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Bekenstein bound in asymptotically free field theory

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For spatially bounded free fields, the Bekenstein bound states that the specific entropy satisfies the inequality $\frac{S}{E} \leq 2\pi R$, where R stands for the radius of the smallest sphere that circumscribes the system. The validity of the Bekenstein bound in the asymptotically free side of the Euclidean $(\lambda\varphi^4)_d$ scalar field theory is investigated. We consider the system in thermal equilibrium with a reservoir at temperature β^{-1} and defined in a compact spatial region without boundaries. Using the effective potential, we discuss the thermodynamic of the model. For low and high temperatures the system presents a condensate. We present the renormalized mean energy E and entropy S for the system and show in which situations the specific entropy satisfies the quantum bound.

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I. INTRODUCTION

There have been a lot of activities discussing classical and quantum fields in the presence of macroscopic boundaries. These subjects raise many interesting questions. A basic question that has been discussed in this scenario, when quantum fields interact with boundaries, is about the issue that these systems may be subjected to certain fundamental bounds. 't Hooft [1] and Susskind [2], combining quantum mechanics and gravity, introduced the holographic entropy bound $S \leq \pi c^3 R^2/\hbar G$ [3]. This holographic bound relates information not with the volume but with the area of surfaces. Another bound is the Bekenstein bound, which relates the entropy S and the mean energy E of a quantum system with the size of the boundaries that confine the fields. It is given by $S \leq 2\pi ER/\hbar c$, where R stands for the radius of the smallest sphere that circumscribes the system [4–8].

The aim of this paper is to investigate the validity of the Bekenstein bound in systems defined in a compact spatial region without boundaries, described by asymptotically free theories. We study the ordinary Euclidean $(\lambda\varphi^4)_d$ massless scalar field theory, with a negative sign of the coupling constant [9–11]. This field theory is renormalizable in a four-dimensional space-time, asymptotically free, and has a nontrivial vacuum expectation value.

Studying the $(\lambda\varphi^p)_d$ self-interacting massless scalar field theory in the strong-coupling regime at finite temperature, and also assuming that the field is confined in a compact spatial region, a generalization for the Bekenstein bound was obtained by Aparicio Alcalde *et al* [12]. The basic problem that arises in theories with nonlinear fields is the possibility of nonlinear interactions changing the energy spectrum of the system invalidating the quantum bound. Previous works studying the bound in weakly coupled fields can be found in Refs. [13,14]. Bekenstein and Guedelman studied the massless charged self-interacting scalar field in a box and proved that in this case nonlinearity does not violate the bound on the specific entropy. In Ref. [12] it was assumed that the fields are defined in a simply connected bounded region, i.e., a hypercube of size L, where the scalar field satisfies Dirichlet boundary conditions. Working in the strong-coupling regime of the $(\lambda\varphi^p)_d$ field theory and making use of the strong-coupling expansion [15–19], the renormalized mean energy and the entropy for the system up to the order $\lambda^{-(2/p)}$ were found, presenting an analytic proof that the specific entropy also satisfies in some situations a quantum bound. Considering the low temperature behavior of the thermodynamic quantities of the system, it was shown that for negative renormalized zero-point energy, the quantum bound can be invalidated. Note that a still open question is how the sign of the renormalized zero-point energy of free fields described by Gaussian functional integrals depends on the topology, dimensionality of the space-time, shape of bounding geometry, or other physical properties of the system [20–23]. For complete reviews discussing the Casimir effect [24], see, for example, Refs. [25–29].

The purpose of this article is to investigate another physical situation that has not been discussed in the literature. We should note that a step that remains to be derived is the validity of the bound for the case of interacting field theory described by asymptotically free models [30–34], at least up to some order of perturbation theory. This situation of a deconfined field theory with asymptotically free behavior, defined in a small compact region of space, may occur in QCD in the confinement-deconfinement phase transition at high temperatures or if usual matter is strongly

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compressed [35–37]. For a complete review of the subject, see Ref. [38]. In ultrarelativistic heavy ion collisions we expect that the plasma of quarks and gluons can be produced, just after the collision, and that hot and compressed nuclear matter is confined in a small region of space. No practical method had been developed to solve QCD, and therefore the basic question we have presented here remains unanswered, unless we try to describe a simpler model that develops asymptotic freedom for some values of the coupling constant.

In order to investigate the Bekenstein bound in this asymptotically free theory, we assume that the scalar field is confined in a bounded region. Working in the weak-coupling perturbative expansion with the $(\lambda\varphi^4)_d$, we assume periodic boundary conditions in all spatial directions, in order to maintain translational invariance of the model. This same approach was used in Ref. [39]. For papers studying nontranslational invariant systems and analyzing the divergences of the theory, see, for example, Refs. [40–46]. We also assume that the system is in thermal equilibrium with a reservoir and investigate the asymptotic free side of the $(\lambda\varphi^4)_d$ [9,47–51]. In order to study the existence of a quantum bound on the specific entropy, we study the behavior of the specific entropy using the effective action method.

We would like to point out that the theory with a negative coupling constant develops a condensate as was shown by Parisi [52]. In the self-interaction $(\lambda\varphi^4)_d$ field theory, it is possible to find the vacuum energy $E(\lambda)$. This quantity is given by the sum of all vacuum-to-vacuum connected diagrams. In the λ complex plane, the function $E(\lambda)$ is analytic for $\mathrm{Re}(\lambda) > 0$ and the discontinuity on the negative real axis is related to the mean life of the vacuum. For a system with N particles, let us define $E_N(\lambda)$ as the energy of such state. For negative λ, there are collapsed states of negative energy. Defining $\max E_N(\lambda) = E_B$, the probability of the vacuum to decay is e^{-E_B}. The particles on the collapsed state will be described by a classical field φ_0.

The organization of the paper is as follows: In Sec. II we study the effective potential of the theory at the one-loop level. Because of the boundary conditions imposed on the field, there is a topological generation of mass. The topological squared mass depends on the ratio $\xi = \beta/L$, and its sign is critical to the profile of the effective potential. In Sec. III we present our results of the thermodynamic functions and study the validity of the Bekenstein bound in the model. To simplify the calculations we assume the units to be such that $\hbar = c = k_B = 1$.

II. THE EFFECTIVE POTENTIAL AT THE ONE-LOOP LEVEL

Let us consider a neutral scalar field with a $(\lambda\varphi^4)$ self-interaction, defined in a d-dimensional Minkowski spacetime. The generating functional of all vacuum expectation values of time-ordered products of the theory has a Euclidean counterpart, that is the generating functional of complete Schwinger functions. The $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z(h)$ is defined by the following functional integral [53,54]:

$$Z(h) = \int [d\varphi] \exp\left(-S_0 - S_{\mathrm{I}} + \int d^d x h(x)\varphi(x)\right), \quad (1)$$

where the action that describes a free scalar field is given by

$$S_0(\varphi) = \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2} m_0^2 \varphi^2\right). \quad (2)$$

The interacting part, defined by the non-Gaussian contribution, is given by the following term in the action:

$$S_{\mathrm{I}}(\varphi) = \int d^d x \frac{\lambda}{4!} \varphi^4(x). \quad (3)$$

In Eq. (1), $[d\varphi]$ is formally given by $[d\varphi] = \prod_x d\varphi(x)$, and m_0^2 and λ are the bare squared mass and coupling constant, respectively. Finally, $h(x)$ is a smooth function that is introduced to generate the Schwinger functions of the theory by functional derivatives.

We are assuming a spatially bounded system in equilibrium with a thermal reservoir at temperature β^{-1}. Assuming that the coupling constant is a small parameter, the weak-coupling expansion can be used to compute the partition function defined by $Z(\beta, \Omega, h)|_{h=0}$, where h is an external source and we are defining the volume of the $(d-1)$ manifold as $V_{d-1} \equiv \Omega$. From the partition function we define the free energy of the system, given by $F(\beta, \Omega) = -\frac{1}{\beta} \ln Z(\beta, \Omega, h)|_{h=0}$. This quantity can be used to derive the mean energy $E(\beta, \Omega)$, defined as

$$E(\beta, \Omega) = -\frac{\partial}{\partial\beta} \ln Z(\beta, \Omega, h)|_{h=0}, \quad (4)$$

and the canonical entropy $S(\beta, \Omega)$ of the system is given by

$$S(\beta, \Omega) = \left(1 - \beta\frac{\partial}{\partial\beta}\right) \ln Z(\beta, \Omega, h)|_{h=0}. \quad (5)$$

Since the scalar theory with the negative coupling constant develops a condensate, it is convenient to work with the effective potential of the system. As was stressed by Bender *et al.* [10], nonperturbative techniques must be used to find the true vacuum of the system. Therefore, let us study first the effective potential at the one-loop level associated to a self-interacting scalar field defined in a d-dimensional Euclidean space.

Let us consider that the system is in thermal equilibrium with a reservoir at temperature β^{-1}. Therefore, we assume the Kubo-Martin-Schwinger condition [55–58]. We will work with a massless scalar field and assume $d = 4$, and in order to simplify the calculations we impose periodic boundary conditions for the field in all three spatial directions, with compactified lengths L_1, L_2, and L_3. The Euclidean effective potential can be written as

$$V_{\rm eff}(\phi;\beta,L_1,L_2,L_3)=\frac{\mu^4}{3}\pi^2 g\phi^4+U+\text{counterterms}+\frac{1}{\beta\Omega}\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{2s}\times g^s\phi^{2s}Z_4(2s,a_1,a_2,a_3,a_4), \quad (6)$$

where we have defined the quantities $\phi=\varphi/\mu$, $g=\lambda/8\pi^2$, $a_i^{-1}=\mu L_i$ $(i=1,\ 2,\ 3)$, $a_4^{-1}=\mu\beta$, $\Omega=L_1L_2L_3$, and finally $Z_4(2s,a_1,\ldots,a_4)$ is the Epstein zeta function [20]. Note that we have introduced a mass parameter μ in order to keep the Epstein zeta function, Z_4, a dimensionless quantity.

The first contribution to the effective potential given in Eq. (6) is the classical potential. The second contribution $U(\beta,L_1,L_2,L_3)$, is given by

$$U(\beta,L_1,L_2,L_3)=\frac{1}{2\beta\Omega}\sum_{n_1,\ldots,n_4=-\infty}^{\infty,}\times\ln\left(\left(\frac{2\pi n_1}{L_1}\right)^2+\left(\frac{2\pi n_2}{L_2}\right)^2+\left(\frac{2\pi n_3}{L_3}\right)^2+\left(\frac{2\pi n_4}{\beta}\right)^2\right). \quad (7)$$

The prime that appears in Eq. (7) indicates that the term for which all $n_i=0$ must be omitted. We can rewrite Eq. (7) as

$$U(\beta,L_1,L_2,L_3)=\frac{1}{\beta\Omega}\sum_{n_1,\ldots,n_3=-\infty}^{\infty,}(\pi\beta\bar{n}+\ln(1-e^{-2\pi\beta\bar{n}}))+\frac{1}{\beta\Omega}J_1, \quad (8)$$

where we are defining the quantity $\bar{n}(L_1,L_2,L_3)$ and the (infinite) constant J_1 as

$$\bar{n}=\sqrt{\left(\frac{n_1}{L_1}\right)^2+\left(\frac{n_2}{L_2}\right)^2+\left(\frac{n_3}{L_3}\right)^2}, \quad (9)$$

and

$$J_1=\sum_{n_1,\ldots,n_3=-\infty}^{\infty,}\sum_{m=-\infty}^{\infty,}\ln(1+(2\pi m)^2)-\sum_{n_1,\ldots,n_3=-\infty}^{\infty,}(1+2\ln(1-e^{-1})). \quad (10)$$

The last term of the Eq. (6) is explicitly the one-loop correction to the effective potential, defined in terms of the homogeneous Epstein zeta function $Z_p(2s,a_1,\ldots,a_p)$ given in Ref. [20] by

$$Z_p(2s,a_1,\ldots,a_p)=\sum_{n_1,\ldots,n_p=-\infty}^{\infty,}((a_1n_1)^2+\ldots+(a_pn_p)^2)^{-s}. \quad (11)$$

The summation give by Eq. (11) is convergent for $s>p/2$. The homogeneous Epstein zeta function has an analytic extension to the complex plane $s\in C$, except for a pole in $s=p/2$. Since the unique polar contribution occurs for the case in $s=2$, the theory can be renormalized using only a unique counterterm, introduced to renormalize the coupling constant of the theory. Because we are assuming periodic boundary conditions for the field in all spatial directions, it appears as a topological generation of mass, coming from the self-energy Feynman diagram [59–61]. The topological mass is defined in terms of the first renormalization condition given by

$$\left.\frac{\partial^2 V_{\rm eff}}{\partial\phi^2}\right|_{\phi=0}=m_T^2\mu^2. \quad (12)$$

Using the Epstein zeta function, the topological squared mass m_T^2 can be written as

$$m_T^2=\frac{g}{\mu^2\beta\Omega}Z_4(2,a_1,a_2,a_3,a_4). \quad (13)$$

The above result was obtained also by Elizalde and Kirsten [62]. As was discussed by these authors, the topological squared mass depends on the values of the compactified lengths and the temperature. For simplicity we will call this quantity a topological mass. The next step is to study the two cases $m_T^2>0$ and $m_T^2<0$ separately.

A. The positive topological squared mass, i.e., $m_T^2>0$

First, let us write the effective potential in the form

$$V_{\rm eff}(\phi;\beta,L_1,L_2,L_3)=\mu^2\frac{m_T^2}{2}\phi^2+\frac{\mu^4}{3}\pi^2 g\phi^4+\mu^4\frac{\delta\lambda}{4!}\phi^4+U+\frac{1}{\beta\Omega}\sum_{s=2}^{\infty}\frac{(-1)^{s+1}}{2s}g^s\phi^{2s}Z_4(2s,a_1,a_2,a_3,a_4). \quad (14)$$

We begin studying the case $m_T^2>0$. We will consider first particular values of the compactified lengths and temperature in such a way that the analytic extension of the homogeneous Epstein zeta function $Z_4(2,a_1,\ldots,a_4)$ takes only negative values. Therefore, we consider that the coupling constant is negative, i.e., $g=-|g|<0$. In this case we will have that the topological squared mass is given by

$$m_T^2=-\frac{|g|}{\mu^2\beta\Omega}Z_4(2,a_1,a_2,a_3,a_4). \quad (15)$$

Therefore, we get a physical mass of a scalar particle confined inside our finite domain. The second renormalization condition, which gives a finite coupling constant, is

$$\frac{\partial^4 V_{\text{eff}}}{\partial \phi^4}\bigg|_{\phi=0} = 8\pi^2 g \mu^4. \tag{16}$$

Using Eq. (16) in Eq. (14) we can find the renormalized effective potential. In this case [for negative coupling constant, $Z_4(2, a_1, \ldots, a_4)$ taking only negative values and hence $m_T^2 > 0$] it can be written as

$$V_{\text{eff}}^R(\phi; \beta, L_1, L_2, L_3) = \mu^2 \frac{m_T^2}{2}\phi^2 - \frac{\mu^4}{3}\pi^2|g|\phi^4 + U - \frac{1}{\beta\Omega}\sum_{s=3}^{\infty}\frac{|g|^s}{2s}\phi^{2s}Z_4(2s, a_1, a_2, a_3, a_4). \tag{17}$$

The renormalized effective potential is presented in Fig. 1. It has a local metastable minimum at the origin and it is not bounded from below. This is an expected result since the model is the asymptotically free side of the Euclidean $(\lambda\varphi^4)_d$ scalar field theory.

Next, let us calculate the specific entropy S/E, where the mean energy E and the entropy S are given by Eqs. (4) and (5). First we should perform and inverse Legendre transform in order to obtain $\ln Z(\beta, \Omega, h)$. Note that these thermodynamics functions are calculated in the absence of the source, i.e., $h = 0$. In terms of the effective potential, we have to find the stationary point of the renormalized effective potential, ϕ_0, defined by the equation

$$\frac{\partial V_{\text{eff}}^R}{\partial \phi}\bigg|_{\phi=\phi_0} = 0. \tag{18}$$

Substituting Eq. (17) in Eq. (18) we obtain that ϕ_0 must satisfy

$$\mu^2 m_T^2 \phi_0 - \frac{4}{3}\mu^4\pi^2|g|\phi_0^3 - \frac{1}{\beta\Omega}\sum_{s=3}^{\infty}|g|^s\phi_0^{2s-1}Z_4(2s, a_1, a_2, a_3, a_4) = 0. \tag{19}$$

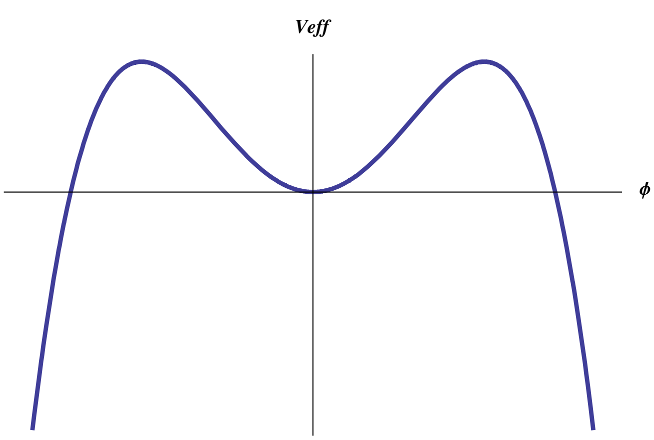

FIG. 1 (color online). The effective potential for the case where $m_T^2 > 0$.

From Fig. 1 we see that Eq. (19) has three solutions. Because we are interested in the configuration which is stable under small external perturbations, we take the solution with the local minimum of the effective potential, i.e., $\phi_0 = 0$. Performing the Legendre transform when the effective potential reaches its metastable stationary point, we get that $\ln Z(\beta, \Omega)$ is given by

$$\ln Z(\beta, \Omega) = \ln Z(\beta, \Omega, h)|_{h=0} = -(\beta\Omega)V_{\text{eff}}^R(\phi; \beta, L_1, L_2, L_3)|_{\phi=\phi_0=0}. \tag{20}$$

Substituting Eqs. (8) and (17) in Eq. (20) we get

$$\ln Z(\beta, \Omega) = -\sum_{n_1,\ldots,n_3=-\infty}^{\infty,}(\pi\beta\tilde{n} + \ln(1 - e^{-2\pi\beta\tilde{n}})) - J_1. \tag{21}$$

The mean energy $E(\beta, \Omega)$ and the canonical entropy $S(\beta, \Omega)$ of the system in equilibrium with a reservoir can be derived using Eqs. (4), (5), and (21). We have

$$E(\beta, \Omega) = \sum_{n_1,\ldots,n_3=-\infty}^{\infty,}\left(\tilde{n}\pi + \frac{2\tilde{n}\pi}{e^{2\tilde{n}\pi\beta} - 1}\right) \tag{22}$$

and

$$S(\beta, \Omega) = \sum_{n_1,\ldots,n_3=-\infty}^{\infty,}\left(\frac{2\tilde{n}\pi\beta}{e^{2\tilde{n}\pi\beta} - 1} - \ln(1 - e^{-2\tilde{n}\pi\beta})\right) - J_1. \tag{23}$$

Note that we have an infinite constant in the definition of the canonical entropy. This ambiguity will be circumvented later using the third law of thermodynamics and assuming the continuity of the entropy. For simplicity we will assume that the lengths of compactification of the spacial coordinates are all equal, $L_i = L$, for $i = 1, 2, 3$, and we will define the dimensionless variable $\xi = \beta/L$. In this case we can write the mean energy and the canonical entropy as

$$E(\xi) = (\varepsilon^{(r)} + P(\xi))/L \tag{24}$$

and

$$S(\xi) = \xi P(\xi) + R(\xi) + \text{cte}. \tag{25}$$

In Eq. (24) the quantity $\varepsilon^{(r)}$ is defined by

$$\varepsilon^{(r)} = \sum_{n_1,\ldots,n_3=-\infty}^{\infty,}\tilde{n}\,\pi, \tag{26}$$

where the variable $\tilde{n}$ is defined as $\tilde{n} = \sqrt{(n_1)^2 + (n_2)^2 + (n_3)^2}$. The term $\varepsilon^{(r)}$ is just the renormalized Casimir energy of the massless scalar field where we impose periodic boundary conditions in the three spatial coordinates. In Ref. [20] it was shown that $\varepsilon^{(r)} = -0.81$. The positive functions $P(\xi)$ and $R(\xi)$ are defined by

$$P(\xi) = \sum_{n_1,\ldots,n_3=-\infty}^{\infty,} \left(\frac{2\tilde{n}\pi}{e^{2\tilde{n}\pi\xi} - 1}\right) \tag{27}$$

and

$$R(\xi) = -\sum_{n_1,\ldots,n_3=-\infty}^{\infty,} \ln(1 - e^{-2\tilde{n}\pi\xi}). \tag{28}$$

Note that where $m_T^2 > 0$ the situation is satisfied only for some specific values of the ratio between β and L, given by ξ. Using the analytic extensions presented in Ref. [62], we can write the topological squared mass as

$$m_T^2 = -\frac{|g|}{L^2}\frac{f_1(\xi)}{\xi}, \tag{29}$$

where the function $f_1(\xi)$ is the analytic extension of $Z_4(2s, 1, 1, 1, \xi^{-1})$ at $s = 1$ and is given by

$$f_1(\xi) = a\xi + \frac{\pi^2}{3}\xi^2 + K(\xi). \tag{30}$$

The coefficient a and the function $K(\xi)$ in Eq. (30) are, respectively, given by

$$\begin{aligned} a = {} & 2\pi\gamma + 2\pi\ln\frac{1}{4\pi} + \frac{\pi^2}{3} \\ & + 8\pi\sum_{n,n_1=1}^{\infty}\left(\frac{n_1}{n}\right)^{1/2} K_{1/2}(2\pi n n_1) \\ & + 4\pi\xi\sum_{n=1}^{\infty}\sum_{n_1,n_2,n_3=-\infty}^{\infty,} K_0\big(2\pi n\sqrt{n_1^2 + n_2^2}\big) \end{aligned} \tag{31}$$

and

$$K(\xi) = 4\pi\xi^{3/2}\sum_{n=1}^{\infty}\sum_{n_1,n_2,n_3=-\infty}^{\infty,}\left(\frac{\tilde{n}}{n}\right)^{-1/2} K_{-1/2}(2\pi n\tilde{n}\xi). \tag{32}$$

The functions $K_r(\xi)$ that appear in Eqs. (31) and (32) are the Kelvin functions [63]. In Fig. 2 the behavior of the topological squared mass is presented. There are three regions of values of ξ where the topological squared mass has a defined sign. They are given, respectively, by $\mathrm{I} = (0, \xi_1)$, $\mathrm{II} = (\xi_1, \xi_2)$, and $\mathrm{III} = (\xi_2, \infty)$, where $\xi_1 = 0.25\,526$ and $\xi_2 = 2.6776$. In the cases I and III the topological squared mass is negative, while in II it is positive. Therefore, only the situation II is consistent. In this case the mean energy and the canonical entropy are given by

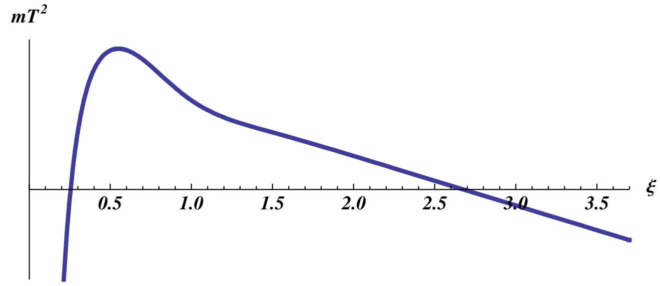

FIG. 2 (color online). Behavior of the m_T^2 with ξ.

$$E_{\mathrm{II}}(\xi) = (\varepsilon^{(r)} + P(\xi))/L \tag{33}$$

and

$$S_{\mathrm{II}}(\xi) = \xi P(\xi) + R(\xi) + \mathrm{cte}_{\mathrm{II}}. \tag{34}$$

B. The negative topological squared mass, i.e., $m_T^2 < 0$

Now let us consider the case where the values of the compactified lengths and temperature give us to the situation where the analytic extension of the homogeneous Epstein zeta function $Z_4(2, a_1, \ldots, a_4)$ has only positive values. In this case the topological squared mass is a negative quantity, since we are considering that the coupling constant is negative $g = -|g| < 0$. In this case we have to impose the second renormalization condition of the effective potential in an arbitrary point $\phi = M$ different from zero. If we take $M = 0$, the effective potential is not only not bounded from below, but also it will not have any local minimum, and in this case the system is unstable under small external perturbations. The second renormalization condition can be written as

$$\left.\frac{\partial^4 V_{\mathrm{eff}}}{\partial\phi^4}\right|_{\phi=M} = 8\pi^2 g\mu^4. \tag{35}$$

Using Eqs. (14) and (35) we get the renormalized effective potential

$$\begin{aligned} & V_{\mathrm{eff}}^R(\phi; \beta, L_1, L_2, L_3) \\ & = \mu^2\frac{m_T^2}{2}\phi^2 - \frac{\mu^4}{3}\pi^2|g|\phi^4 + U \\ & \quad - \frac{1}{\beta\Omega}\sum_{s=3}^{\infty}\alpha(\phi, s) Z_4(2s, a_1, a_2, a_3, a_4). \end{aligned} \tag{36}$$

In Eq. (36) the quantity U is defined in Eq. (8) and α is given by

$$\alpha(\phi, s) = |g|^s\left(\frac{\phi^{2s}}{2s} - \frac{\phi^4}{4!}(2s-1)(2s-2)(2s-3)M^{2s-4}\right). \tag{37}$$

The renormalized effective potential Eq. (36) can be rewritten as

$$V_{\rm eff}^{R}(\phi;\beta,L_1,L_2,L_3) = -\frac{\mu^2 m_T^2}{2} F(\phi;\beta,L_1,L_2,L_3) + U, \tag{38}$$

where we have defined the function $F(\phi;\beta,L_1,L_2,L_3)$ as

$$F(\phi;\beta,L_1,L_2,L_3) = -\phi^2 + A\phi^4 - \sum_{s=3}^{\infty} C_s \phi^{2s}. \tag{39}$$

The coefficients C_s, independent of the field ϕ, are defined, for $s = 3, 4, \ldots$, by

$$C_s = \frac{|g|^{s-1}}{s}\frac{Z_4(2s,a)}{Z_4(2,a)}, \tag{40}$$

and the coefficient A is defined by the expression

$$A = A_o + \frac{1}{4!}\sum_{s=3}^{\infty} C_s(2s)(2s-1)(2s-2)(2s-3)M^{2s-4} \tag{41}$$

where

$$A_o = -\frac{2\beta\Omega\mu^4\pi^2}{3Z_4(2,a)}. \tag{42}$$

We have denoted for simplicity $Z_4(2s,a) = Z_4(2s,a_1,a_2,a_3,a_4)$. Note that the coefficients C_s are defined in the domain of convergence of $Z_4(2s,a)$, i.e., $s = 3, 4, \ldots$; therefore, we have that $Z_4(2s,a) > 0$ and as we are considering the case where $Z_4(2,a) > 0$, the coefficients C_s are positive. If we take the second renormalization condition in a point $M = 0$, the coefficient of the fourth power of the field in Eq. (39) would be negative. In this case it is not possible to find a local minimum of the effective potential. One way to circumvent this situation is to choose M where the coefficient A assumes a positive value. In Fig. 3 the behavior of the effective potential for different values of M, and consequently, for different values of A, is presented for small values of the field ϕ and of the coupling constant. This behavior depends on the first terms of Eq. (39). In this approximation we are taking into account only the first three terms in Eq. (39), where we are denoting the third coefficient $C = C_3 > 0$, i.e., we are taking

$$F(\phi;\beta,L_1,L_2,L_3) = -\phi^2 + A\phi^4 - C\phi^6. \tag{43}$$

From Fig. 3 we show that the only situation where the effective potential has a local minimum and the theory is metastable is by taking M where A is positive and $A^2 > 3C$. This case is the only one where we can find a local

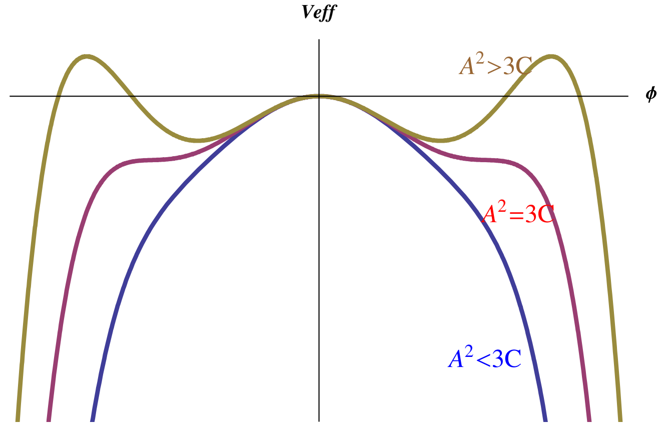

FIG. 3 (color online). Behavior of the effective potential in the case $m_T^2 < 0$ for different values of A.

minimum of the effective potential when the topological squared mass satisfies the inequality $m_T^2 < 0$. This minimum is localized outside the origin and the system develops a condensate. We conclude that we have to take M in such a way that $A > \sqrt{3C_3} > 0$. In terms of M this inequality can be written as

$$-\frac{2\beta\Omega\mu^4\pi^2}{3Z_4(2,a)} + \frac{1}{4!}\sum_{s=3}^{\infty} C_s(2s)(2s-1)(2s-2)(2s-3) \times M^{2s-4} > \sqrt{\frac{|g|^2 Z_4(6,a)}{Z_4(2,a)}}. \tag{44}$$

We will show later that for a given coupling constant and volume of the compact domain, we can always find M that satisfies Eq. (44) for any temperature. We can make an approximation in the series given by Eq. (41) taking only the term $s = 3$. The coefficient A would be

$$A = A_o + 15CM^2. \tag{45}$$

From now on we will consider that the lengths of our compact domain are the same, $L_1 = L_2 = L_3 = L$, and we will define $\xi = \beta/L$. It is easy to show that

$$Z_4(2s,a) = (\mu L)^{2s} f_s(\xi), \tag{46}$$

where the function $f_s(\xi)$ is defined by

$$f_s(\xi) = Z_4(2s,1,1,1,\xi^{-1}). \tag{47}$$

Considering Eqs. (45) and (46) the condition Eq. (44) can be rewritten as

$$M^2(\mu L)^2 > \frac{2\pi^2}{15|g|^2}\frac{\xi}{f_3(\xi)} + \frac{1}{5|g|}\sqrt{\frac{f_1(\xi)}{f_3(\xi)}}. \tag{48}$$

From Fig. 4 we see that the functions $\xi/f_3(\xi)$ and

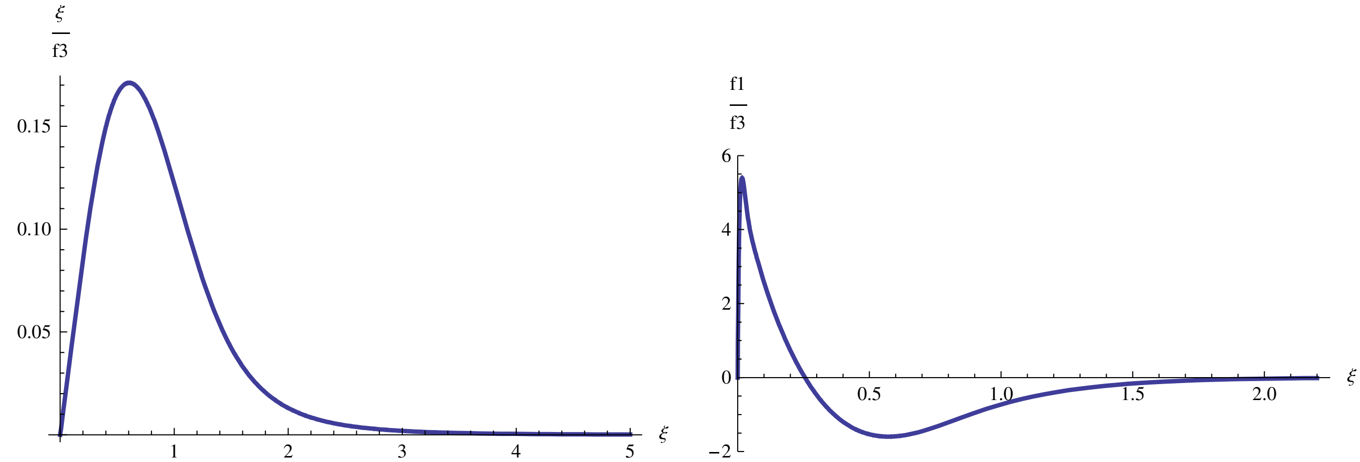

FIG. 4 (color online). The functions $\xi/f_3(\xi)$ and $f_1(\xi)/f_3(\xi)$.

$f_1(\xi)/f_3(\xi)$ are bounded from above and then we always can find a value of M that satisfies Eq. (48). Note that because we are considering a negative topological squared mass, we are taking values of ξ such that $f_1(\xi)$ is positive and as $f_3(\xi)$ is always positive. Consequently, we are able to take the square root of $f_1(\xi)/f_3(\xi)$ in the domain where we are working now. Defining v_1 and v_2 as upper bounds of the functions $\xi/f_3(\xi)$ and $\sqrt{f_1(\xi)/f_3(\xi)}$, respectively, Eq. (48) can be satisfied by taking

$$M^2(\mu L)^2 = \frac{2\pi^2 v_1}{15|g|^2} + \frac{v_2}{5|g|}. \tag{49}$$

Using Eqs. (43) and (45), we can find the local minimum of the renormalized effective potential ϕ_0 given by

$$\phi_0^2 = \frac{A - \sqrt{A^2 - 3C}}{3C}. \tag{50}$$

It is better to define $\Theta = \phi_0^2(\mu L)^2$. We have

$$\Theta(\xi) = \frac{2\pi^2 v_1}{3|g|^2} + \frac{v_2}{|g|} - \frac{2\pi^2}{3|g|^2}\frac{\xi}{f_3(\xi)} - \frac{1}{3}\left(\frac{4\pi^2}{|g|^2}\left(\frac{\xi}{f_3(\xi)}\right)^2 - \frac{12\pi^2}{|g|^2}\left(\frac{2\pi^2 v_1}{15|g|^2} + \frac{v_2}{5|g|}\right)\frac{\xi}{f_3(\xi)} + \left(\frac{2\pi^2 v_1}{|g|^2} + \frac{3v_2}{|g|}\right)^2 - \frac{9}{|g|^2}\frac{f_1(\xi)}{f_3(\xi)}\right)^{1/2}. \tag{51}$$

Considering the parameters v_1, v_2, and g as constants, we analyzed the behavior of Θ with respect to ξ. Performing the Legendre transform in the metastable stationary point of the renormalized effective potential, we get

$$\ln Z(\beta, \Omega) = \ln Z(\beta, \Omega, h)|_{h=0} = -(\beta\Omega)V_{eff}^R(\phi; \beta, L_1, L_2, L_3)|_{\phi=\phi_0}. \tag{52}$$

Substituting Eqs. (8), (38), (43), (45), and (49), in Eq. (52) we have

$$\ln Z(\xi) = \frac{|g|}{2}f_1(\xi)\Theta(\xi) + \frac{|g|}{2}\left\{\frac{2\pi^2\xi}{3} - \left(\frac{2\pi^2 v_1}{3} + |g|v_2\right)f_3(\xi)\right\}\Theta^2(\xi) + \frac{|g|^4}{3}f_3(\xi)\Theta^3(\xi) - \sum_{n_1,\ldots,n_3=-\infty}^{\infty,}(\tilde{n}\pi\xi + \ln(1 - e^{-2\tilde{n}\pi\xi})) - J_1. \tag{53}$$

Using Eqs. (4), (5), and (53) we obtain the mean energy

$$E(\xi) = (\varepsilon^{(r)} + P(\xi) + \chi(\xi))/L \tag{54}$$

and the canonical entropy

$$S(\xi) = \xi P(\xi) + R(\xi) + \psi(\xi) + \text{cte}, \tag{55}$$

where the functions $\varepsilon^{(r)}$, $P(\xi)$, and $R(\xi)$ are defined in Eqs. (26)–(28), respectively. The functions $\chi(\xi)$ and $\psi(\xi)$ are given by the expressions

$$\chi(\xi) = -\frac{|g|}{2}\left\{f_1'(\xi)\Theta(\xi) + \left(\frac{2\pi^2}{3} - \left(\frac{2\pi v_1}{3} + |g|v_2\right)f_3'(\xi)\right)\Theta^2(\xi) + \frac{|g|^2}{3}f_3'(\xi)\Theta^3(\xi)\right\} \tag{56}$$

and

$$\psi(\xi) = \frac{|g|}{2}\left\{g_1(\xi)\Theta(\xi) + \left(\frac{2\pi v_1}{3} + |g|v_2\right)g_3(\xi)\Theta^2(\xi) + \frac{|g|^2}{3}g_3(\xi)\Theta^3(\xi)\right\}. \tag{57}$$

Since we are considering here that the topological squared mass is negative, these results are valid only in the intervals $\mathrm{I} = (0, \xi_1)$ and $\mathrm{III} = (\xi_2, \infty)$ of the variable ξ. These results can be expressed in the following way. We have

$$E_{(\mathrm{I,III})}(\xi) = (\varepsilon^{(r)} + P(\xi) + \chi(\xi))/L \tag{58}$$

and

$$S_{(\mathrm{I,III})}(\xi) = \xi P(\xi) + R(\xi) + \psi(\xi) + \mathrm{cte}_{(\mathrm{I,III})}. \tag{59}$$

In Eqs. (58) and (59) we see explicitly that the form of the mean energy is the same in regions I and III, but the form of the canonical entropy is different in each of these intervals. This discrepancy is due to certain constants, $\mathrm{cte}_{\mathrm{I}}$ and $\mathrm{cte}_{\mathrm{III}}$, that will be fixed with the help of the third law of thermodynamics and assuming the continuity of the entropy.

III. ANALYSIS OF THE RESULTS

We have found that due to the boundary conditions imposed on the field and the presence of a thermal reservoir, there is a topological and thermal generation on mass. This topological mass depends on the lengths of the compactification of the spatial coordinates and on the temperature. It was shown that the sign of the topological squared mass is crucial to determine the profile of the effective potential. Then we obtained two different physical situations: the case where the topological squared mass is positive and the case where it is negative. We have shown that when the topological squared mass is negative the system develops a condensate. In this case, the minimum of the effective potential is not localized at the origin and it is given by the function $\Theta(\xi)$ defined in Eq. (51). We would like to stress that only in the intervals I and III of the variable ξ the topological squared mass is negative. In the interval II of ξ the topological squared mass is positive and the effective potential has a trivial minimum. Figure 5 shows the minimum of the effective potential, Θ, as a function of ξ, for the values $v_1 = 100$, $v_2 = 100$, and $|g| = 0.13$. Also presented is the form of the effective potential in each of the three ranges of values of ξ. From Fig. 5 we see that the minimum of the effective potential is at the origin when we are considering very high temperatures, $\xi \to 0$, or when we are considering very low temperature, $\xi \to \infty$. From this last result we have that the function $\psi(\xi)$ goes to zero when the temperature tends to zero.

We have found the entropy formulas in each interval of values of ξ up to certain constants

$$\begin{aligned} S_{\mathrm{I}}(\xi) &= \xi P(\xi) + R(\xi) + \psi(\xi) + \mathrm{cte}_{\mathrm{I}}, \\ S_{\mathrm{II}}(\xi) &= \xi P(\xi) + R(\xi) + \mathrm{cte}_{\mathrm{II}}, \\ S_{\mathrm{III}}(\xi) &= \xi P(\xi) + R(\xi) + \psi(\xi) + \mathrm{cte}_{\mathrm{III}}. \end{aligned} \tag{60}$$

Using the third law of thermodynamics, $\lim_{\xi\to\infty} S_{\mathrm{III}} = 0$, assuming the continuity of the entropy with the parameter ξ: $S_{\mathrm{I}}(\xi_1) = S_{\mathrm{II}}(\xi_1)$ and $S_{\mathrm{II}}(\xi_2) = S_{\mathrm{III}}(\xi_2)$, and using the fact that the functions $P(\xi)$, $R(\xi)$, and $\psi(\xi)$ go to zero when $\xi \to \infty$, we can fix the constants that appear in the formulas of the entropies

$$\mathrm{cte}_{\mathrm{I}} = \psi(\xi_2) - \psi(\xi_1), \qquad \mathrm{cte}_{\mathrm{II}} = \psi(\xi_2), \qquad \mathrm{cte}_{\mathrm{III}} = 0. \tag{61}$$

For generic values of the parameters (v_1, v_2, g), the function $\psi(\xi)$ is not positive defined and the entropy can be negative for some values of ξ. For large values of v_1 and v_2 and small g this situation is excluded.

With the thermodynamics quantities, the validity of the Bekenstein bound can be verified for the system. The Bekenstein bound states that $S/E \leq 2\pi R$, where R is the smallest ratio of the sphere that circumscribes our finite spatial domain. Because we are considering that all our compactified lengths are equal to L, we have that $R = \sqrt{3}L/2$. Defining the function $T = S/2\pi RE$ on each of the intervals I, II, and III and using Eqs. (33), (58), (60), and (61), we have that

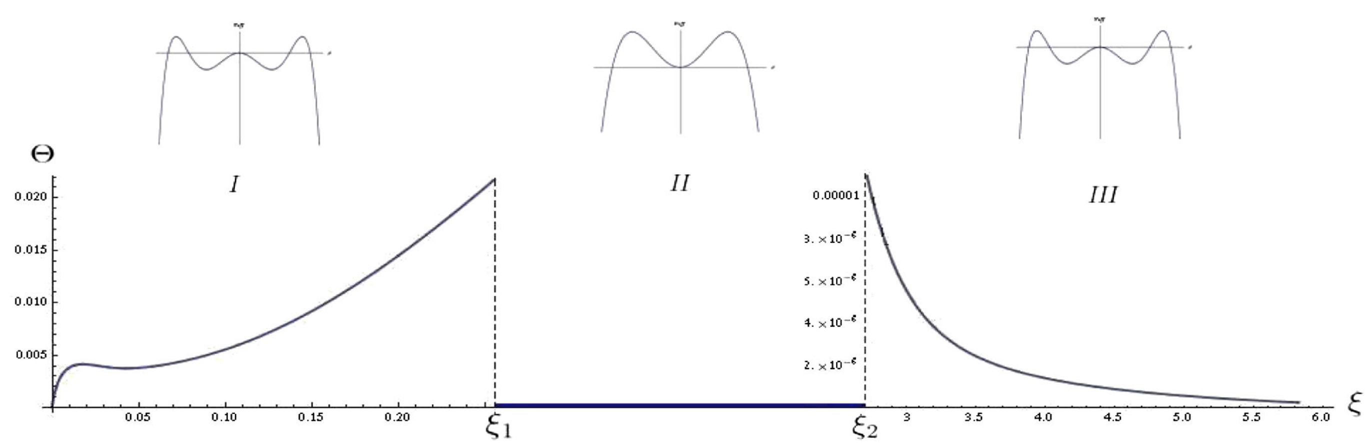

FIG. 5 (color online). The minimum of the effective potential $\Theta(\xi)$ and the form of the effective potential for different values of ξ.

$$T_{\mathrm{I}}(\xi) = \frac{1}{\sqrt{3}\pi} \frac{\xi P(\xi) + R(\xi) + \psi(\xi) + \psi(\xi_2) - \psi(\xi_1)}{\varepsilon^{(r)} + P(\xi) + \chi(\xi)}, \tag{62}$$

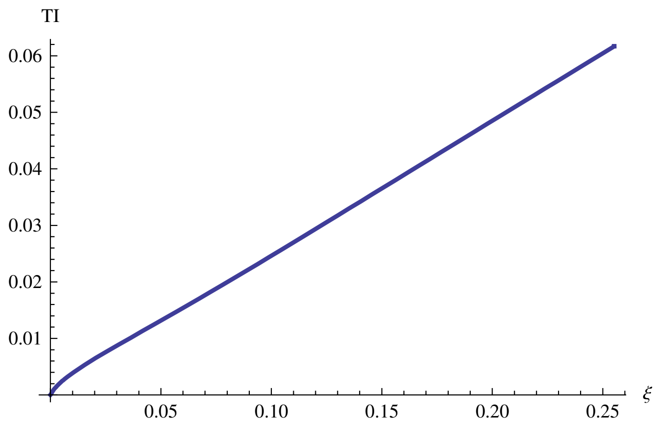

FIG. 6 (color online). The function $T_{\mathrm{I}}(\xi)$ in its domain $\xi \in \mathrm{I} = (0, \xi_1)$.

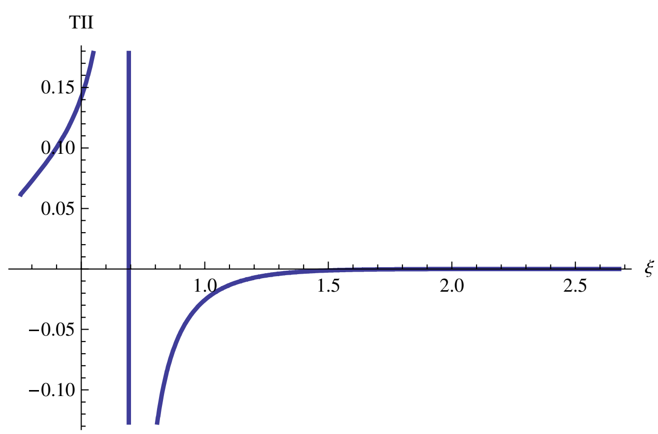

FIG. 7 (color online). The function $T_{\mathrm{II}}(\xi)$ in its domain $\xi \in \mathrm{II} = (\xi_1, \xi_2)$.

$$T_{\mathrm{II}}(\xi) = \frac{1}{\sqrt{3}\pi} \frac{\xi P(\xi) + R(\xi) + \psi(\xi_2)}{\varepsilon^{(r)} + P(\xi)}, \tag{63}$$

$$T_{\mathrm{III}}(\xi) = \frac{1}{\sqrt{3}\pi} \frac{\xi P(\xi) + R(\xi) + \psi(\xi)}{\varepsilon^{(r)} + P(\xi) + \chi(\xi)}. \tag{64}$$

Each of these functions are valid only when ξ is defined in the domains I, II, and III, respectively. In Fig. 6 we have the function $T_{\mathrm{I}}(\xi)$ for $\xi \in \mathrm{I} = (0, \xi_1)$; there we have used the values $v_1 = 100$, $v_2 = 100$, and $|g| = 0.13$. In this situation we have that the field exhibits a condensate. In this regime of high temperatures, we expected that the negative Casimir energy of the system would be irrelevant to the Bekenstein bound, since as we can see in Fig. 6, the thermal fluctuations dominate over any quantum contributions and the Bekenstein bound is satisfied in this situation.

In Fig. 7 we have the function $T_{\mathrm{II}}(\xi)$ in the region $\xi \in \mathrm{II} = (\xi_1, \xi_2)$. In this regime the renormalized effective potential has a trivial minimum and the system behaves as a free bosonic gas. Since we are considering a compact domain with periodic boundary conditions on the spatial coordinates, we have that the renormalized Cassimir energy is negative, $\varepsilon^{(r)} = -0.81$. From Fig. 7 we see that from some value ξ', defined by the equation $\varepsilon^{(r)} + P(\xi') = 1$, the function $T_{\mathrm{II}}(\xi)$ begins to take values greater than one and the Bekenstein bound is violated. It was found that $\xi' = 0.6720$. In Fig. 7 one can also see a divergent point ξ_d given by $\varepsilon^{(r)} + P(\xi_d) = 0$. Because the sign of the Casimir energy is negative, the Bekenstein bound is violated.

In the domain III our theory also exhibits a condensate. Since in this regime we are considering low temperatures, the quantum fluctuations dominate over the thermal one. Figure 8 shows that $T_{\mathrm{III}}(\xi)$ is negative; this is because the negative Casimir prevails over the condensate contribution making the total mean energy of the system negative. Since the entropy is always positive, the Bekenstein bound is also violated in this situation.

Then we shown that there is an intrinsically information storage capacity limit for the $(\lambda\varphi^4)_d$ field theory with the

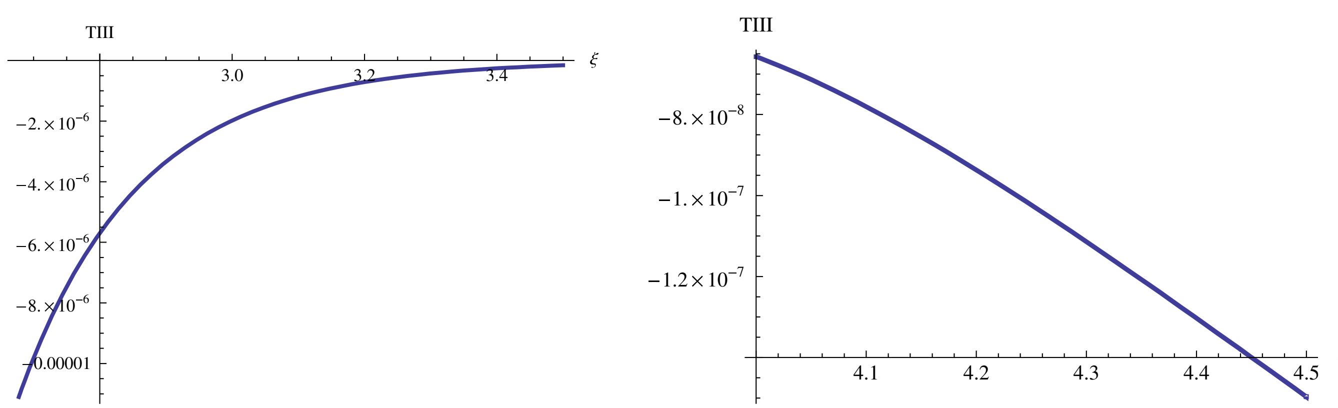

FIG. 8 (color online). The function $T_{\mathrm{III}}(\xi)$ in its domain $\xi \in \mathrm{III} = (\xi_2, \infty)$.

negative sign of the coupling constant, for values of the temperature greater that certain critical temperatures given by $T_{\rm cr} = 1/L\xi'$. For temperatures lower than $T_{\rm cr}$ the Bekenstein bound in invalidated mainly due to the negative Casimir energy. The asymptotically freedom of the model and the presence of the condensate do not change the discussion about the quantum bound. In conclusion, the main feature in the discussion of the validity of the Bekenstein bound is related to the sign of the zero-point energy of the system.

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