason a gauge fixing is needed. Two of the most common gauge fixings are the Euclidean Lorenz gauge and the Coulomb gauge, respectively, given by

$$\partial_\mu A^\mu = 0; \qquad \partial_i A^i = 0, \quad (2)$$

where $\mu = 0, \ldots, 3$, while the spatial indices go from $i = 1, \ldots, 3$. A natural question then arises: does condition (2) uniquely fix the vector potential for each class of configurations which are related by gauge transformations with suitable falloff conditions at infinity? As it was shown by Gribov, the answer is in general negative and many gauge-equivalent fields can satisfy the same Coulomb, Euclidean Lorenz, or Landau gauge condition. As stated above, these gauge fields are related by gauge transformations which are well defined, and this will produce an overcounting of configurations in the path integral approach.

Let us consider the Coulomb gauge. In order for two gauge-equivalent fields $A^\mu_{(1)}$ and $A^\mu_{(2)}$ to satisfy the gauge-fixing condition, a group-valued scalar function U has to exist and it should fulfill the following equations:

$$A_i^{(2)} = U^\dagger A_i^{(1)} U + U^\dagger \partial_i U \quad (3)$$

$$\partial^i A_i^{(2)} = \partial^i A_i^{(1)} = 0. \quad (4)$$

On flat spacetime, in the case of non-Abelian gauge theories, even when $A_{(1)}$ vanishes, there could exist solutions of Eqs. (3) and (4), giving rise to $A_{(2)}$ whose spatial components decrease at spatial infinity as $r^{-\alpha}$ (r being the Euclidean distance and $\alpha > 0$). Anyway these solutions do not decrease fast enough to define a proper gauge transformation. Even more, a proper gauge transformation must not only decrease fast enough but must also be nonsingular everywhere. However, if $A_{(1)}$ is large enough:

$$\|A_{(1)}\| \gtrsim \|A_G\|, \quad (5)$$

where A_G^μ is the critical Gribov field and $\|.\|$ is a suitable norm,[5] then Eqs. (3) and (4) have smooth solutions in which $A_{(2)}$ decreases rapidly at infinity and therefore the gauge field is affected by the Gribov ambiguity. As s well known, the Gribov ambiguity affects a proper definition of the path integral, since due to the presence of nontrivial

[4]In this section the $SU(N)$ case will be considered and the spacetime will be considered flat and Euclidean.

[5]This norm is discussed in detail in the classic papers [15–18].

Gribov copies, the FP operator acquires zero modes and then the FP determinant vanishes.

In the path integral approach, a possible solution for this ambiguity is to restrict the integration range in the functional space to a region in which the FP operator is positive definite. In such a way the gauge related configurations would be counted only once (a detailed analysis of this approach can be found in [15–18]).

On the other hand, in the Abelian case on flat backgrounds, being

$$U = \exp(i\phi), \tag{6}$$

the condition for the appearance of zero modes of the FP operator in the Euclidean Lorenz gauge reduces to

$$\Box_E \phi = 0, \tag{7}$$

while in the Coulomb gauge reduces to

$$\Delta\phi = 0. \tag{8}$$

Here Δ is the three-dimensional Laplace operator and $\Box_E$ is the Euclidean d'Alambert operator, in which the time has been Wick-rotated $t \rightarrow it$.

This means that the gauge field does not enter explicitly in the equation for the Gribov copies, so that in the Abelian case the existence of the Gribov ambiguities cannot be eliminated by restricting the path integral to a given region as it occurs in the non-Abelian case. As is well known, on flat spacetime, suitable boundary conditions make invertible both $\Box_E$ and Δ, or in other words, there are no nonsingular solutions with a strong enough falloff condition of the Laplace equation, avoiding the appearance of copies in Coulomb and Euclidean Lorenz gauges. However, on curved spacetimes, and depending on the exact form of the metric, nonsingular and normalizable copies may exist. This is shown in the next section.

In the Coulomb gauge, suitable boundary conditions (see, for instance, [15–19]; for two detailed reviews, see also [3,4]) on the gauge transformation are

$$\phi \underset{r\rightarrow\infty}{\longrightarrow} 0, \tag{9}$$

$$\mathcal{N}(\phi) := \int_\Sigma \sqrt{\sigma} d^3\Sigma (\nabla_i\phi)(\nabla^i\phi) < \infty, \tag{10}$$

where σ is the determinant of the induced metric on the spatial section Σ. On flat spacetimes, ∇_i reduces to the partial derivative ∂_i and the indices are raised and lowered a with Minkowski metric. On the other hand on a curved background, the metric $g_{\mu\nu}$ and the corresponding covariant derivatives have to be used. The above condition (10) on the norm is derived from the requirement that, if before the gauge transformation one has

$$\int_\Sigma \sqrt{\sigma} d^3\Sigma A_i A^i < \infty, \tag{11}$$

then, after the gauge transformation,

$$A_i \rightarrow (A^\phi)_i = A_i + \partial_i\phi, \tag{12}$$

and one should also have

$$\int_\Sigma \sqrt{\sigma} d^3\Sigma (A^\phi)_i (A^\phi)^i < \infty. \tag{13}$$

In other words, one should ask for $\nabla_i\phi$ to satisfy the same falloff conditions that one requires for A. Unlike the non-Abelian case, to avoid zero modes of the FP operator in the Abelian case, it is enough to consider nonsingular fields decreasing at infinity and to notice that, on flat backgrounds, the Laplace operator does not possess everywhere smooth and nonsingular solutions, decreasing fast enough at infinity.

On the other hand, when defining the path integral on a curved background in the Coulomb gauge for an Abelian gauge theory, the FP determinant explicitly appears and is given by

$$J = \det\nabla_i\nabla^i, \tag{14}$$

which represents the Jacobian of the change of coordinates in the functional space. Therefore, if there exist nontrivial and everywhere smooth regular solutions of the equation

$$\nabla_i\nabla^i\phi = 0, \tag{15}$$

then the determinant in Eq. (14) vanishes and the path integral in the Coulomb gauge cannot be defined properly.[6] On curved backgrounds, the operator ∇_i is the covariant derivative along the spatial directions on which one would like to define a path integral quantization of an Abelian gauge theory. Therefore, in some interesting cases nontrivial smooth-everywhere regular solutions of Eq. (15) do indeed exist as will be shown in the next sections. One could still try to insist in factoring out the zero modes of the FP operator $\nabla_i\nabla^i$ when computing the determinant: this procedure seems to be reasonable when the operator just has one or few zero modes. Nevertheless in what follows, we will construct examples in which the FP operator $\nabla_i\nabla^i$ has infinitely many zero modes and, in these cases, the factorizing-out procedure appears quite unnatural. It is worth emphasizing that one cannot exclude the existence of zero modes by just imposing stronger asymptotic falloff conditions on the gauge potential since this would correspond to arbitrarily "mutilating" the solution space of the gauge field [20].

III. COULOMB GAUGE IN STATIC SPHERICALLY SYMMETRIC SPACETIMES

Let us consider a static spherically symmetric metric in d dimensions of the form

[6] In flat spacetimes the operators ∇_i are nothing but the usual partial derivatives ($\nabla_i = \partial_i$) so that, as has been already discussed, the FP determinant $J = \det\partial_i\partial^i$ does not vanish.

$$ds_d^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2, \tag{16}$$

where the spatial section is the warped product of the real line and a $d-2$-dimensional sphere S^{d-2}.

The Coulomb gauge condition reads

$$\nabla^i A_i = 0, \tag{17}$$

where ∇^i is the covariant derivative along the spatial directions ($i = 1, 2, \ldots, d-1$) of the metric (16), and the A_i are the spatial components of the Abelian vector potential. Therefore, the equation which determines the appearance of a zero mode ϕ of the FP operator in the Coulomb gauge simply reduces to the Laplace equation on the spatial section of (16), i.e.,

$$\nabla^i \nabla_i \phi = 0. \tag{18}$$

Let us now decompose the gauge parameter ϕ as follows:

$$\phi = F(r)Y(S^{d-2}), \tag{19}$$

where $Y(S^{d-2})$ is a generalized spherical harmonic on the $d-2$-dimensional sphere

$$\nabla^2_{S^{d-2}} Y = -QY, \tag{20}$$

with Q being the corresponding eigenvalue $l(l+d-3)$. Equation (18) reduces to

$$2fF'' + (2(d-2)f + rf')r^{-1}F' - 2Qr^{-2}F = 0, \tag{21}$$

where prime denotes derivative with respect to the argument.

Interestingly enough, Eq. (21) allows one to integrate the metric function $f(r)$ in terms of the radial part $F(r)$ of the zero mode:

$$f(r) = \frac{r^{-2d+4}(\int^r Q(F^2(x))' x^{2d-6} dx + I_1)}{(F')^2}, \tag{22}$$

where I_1 is an integration constant. This simple relation allows one to draw some general conclusions about how a static spherically symmetric gravitational field has to behave, in order to induce zero modes of the FP operator in the Coulomb gauge. In other words, assuming that ϕ is an everywhere-regular smooth gauge transformation, it is possible to determine the asymptotic behavior of the metric for $r \to \infty$. It is worth stressing here the close similarity of the present procedure, which allows one to express the gravitational field in terms of the zero modes of the FP operator, with the well-known technique in supersymmetric quantum mechanics, which allows one to write the quantum Hamiltonian in terms of the corresponding zero mode (see, in particular, [21–23]).

IV. SOME COMMENTS ON THE GENERIC BEHAVIOR OF BACKGROUNDS INDUCING ZERO MODES OF THE FP OPERATOR

In this section we will concentrate on the asymptotic behavior of the metric function $f(r)$ obtained from Eq. (22) at infinity, i.e., when $r \to \infty$.

Since the hypothesis is that the zero mode $\phi = F(r)Y(S^{d-2})$ is everywhere regular and smooth, let us assume that for $r \to \infty$, the behavior of $F(r)$ is

$$F(r) \underset{r\to\infty}{\approx} c_0 + \frac{c_1}{r^n} + \cdots, \qquad n > 0, \qquad c_0 \neq 0, \qquad c_1 \neq 0, \tag{23}$$

where $(\cdots)$ denotes subleading terms and c_0, c_1 are constants (the analysis in the cases in which c_0 and/or c_1 vanish/es follows along the same lines). Then

$$F^2 \underset{r\to\infty}{\approx} (c_0)^2 + \frac{2c_0c_1}{r^n} + \cdots, \qquad F' \underset{r\to\infty}{\approx} -\frac{nc_1}{r^{n+1}} + \cdots. \tag{24}$$

Therefore, in the asymptotic region $r \to \infty$ using (22) when $c_0 \neq 0$, one obtains the following expression for $f(r)$:

$$f(r) \underset{r\to\infty}{\approx} \frac{I_1}{c_1^2 n^2} r^{6+2n-2d} - \frac{2Qc_0}{nc_1(2d-n-6)} r^n - \frac{Q}{n(d-n-3)}, \quad n \neq 2d-6, \text{ and } n \neq d-3. \tag{25}$$

Then this is the generic expression for the metric function $f(r)$ when the zero mode of the FP operator takes the form (23).[7]

On the other hand, when $c_0 = 0$ in Eq. (23), one obtains

$$f(r) \underset{r\to\infty}{\approx} \frac{I_1}{c_1^2 n^2} r^{6+2n-2d} - \frac{Q}{n(d-n-3)}, \qquad n \neq d-3. \tag{26}$$

The requirement that on the spacetime (16), the norm (10) of the gauge transformation $\phi = F(r)Y(S^{d-2})$ has to remain finite reduces to

$$\mathcal{N}(\phi) := \left[\int dr r^{d-2} f^{1/2}(F')^2 + r^{d-4} f^{-1/2} F^2 Q\right] \times \left[\int_{S^{d-2}} Y^* Y \sqrt{\Omega_{d-2}} d^{d-2}x\right] < \infty. \tag{27}$$

The angular integral is finite and so one needs to take care only of the radial integral. When $c_0 \neq 0$, replacing in the expressions (23) and (25) in the norm (27), one concludes that the convergence of the latter is guaranteed when

$$n > 2(d-3) \quad \text{or} \tag{28}$$

[7] For the special values $n = 2(d-3)$ or $n = d-3$, logarithmic branches may appear, and the analysis follows along the same lines.

$$n \leq 2(d-3) \quad \text{and} \quad Q = 0 \quad \text{with} \quad I_1 \neq 0. \tag{29}$$

For $c_0 = 0$, the expression for the metric function $f(r)$ is given by Eq. (26) and the norm (27) converges if

$$n > \frac{d-3}{2}. \tag{30}$$

The case with $d = 3$ must be analyzed separately: the metric function $f(r)$ and the radial part $F(r)$ of the zero mode are given by (22) and (23), respectively. In this case, the norm (27) always converges at infinity. In the next section, some explicit three-dimensional spacetimes which induce zero modes of the FP operator will be constructed.

Based on the above discussion, it is possible to draw some interesting conclusions.

If the spacetime is asymptotically locally flat, $f(r)$ in (16) should behave as follows:

$$f \underset{r\to\infty}{\approx} 1 + \frac{c}{r^p} + \cdots, \qquad p > 0, \tag{31}$$

with c and p being constants. Thus, in such cases, induced zero modes of the FP operator in the Coulomb gauge could appear only when $c_0 = 0$ and $n < d - 3$ [see Eq. (26)]. However when $c_0 = 0$, the finiteness of the norm $\mathcal{N}(\phi)$ in (27) implies Eq. (30), and consequently[8] in this case zero modes of the FP operator would exist provided

$$(d-3)/2 < n < d - 3. \tag{32}$$

From this analysis it is also clear that an asymptotically Schwarzschild spacetime, for which the asymptotic behavior of $f(r)$ is given by (31) with $p = d - 3$, will not generate zero modes of the FP operator. This is due to the fact that comparing (26) and (31) one obtains $n = (d - 3)/2$, giving rise to a non-normalizable zero mode [see Eq. (32)].

Another interesting asymptotic behavior is defined by the AdS spacetimes. In AdS and asymptotically AdS metrics, fields propagating in the bulk have been conjectured to be dual to a conformal field theory in the boundary [11]: the so-called AdS/CFT correspondence opens the remarkable possibility to explore the nonperturbative regime of supersymmetric Yang-Mills theories by performing semiclassical computations in the bulk of asymptotically AdS background (see also [24]). Furthermore, this idea also has been used to give a consistent microscopic interpretation of the black hole entropy. In particular, unlike the higher dimensional cases, in $2 + 1$ dimensions the algebra of the asymptotic symmetries of an asymptotically AdS spacetime is enlarged to two copies of the infinite dimensional Virasoro algebra, whose central charge can be used to compute the entropy of the BTZ black hole [12] by means of the Cardy formula [25,26]. These reasons make AdS_3 as well as the BTZ black hole two of the most important curved geometries in high-energy physics. In the next section, it will be shown that the problem of finding normalizable zero modes of the FP operator both on AdS_3 and on the BTZ black hole can be solved in a quite elegant and simple way. It also will be shown how to construct normalizable zero modes of the FP operator on an interesting four-dimensional wormhole spacetime.

[8]Note that again these bounds suggest that the case $d = 3$ should be analyzed separately.

V. EXPLICIT EXAMPLES OF BACKGROUNDS GENERATING ZERO MODES OF THE FP OPERATOR

A. AdS_3 spacetime

As mentioned above, the three-dimensional AdS spacetime is of special interest for high-energy physics in view of the AdS/CFT correspondence [11]. The AdS_3 metric is given by

$$ds^2 = -\left(\frac{r^2}{l^2} + 1\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + 1} + r^2 d\psi^2, \tag{33}$$

where $-\infty < t < +\infty$, the radial coordinate runs in the range $0 \leq r < +\infty$, and the angular coordinates fulfill $0 \leq \psi \leq 2\pi$, 0 being identified with 2π. This spacetime is of constant and negative curvature. It solves the Einstein equations with a negative cosmological constant $\Lambda = -\frac{1}{l^2}$, where l is the AdS radius. Assuming the separation (19) for the zero mode, where now the spherical harmonics are

$$Y(S^1) \sim e^{im\psi}, \tag{34}$$

with m an integer, Eq. (18) reduces to

$$2r^2 f F'' + r(2f + rf')F' - 2m^2 F = 0, \tag{35}$$

where $f(r) = \frac{r^2}{l^2} + 1$. For positive m, the regular solution at the origin reads

$$F(r) = C\frac{(r/l)^m}{\left(1 + \sqrt{\frac{r^2}{l^2} + 1}\right)^m}, \tag{36}$$

C being an integration constant.[9] This solution is regular everywhere and when r goes to infinity $F(r)$ tends to the constant C. The norm (27) in this case reduces to

$$\mathcal{N}(\phi) = 4\pi C^2 m^2 \int_0^\infty \frac{l r^{2m-1}(r^2 + 2l^2 + 2l\sqrt{r^2 + l^2})}{\sqrt{r^2 + l^2}(l + \sqrt{r^2 + l^2})^{2m+2}} dr. \tag{37}$$

This integral converges, since the integrand (denoted by H_{AdS_3}) is smooth, it goes to zero at the origin and also goes to zero when r goes to infinity as

[9]For negative m the solution which is regular at the origin is precisely the one we discarded in the case with m positive, and the solution takes the same expression.

$$H_{\mathrm{AdS}_3} \sim_{r\to\infty} \frac{1}{r^2} + O(1/r^3). \tag{38}$$

Thus AdS_3 generates infinite zero modes of the FP operator. This result is particularly relevant for the AdS/CFT conjecture, and it would be nice to further analyze the consequences of the existence of these zero modes from the point of view of the dual-boundary CFT.

B. Zero modes of the FP operator on the BTZ black hole

The metric for the static BTZ black hole is given by [12]

$$ds^2 = -\left(\frac{r^2}{l^2} - \mu\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - \mu} + r^2 d\psi^2, \tag{39}$$

with $0 < \psi \leq 2\pi$, and 0 identified with 2π. This spacetime describes an asymptotically AdS black hole whose mass is proportional to the parameter μ. The corresponding event horizon is located at $r = r_+ = l\sqrt{\mu}$. The metric (39) can be also obtained by an identification of the three-dimensional AdS_3 spacetime (33) [27]. For our purposes, we will consider only the exterior part of the black hole (39),

$$l\sqrt{\mu} < r < +\infty, \tag{40}$$

since the Euclidean continuation of the black hole metric (which is necessary to achieve a proper definition of black hole thermodynamics) covers only the region outside of the event horizon. It is useful to perform the change of coordinates

$$r = \frac{l\sqrt{\mu}}{\sqrt{1-x^2}}, \tag{41}$$

with $0 < x < 1$, the surfaces $x = 0$ and $x = 1$ being the horizon and spatial infinity, respectively. With this change of coordinates the BTZ metric (39) reduces to

$$ds^2 = -\frac{\mu x^2}{1-x^2}dt^2 + \frac{l^2 dx^2}{(1-x^2)^2} + \frac{l^2\mu}{1-x^2}d\psi^2. \tag{42}$$

Assuming that the angular part of the zero mode is given by (34), Eq. (18) for the corresponding radial part reads

$$\mu(1-x^2)F''(x) - \mu x F'(x) - m^2 F(x) = 0. \tag{43}$$

The solution of the above equation is given by

$$F(x) = \alpha_1(x + \sqrt{x^2-1})^{(im/\sqrt{\mu})} + \alpha_2(x + \sqrt{x^2-1})^{-(im/\sqrt{\mu})}, \tag{44}$$

α_1 and α_2 being integration constants. The norm in this case reduces to

$$\mathcal{N}(\phi) = 4\pi^2\alpha_1\alpha_2\mu^{-1/2}m^2, \tag{45}$$

being finite for any value of[10] m.

C. FP zero modes in a four-dimensional wormhole spacetime

Wormholes are some of the most fascinating curved backgrounds since they possess two (or more) asymptotic regions connected by one (or more) throat(s). Let us consider the four-dimensional spherically symmetric wormhole metric

$$ds^2 = -dt^2 + d\rho^2 + \cosh^2\rho d\Omega_2^2, \tag{46}$$

where $d\Omega_2^2$ is the line element for the two-sphere. The two asymptotic regions possess geometries locally equivalent to $\mathcal{R} \times H_3$ (H_3 being a negative three-dimensional constant curvature manifold) and are located at $\rho \to \pm\infty$. These boundaries are connected by the throat being located at $\rho = 0$. As shown in Ref. [13], this spacetime is a vacuum solution of Weyl conformal gravity in four dimensions[11]: in this theory the field equations reduce to the vanishing of the Bach tensor (for some relevant references, see also [32–34]). It is interesting to note that wormhole geometries in four dimensions can also arise in general relativity with matter sources localized around the throat and violating standard energy conditions (this type of source can arise from quantum fluctuations [35]). The metric (46) provides a simple background generating zero modes of the FP operator.

After the change of coordinates $\rho = \tanh^{-1}x$, the equation for the (radial part of the) FP zero mode in this case acquires a very simple form,

$$(1-x^2)F''(x) - QF(x) = 0, \tag{47}$$

which is integrated in terms of hypergeometric functions. The norm of the zero mode is finite at both asymptotic regions $x \to \pm 1$ only when $Q = 0$ (the "S-wave"). In this case, the normalizable zero mode is given by

$$F(x) = \beta(1-x), \tag{48}$$

β being an integration constant. In the original coordinate system the FP zero mode reads

$$\phi = \frac{\beta}{\sqrt{4\pi}}(1-x) = \frac{\beta}{\sqrt{4\pi}}(1-\tanh\rho), \tag{49}$$

whose norm reduces to

[10] From Eq. (45), one may think that the massless BTZ black hole, in which $\mu \to 0$, does not possess any normalizable zero mode. Nevertheless analyzing the problem from scratch in such a geometry one is able to show that also in this case infinitely many normalizable zero modes exist.

[11] This wormhole has been extended to diverse dimensions within certain gravity theories in [28–31].

$$\mathcal{N}(\phi) = 2\beta^2. \qquad (50)$$

VI. POSSIBLE SOLUTIONS TO GRAVITATIONALLY INDUCED GAUGE-FIXING PROBLEMS AND FURTHER COMMENTS

In the previous sections, we have discussed the asymptotic behavior of static spherically symmetric curved backgrounds which induces zero modes of the FP operator in the Coulomb gauge and we have also constructed interesting concrete examples. The aim of the previous sections was first to show that this phenomenon is quite generic and second to show that curved backgrounds of great theoretical interest are affected by this problem. The question of how to solve this problem then naturally arises. A possible point of view is that this is not a problem at all since in the present case the metric-dependent FP determinant $\det(-\nabla_i\nabla^i)$ does not depend on the gauge field and, therefore, it factorizes out from the path integral on the gauge field:

$$Z = \int DA_\mu \det(-\nabla_i\nabla^i)\exp(S_{\rm QED}) = \det(-\nabla_i\nabla^i)\int DA_\mu \exp(S_{\rm QED}), \qquad (51)$$

where $S_{\rm QED}$ is the standard QED action. Nevertheless, this point of view is not very appealing for the following reasons.

The first and main reason arises when one considers the semiclassical quantization of gravity[12]; whatever the final theory of quantum gravity is, a semiclassical regime in which the weakly coupled quantum fluctuations of gravity can be treated perturbatively with the usual method of quantum field theory has to exist. In this regime, in which the quantum gravitational effects are not strong (as, for instance, in the first stages of the Hawking evaporation process), one should treat the graviton ($h_{\mu\nu}$) as a quantum fluctuation propagating on a classical gravitational background. This makes it compulsory that, when dealing with QED on curved backgrounds, one should consider the quantum fluctuations of the graviton as well. This point of view is mandatory when one analyzes, for instance, the perturbative regime of supergravity theories in which the graviton $h_{\mu\nu}$ and the photon A_μ are in the same supermultiplet or in the already mentioned black hole evaporation process. Obviously, in this situation, the FP operator does not factor out from the path integral (on both A_μ and $h_{\mu\nu}$) anymore. The second reason is that, even in the context of QED on a fixed curved background when Eq. (51) holds, the proper definition of the FP determinant may be problematic. Indeed, some regularization procedure is needed otherwise the path integral would be just multiplied by zero. Furthermore, already at classical level the proper definition of the canonical formalism through the Dirac brackets would be ill defined since the inverse of the spacelike Laplacian appears in the denominator of the right-hand side of the Dirac brackets [37]. The usual procedure to achieve this for the Coulomb gauge on flat spacetime would be to regularize the FP determinant by considering the functional space of functions orthogonal to the zero mode (which on flat space is the constant function). Such a procedure is viable since it does not imply any unphysical mutilation of the space of solutions. On the other hand, if the FP operator admits nontrivial zero modes, this procedure is quite subtle since by restricting the functional space to functions orthogonal to the nontrivial zero modes one could discard a physical part of the functional space. Careless mutilations of the space of solutions are known to lead to unphysical results: the soundest point of view is to take as physical boundary conditions the ones that lead to a well-defined variational principle [20]. It seems that this criterion is not enough to eliminate from the functional space the zero modes which have been presented here since they satisfy all the required boundary conditions. Eventually, when a classical background induces an infinite number of nontrivial zero modes of the FP operator it appears that a suitable definition of the path integral along these lines would not be possible at all.

The first and most obvious solution to this problem would be to follow the same procedure as in non-Abelian gauge theories, i.e., to restrict the path integral to a copy-free region. In the context of the present paper, this means to exclude spacetimes which induce zero modes of the Coulomb gauge FP operator from the path integral.[13] This point of view is strongly supported by the need to take into account the effects of gravitational fluctuations in many situations of interest (such as perturbative quantization of supergravity or the analysis of the Hawking process) as it has been explained in the discussion above. From the point of view of quantum field theory, this does not seem to produce any conceptual difficulty as has been explained in Sec. II. Nevertheless from the point of view of gravitational physics the need to declare, for instance, the AdS_3 and BTZ spacetimes unphysical is extremely problematic. This is due to the fact that one of the most powerful tools for making predictions both on a possible quantum theory of gravity and on a nonperturbative regime in Yang-Mills theory is given by the AdS/CFT correspondence where the fields living in an AdS spacetime (including Abelian fields) can be related to a conformal field theory on the boundary. In the context of black hole

[12] We will not dwell here on which is the final theory of quantum gravity; nevertheless, it is worth noting that the gravitational action in three dimensions is renormalizable (see e.g. [36]).

[13] As is well known, if the Coulomb gauge is pathological, usually other covariant gauges manifest some kinds of pathologies as well (see, for instance, [14,38]).

physics, the case of a three-dimensional bulk spacetime turns out to be especially important. The AdS_3/CFT_2 correspondence is one of the most-used tools to describe the gravitational microstates responsible for the black hole entropy. Therefore, the solution to declare, for instance, the three-dimensional AdS spacetime and the BTZ black hole as unphysical would have highly nontrivial consequences. This issue is worth further investigation.

The second and more ambitious solution to gravitationally induced gauge-fixing problems would be to formulate from the very beginning QED in terms of gauge-invariant variables (such as the Wilson loops) which are not affected by gauge-fixing problems. Even if this would be the most elegant possibility, it is not very practical yet since, up to now, explicit nontrivial computations in field theory in terms of Wilson loops have been performed only in topological field theories.

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