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Black hole entropy from the $SU(2)$-invariant formulation of type I isolated horizons

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A detailed analysis of the spherically symmetric isolated horizon system is performed in terms of the connection formulation of general relativity. The system is shown to admit a manifestly $SU(2)$ invariant formulation where the (effective) horizon degrees of freedom are described by an $SU(2)$ Chern-Simons theory. This leads to a more transparent description of the quantum theory in the context of loop quantum gravity and modifications of the form of the horizon entropy.

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I. INTRODUCTION

Black holes (BH) are intriguing solutions of classical general relativity describing important aspects of the physics of gravitational collapse. Their existence in our nearby universe is by now supported by a great amount of observational evidence [1]. When isolated, these systems are remarkably simple for late and distant observers: once the initial very dynamical phase of collapse is passed the system is expected to settle down to a stationary situation completely described (as implied by the famous results by Carter, Israel, and Hawking [2]) by the three extensive parameters (mass M, angular momentum J, electric charge Q) of the Kerr-Newman family [3].

However, the great simplicity of the final stage of an isolated gravitational collapse for late and distant observers is in sharp contrast with the very dynamical nature of the physics seen by in-falling observers which depends on all the details of the collapsing matter. Moreover, this dynamics cannot be consistently described for late times (as measured by the in-falling observers) using general relativity due to the unavoidable development, within the classical framework, of unphysical pathologies of the gravitational field. Concretely, the celebrated singularity theorems of Hawking and Penrose [4] imply the breakdown of predictability of general relativity in the black hole interior. Dimensional arguments imply that quantum effects cannot be neglected near the classical singularities. Understanding of physics in this extreme regime requires a quantum theory of gravity. BH provide, in this precise sense, the most tantalizing theoretical evidence for the need of a more fundamental (quantum) description of the gravitational field.

Extra motivation for the quantum description of gravitational collapse comes from the physics of black holes available to observers outside the horizon. As for the interior physics, the main piece of evidence comes from the classical theory itself which implies an (at first only) apparent relationship between the properties of idealized black hole systems and those of thermodynamical systems. On the one hand, black hole horizons satisfy the very general Hawking area theorem (the so-called *second law*) stating that the black hole horizon area a_H can only increase, namely

$$\delta a_H \geq 0. \tag{1}$$

On the other hand, the uniqueness of the Kerr-Newman family, as the final (stationary) stage of the gravitational collapse of an isolated gravitational system, can be used to prove the first and zeroth laws: under external perturbation the initially stationary state of a black hole can change but the final stationary state will be described by another Kerr-Newman solution whose parameters readjust according to the *first law*

$$\delta M = \frac{\kappa_H}{8\pi G}\delta a_H + \Phi_H \delta Q + \Omega_H \delta J, \tag{2}$$

where κ_H is the surface gravity, Φ_H is the electrostatic potential at the horizon, and Ω_H the angular velocity of the horizon. There is also the *zeroth law* stating the uniformity of the surface gravity κ_H on the event horizon of stationary black holes, and finally the *third law* precluding the possibility of reaching an extremal black hole (for which $\kappa_H = 0$) by means of any physical process.[1] The validity of these classical laws motivated Bekenstein to put forward the idea that black holes may behave as thermodynamical systems with an entropy $S = \alpha a/\ell_p^2$ and a temperature $kT = \hbar\kappa_H/(8\pi\alpha)$, where α is a dimensionless constant and the dimensionality of the quantities involved require the introduction of $\hbar$ leading in turn to the appearance of the Planck length ℓ_p, even though in his first paper [5] Bekenstein states "that one should not regard T as the

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[1]The third law can only be motivated by a series of examples. Extra motivation comes from the validity of the cosmic censorship conjecture.

temperature of the black hole; such identification can lead to all sorts of paradoxes, and is thus not useful.'' The key point is that the need of $\hbar$ required by the dimensional analysis involved in the argument called for the investigation of black hole systems from a quantum perspective. In fact, not long after, the semiclassical calculations of Hawking [6]—that studied particle creation in a quantum test field (representing quantum matter and quantum gravitational perturbations) on the space-time background of the gravitational collapse of an isolated system described for late times by a stationary black hole—showed that once black holes have settled to their stationary (classically) final states, they continue to radiate as perfect black bodies at temperature $kT = \kappa_H \hbar/(2\pi)$. Thus, on the one hand, this confirmed that black holes are indeed thermal objects that radiate at a given temperature and whose entropy is given by $S = a/(4\ell_p^2)$, while, on the other hand, this raised a wide range of new questions whose proper answer requires a quantum treatment of the gravitational degrees of freedom.

Among the simplest questions is the issue of the statistical origin of black hole entropy. In other words, what is the nature of the large amount of microstates responsible for black hole entropy? This simple question cannot be addressed using semiclassical arguments of the kind leading to Hawking radiation and requires a more fundamental description. In this way, the computation of black hole entropy from basic principles became an important test for any candidate quantum theory of gravity. In string theory it has been computed using dualities and nonormalization theorems valid for extremal black holes [7]. There are also calculations based on the effective description of near horizon quantum degrees of freedom in terms of effective 2-dimensional conformal theories [8]. In loop quantum gravity the first computations (valid for physical black holes) were based on general considerations and the fact that the area spectrum in the theory is discrete [9]. The calculation was later refined by quantizing a sector of the phase space of general relativity containing a horizon in ''equilibrium'' with the external matter and gravitational degrees of freedom [10]. In all cases agreement with the Bekenstein-Hawking formula is obtained with logarithmic corrections in a/ℓ_p^2.

In this work we concentrate and further develop the theory of isolated horizons in the context of loop quantum gravity. Recently, we have proposed a new computation of BH entropy in loop quantum gravity (LQG) that avoids the internal gauge fixing used in prior works [11] and makes the underlying structure more transparent. We show, in particular, that the degrees of freedom of type I isolated horizons can be encoded (along the lines of the standard treatment) in an $SU(2)$ boundary connection. The results of this work clarify the relationship between the theory of isolated horizons and $SU(2)$ Chern-Simons (CS) theory first explored in [12], and makes the relationship with the usual treatment of degrees of freedom in loop quantum gravity clear-cut. In the present work, we provide a full detailed derivation of the result of our recent work and discuss several important issues that were only briefly mentioned then.

An important point should be emphasized concerning the logarithmic corrections mentioned above. The logarithmic corrections to the Bekenstein-Hawking area formula for black hole entropy in the loop quantum gravity literature were thought to be of the (universal) form $\Delta S = -1/2 \log(a_H/\ell_p^2)$ [13]. In [14] Kaul and Majumdar pointed out that, due to the necessary $SU(2)$ gauge symmetry of the isolated horizon system, the counting should be modified leading to corrections of the form $\Delta S = -3/2 \log(a_H/\ell_p^2)$. This suggestion is particularly interesting because it would eliminate the apparent tension with other approaches to entropy calculation. In particular their result is in complete agreement with the seemingly very general treatment (which includes the string theory calculations) proposed by Carlip [15]. Our analysis confirms Kaul and Majumdar's proposal and eliminates in this way the apparent discrepancy between different approaches.

The article is organized as follows. In the following section we review the formal definition of isolated horizons. In Sec. III we state the main equations implied by the isolated horizon boundary conditions for fields at a spherically symmetric isolated horizon. In Sec. IV we prove a series of propositions that imply the main classical part of our results: we derive the form of the conserved presymplectic structure of spherically symmetric isolated horizons, and we show that degrees of freedom at the horizon are described by an $SU(2)$ Chern-Simons presymplectic structure. In Sec. V we briefly review the derivation of the zeroth and first law of isolated horizons. In Sec. VI we study the gauge symmetries of the type I isolated horizon and explicitly compute the constraint algebra. In Sec. VII we review the quantization of the spherically symmetric isolated horizon phase space and present the basic formulas necessary for the counting of states that leads to the entropy. We close with a discussion of our results in Sec. VIII. The Appendix contains an analysis of type I isolated horizons from a concrete (and intuitive) perspective that makes use of the properties of stationary spherically symmetric black holes in general relativity.

II. DEFINITION OF ISOLATED HORIZONS

The standard definition of a BH as a space-time region from which no information can reach idealized observers at (future null) infinity is a global definition. This notion of BH requires a complete knowledge of a space-time geometry and is therefore not suitable for describing local physics. The physically relevant definition used, for instance, when one claims there is a black hole in the center of the galaxy, must be local. One such local definition was introduced in [16–18] with the name of isolated horizons

(IH). Here we present this definition according to [18–21]. This discussion will also serve to fix our notation. In the definition of an isolated horizon below, we allow general matter, subject only to conditions that we explicitly state.

Definition: The internal boundary Δ of a history $(\mathcal{M}, g_{ab})$ will be called an *isolated horizon* provided the following conditions hold:

(i) Manifold conditions: Δ is topologically $S^2 \times R$, foliated by a (preferred) family of 2-spheres S and equipped with an equivalence class $[\ell^a]$ of transversal future pointing vector fields whose flow preserves the foliation, where ℓ^a is equivalent to ℓ'^a if $\ell^a = c\ell'^a$ for some positive real number c.

(ii) Dynamical conditions: All field equations hold at Δ.

(iii) Matter conditions: On Δ the stress-energy tensor T_{ab} of matter is such that $-T^a{}_b\ell^b$ is causal and future directed.

(iv) Conditions on the metric g determined by e, and on its Levi-Civita derivative operator ∇: (iv.a) The expansion of ℓ^a within Δ is zero. This, together with the energy condition (iii) and the Raychaudhuri equation at Δ, ensures that ℓ^a is additionally shear free. This in turn implies that the Levi-Civita derivative operator ∇ naturally determines a derivative operator D_a intrinsic to Δ via $X^a D_a Y^b := X^a \nabla_a Y^b$, X^a, Y^a tangent to Δ. We then impose (iv.b) $[\mathcal{L}_\ell, D] = 0$.

(v) Restriction to "good cuts." One can show furthermore that $D_a \ell^b = \omega_a \ell^b$ for some ω_a intrinsic to Δ. A 2-sphere cross-section S of Δ is called a good cut if the pullback of ω_a to S is divergence free with respect to the pullback of g_{ab} to S. As shown in [20], every horizon satisfying (i)–(iv) above possesses at least one foliation into good cuts; this foliation is furthermore generically unique. We require that the fixed foliation coincide with a foliation into good cuts.

Let us discuss the physical meaning of these conditions. The first two conditions are rather weak. The third condition is satisfied by all matter fields normally used in general relativity. The fifth condition is a partial gauge fixing of diffeomorphisms in the "time" direction. The main condition is therefore the fourth condition. (iv.a) requires that ℓ^a be expansion free. This is equivalent to asking that the area 2-form of the 2-sphere cross sections of Δ be constant along generators $[\ell^a]$. This combined with the matter condition (iii) and the Raychaudhuri equation implies that in fact the *entire* pullback q_{ab} of the metric to the horizon is Lie dragged by ℓ^a. Condition (iv.b) further stipulates that the derivative operator D_a be Lie dragged by ℓ^a. This implies, among other things, an analog of the zeroth law of black hole mechanics: conditions (i) and (iii) imply that ℓ^a is geodesic—$\ell^b \nabla_b \ell_a \propto \ell_a$. The proportionality constant is called the *surface gravity*, and condition (iv.b) ensures that it is constant on the horizon for any given $\ell^a \in [\ell^a]$. Furthermore, if we had not fixed $[\ell^a]$, but only required that an $[\ell^a]$ exist such that the isolated horizon boundary conditions hold, then condition (iv.b) would ensure that this ℓ^a is generically unique [20]. From the above discussion, one sees that the geometrical structures on Δ that are time independent are precisely the pullback q_{ab} of the metric to Δ, and the derivative operator D. In fact, the main conditions (iv.a) and (iv.b) are equivalent to requiring $\mathcal{L}_\ell q_{ab} = 0$ and $[\mathcal{L}_\ell, D] = 0$. For this reason it is natural to define (q_{ab}, D) as the *horizon geometry.*

Let us summarize. Isolated horizons are null surfaces, foliated by a family of marginally trapped 2-spheres such that certain geometric structures intrinsic to Δ are time independent. The presence of trapped surfaces motivates the term "horizon" while the fact that they are *marginally* trapped—i.e., that the expansion of ℓ^a vanishes—accounts for the adjective "isolated." The definition extracts from the definition of Killing horizon just that the "minimum" of conditions necessary for analogs of the laws of black hole mechanics to hold. Boundary conditions refer only to behavior of fields at Δ and the general spirit is very similar to the way one formulates boundary conditions at null infinity.

Remarks:

(1) All the boundary conditions are satisfied by stationary black holes in the Einstein-Maxwell-dilaton theory possibly with cosmological constant. Note however that, in the nonstationary context, there still exist physically interesting black holes satisfying our conditions: one can solve for all our conditions and show that the resulting 4-metric need not be stationary on Δ [22].

(2) In the choice of boundary conditions, we have tried to strike the usual balance: On the one hand the conditions are strong enough to enable one to prove interesting results (e.g., a well-defined action principle, a Hamiltonian framework, and a realization of black hole mechanics) and, on the other hand, they are weak enough to allow a large class of examples. As we already remarked, the standard black holes in the Einstein-Maxwell-dilatonic systems satisfy these conditions. More importantly, starting with the standard stationary black holes, and using known existence theorems one can specify procedures to construct new solutions to field equations which admit isolated horizons as well as radiation at null infinity [22]. These examples already show that, while the standard stationary solutions have only a finite parameter freedom, the space of solutions admitting isolated horizons is *infinite* dimensional. Thus, in the Hamiltonian picture, even the reduced phase space is infinite dimensional; the conditions thus admit a very large class of examples.

(3) Nevertheless, space-times admitting isolated horizon are very special among generic members of the

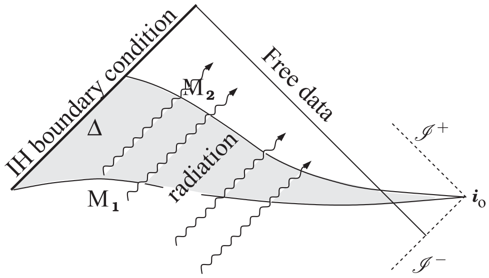

FIG. 1. The characteristic data for a (vacuum) spherically symmetric isolated horizon corresponds to Reissner-Nordstrom data on Δ, and free radiation data on the transversal null surface with suitable falloff conditions. For each mass, charge, and radiation data in the transverse null surface there is a unique solution of Einstein-Maxwell equations locally in a portion of the past domain of dependence of the null surfaces. This defines the phase space of type I isolated horizons in Einstein-Maxwell theory. The picture shows two Cauchy surfaces M_1 and M_2 "meeting" at spacelike infinity i_0. Portions of $\mathcal{I}^+$ and $\mathcal{I}^-$ are shown; however, no reference to future timelike infinity i^+ is made as the isolated horizon need not coincide with the black hole event horizon.

full phase space of general relativity. The reason is apparent in the context of the characteristic formulation of general relativity where initial data are given on a set (pairs) of null surfaces with nontrivial domain of dependence. Let us take an isolated horizon as one of the surfaces together with a transversal null surface according to the diagram shown in Fig. 1. Even when the data on the isolated horizon may be infinite dimensional (for type II and III isolated horizons, see below), in all cases no transversing radiation data are allowed by the IH boundary condition. Roughly speaking the isolated horizon boundary condition reduces to one-half the number of local degrees of freedom.

(4) Notice that the above definition is completely geometrical and does not make reference to the tetrad formulation. There is no reference to any internal gauge symmetry. In what follows we will deal with general relativity in the first order formulation which will introduce, by the choice of variables, an internal gauge group corresponding to local $SL(2,\mathbb{C})$ transformations (in the case of Ashtekar variables) or $SU(2)$ transformations (in the case of real Ashtekar-Barbero variables). It should be clear from the purely geometric nature of the above definition that the IH boundary condition cannot break by any means these internal symmetries.

Isolated horizon classification according to their symmetry groups

Next, let us examine symmetry groups of isolated horizons. A *symmetry* of $(\Delta, q, D, [\ell^a])$ is a diffeomorphism on Δ which preserves the horizon geometry (q, D) and at most rescales elements of $[\ell^a]$ by a positive constant. It is clear that diffeomorphisms generated by ℓ^a are symmetries. So, the symmetry group G_Δ is at least one dimensional. In fact, there are only three possibilities for G_Δ:

(a) Type I: the isolated horizon geometry is spherical; in this case, G_Δ is four dimensional [$SO(3)$ rotations plus rescaling translations[2] along ℓ].

(b) Type II: the isolated horizon geometry is axisymmetric; in this case, G_Δ is two dimensional (rotations around the symmetry axis plus rescaling translations along ℓ).

(c) Type III: the diffeomorphisms generated by ℓ^a are the only symmetries; G_Δ is one dimensional.

Note that these symmetries refer only to the horizon geometry. The full space-time metric need not admit any isometries even in a neighborhood of the horizon. In this paper, as in the classic works [10,16], we restrict ourselves to the type I case. Although a revision would be necessary in light of the results of our present work, the quantization and entropy calculation in the context of types II and III isolated horizons has been considered in [23].

III. SOME EXTRA DETAILS FOR TYPE I ISOLATED HORIZONS

In this section we first list the main equations satisfied by fields at an isolated horizon of type I. The equations presented here can be directly derived from the IH boundary conditions implied by the definition of type I isolated horizons given above. Most of the equations presented here can be found in [16]. For completeness we prove these equations at the end of this section. As we shall see in Sec. III B, some of the coefficients entering the form of these equations depend on the representative chosen among the equivalence class of null generators $[\ell]$. Throughout this paper we shall fix a null generator $\ell \in [\ell]$ by the requirement that the surface gravity $\ell \lrcorner \omega = \kappa$ matches the one corresponding to the stationary black hole with the same macroscopic parameters as the type I isolated horizon under consideration. This choice makes the first law of IH take the form of the usual first law of stationary black holes (see Sec. VI).

A. The main equations

When written in connection variables, the isolated horizon boundary condition implies the following relationship between the curvature of the Ashtekar connection $A^i_+ = \Gamma^i + iK^i$ at the horizon and the 2-form $\Sigma^i = \epsilon^i{}_{jk} e^j \wedge e^k$ (in the time gauge)

[2]In a coordinate system where $\ell^a = (\partial/\partial v)^a$ the rescaling translation corresponds to the affine map $v \to cv + b$ with c, $b \in \mathbb{R}$ constants.

$$\underset{\Leftarrow}{F}_{ab}{}^{i}(A^{+}) = -\frac{2\pi}{a_H}\underset{\Leftarrow}{\Sigma}_{ab}{}^{i}, \tag{3}$$

where a_H is the area of the IH, the double arrows denote the pullback to $H = \Delta \cap M$ with M a Cauchy surface with normal $\tau^a = (\ell^a + n^a)/\sqrt{2}$ at H, and n^a null and normalized according to $n \cdot \ell = -1$. Notice that the imaginary part of the previous equation implies that

$$\underset{\Leftarrow}{d}_{\Gamma} K^i = 0. \tag{4}$$

Another important equation is

$$\epsilon^{i}{}_{jk}\underset{\Leftarrow}{K}^{j} \wedge \underset{\Leftarrow}{K}^{k} = \frac{2\pi}{a_H}\underset{\Leftarrow}{\Sigma}^{i}. \tag{5}$$

The previous equations follow from Eqs. (3.12) and (B.7) of Ref. [16]. Nevertheless, they also follow from the abstract definition given in the Introduction. From the previous equations, only Eq. (5) is not explicitly proven from the definition of IH in the literature. Therefore, we give here an explicit prove at the end of this section. For concreteness, as we think it is helpful for some readers to have a concrete less abstract treatment, another derivation using directly the Schwarzschild geometry is given in the Appendix. The previous equations imply in turn that

$$\underset{\Leftarrow}{F}_{ab}{}^{i}(A^{\beta}) = -\frac{\pi(1-\beta^2)}{a_H}\underset{\Leftarrow}{\Sigma}_{ab}{}^{i}, \tag{6}$$

where $A^i_\beta = \Gamma^i + \beta K^i$ is the Ashtekar-Barbero connection.[3]

B. Proof of the main equations

In this section we use the definition of isolated horizons provided in the previous section to prove some of the equations stated above. We will often work in a special gauge where the tetrad (e^I) is such that e^1 is normal to H and e^2 and e^3 are tangent to H. This choice is only made for convenience, as the equations presented in the previous section are all gauge covariant; their validity in one frame implies their validity in all frames. The following lemma is key in the $U(1)$ treatment to prove the reducibility of the $SU(2)$ connection, and the reducibility is viable for non-expanding horizons as shown in [24].

Lemma 1: In the gauge where the tetrad is chosen so that $\ell^a = 2^{-1/2}(e^a_0 + e^a_1)$ [which can be completed to a null tetrad $n^a = 2^{-1/2}(e^a_0 - e^a_1)$, and $m^a = 2^{-1/2}(e^a_2 + ie^a_3)$], the shear-free and vanishing expansion [conditions (iv.a) in the definition of IH] imply

$$\underset{\Leftarrow}{\omega}^{21} = \underset{\Leftarrow}{\omega}^{20} \quad \text{and} \quad \underset{\Leftarrow}{\omega}^{31} = \underset{\Leftarrow}{\omega}^{30}. \tag{7}$$

Proof: The expansion ρ and shear σ of the null congruence of generators ℓ of the horizon are given by

$$\rho = m^a\bar{m}^b\nabla_a\ell_b, \qquad \sigma = m^a m^b \nabla_a \ell_b. \tag{8}$$

This implies

$$0 = \rho = \frac{1}{2\sqrt{2}}m^a(e^b_2 - ie^b_3)\nabla_a(e^1_b - ie^0_b) = \tag{9}$$

$$= \frac{1}{2\sqrt{2}}m^a((\omega^{21}_a - \omega^{20}_a) - \imath(\omega^{31}_a - \omega^{30}_a)), \tag{10}$$

where we have used the definition of the spin connection $\omega^{IJ}_a = e^{Ib}\nabla_a e^J_b$. Similarly we have

$$0 = \sigma = \frac{1}{2\sqrt{2}}m^a(e^b_2 + ie^b_3)\nabla_a(e^1_b - ie^0_b) = \tag{11}$$

$$= \frac{1}{2\sqrt{2}}m^a((\omega^{21}_a - \omega^{20}_a) + \imath(\omega^{31}_a - \omega^{30}_a)). \tag{12}$$

As e^a_2 and e^a_3 form a nondegenerate frame for $H = \Delta \cap M$, and from the definition of pullback, the previous two equations imply the statement of our lemma. ■

The previous lemma has an immediate consequence on the form of Eq. (4) for the component $i = 1$ in the frame of the previous lemma. More precisely it says that $d\underset{\Leftarrow}{K}^1 = 0$. The good-cut condition (v) in the definition implies then that

$$\underset{\Leftarrow}{K}^1 = 0. \tag{13}$$

Another important consequence of the previous lemma is Eq. (3), also derived in [16]. We give here for completeness and self-consistency a sketch of its derivation. This equation follows from identity

$$F_{ab}{}^{i}(A^+) = -\tfrac{1}{4}R_{ab}{}^{cd}\Sigma^{+i}_{cd}, \tag{14}$$

where R_{abcd} is the Riemann tensor and $\Sigma^{+i} = \epsilon^i{}_{jk}e^j \wedge e^k + i2e^0 \wedge e^i$, which can be derived using Cartan's structure equations. A simple algebraic calculation using the null tetrad formalism (see for instance [25], page 43) with the null tetrad of Lemma 1, and the definitions $\Psi_2 = C_{abcd}\ell^a m^b \bar{m}^c n^d$ and $\Phi_{11} = R_{ab}(\ell^a n^b + m^a \bar{m}^b)/4$, where R_{ab} is the Ricci tensor and C_{abcd} the Weyl tensor, yields

$$\underset{\Leftarrow}{F}_{ab}{}^{i} = \left(\Psi_2 - \Phi_{11} - \frac{R}{24}\right)\underset{\Leftarrow}{\Sigma}^{i}, \tag{15}$$

where $\Sigma^i = \mathrm{Re}[\Sigma^{+i}] = \epsilon^i{}_{jk}e^j \wedge e^k$. An important point here is that the previous expression is valid for any two sphere S^2 embedded in space-time in an adapted null tetrad where ℓ^a and n^a are normal to S^2. However, in the special case where $S^2 = H$ (where $H = \Delta \cap M$ with Δ a type I isolated horizon) it follows from spherical symmetry that $(\Psi_2 - \Phi_{11} - \frac{R}{24}) = C$ with C a constant on the horizon H. Moreover, in the gauge defined in the statement of Lemma 1, the only nonvanishing component of the pre-

[3] In our convention the $so(3) \to \mathbb{R}^3$ isomorphism is defined by $\lambda^i = -\frac{1}{2}\epsilon^i{}_{jk}\lambda^{jk}$ which implies that $F^i = dA^i + \frac{1}{2}\epsilon^i{}_{jk}A^j \wedge A^k$ and $d_A\lambda^i = d\lambda^i + \epsilon^i{}_{jk}A^j \wedge \lambda^k$.

vious equation is the $i = 1$ component for which (using Lemma 1) we get

$$dA^1_+ = C\epsilon, \tag{16}$$

with ϵ the area element of H. Integrating the previous equation on H one can completely determine the constant C, namely

$$C = \left(\Psi_2 - \Phi_{11} - \frac{R}{24}\right) = -\frac{2\pi}{a_H}, \tag{17}$$

from where Eq. (3) immediately follows.

Lemma 2: For type I isolated horizons

$$\underset{\Leftarrow}{K}^j \wedge \underset{\Leftarrow}{K}^k \epsilon_{ijk} = c_0 \frac{2\pi}{a_H} \underset{\Leftarrow}{\Sigma}^i, \tag{18}$$

for some constant c_0. One can choose a representative from the equivalence class $[\ell]$ of null normals to the isolated horizon in order to fix $c_0 = 1$ by making use of the translation symmetry of IH along ℓ. By studying the stationary spherically symmetric back hole solutions one can show that this corresponds to the choice where the surface gravity of the IH matches the stationary surface gravity (see the Appendix A).

Proof: In order to simplify the notation all free indices associated with forms that appear in this proof are pulled back to H (this allows us to drop the double arrows from equations). In the frame of Lemma 1, where e^1 is normal to H, the only nontrivial component of the equation we want to prove is the $i = 1$ component, namely:

$$K^A \wedge K^B \epsilon_{AB} = c_0 \frac{2\pi}{a_H} \Sigma^1, \tag{19}$$

where $A, B = 2, 3$ and $\epsilon^{AB} = \epsilon^{1AB}$. Now, in that gauge, we have that $K^A = c^A{}_B e^B$ for some matrix of coefficients $c^A{}_B$. Notice that the left-hand side of the previous equation equals $\det(c) e^A \wedge e^B \epsilon_{AB}$. We first prove that $\det(c)$ is time independent, i.e. that $\ell(\det c) = 0$. We need to use the isolated horizon boundary condition

$$[\mathcal{L}_\ell, D_b] v^a = 0 \qquad v^a \in T(\Delta), \tag{20}$$

where D_a is the derivative operator determined on the horizon by the Levi-Civita derivative operator ∇_a. One important property of the commutator of two derivative operators is that it also satisfies the Leibnitz rule (it is itself a new derivative operator). Therefore, using the fact that the null vector n^a is normalized so that $\ell \cdot n = -1$ we get

$$0 = [\mathcal{L}_\ell, D_b]\ell^a n_a = n_a [\mathcal{L}_\ell, D_b]\ell^a + \ell^a [\mathcal{L}_\ell, D_b] n_a$$
$$\Rightarrow \ell^a [\mathcal{L}_\ell, D_b] n_a = 0, \tag{21}$$

where we have also used that $\ell^a \in T(\Delta)$. Evaluating the equation on the right-hand side explicitly, and using the fact that $\mathcal{L}_\ell n = \ell \lrcorner dn + d(\ell \lrcorner n) = 0$[4] we get

$$\begin{aligned}0 &= \ell^a [\mathcal{L}_\ell, D_b] n_a = \ell^a \mathcal{L}_\ell (D_b n_a) = -\frac{1}{\sqrt{2}} \ell^a \mathcal{L}_\ell (D_b [e^1_a + e^0_a]) \\ &= \frac{1}{\sqrt{2}} \ell^a \mathcal{L}_\ell (\omega^1_{b\mu} e^\mu_a + \omega^0_{b\mu} e^\mu_a) = -\frac{1}{\sqrt{2}} \ell^a \mathcal{L}_\ell (\omega^{10}_b [e^0_a + e^1_a]) + \frac{1}{\sqrt{2}} \ell^a \mathcal{L}_\ell (\omega^1_{bA} e^A_a + \omega^0_{bA} e^A_a) \\ &= \ell^a \mathcal{L}_\ell (\omega^{10}_b) n_a,\end{aligned}$$

where in the second line we have used the fact that $D_b e^\nu_a = -\omega^\nu_{b\mu} e^\mu_a$ plus the fact that as $\mathcal{L}_\ell q_{ab} = 0$ the Lie derivative $\mathcal{L}_\ell e^A = \alpha \epsilon^{AB} e_B$ for some α (moreover, one can even fix $\alpha = 0$ if one wanted to by means of internal gauge transformations). Then it follows that

$$\mathcal{L}_\ell K^1 = 0, \tag{22}$$

a condition that is also valid for the so-called *weakly isolated horizons* [17]. A similar argument to the one given in Eq. (21)—but now replacing ℓ^a by $e^a_B \in T(\Delta)$ for $B = 2, 3$—leads to

$$\begin{aligned}0 &= e^a_B [\mathcal{L}_\ell, D_b] n_a = e^a_B \mathcal{L}_\ell (D_b n_a) = -\frac{1}{\sqrt{2}} e^a_B \mathcal{L}_\ell (D_b [e^1_a + e^0_a]) \\ &= \frac{1}{\sqrt{2}} e^a_B \mathcal{L}_\ell (\omega^1_{b\mu} e^\mu_a + \omega^0_{b\mu} e^\mu_a) = -\frac{1}{\sqrt{2}} e^a_B \mathcal{L}_\ell (\omega^{10}_b [e^0_a + e^1_a]) + \frac{1}{\sqrt{2}} e^a_B \mathcal{L}_\ell (\omega^1_{bA} e^A_a + \omega^0_{bA} e^A_a) \\ &= \sqrt{2} e^a_B \mathcal{L}_\ell (\omega^0_{bA} e^A_a) = \sqrt{2} [\mathcal{L}_\ell (\omega^{0B}_b) + \alpha \epsilon^{BA} \omega^{0A}_b],\end{aligned}$$

[4] Here we used that $dn = 0$ which comes from the restriction to good cuts in the definition of Sec. II. More precisely, if one introduces a coordinate v on Δ such that $\ell^a \partial_a v = 1$ and $v = 0$ on some leaf of the foliation, then it follows—from the fact that ℓ is a symmetry of the horizon geometry (q, D), and the fact that the horizon geometry uniquely determines the foliation into good cuts—that v will be constant on all the leaves of the foliation. As n must be normal to the leaves one has $n = -dv$, whence $dn = 0$.

where, in addition to previously used identities, we have made use of Lemma 1, Eq. (7). The previous equations imply that the left-hand side of Eq. (19) is Lie dragged along the vector field ℓ, and since Σ^i is also Lie dragged (in this gauge), all this implies that

$$\mathcal{L}_{\ell}(\det(c)) = \ell(\det(c)) = 0. \tag{23}$$

Now we must use the rest of the symmetry group of type I isolated horizons. We denote by $j_i \in T(H)$ $(i = 1, 2, 3)$ the three Killing vectors corresponding to the $SO(3)$ symmetry group of type I isolated horizons. Spherical symmetry of the horizon geometry (q, D) implies

$$\mathcal{L}_{j_i} q = 0 \quad \text{and} \quad [\mathcal{L}_{j_i}, D_b]v^a = 0 \quad \forall\ v^a \in T(\Delta), \tag{24}$$

which, through similar manipulations as the one used above, lead to

$$j_i(\det c) = 0 \tag{25}$$

which completes the prove that $\det c$ is constant on Δ. We can now introduce the dimensionless constant $2\pi c_0 := a_H \det(c)$. Finally one can fix $c_0 = 1$ by choosing the appropriate null generator from the equivalence class $[\ell]$. ■

IV. THE CONSERVED PRESYMPLECTIC STRUCTURE

In this section we show in detail how the IH boundary condition implies the appearance of an $SU(2)$ Chern-Simons boundary term in the symplectic structure describing the dynamics of type I isolated horizons. This result is key for the quantization of the system described in Sec. VII.

A. The action principle

The conserved presymplectic structure in terms of Ashtekar variables can be easily obtained in the covariant phase space formalism. The action principle of general relativity in self-dual variables containing an inner boundary satisfying the IH boundary condition (for asymptotically flat space-times) takes the form

$$S[e, A_+] = -\frac{i}{\kappa}\int_{\mathcal{M}} \Sigma_i^+(e) \wedge F^i(A_+) + \frac{i}{\kappa}\int_{\tau_\infty} \Sigma_i^+(e) \wedge A_+^i, \tag{26}$$

where $\Sigma_i^+(e) = \epsilon^i{}_{jk} e^j \wedge e^k + i2e^0 \wedge e^i$ and A_+^i is the self-dual connection, and a boundary contribution at a suitable time cylinder τ_∞ at infinity is required for the differentiability of the action. No boundary term is necessary if one allows variations that fix an isolated horizon geometry up to diffeomorphisms and Lorentz transformations. This is a very general property and we shall prove it in the next section as we need a little bit of notation that is introduced there.

The first variation of the action yields

$$\delta S[e, A_+] = \frac{-i}{\kappa}\int_{\mathcal{M}} \delta\Sigma_i^+(e) \wedge F^i(A_+) - d_{A_+}\Sigma_i^+ \wedge \delta A_+^i + d(\Sigma_i^+ \wedge \delta A_+^i) + \frac{i}{\kappa}\int_{\tau_\infty} \delta(\Sigma_i^+(e) \wedge A_+^i), \tag{27}$$

from which the self-dual version of Einstein's equations follow

$$\begin{aligned} &\epsilon_{ijk} e^j \wedge F^i(A_+) + ie^0 \wedge F_k(A_+) = 0 \\ &e_i \wedge F^i(A_+) = 0 \qquad d_{A_+}\Sigma_i^+ = 0, \end{aligned} \tag{28}$$

as the boundary terms in the variation of the action cancel.

B. The classical results in a nutshell

In the following sections a series of technical results are explicitly proven. Here we give an account of these results. The reader who is not interested in the explicit proofs can jump directly to Sec. V after reading the present section. In this work we study general relativity on a space-time manifold with an internal boundary satisfying the isolated boundary condition corresponding to type I isolated horizons, and asymptotic flatness at infinity. The phase space of such a system is denoted by Γ and is defined by an infinite dimensional manifold where points $p \in \Gamma$ are given by solutions to Einstein's equations satisfying the type I IH boundary condition. Explicitly a point $p \in \Gamma$ can be parametrized by a pair $p = (\Sigma^+, A_+)$ satisfying the field equations (28) and the requirements of definition II. In particular fields at the boundary satisfy Einstein's equations and the constraints given in Sec. III. Let $T_p(\Gamma)$ denote the space of variations $\delta = (\delta\Sigma^+, \delta A_+)$ at p [in symbols $\delta \in T_p(\Gamma)$]. A very important point is that the IH boundary conditions severely restrict the form of field variations at the horizon. Thus we have that variations $\delta = (\delta\Sigma^+, \delta A_+) \in T_p(\Gamma)$ are such that for the pullback of fields on the horizon they correspond to linear combinations of $SL(2, \mathbb{C})$ internal gauge transformations and diffeomorphisms preserving the preferred foliation of Δ. In equations, for α: $\Delta \to sl(2, C)$ and v: $\Delta \to T(H)$ we have that

$$\begin{aligned} \delta\underleftarrow{\Sigma}^+ &= \delta_\alpha\underleftarrow{\Sigma}^+ + \delta_v\underleftarrow{\Sigma}^+, \\ \delta\underleftarrow{A}_+ &= \delta_\alpha\underleftarrow{A}_+ + \delta_v\underleftarrow{A}_+, \end{aligned} \tag{29}$$

where the arrows denote pullback to Δ, and the infinitesimal $SL(2, C)$ transformations are explicitly given by

$$\delta_\alpha\Sigma^+ = [\alpha, \Sigma^+], \qquad \delta_\alpha A_+ = -d_{A_+}\alpha, \tag{30}$$

while the diffeomorphisms tangent to H take the following form:

$$\delta_v \Sigma_i^+ = \mathcal{L}_v \Sigma_i^+ = \underbrace{v \lrcorner d_{A_+} \Sigma_i^+}_{=0\,(\text{Gauss})} + d_{A_+}(v \lrcorner \Sigma^+)_i - [v \lrcorner A_+, \Sigma^+]_i$$

$$\delta_v A_+^i = \mathcal{L}_v A_+^i = v \lrcorner F_+^i + d_{A_+}(v \lrcorner A_+)^i, \quad (31)$$

where $(v \lrcorner \omega)_{b_1 \cdots b_{p-1}} \equiv v^a \omega_{ab_1 \cdots b_{p-1}}$ for any p form $\omega_{b_1 \cdots b_p}$, and the first term in the expression of the Lie derivative of Σ_i^+ can be dropped due to the Gauss constraint $d_A \Sigma_i^+ = 0$.

So far we have defined the covariant phase space as an infinite dimensional manifold. For it to become a phase space it is necessary to provide it with a presymplectic structure. As the field equations, the presymplectic structure can be obtained from the first variation of the action (27). In particular a symplectic potential density for gravity can be directly read off from the total differential term in (27) [26]. The symplectic potential density is therefore

$$\theta(\delta) = \frac{-i}{\kappa} \Sigma_i^+ \wedge \delta A_+^i, \qquad \forall \delta \in T_p\Gamma, \quad (32)$$

and the symplectic current takes the form

$$J(\delta_1, \delta_2) = -\frac{2i}{\kappa} \delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i \qquad \forall \delta_1, \delta_2 \in T_p\Gamma. \quad (33)$$

Einstein's equations imply $dJ = 0$. Therefore, applying Stokes theorem to the four-dimensional (shaded) region in Fig. 1 bounded by M_1 in the past, M_2 in the future, a timelike cylinder at spacial infinity on the right, and the isolated horizon Δ on the left we obtain

$$\int_{M_1} \delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i - \int_{M_2} \delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i + \int_{\Delta} \delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i = 0. \quad (34)$$

Now it turns out that the horizon integral in this expression is a pure boundary contribution: the symplectic flux across the horizon can be expressed as a sum of two terms corresponding to the two-spheres $H_1 = \Delta \cap M_1$ and $H_2 = \Delta \cap M_2$. Explicitly (see Proposition 1 proven below), the symplectic flux across the horizon Δ factorizes into two contributions on $\partial\Delta$ given by $SU(2)$ Chern-Simons presymplectic terms according to

$$\int_{\Delta} 2\delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i = \frac{a_H}{2\pi} \Big[\int_{H_2} - \int_{H_1} \Big] \delta_{[1} A_{+i} \wedge \delta_{2]} A_+^i. \quad (35)$$

Thus

$$\int_{M_1} 2\delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i - \frac{a_H}{2\pi} \int_{H_1} \delta_{[1} A_{+i} \wedge \delta_{2]} A_+^i = \int_{M_2} 2\delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i - \frac{a_H}{2\pi} \int_{H_2} \delta_{[1} A_{+i} \wedge \delta_{2]} A_+^i \quad (36)$$

which implies that the following presymplectic structure is conserved:

$$i\kappa \Omega_M(\delta_1, \delta_2) = \int_M 2\delta_{[1} \Sigma_i^+ \wedge \delta_{2]} A_+^i - \frac{a_H}{2\pi} \int_H \delta_{[1} A_{+i} \wedge \delta_{2]} A_+^i, \quad (37)$$

or in other words is independent of M. The presence of the boundary term in the presymplectic structure might seem at first sight peculiar; however, we will prove in the following section that the previous symplectic structure can be written as

$$\kappa \Omega_M(\delta_1, \delta_2) = \int_M 2\delta_{[1} \Sigma_i \wedge \delta_{2]} K^i, \quad (38)$$

where we are using the fact that, in the time gauge where e^0 is normal to the space slicing, $\Sigma^{+i} = \mathrm{Re}[\Sigma^{+i}] = \Sigma^i$ when pulled back on M. The previous equation is nothing else but the familiar presymplectic structure of general relativity in terms of the Palatini $\Sigma - K$ variables. In essence the boundary term arises when connection variables are used in the parametrization of the gravitational degrees of freedom.

Finally, as shown in Sec. IV D, the key result for the quantization of type I IH phase space: the presymplectic structure in Ashtekar-Barbero variables takes the form

$$\kappa\beta \Omega_M(\delta_1, \delta_2) = \int_M 2\delta_{[1} \Sigma^i \wedge \delta_{2]} A_i - \frac{a_H}{\pi(1 - \beta^2)} \times \int_H \delta_1 A_i \wedge \delta_2 A^i. \quad (39)$$

The above equation is the main result of the classical analysis of this paper. It shows that the conserved presymplectic structure of type I isolated horizons acquires a boundary term given by an $SU(2)$ Chern-Simons presymplectic structure when the unconstrained phase space is parametrized in terms of Ashtekar-Barbero variables. In the following section we prove this equation.

On the absence of boundary term on the internal boundary

Before getting involved with the construction of the conserved presymplectic structure let us come back to the issue of the differentiability of the action principle. In the isolated horizon literature it is argued that the IH boundary condition guarantees the differentiability of the action principle without the need of the addition of any boundary term (see [19]). As we show here, this property is satisfied by more general kind of boundary conditions. As mentioned above, the allowed variations are such that the IH geometry is fixed up to diffeomorphisms of Δ and gauge transformations. This is enough for the boundary term arising in the first variation of the action (26) to vanish. The boundary term arising on Δ upon first variation of the action is

$$B(\delta) = -\frac{i}{\kappa}\int_\Delta \Sigma_i \wedge \delta A^i. \tag{40}$$

First let us show that $B(\delta_\alpha) = 0$ for δ_α as given in (30). We get

$$\begin{aligned} B(\delta_\alpha) &= \frac{i}{\kappa}\int_\Delta \Sigma_i \wedge d_A\alpha^i \\ &= -\frac{i}{\kappa}\int_\Delta (d_A\Sigma_i)\alpha^i - \int_{\partial\Delta}\Sigma_i\alpha^i = 0, \end{aligned} \tag{41}$$

where we integrated by parts in the first identity, the first term in the second identity vanishes due to Eisntein's equations while the second term vanishes due to the fact that fields are held fixed at the initial and final surfaces M_1 and M_2 and so $\alpha = 0$ when evaluated at $\partial\Delta$. Similarly we can prove that $B(\delta_v) = 0$ for δ_v as given in (31) with (this is the only difference) $v \in T(\Delta)$. We get

$$\begin{aligned} B(\delta_v) &= -\frac{i}{\kappa}\int_\Delta \Sigma_i \wedge (v\lrcorner F^i(A) + d_A(v\lrcorner A^i)) \\ &= -\frac{i}{\kappa}\int_\Delta \Sigma_i \wedge (v\lrcorner F^i(A)) + \frac{i}{\kappa}\int_\Delta d_A\Sigma_i(v\lrcorner A^i) \\ &\quad + \int_{\partial\Delta}\Sigma_i(v\lrcorner A^i) = 0, \end{aligned} \tag{42}$$

where in the last line the first and second terms vanish due to Einstein's equations, and the last term vanishes because variations are such that the vector field v vanishes at $\partial\Delta$. Notice that we have not made use of the IH boundary condition.

C. The presymplectic structure in self-dual variables

In this section we prove a series of propositions implying that the presymplectic structure of type I isolated horizons is given by Eq. (37). In addition, we will prove that the symplectic structure is real and takes the simple form (38) in terms of Palatini variables.

Proposition 1: The symplectic flux across a type I isolated horizon Δ factorizes into boundary contributions at $H_1 = \Delta \cap M_1$ and $H_2 = \Delta \cap M_2$ according to

$$\int_\Delta \delta_{[1}\Sigma_i^+ \wedge \delta_{2]}A^i_+ = \frac{a_H}{2\pi}\left[\int_{H_2} - \int_{H_1}\right]\delta_{[1}A_{+i} \wedge \delta_{2]}A^i_+. \tag{43}$$

Proof: On Δ all variations are linear combinations of $SL(2, C)$ transformations and tangent diffeos as stated in (29)–(31).

$$\delta = \delta_\alpha + \delta_v$$

for α: $\Delta \to sl(2, C)$ and v: $\Delta \to T(H)$. Let us start with $SL(2, \mathbb{C})$ transformations. Using (30) we get

$$\begin{aligned} i\kappa\Omega_\Delta(\delta_\alpha, \delta) &= \int_\Delta [\alpha, \Sigma]^i \wedge \delta A^i_+ + \delta\Sigma_i \wedge d_A(\alpha)^i \\ &= \int_\Delta -\alpha_i\delta(d_A\Sigma^i) + d(\delta\Sigma_i\alpha^i) \\ &= \int_{\partial\Delta}\delta\Sigma^i\alpha_i = -\frac{a_H}{2\pi}\int_{\partial\Delta}\delta F^i_+\alpha_i \\ &= -\frac{a_H}{2\pi}\int_{\partial\Delta} d_A(\delta A^i_+)\alpha_i \\ &= -\frac{a_H}{2\pi}\int_{\partial\Delta}\delta A^i_+ \wedge d_A\alpha_i \\ &= \frac{a_H}{2\pi}\int_{\partial\Delta}\delta A^i_+ \wedge \delta_\alpha A_i, \end{aligned} \tag{44}$$

where in the first line we used the equations of motion $d_A\Sigma^i = 0$ and in the second line we used the IH boundary condition (3). We have therefore shown that

$$i\kappa\Omega_\Delta(\delta_\alpha, \delta) = -\frac{a_H}{2\pi}\int_{\partial\Delta}\delta_\alpha A_{+i} \wedge \delta A^i_+.$$

Similarly, for diffeomorphisms we first notice that (31) implies that

$$\delta_v = \delta^\star_v + \delta_{\alpha(A,v)},$$

where $\alpha(A, v) = v\lrcorner A_+$ and the explicit form of $\delta^\star_v$ is defined as

$$\delta^\star_v\Sigma_i = d_A(v\lrcorner\Sigma)^i,\ \delta^\star_v A^i_+ = v\lrcorner F^i_+.$$

We have that

$$\begin{aligned} i\kappa\Omega_\Delta(\delta^\star_v, \delta) &= \int_\Delta d_A(v\lrcorner\Sigma)^i \wedge \delta A^i_+ - \delta\Sigma_i \wedge (v\lrcorner F^i_+) \\ &= \int_\Delta d((v\lrcorner\Sigma)^i \wedge \delta A^i_+) \\ &\quad + (v\lrcorner\Sigma)^i \wedge d_A(\delta A^i_+) - \delta\Sigma_i \wedge (v\lrcorner F^i_+) \\ &= \int_\Delta d((v\lrcorner\Sigma)^i \wedge \delta A^i_+) + (v\lrcorner\Sigma)^i \wedge \delta F^i_+ \\ &\quad - \delta\Sigma_i \wedge (v\lrcorner F^i_+) \\ &= \int_\Delta d((v\lrcorner\Sigma)^i \wedge \delta A^i_+) + \delta(\Sigma_i[v\lrcorner F^i(A_+)]) \\ &= \int_{\partial\Delta}(v\lrcorner\Sigma_+)^i \wedge \delta A^i_+ \\ &= -\frac{a_H}{2\pi}\int_{\partial\Delta}\delta^\star_v A^i_+ \wedge \delta A^i_+, \end{aligned} \tag{45}$$

where in the third line we used the vector constraint $\Sigma_i[v\lrcorner F^i(A_+)] = 0$, while in the last line we have used the equations of motion and Eq. (3). Notice that the calculation leading to Eq. (44) is also valid for a field dependent α such as $\alpha(A, v)$. This plus the linearity of the presymplectic structure leads to

$$i\kappa\Omega_\Delta(\delta_v, \delta) = -\frac{a_H}{2\pi}\int_{\partial\Delta}\delta_v A_{+i}\wedge\delta A^i_+ \tag{46}$$

and concludes the proof of our proposition. ■

The previous proposition implies that the presymplectic structure (37) is indeed conserved by evolution in Γ. Now we are ready to state the next important proposition.

Proposition 2: The presymplectic form $\Omega_M(\delta_1, \delta_2)$ given by

$$i\kappa\Omega_M(\delta_1, \delta_2) = \int_M 2\delta_{[1}\Sigma^+_i\wedge\delta_{2]}A^i_+ - \frac{a_H}{2\pi}\int_H \delta_{[1}A_{+i}\wedge\delta_{2]}A^i_+$$

is independent of M and *real*. Moreover, the symplectic structure can be described entirely in terms of variables $K\equiv \mathrm{Im}(A_+)$ and Σ taking the familiar form

$$\kappa\Omega_M(\delta_1, \delta_2) = \int_M 2\delta_{[1}\Sigma_i\wedge\delta_{2]}K^i, \tag{47}$$

which is manifestly real and has no boundary contribution.

Proof: The independence of the symplectic structure on M follows directly from Proposition 2 and the argument presented at the end of the previous section. Now let us analyze the reality of the presymplectic structure. The symplectic potential for Ω written in terms of self-dual variables is

$$i\kappa\Theta(\delta) = \int_M \Sigma_i\wedge\delta A^i_+ - \frac{a_H}{4\pi}\int_H A_{+i}\wedge\delta A^i_+. \tag{48}$$

Using $A^i_+ = \Gamma^i + iK^i$ we get

$$\kappa\Theta(\delta) = \int_M \Sigma_i\wedge\delta K^i - i\bigg(\int_M \Sigma_i\wedge\delta\Gamma^i - \frac{a_H}{4\pi}\int_H A_{+i}\wedge\delta A^i_+\bigg). \tag{49}$$

Using a well-known property of the spin connection [27], and denoting by $\theta_0(\delta)$ the term in parentheses in the previous equation, we have

$$\Theta_0(\delta) \equiv \int_M \Sigma_i\wedge\delta\Gamma^i - \frac{a_H}{4\pi}\int_H A_{+i}\wedge\delta A^i_+ = \int_H -e_i\wedge\delta e^i - \frac{a_H}{4\pi}\int_H A_{+i}\wedge\delta A^i_+.$$

The proposition follows from the fact that $\Theta_0(\delta)$ vanishes as proven in the following lemma. ■

Lemma 3: The phase space one-form $\Theta_0(\delta)$ defined by

$$\Theta_0(\delta) \equiv \int_H -e_i\wedge\delta e^i - \frac{a_H}{4\pi}\int_H A_{+i}\wedge\delta A^i_+ \tag{50}$$

is closed.

Proof: From the definition of the phase space Γ given in Sec. IV B, in particular, from Eqs. (29) we know that

$$\delta\underleftarrow{e} = \delta_\alpha\underleftarrow{e} + \delta_v\underleftarrow{e}, \qquad \delta\underleftarrow{A}_+ = \delta_\alpha\underleftarrow{A}_+ + \delta_v\underleftarrow{A}_+. \tag{51}$$

Let us denote by

$$\mathfrak{d}\,\Theta_0(\delta_1, \delta_2) = \delta_1(\Theta_0(\delta_2)) - \delta_2(\Theta_0(\delta_1))$$

the exterior derivative of Θ_0. For infinitesimal $SL(2, C)$ transformations we have

$$\delta_\alpha e = [\alpha, e], \qquad \delta_\alpha A = -d_A\alpha, \tag{52}$$

from which it follows

$$\begin{aligned}\mathfrak{d}\Theta_0(\delta, \delta_\alpha) &= \int_H -2\delta e_i\wedge[\alpha, e]^i + \frac{a_H}{2\pi}\delta A_{+i}\wedge d_A\alpha^i \\ &= \int_H -2\delta e_i\wedge[\alpha, e]^i + \frac{a_H}{2\pi}\delta F^i(A_+)\alpha_i \\ &= \int_H \delta(e^j\wedge e^k)\alpha^i\epsilon_{ijk} + \frac{a_H}{2\pi}\delta F^i(A_+)\alpha_i \\ &= \int_H \delta\Big[\Sigma^i + \frac{a_H}{2\pi}F^i(A_+)\Big]\alpha_i = 0,\end{aligned} \tag{53}$$

where in the first line we have integrated by parts, and in the second line we used the IH boundary condition. The action of diffeomorphisms tangent to H on the connection and triad take the following form:

$$\begin{aligned}\delta_v e^i &= \mathcal{L}_v e^i = d(v\lrcorner e^i) + v\lrcorner de^i \\ \delta_v A^i_+ &= \mathcal{L}_v A^i_+ = v\lrcorner F^i(A_+) + d_{A_+}(v\lrcorner A^i_+).\end{aligned} \tag{54}$$

Now we have

$$\begin{aligned}\mathfrak{d}\Theta_0(\delta, \delta_v) &= \int_H -2\delta e_i\wedge\mathcal{L}_v e^i - \frac{a_H}{2\pi}\delta A_{+i}\wedge\mathcal{L}_v A^i_+ \\ &= \int_H -2\delta e_i\wedge d(v\lrcorner e^i) - 2\delta e_i\wedge v\lrcorner de^i \\ &\quad - \frac{a_H}{2\pi}[\delta A_{+i}\wedge v\lrcorner F^i(A_+) \\ &\quad + \delta A_{+i}\wedge d_{A_+}(v\lrcorner A^i_+)] \\ &= \int_H -2\delta e_i\wedge d(v\lrcorner e^i) - 2v\lrcorner\delta e_i\wedge de^i \\ &\quad - \frac{a_H}{2\pi}[\delta(v\lrcorner A_{+i})\wedge F^i(A_+) \\ &\quad + \delta F_i(A_+)\wedge v\lrcorner A^i_+] \\ &= \int_H -2\delta(de_i)\wedge v\lrcorner e^i - 2v\lrcorner\delta e_i\wedge de^i \\ &\quad - \frac{a_H}{2\pi}\delta[v\lrcorner A_{+i}\wedge F^i(A_+)] \\ &= \int_H \delta[v\lrcorner\Gamma^i\wedge\Sigma_i - v\lrcorner(\Gamma^i + iK^i)\wedge\Sigma_i] = 0,\end{aligned} \tag{55}$$

where in addition to integrating by parts and using that $\partial H = 0$, we have used the identity $A\wedge(v\lrcorner B) - (v\lrcorner A)\wedge B = 0$ for A a 1-form and B a 2-form in a two-dimensional manifold, and Cartan's structure equation $de^i + \epsilon_{ijk}\Gamma^j e^k = 0$. In the last line we used Eq. (3), and Eq. (5)—which implies that $K^i \Sigma_i = 0$. ■

D. Presymplectic structure in Ashtekar-Barbero variables

In the previous section (Proposition 2) we have shown how the presymplectic structure

$$\Omega_M(\delta_1, \delta_2) = \frac{1}{\kappa} \int_M [\delta_1 \Sigma^i \wedge \delta_2 K_i - \delta_2 \Sigma^i \wedge \delta_1 K_i] \quad (56)$$

is indeed preserved in the presence of an IH. More precisely in the shaded space-time region in Fig. 1 one has

$$\Omega_{M_2}(\delta_1, \delta_2) = \Omega_{M_1}(\delta_1, \delta_2). \quad (57)$$

That is, the symplectic flux across the isolated horizon Δ *vanishes* due to the isolated horizon boundary condition [16]. We will show now, how the very same presymplectic structure takes the form (39) when written in terms of the Ashtekar-Barbero connection variables. For this we need to prove the following lemma:

Lemma 4: The phase space one-form $\Theta_0^\beta(\delta)$ defined by

$$\beta \Theta_0^\beta(\delta) \equiv \int_H -e_i \wedge \delta e^i - \frac{a_H}{2\pi(1-\beta^2)} \int_H A_i \wedge \delta A^i \quad (58)$$

is closed.

Proof: from the definition of the phase space (Sec. IV B) we have

$$\delta \underleftarrow{e} = \delta_\alpha \underleftarrow{e} + \delta_v \underleftarrow{e}, \qquad \delta \underleftarrow{A}_+ = \delta_\alpha \underleftarrow{A}_+ + \delta_v \underleftarrow{A}_+. \quad (59)$$

Let us first study $\Theta_0^\beta(\delta_\alpha)$ where the infinitesimal $SU(2)$ transformations are explicitly given by

$$\delta_\alpha e = [\alpha, e], \qquad \delta_\alpha A = -d_A \alpha. \quad (60)$$

We have

$$\begin{aligned} \beta \mathfrak{d} \Theta_0^\beta(\delta, \delta_\alpha) &= \int_H -2\delta e_i \wedge [\alpha, e]^i + \frac{a_H}{\pi(1-\beta^2)} \delta A_i \wedge d_A \alpha^i \\ &= \int_H -2\delta e_i \wedge [\alpha, e]^i + \frac{a_H}{\pi(1-\beta^2)} \delta F^i(A) \alpha_i \\ &= \int_H \delta\Big[\Sigma^i + \frac{a_H}{\pi(1-\beta^2)} F^i(A)\Big] \alpha_i = 0, \end{aligned} \quad (61)$$

where in the first line we have integrated by parts, and in the second line we used the IH boundary condition. The proof that the presymplectic potential vanishes for δ_v mimics exactly the corresponding part of the proof of Lemma3. ■

The next step is to write the symplectic structure in terms of Ashtekar-Barbero connection variables necessary for the quantization program of LQG. When there is no boundary the $SU(2)$ connection

$$A_a^i = \Gamma^i + \beta K_a^i \quad (62)$$

is canonically conjugate to $\epsilon^{abc}\beta^{-1}\Sigma_{bc}^i/4$ where β is the so-called Immirzi parameter. In the presence of a boundary the situation is more subtle: the symplectic structure acquires a boundary term.

Proposition 3: In terms of Ashtekar-Barbero variables the presymplectic structure of the spherically symmetric isolated horizon takes the form

$$\kappa\beta\Omega_M = \int_M 2\delta_{[1}\Sigma^i \wedge \delta_{2]} A_i - \frac{a_H}{\pi(1-\beta^2)} \int_H \delta_1 A_i \wedge \delta_2 A^i, \quad (63)$$

where $\kappa = 32\pi G$.

Proof: The result follows from the variation of the presymplectic potential

$$\begin{aligned} \kappa\beta\Theta(\delta) &= \int_M \Sigma_i \wedge (\beta\delta K^i) + \beta\Theta_0^\beta(\delta) \\ &= \int_M \Sigma_i \wedge \delta(\Gamma^i + \beta K^i) - \frac{a_H}{2\pi(1-\beta^2)} \int_H A_i \wedge \delta A^i, \end{aligned}$$

which is simply the presymplectic potential leading to the conserved presymplectic structure in $\Sigma - K$ variables [in Eq. (47)] to which we have added a term proportional to $\Theta_0^\beta(\delta)$; a closed term which does not affect the presymplectic structure according to Lemma 4. ■

Remark: Notice that one could have introduced a new connection $\bar{A}^i = \Gamma^i + \bar{\beta} K^i$ with a new parameter $\bar{\beta}$ independent of the Immirzi parameter. The statement of the previous lemma would have remained true if on the right-hand side of Eq. (58) one would have replaced β by $\bar{\beta}$ and A^i by $\bar{A}^i$. Consequently, the presymplectic structure can also be parametrized in terms of the analog of Eq. (63) with a boundary term where A^i is replaced by $\bar{A}^i$ and β on the prefactor of the boundary term is replaced by $\bar{\beta}$. This implies that the description of the boundary term in terms of Chern-Simons theory allows for the introduction of a new independent parameter $\bar{\beta}$ which has the effect of modifying the Chern-Simons level. This ambiguity in the description of the boundary degrees of freedom has however no effect in the value of the entropy.

E. A side remark on the triad as the boundary degrees of freedom

Here we show that one can write the presymplectic structure

$$\Omega_M(\delta_1, \delta_2) = \frac{1}{\kappa}\int_M [\delta_1 \Sigma^i \wedge \delta_2 K_i - \delta_2 \Sigma^i \wedge \delta_1 K_i] \quad (64)$$

in a way such that a surface term depends only on the pullback of the triad field while the bulk term coincides with the one obtained in the previous section in terms of real connection variables. In order to do this we rewrite the symplectic potential as follows:

$$\begin{aligned}\kappa\beta\Phi(\delta) &= \int_M \Sigma_i \wedge \delta(\beta K^i)\\ &= \int_M \Sigma_i \wedge \delta(\beta K^i + \Gamma^i) - \int_H \Sigma_i \wedge \delta\Gamma^i\\ &= \int_M \Sigma_i \wedge \delta A^i + \int_H e_i \wedge \delta e^i. \end{aligned} \quad (65)$$

As a result the symplectic structure becomes [28] (and independently [29])

$$\begin{aligned}\Omega_M(\delta_1, \delta_2) = \frac{1}{\kappa\beta}\int_M [\delta_1 \Sigma^i \wedge \delta_2 A_i - \delta_2 \Sigma^i \wedge \delta_1 A_i]\\ + \frac{2}{\beta\kappa}\int_H \delta_1 e_i \wedge \delta_2 e^i. \end{aligned} \quad (66)$$

The previous equation shows that the boundary degrees of freedom could be described in terms of the pullback of the triad on the horizon. One could try to quantize the IH system in this formulation in order to address the question of black hole entropy calculation. Such a project would be certainly interesting. However, the treatment is clearly not immediate as it would require the background independent quantization of the triad field on the boundary for which the usual available techniques do not seem to naturally apply. Nevertheless, the previous equations provide an interesting insight already at the classical level, as the boundary symplectic structure, written in this way, has a remarkable implication for geometric quantities of interest in the first order formulation. To see this let us take $S \subset H$ and α: $H \to su(2)$ so that we can introduce the fluxes $\Sigma(S, \alpha)$ according to

$$\Sigma(S, \alpha) = \int_{S\subset H} \mathrm{Tr}[\alpha\Sigma], \quad (67)$$

where $\mathrm{Tr}[\alpha\Sigma] = \epsilon_{ijk}\alpha^i e^j \wedge e^k$. Now (66) implies the Poisson bracket $\{e^i_a(x), e^j_b(y)\} = \epsilon_{ab}\delta^{ij}\delta(x, y)$ from which the following remarkable equation follows:

$$\{\Sigma(S, \alpha), \Sigma(S', \beta)\} = \Sigma(S \cap S', [\alpha, \beta]). \quad (68)$$

The Poisson brackets among surface fluxes are nonvanishing and reproduce the $su(2)$ Lie algebra. This is an interesting property that we find entirely in terms of classical considerations using smooth field configurations. However, compatibility with the bulk fields seems to single out the treatments of kinematical degrees of freedom in terms of the so-called *holonomy-flux* algebra of classical observables for which flux variables satisfy the exact analog of (68) as described in [30]. This fact strengthens even further the relevance of the uniqueness theorems [31], as they assume the use of the holonomy-flux algebra as the starting point for quantization.

V. GAUGE SYMMETRIES

In this section we rederive the form of the presymplectic symplectic structure written in Ashtekar-Barbero variables by means of a gauge symmetry argument. The idea is to first study the gauge symmetries of the presymplectic structure when written in Palatini variables, as in Eq. (38). We will show that, due to the nature of variations at the horizon, the boundary term in Eq. (39) is completely fixed by the requirement that the gauge symmetry content is unchanged when the presymplectic structure is parametrized by Ashtekar-Barbero variables. This argument is completely equivalent to the content of the previous section and was used in [11] as a shortcut construction of the presymplectic structure for type I isolated horizons in terms of real connection variables. Another important result of this section is the computation of the classical constraint algebra in Sec. VA which is essential for clarifying the quantization strategy implemented in Sec. VIII.

The gauge symmetry content of the phase space Γ is implied by the following proposition.

Proposition 4: Phase space tangent vectors δ_α, $\delta_v \in T_p\Gamma$ of the form

$$\begin{aligned}\delta_\alpha\Sigma = [\alpha, \Sigma], \qquad \delta_\alpha K = [\alpha, K];\\ \delta_v\Sigma = \mathcal{L}_v\Sigma = v\lrcorner d\Sigma + d(v\lrcorner\Sigma),\\ \delta_v K = \mathcal{L}_v K = v\lrcorner dK + d(v\lrcorner K)\end{aligned} \quad (69)$$

for α: $M \to \mathfrak{su}(2)$ and $v \in \mathrm{Vect}(M)$ tangent to the horizon, are degenerate directions of Ω_M.

Proof: The proof follows from manipulations very similar in spirit to the ones used for proving the previous propositions. We start with the $SU(2)$ transformations δ_α, and we get

$$\begin{aligned}\kappa\Omega_M(\delta_\alpha, \delta) &= \int_M [\alpha, \Sigma]_i \wedge \delta K^i - \delta\Sigma_i \wedge [\alpha, K]^i\\ &= \int_M \delta(\epsilon_{ijk}\alpha^j\Sigma^k \wedge K^i)) = 0, \end{aligned} \quad (70)$$

where we used the Gauss constraint $\epsilon_{ijk}\Sigma^k \wedge K^i = 0$. In order to treat the case of the infinitesimal diffeomorphisms tangent to the horizon H it will be convenient to first write the form of the vector constraint V_a in terms of $\Sigma - K$ variables [32]. We have

$$v\lrcorner V = dK^i \wedge v\lrcorner\Sigma_i + v\lrcorner K^i d\Sigma_i \approx 0. \quad (71)$$

Variations of the previous equation yield

$$\begin{aligned} v\lrcorner\delta V &= d(\delta K)^i \wedge v\lrcorner\Sigma_i + dK^i \wedge v\lrcorner\delta\Sigma_i + v\lrcorner\delta K^i d\Sigma_i \\ &\quad + v\lrcorner K^i d(\delta\Sigma)_i \\ &= v\lrcorner\Sigma_i \wedge d(\delta K)^i - \delta\Sigma_i \wedge v\lrcorner dK^i + v\lrcorner d\Sigma_i \wedge \delta K^i \\ &\quad + d(\delta\Sigma)_i v\lrcorner K^i = 0, \end{aligned} \tag{72}$$

where in the second line we have put all the K's to the right, and modified the second and third terms using the identities $A \wedge (v\lrcorner B) + (v\lrcorner A) \wedge B = 0$ that is valid for any two 2-forms A and B on a 3-manifold, and $A \wedge (v\lrcorner B) - (v\lrcorner A) \wedge B = 0$ for a 1-form A and a 3-form B on a 3-manifold, respectively. We are now ready to show that δ_v is a null direction of Ω_M. Explicitly:

$$\begin{aligned} \kappa\Omega_M(\delta_v, \delta) &= \int_M (v\lrcorner d\Sigma + d(v\lrcorner\Sigma))_i \wedge \delta K^i - \delta\Sigma_i \wedge (v\lrcorner dK + d(v\lrcorner K))^i \\ &= \int_M v\lrcorner d\Sigma_i \wedge \delta K^i + d(v\lrcorner\Sigma)_i \wedge \delta K^i - \delta\Sigma^i \wedge v\lrcorner dK_i - \delta\Sigma_i \wedge d(v\lrcorner K)^i \\ &= \int_M \underbrace{v\lrcorner d\Sigma_i \wedge \delta K^i + v\lrcorner\Sigma_i \wedge d(\delta K)^i - \delta\Sigma^i \wedge v\lrcorner dK_i + d(\delta\Sigma)^i \wedge v\lrcorner K_i}_{v\lrcorner\delta V=0} + \int_{\partial M} v\lrcorner\Sigma_i \wedge \delta K^i - \delta\Sigma_i \wedge v\lrcorner K^i \\ &= \int_{\partial M} \delta(v\lrcorner\Sigma_i \wedge K^i) = 0, \end{aligned} \tag{73}$$

where in the last line we have used the identity $v\lrcorner A \wedge B + A \wedge v\lrcorner B = 0$ valid for an arbitrary 2-form A and arbitrary 1-form B on a 2-manifold, the fact that v is tangent to H, and the IH boundary condition Eq. (5) implying $\Sigma_i \wedge K^i = 0$ when pulled back on H. ■

The previous proposition shows that the IH boundary condition breaks neither the symmetry under diffeomorphisms preserving H nor the $SU(2)$ internal gauge symmetry introduced by the use of triad variables.

The gauge invariances of the IH system have been made explicit in the $\Sigma - K$ parametrization of the presymplectic structure. However, due to the results of Propositions 2 and 3, these can also be made explicit in the parametrization of the presymplectic structure using either self-dual connection variables or real Ashtekar-Barbero variables. It is in fact possible to uniquely determine the horizon contributions to the presymplectic structure in connection variables entirely in terms of the requirement that the transformations (69) be gauge invariances of the standard bulk presymplectic contribution plus a suitable boundary term. More precisely, the requirement of $SU(2)$ local invariance becomes

$$\begin{aligned} 0 &= \kappa\beta\Omega_M(\delta_\alpha, \delta) \\ &= \int_M \delta_\alpha\Sigma_i \wedge \delta A^i - \delta\Sigma_i \wedge \delta_\alpha A^i + \kappa\beta\Omega_H \\ &\quad \forall\ \delta \in T_p(\Gamma), \end{aligned} \tag{74}$$

for an (in principle) unknown horizon contribution to the presymplectic structure Ω_H. This gives some information about the nature of the boundary term, namely

$$\begin{aligned} -\kappa\beta\Omega_H &= \int_M \delta_\alpha\Sigma_i \wedge \delta A^i - \delta\Sigma_i \wedge \delta_\alpha A^i \\ &= \int_M [\alpha, \Sigma]_i \wedge \delta A^i + \delta\Sigma_i \wedge d_A\alpha^i \\ &= \int_M d(\alpha_i\delta\Sigma^i) - \alpha_i\delta(d_A\Sigma^i) \\ &= -\frac{a_H}{\pi(1-\beta^2)} \int_H \alpha_i \delta F^i(A) \\ &= \frac{a_H}{\pi(1-\beta^2)} \int_H \delta_\alpha A_i \wedge \delta A^i, \end{aligned}$$

where we used the Gauss law $\delta(d_A\Sigma) = 0$, condition (6), and that boundary terms at infinity vanish. A similar calculation for diffeomorphisms tangent to the horizon gives an equivalent result. This together with the nature of variations at the horizon [see Eqs. (29)] provides an independent way of establishing the results of Proposition 3. This alternative derivation of the conserved presymplectic structure was used in [11].

A. On the first-class nature of the IH constraints

The previous equation above can be written as

$$\begin{aligned} \kappa\beta\Omega(\delta_\alpha, \delta) &= -\int_M \alpha_i \delta(d_A\Sigma^i) \\ &\quad - \int_H \alpha_i \left[\frac{a_H}{\pi(1-\beta^2)} \delta F^i + \delta\Sigma^i\right], \end{aligned} \tag{75}$$

or equivalently

$$\Omega(\delta_\alpha, \delta) + \delta G[\alpha, A, \Sigma] = 0, \tag{76}$$

where

$$G[\alpha, A, \Sigma] = \int_M \alpha_i (d_A \Sigma^i/(\kappa\beta)) + \int_H \alpha_i \Big[\frac{a_H}{\pi\kappa\beta(1-\beta^2)} F^i + \frac{1}{\kappa\beta}\Sigma^i\Big]. \tag{77}$$

In the canonical framework Eq. (76) implies that local $SU(2)$ transformations δ_α are Hamiltonian vector fields generated by the "Hamiltonian" $G[\alpha, A, \Sigma]$. It follows immediately from the definition of Poisson brackets that the Poisson algebra of $G[\alpha, A, \Sigma]$ closes. More precisely, one has

$$\{G[\alpha, A, \Sigma], G[\beta, A, \Sigma]\} = \Omega(\delta_\alpha, \delta_\beta) = \delta_\beta G(\alpha, A, \Sigma) \tag{78}$$

from where we get

$$\{G[\alpha, A, \Sigma], G[\beta, A, \Sigma]\} = G([\alpha, \beta], A, \Sigma). \tag{79}$$

Not surprisingly we get the $SU(2)$ Lie algebra a local $SU(2)$ transformation. In the previous section we showed that these local transformations are indeed gauge transformations. This implies, in the canonical picture, that canonical variables must satisfy the constraints

$$G(\alpha, A, \Sigma) \approx 0 \quad \forall\ \alpha: H \cup M \to su(2). \tag{80}$$

Now let us look at diffeomorphisms. A calculation based on the analog of Eq. (74) for an infinitesimal diffeormorphism preserving H yields

$$\Omega(\delta_v, \delta) + \delta V[v, A, \Sigma] = 0, \tag{81}$$

where

$$V[v, A, \Sigma] = \int_M \frac{1}{\kappa\beta}[\Sigma_i \wedge v\lrcorner F^i - v\lrcorner A_i d_A \Sigma^i] - \int_H v\lrcorner A_i \Big[\frac{a_H}{\pi\kappa\beta(1-\beta^2)} F^i + \frac{1}{\kappa\beta}\Sigma^i\Big]. \tag{82}$$

Finally, a simple calculation as the one leading to (79) leads to the following first-class constraint algebra

$$\begin{aligned} &\{G[\alpha, A, \Sigma], G[\beta, A, \Sigma]\} = G([\alpha, \beta], A, \Sigma),\\ &\{G[\alpha, A, \Sigma], V[v, A, \Sigma]\} = G(\mathcal{L}_v \alpha, A, \Sigma),\\ &\{V[v, A, \Sigma], V[w, A, \Sigma]\} = V([v, w], A, \Sigma), \end{aligned} \tag{83}$$

where we have ignored the Poisson brackets involving the scalar constraint.[5] Using α and v with support only on the horizon H we can now conclude that the IH boundary condition is first class which justifies the Dirac implementation that will be carried out in the quantum theory.

[5] Recall that the smearing of the scalar constraints must vanish on H and hence the full constraint algebra including the scalar constraint will remain first class.

VI. THE ZEROTH AND FIRST LAWS OF BH MECHANICS FOR (SPHERICAL) ISOLATED HORIZONS

The definition given in Sec. II implies automatically the zeroth law of BH mechanics as κ_H is constant on Δ. In turn, the first law cannot be tested unless a definition of energy of the IH is given. Because of the fully dynamical nature of the gravitation field in the neighborhood of the horizon this might seem problematic. Of course one can in addition postulate an energy formula for the IH in order to satisfy *de facto* the first law. Fortunately, there is a more elegant way. This consists of requiring the time evolution along vector fields $t^a \in T(\mathcal{M})$ which are time translations at infinity and proportional to the null generators ℓ at the horizon to correspond to a Hamiltonian time evolution [19]. More precisely, denote by δ_t: $\Gamma \to T(\Gamma)$ the phase space tangent vector field associated with an infinitesimal time evolution along the vector field t^a (which we allow to depend on the phase space point). Then δ_t is Hamiltonian if there exists a functional H such that

$$\delta H = \Omega_M(\delta, \delta_t). \tag{84}$$

This requirement fixes a family of good energy formula and translates into the first law of isolated horizons

$$\delta E_H = \frac{\kappa_H}{\kappa}\delta a_H + \Phi_H \delta Q_H + \text{other work terms}, \tag{85}$$

where we have put the explicit expression of the electromagnetic work term where Φ_H is the electromagnetic potential (constant due to the IH boundary condition) and Q_H is the electric charge. The above equation implies that κ_H and Φ_H are functions of the IH area a_H and charge Q_H alone. A unique energy formula is singled out if we require κ_H to coincide with the surface gravity of type I stationary BHs, i.e., those in the Reissner-Nordstrom family:

$$\kappa_H = \frac{\sqrt{(M^2 - Q^2)}}{2M[M + \sqrt{(M^2 - Q^2)}] - Q^2}. \tag{86}$$

Here we can explicitly prove the above statement in terms of our variables. We shall make here the simplifying assumption that there are no matter fields, i.e., we work in the vacuum case. The explicit form of δ_t is given by

$$\begin{aligned} \delta_t\Sigma &= \mathcal{L}_t\Sigma = t\lrcorner d\Sigma + d(t\lrcorner\Sigma),\\ \delta_t K &= \mathcal{L}_t K = t\lrcorner dK + d(t\lrcorner K). \end{aligned} \tag{87}$$

We can now explicitly write the main condition, namely

$$
\begin{aligned}
16\pi G\Omega_M(\delta_t,\delta) &= \int_M (t\lrcorner d\Sigma + d(t\lrcorner\Sigma))_i \wedge \delta K^i \\
&\quad - \delta\Sigma_i \wedge (t\lrcorner dK + d(t\lrcorner K))^i \\
&= \int_M t\lrcorner d\Sigma_i \wedge \delta K^i + d(t\lrcorner\Sigma)_i \wedge \delta K^i \\
&\quad - \delta\Sigma^i \wedge t\lrcorner dK_i - \delta\Sigma_i \wedge d(t\lrcorner K)^i \\
&= \int_{\partial M} \ell\lrcorner\Sigma_i \wedge \delta K^i - \delta\Sigma_i \wedge \ell\lrcorner K^i \\
&= -\int_{\partial M} \delta\Sigma_i \wedge \ell\lrcorner K^i \\
&= 2\kappa_H \delta a_H + \delta E_{\mathrm{ADM}},
\end{aligned} \tag{88}
$$

where we have used the same kind of manipulations used in Eq. (73) paying special attention to the fact that the relevant vector field $t = \ell$ is (at the horizon) no longer tangent to the horizon cross section, and the fact that the first term in the third line vanishes due to the IH boundary condition.[6]

The condition $\delta H_t = \Omega_M(\delta_t, \delta)$ is solved by $H_t = E_{\mathrm{ADM}} - E_H$ with

$$\delta E_H = \frac{\kappa_H}{\kappa}\delta a_H. \tag{89}$$

Demanding time evolution to be Hamiltonian singles out a notion of isolated horizon energy which automatically satisfies, by this requirement, the first law of black hole mechanics (now extended from the static or locally static context to the isolated horizon context). The general treatment and derivation of the first law can be found in [19,21].

VII. QUANTIZATION

The form of the symplectic structure motivates one to handle the quantization of the bulk and horizon degrees of freedom (d.o.f.) separately. We first discuss the bulk quantization. As in standard LQG [8] one first considers (bulk) Hilbert spaces $\mathcal{H}^B_\gamma$ defined on a graph $\gamma \subset M$ and then takes the projective limit containing the Hilbert spaces for arbitrary graphs. Along these lines let us first consider $\mathcal{H}^B_\gamma$ for a fixed graph $\gamma \subset M$ with end points on H, denoted $\gamma \cap H$. The quantum operator associated with Σ in (6) is

$$\epsilon^{ab}\hat{\Sigma}^i_{ab}(x) = 16\pi G\beta \sum_{p\in\gamma\cap H} \delta(x, x_p)\hat{J}^i(p), \tag{90}$$

where $[\hat{J}^i(p), \hat{J}^j(p)] = \epsilon^{ij}{}_k \hat{J}^k(p)$ at each $p \in \gamma \cap H$.

[6]This follows from Eqs. (5), (A15), and (A18) implying that

$$
\begin{aligned}
\ell\lrcorner\Sigma_i \wedge \delta K^i &= -e^\alpha e^3 \wedge \delta\left(e^2\sqrt{\frac{2\pi}{a_H}}\right) + e^\alpha e^2 \wedge \delta\left(e^3\sqrt{\frac{2\pi}{a_H}}\right) \\
&= e^\alpha\left(\sqrt{\frac{2\pi}{a_H}}\delta(e^2\wedge e^3) + 2e^2\wedge e^3\delta\left(\sqrt{\frac{2\pi}{a_H}}\right)\right).
\end{aligned}
$$

Integrating the previous expression on the horizon gives zero.

Consider a basis of $\mathcal{H}^B_\gamma$ of eigenstates of both $J_p \cdot J_p$ as well as J^3_p for all $p \in \gamma \cap H$ with eigenvalues $\hbar^2 j_p(j_p + 1)$ and $\hbar m_p$, respectively. These states are spin network states, here denoted $|\{j_p, m_p\}^n_1; \cdots\rangle$, where j_p and m_p are the spins and magnetic numbers labeling the n edges puncturing the horizon at points x_p (other labels are left implicit). They are also eigenstates of the horizon area operator $\hat{a}_H$

$$
\begin{aligned}
\hat{a}_H|\{j_p, m_p\}^n_1; \cdots\rangle &= 8\pi\beta\ell_p^2 \sum_{p=1}^n \sqrt{j_p(j_p+1)} \\
&\quad \times |\{j_p, m_p\}^n_1; \cdots\rangle.
\end{aligned}
$$

Now substituting the expression (90) into the quantum version of (6) we get

$$-\frac{a_H}{\pi(1-\beta^2)}\epsilon^{ab}\hat{F}^i_{ab} = 16\pi G\beta \sum_{p\in\gamma\cap H}\delta(x, x_p)\hat{J}^i(p). \tag{91}$$

As we will show now, the previous equation tells us that the surface Hilbert space $\mathcal{H}^H_{\gamma\cap H}$ that we are looking for is precisely the one corresponding to (the well-studied) CS theory in the presence of particles. More precisely, consider the $SU(2)$ Chern-Simons action

$$S_{\mathrm{CS}}[A] = \frac{-a_H}{32\pi^2 G\beta(1-\beta^2)}\int_\Delta A_i \wedge dA^i + \frac{1}{3}A_i \wedge [A, A]^i,$$

to which we couple a collection of particles by adding the following source term:

$$
\begin{aligned}
&S_{\mathrm{int}}[A, \Lambda_1 \cdots \Lambda_n] \\
&\quad = \sum_{p=1}^n \lambda_p \int_{c_p} \mathrm{tr}[\tau_3(\Lambda_p^{-1} d\Lambda_p + \Lambda_p^{-1} A \Lambda_p)],
\end{aligned}
$$

where $c_p \subset \Delta$ are the particle worldlines, λ_p coupling constants, and $\Lambda_p \in SU(2)$ are group valued d.o.f. of the particles. The particle d.o.f. being added will turn out to correspond precisely to the d.o.f. associated with the bulk $\hat{J}(p)^i$ appearing in (90). The horizon and bulk will then be coupled by identifying these d.o.f. The gauge symmetries of the full action are

$$A \to gAg^{-1} + gdg^{-1}, \qquad \Lambda_p \to g(x_p)\Lambda_p, \tag{92}$$

where $g \in C^\infty(\Delta, SU(2))$, and

$$\Lambda_p \to \Lambda_p \exp(\phi\tau^3), \tag{93}$$

where $\phi \in C^\infty(c_p, [0, 2\pi])$.

In order to perform the canonical analysis we assume that $\Lambda_p(r) = \exp(-r^\alpha_p \tau_\alpha)$ ($\alpha = 1, 2, 3$). Under the left action of the group we have

$$\exp(-\kappa^e\tau_e)\Lambda_p(r) = \Lambda_p(f(r, \kappa)) \tag{94}$$

for a function $f(r, \kappa)$ whose explicit form will not play any role in what follows. The infinitesimal version of the previous action is

$$-\tau_e\Lambda_p(r)=\frac{\partial\Lambda_p}{\partial r^\alpha}\frac{\partial f^\alpha}{\partial\kappa^\epsilon}. \tag{95}$$

If we define the (spin) momentum S_p^i as

$$S_p^i=-\pi_\alpha^r\frac{\partial f^\alpha}{\partial\kappa^i}, \tag{96}$$

where π_α^r are the conjugate momenta of r^α then it is easy to recover the following simple Poisson brackets:

$$\{S_p^\alpha,\Lambda_{p'}\}=-\tau^\alpha\Lambda_p\delta_{pp'}\qquad\{S_p^\alpha,S_{p'}^\beta\}=\epsilon^{\alpha\beta}{}_\gamma S_p^\gamma\delta_{pp'}, \tag{97}$$

where the last equation follows from the Jacobi identity. Explicit computation shows that $S_p^i=\lambda_p\,\mathrm{Tr}[\tau^i\Lambda_p\tau_3\Lambda_p^{-1}]$. Therefore, we have three primary constraints per particle

$$\Psi^i(S_p,\Lambda_p)\equiv S_p^i-\lambda_p\,\mathrm{Tr}[\tau^i\Lambda_p\tau_3\Lambda_p^{-1}]\approx 0. \tag{98}$$

The primary Hamiltonian is simply given by

$$H(\{S_p\},\{\Lambda_p\})=\sum_p\eta_i^p\Psi^i(S_p,\Lambda_p).$$

The requirement that the constraints be preserved by the time evolution reads

$$\{\Psi^i(S_p,\Lambda_p),H\}\approx-\epsilon_{ij}{}^k\,\mathrm{Tr}[\tau_i\Lambda_p\tau_3\Lambda_p^{-1}]\eta_p^j \tag{99}$$

and the constraint algebra is

$$\{\Psi^i(S_p,\Lambda_p),\Psi_j(S_{p'},\Lambda_{p'})\}\approx\epsilon_{ij}{}^k(\Psi_k(S_p,\Lambda_p)-\lambda_p\,\mathrm{Tr}[\tau_i\Lambda_p\tau_3\Lambda_p^{-1}])\delta_{pp'}.$$

If we write $\eta^p=\eta_\perp^p+\eta_\parallel^p$, where $\eta_\perp^p$ is the component normal to $\Lambda_p\tau_3\Lambda_p^{-1}$ while $\eta_\parallel^p$ is the parallel one, Eqs. (99) completely fix the Lagrange multipliers $\eta_\perp^p$. This means that, per particle, two (out of three) constraints are second class. The fact that $\eta_\parallel^p$ remains unfixed by the equations of motion implies the presence of first-class contraints which are in fact given by

$$S_p\cdot S_p-\lambda_p^2\approx 0. \tag{100}$$

This constraint generates rotations $\Lambda\to\exp\phi\tau_3\Lambda$ which conserve the quantity $\mathrm{Tr}[\tau_i\tau_3\Lambda^{-1}]$. Now, the fact that there are second class constraints implies that in order to quantize the theory one has to either work with Dirac brackets, solve the constraints classically before quantizing, or parametrize the phase space in terms of Dirac observables. In this case the third option turns out to be immediate. The reason is that the S_p turn out to be Dirac observables of the particle system as far as the constraints (98) are concerned, namely

$$\{S_p^i,\Psi^j(S_{p'},\Lambda_{p'})\}=\epsilon^{ijk}\Psi_k(S_p,\Lambda_p)\delta_{pp'}\approx 0. \tag{101}$$

This implies that the Poisson bracket relations (97) remain unchanged when one replaces the brackets $\{,\}$ by Dirac brackets $\{,\}_D$. Because of this fact and for notational simplicity we shall keep using the standard Poisson bracket notation.

In summary, the phase space of each particle is $T^*(SU(2))$, where the momenta conjugate to Λ_p are given by S_p^i satisfying the Poisson bracket relations

$$\{S_p^i,\Lambda_{p'}\}=-\tau^i\Lambda_p\delta_{pp'}\quad\text{and}\quad\{S_p^i,S_{p'}^j\}=\epsilon^{ij}{}_kS_p^k\delta_{pp'}. \tag{102}$$

Explicit computation shows that $S_p^i+\lambda_p\mathrm{tr}[\tau^i\Lambda_p\tau^3\Lambda_p^{-1}]=0$ are primary constraints (two of which are second class). In the Hamiltonian framework we use $\Delta=H\times\mathbb{R}$, and the symmetries (92) and (93) arise from (and imply) the following set of first-class constraints on H:

$$-\frac{a_H}{\pi(1-\beta^2)}\epsilon^{ab}F_{ab}(x)=16\pi G\beta\sum_{p=1}^n\delta(x,x_p)S_p, \tag{103}$$

$$S_p\cdot S_p-\lambda_p^2=0. \tag{104}$$

The first constraint tells us that the level of the Chern-Simons theory is[7]

$$k\equiv a_H/(4\pi\ell_p^2\beta(1-\beta^2)), \tag{105}$$

and that the curvature of the Chern-Simons connection vanishes everywhere on H except at the position of the defects where we find conical singularities of strength proportional to the defects' momenta.

The theory is topological which means in our case that nontrivial d.o.f. are only present at punctures. Note that due to (102) and (104) the λ_p are quantized according to $\lambda_p=\sqrt{s_p(s_p+1)}$ where s_p is a half integer labeling a unitary irreducible representation of $SU(2)$.

From now on we denote $\mathcal{H}^{\mathrm{CS}}(s_1\dots s_n)$ the Hilbert space of the CS theory associated with a fixed choice of spins s_p at each puncture $p\in\gamma\cap H$. This will be a proper subspace of the "kinematical" Hilbert space $\mathcal{H}^{\mathrm{CS}}_{\mathrm{kin}}(s_1\dots s_n):=s_1\otimes\cdots\otimes s_n$. In particular there is an important global constraint that follows from (103) and the fact that the holonomy around a contractible loop that goes around all particles is trivial. It implies

$$\mathcal{H}^{\mathrm{CS}}(s_1\dots s_n)\subset\mathrm{Inv}(s_1\otimes\cdots\otimes s_n). \tag{106}$$

Moreover, the above containment becomes an equality in the limit $k\equiv a_H/(4\pi\ell_p^2\beta(1-\beta^2))\to\infty$ [33], i.e. in the large BH limit. In that limit we see that the constraint (103)

[7] If we use the connection $\bar{A}^i$ introduced in the remark below (63) then the level takes the form $k\equiv a_H/(4\pi\ell_p^2\beta(1-\bar{\beta}^2))$.

has the simple effect of projecting the particle kinematical states in $s_1 \otimes \cdots \otimes s_n$ into the $SU(2)$ singlet.

To make contact with the bulk theory, we first note that the bulk Hilbert space $\mathcal{H}^B_\gamma$ can be written

$$\mathcal{H}^B_\gamma = \bigoplus_{\{j_p\}_{p\in\gamma\cap H}} \mathcal{H}_{\{j_p\}} \otimes (\otimes_{p\in\gamma\cap H} j_p) \tag{107}$$

for certain spaces $\mathcal{H}_{\{j_p\}}$, and where, for each p, the generators $\hat{J}(p)^i$ appearing in (90) act on the representation space j_p. If we now identify $(\otimes_p j_p)$ with the kinematical Chern-Simons Hilbert space $\mathcal{H}^{\rm CS}_{\rm kin}(j_1 \cdots j_n)$, the $J^i(p)$ operators in (91) are identified with the $S^i(p)$ of (103). The constraints of the CS theory then restrict $\mathcal{H}^{\rm CS}_{\rm kin}$ to $\mathcal{H}^{\rm CS}$ yielding

$$\mathcal{H}_\gamma = \bigoplus_{\{j_p\}_{p\in\cap H}} \mathcal{H}_{\{j_p\}} \otimes \mathcal{H}^{\rm CS}(j_1 \cdots j_n), \tag{108}$$

as the full kinematical Hilbert space for fixed γ.

So far we have dealt with a fixed graph. The Hilbert space satisfying the quantum version of (6) is obtained as the projective limit of the spaces $\mathcal{H}_\gamma$. The Gauss and diffeomorphism constraints are imposed in the same way as in [10,16]. The IH boundary condition implies that lapse must be zero at the horizon so that the Hamiltonian constraint is only imposed in the bulk.

State counting

The entropy of the IH is computed by the formula $S = \mathrm{tr}(\rho_{\rm IH} \log \rho_{\rm IH})$ where the density matrix $\rho_{\rm IH}$ is obtained by tracing over the bulk d.o.f., while restricting to horizon states that are compatible with the macroscopic area parameter a_H. Assuming that there exists at least one solution of the bulk constraints for every state in the CS theory, the entropy is given by $S = \log(\mathcal{N})$, where $\mathcal{N}$ is the number of horizon states compatible with the given macroscopic horizon area a_H. After a moment of reflection one sees that

$$N = \sum_{n;(j)_1^n} \dim[\mathcal{H}^{\rm CS}(j_1 \cdots j_n)], \tag{109}$$

where the labels $j_1 \cdots j_p$ of the punctures are constrained by the condition

$$a_H - \epsilon \leq 8\pi\beta\ell_p^2 \sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \leq a_H + \epsilon. \tag{110}$$

Similar formulas, with a different k value, were first used in [14].

Notice that due to (110) we can compute the entropy for $a_H \gg \beta\ell_p^2$ (not necessarily infinite). The reason is that the representation theory of $U_q(SU(2))$—describing $\mathcal{H}_{\rm CS}$ for finite k—implies

$$\dim[\mathcal{H}_{\rm CS}(j_1 \cdots j_n)] = \dim[\mathrm{Inv}(\otimes_p j_p)], \tag{111}$$

as long as all the j_p as well as the interwining internal spins are less than $k/2 = a_H/(8\pi\beta(1-\beta^2)\ell_p^2)$. But for the Immirzi parameter in the range $|\gamma| \leq \sqrt{2}$ this is precisely granted by (110) according to the lemma below. All this simplifies the entropy formula considerably. The previous dimension corresponds to the number of independent states one has if one models the black hole by a single $SU(2)$ intertwiner!

Lemma 5: The Hilbert spaces $\mathcal{H}^{\rm CS}(j_1 \cdots j_n)$ of Chern-Simons theory with level k selected by the restriction

$$\sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \leq \frac{k}{2} \tag{112}$$

are isomorphic to $\mathrm{Inv}[(j_1 \cdots j_n)]$.

Proof: The Chern-Simons Hilbert space $\mathcal{H}^{\rm CS}(j_1 \cdots j_n)$ will be isomorphic to $\mathrm{Inv}[(j_1 \cdots j_n)]$ if for instance all elements of a given basis (of intertwiners) of $\mathrm{Inv}[(j_1 \cdots j_n)]$ if (see for instance [34])

$$j_p \leq \frac{k}{2} \quad \forall\ p = 1, \ldots, n \tag{113}$$

and

$$\iota_a \leq \frac{k}{2} \quad \forall\ a = 1, \ldots, n-3. \tag{114}$$

Equation (113) is immediately implied by (112) as the latter implies

$$\sum_p j_p \leq \frac{k}{2}. \tag{115}$$

The condition (114) requires a more precise analysis. Notice the fact that, being intertwining spins, the ι_a satisfy the following set of nested restrictions which imply the result:

$$0 \leq \iota_1 \leq \min[j_1 + j_2, j_3 + \iota_2] \leq j_1 + j_2 \leq \frac{k}{2},$$
$$0 \leq \iota_2 \leq \min[j_3 + \iota_1, j_4 + \iota_3] \leq j_1 + j_2 + j_3 \leq \frac{k}{2},$$
$$\cdots$$
$$0 \leq \iota_{n-4} \leq \min[j_{n-3} + \iota_{n-5}, j_{n-2} + \iota_{n-3}] \leq \sum_{p=1}^{n-3} j_p \leq \frac{k}{2},$$
$$0 \leq \iota_{n-3} \leq \min[j_{n-2} + \iota_{n-4}, j_n + j_{n-1}] \leq \sum_{p=1}^{n-2} j_p \leq \frac{k}{2},$$

where in each line we have used (115). ■

Remark: An interesting point can be made here as a further development of the remark below Eq. (63).

Notice that if we had worked with the connection $\bar{A}^i = \Gamma^i$ as our boundary field degree of freedom—corresponding to the choice $\bar{\beta} = 0$ in the notation of the remark below (63)—then the boundary Chern-Simons level would be $k = a_H/(4\pi\ell_p^2\beta)$ (see footnote 7). This implies that the condition (113) imposed on representations labeling the punctures would take the simple form

$$j_p \leq j_{\max} \equiv \frac{a_H}{8\pi\ell_p^2\beta}, \tag{116}$$

or equivalently

$$j \leq j_{\max} \quad \text{such that}$$
$$a^{(1)}_{\max} = 8\pi\ell_p^2\beta\sqrt{j_{\max}(j_{\max}+1)} \approx 8\pi\ell_p^2\beta j_{\max} = a_H, \tag{117}$$

where $a^{(1)}_{\max}$ is the maximum single-puncture eigenvalue allowed. Our effective treatment depends on a classical input: the macroscopic area. One would perhaps hope that this effective treatment would only allow for states where the microscopic area is *close* to a_H; unfortunately such a strong requirement is not satisfied as the allowed eigenvalues can be very far away from a_H. However, the effective theory at least forbids quantum states where individual area quanta are larger than a_H. This is a nice interplay between the classical input and the associated effective quantum description. Of course this interplay is still qualitatively valid for the case in which one works with the Ashtekar-Barbero connection on the boundary (i.e., $\beta = \bar{\beta}$).

VIII. CONCLUSION

We have shown that the spherically symmetric isolated horizon (or type I isolated horizon) can be described as a dynamical system by a presymplectic form Ω_M that, when written in the (connection) variables suitable for quantization, acquires a horizon contribution corresponding to an $SU(2)$ Chern-Simons theory. There are different ways to prove this important statement. In [11] we first observed that $SU(2)$ gauge transformations and diffeomorphism preserving H are not broken by the IH boundary condition. Moreover, infinitesimal diffeomorphisms tangent to H and $SU(2)$ local transformations continue to be degenerate directions of Ω_M on shell. This by itself is then sufficient for deriving the boundary term that arises when writing the symplectic structure in terms of Ashtekar-Barbero connection variables. Here we have reviewed this construction in Sec. V. A result that was not explicitly presented in [11] is the precise form of the constraint algebra found in Sec. VA. There we see in a precise way how the canonical gauge symmetry structure of our system is precisely that of an $SU(2)$ Chern-Simons theory: in particular, at the boundary, infinitesimal diffeomorphisms, preserving H, form a subalgebra of $SU(2)$ gauge algebra, as in the topological theory.

A different, more direct approach is based on a subtle fact about the canonical transformation that takes us from the Palatini (Σ^i_{ab}, K^i_a) phase space parametrization to the Ashtekar-Barbero (Σ^i_{ab}, A^i_a) connection formulation, in the presence of an internal boundary. In the case of type I isolated horizons, the term to be added to the symplectic potential producing the above transformation gives rise to a boundary contribution that eventually leads to a boundary Chern-Simons term in the presymplectic structure. This is the content of Sec. IV D. The boundary Chern-Simons term appears due to the use of connection variables which in turn are the ones in terms of which the quantization program of loop quantum gravity is applicable.

Finally, at a fundamental level, what actually fixes the surface term in the symplectic structure is the requirement that it be conserved in time. The above mentioned proofs show that the various expressions for the symplectic structure using different variables are in fact one and the same symplectic structure. That this symplectic structure is preserved in time was proven in Sec. IV.

There is a certain freedom in the choice of boundary variables leading to different parametrizations of the boundary degrees of freedom. The most direct description would appear, at first sight, to be the one defined simply in terms of the triad field (pulled back on H) along the lines exhibited in Sec. IV E. Such a parametrization is however less preferable from the point of view of quantization as one is confronted to the background independent quantization of form fields for which the usual techniques are not directly applicable. In contrast, the parametrization of the boundary degrees of freedom in terms of a connection directly leads to a description in terms of $SU(2)$ Chern-Simons theory which, being a well-studied topological field theory, drastically simplifies the problem of quantization. However, such a description comes with the freedom of the introduction of an extra dimensionless parameter $\bar{\beta}$ [as pointed out in the remark below Eq. (63)]. Such an appearance of extra parameters is very much related to what happens in the general context of the canonical formulation of gravity in terms of connections (see the Appendix in [35]). Therefore, this observation is by no means a new feature proper of IHs. The existence of this extra parameter has a direct influence on the value of the Chern-Simons level; however, the value of the entropy is independent of this extra parameter [36].

Note that no d.o.f. is available at the horizon in the classical theory as the IH boundary condition completely fixes the geometry at Δ [the IH condition allows a single (characteristic) initial data once a_H is fixed (see Fig. 1)]. Nevertheless, nontrivial d.o.f. arise as *would be gauge* d.o.f. upon quantization. These are described by $SU(2)$ Chern-Simons theory coupled to (an arbitrary number of) defects through a dimensionless parameter proportional to $4\pi(1-\beta^2)a_j/a_H$, where $a_j = 8\pi\ell_p^2\sqrt{j(j+1)}$ is the basic quantum of area carried by the defect. These would be gauge excitations are entirely responsible for the entropy in this approach.[8]

We obtain a remarkably simple formula for the horizon entropy: the number of states of the horizon is simply given in terms of the (well-studied) dimension of the Hilbert spaces of Chern-Simons theory with punctures labeled by spins. In the large a_H limit the latter is simply equal to the dimension of the singlet component in the tensor product of the representations carried by punctures. In this limit the black hole density matrix ρ_{IH} is the identity on $\mathrm{Inv}(\otimes_p j_p)$ for admissible j_p. Similar counting formulas have been proposed in the literature [14,38]. Our derivation from first principles clarifies these previous proposals.

Remarkably, the counting of states necessary to compute the entropy of the above yype I isolated horizons can be exactly done [39] using the novel counting techniques introduced in [40]. It turns out to be $S_{\mathrm{BH}} = \beta_0 a_H/(4\beta\ell_p^2)$, where $\beta_0 = 0.274\,067\cdots$. However, the subleading corrections turn out to have the form $\Delta S = -\frac{3}{2}\log a_H$ (instead of the $\Delta S = -\frac{1}{2}\log a_H$ that follows the classic treatment [10,13]) matching other approaches [14]. This is due to the full $SU(2)$ nature of the IH quantum constraints imposed here. We must mention that the proposal of Kaul and Majumdar [14] is most closely related to our result. Their intuition was particularly insightful as it yielded a universal form of logarithmic corrections in agreement with those found in different quantum gravity formulations [15]. Our work clarifies the relevance of their proposal.

In Ref. [16] the classical description of the IH was first done in terms of the null tetrad formalism. In this case the null surface defining the horizon provides the natural structure for a partial gauge fixing from the internal gauge $SL(2,\mathbb{C})$ to $U(1)$. In this setting one fixes an internal direction $r^i \in \mathfrak{su}(2)$ and the IH boundary condition (6) becomes

$$dV + \frac{2\pi}{a_H}\Sigma^i r_i = 0, \qquad \Sigma^i x_i = 0, \qquad \Sigma^i y_i = 0, \quad (118)$$

where x^i, $y^i \in \mathfrak{su}(2)$ are arbitrary vectors completing an internal triad. In the quantum theory [10] only the first of the previous constraints is imposed strongly, while—due to the noncommutativity of Σ^i in LQG—the other two can only be imposed weakly, namely, in [10] one has $\langle\Sigma^i x_i\rangle = \langle\Sigma^i y_i\rangle = 0$. However, this leads to a larger set of admissible states (overcounting). To solve this problem, within the $U(1)$ model, one would have to solve the two constraints $\Sigma^i x_i = 0 = \Sigma^i y_i$ at the classical level first, implementing the reduction also on the pullback of two forms Σ^i on H. However, this would introduce formidable complications for the quantization of the bulk degrees of freedom in terms of LQG techniques. Our $SU(2)$ treatment resolves this problem as now the three components of (6) are first-class constraints. Dirac implementation leads to a smaller subset of admissible surface states that are relevant in the entropy calculation.

We have concentrated in this work on type I isolated horizons. The natural question that follows from this analysis is whether we can generalize the $SU(2)$ invariant treatment in order to include distortion. The classical formulation and quantization of type II isolated horizons in the $U(1)$ (gauge fixed) treatment has been studied in [23]. Work in progress [41] shows that, in the $SU(2)$ invariant formulation, it is possible to include distortion in a simple way as long as the isolated horizon is nonrotating (i.e., when $\mathrm{Im}[\Psi_2] = 0$). The rotating case is more subtle but we believe that there are no insurmountable obstacles to its $SU(2)$ invariant treatment (this will be studied elsewhere).

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APPENDIX A: TYPE I ISOLATED HORIZONS: HORIZON GEOMETRY FROM THE REISSNER-NORDSTROM FAMILY

The spherically symmetric isolated horizons or type I isolated horizons are easy to visualize in terms of the characteristic formulation of general relativity with initial data given on null surfaces [42]. This observation is very useful if one is looking for a concrete visualization of the horizon geometry and properties of the matter fields at the horizon. In this Appendix we chose to derive the main properties of type I isolated horizons by studying their geometry in the context of Einstein-Maxwell theory (which is general enough for the most relevant applications of the formalism). An additional motivation for the explicit approach presented here is its complementarity with more abstract discussions available in the literature [16–18]. In the context of Einstein-Maxwell theory, space-times with a type I IH are solutions to Einstein-Maxwell equations where Reissner-Nordstrom horizon data are given on a

[8] More insight on the nature of these degrees of freedom could be gained by studying simpler models. In [37] a theory with no local degrees of freedom has been introduced. The attractive feature of this model is that it admits an (unconstrained) phase parametrization in terms of the same field content as gravity. Moreover, one can argue that it contains the minimal structure to serve as a toy model to study some generic features of the type I isolated horizon quantization.

null surface $\Delta = S^2 \times \mathbb{R}$ and suitable free radiation is given at the transversal null surface for both geometric as well as electromagnetic degrees of freedom. This allows one to derive the main equations of IH directly from the Reissner-Nordstrom geometry as far as we are careful enough only to use the information that is intrinsic to the IH geometry.

1. The Reissner-Nordstrom solution in Kruskal-like coordinates

The Reissner-Nordstrom metric can be written in Kruskal-like coordinates [25] as

$$ds^2 = \Omega^2(x,t)(-dt^2 + dx^2) + r^2(d\theta^2 + \sin(\theta)d\phi^2), \tag{A1}$$

where

$$\Omega(x,t) = \frac{(r - r_-)^{1+b/2} e^{-ar}}{ar}, \tag{A2}$$

with $a = (r_+ - r_-)/(2r_+^2)$, $b = r_-^2/r_+^2$, and the function $r(x,t)$ is determined by the following implicit equation:

$$F(r) = x^2 - t^2, \quad \text{with} \quad F(r) = \frac{(r - r_+)e^{2ar}}{(r - r_-)^b}. \tag{A3}$$

The previous Kruskal-like coordinates are valid for the external region $r \geq r_+$. The metric is smooth at the horizon $r = r_+$ which in the new coordinates corresponds to the null surface $x = t$. An important identity is

$$dr|_\Delta = \frac{2x}{F'}(dx - dt), \tag{A4}$$

where $|_\Delta$ denotes that the equality holds at the horizon Δ for which $x = t$. Here we are interested in the first order formalism. Thus we are interested in an associated tetrad e^I_μ with $I = 0, 1, 2, 3$. It is immediate to verify that a possible such tetrad is given by

$$e^0 = \Omega(x,t)dt, \qquad e^1 = \Omega(x,t)dx, \qquad e^2 = rd\theta, \qquad e^3 = r\sin(\theta)d\phi. \tag{A5}$$

We now want to compute the components of the spin connection ω^{IJ}_a at the horizon. Therefore, we will use Cartan's first structure equations $de + \omega \wedge e = 0$ at Δ. The solution is (all details are given in Sec. A 3)

$$\begin{aligned} \omega^{01}|_\Delta &= \frac{2x\Omega'}{F'\Omega}(dt - dx), & \omega^{02}|_\Delta &= -\frac{2x}{F'\Omega}d\theta, \\ \omega^{03}|_\Delta &= -\frac{2x}{F'\Omega}\sin(\theta)d\phi, & \omega^{12}|_\Delta &= -\frac{2x}{F'\Omega}d\theta, \\ \omega^{13}|_\Delta &= -\frac{2x}{F'\Omega}\sin(\theta)d\phi, & \omega^{23}|_\Delta &= -\cos(\theta)d\phi. \end{aligned} \tag{A6}$$

At this stage we consider a Lorentz transformation of the form

$$\Lambda^I_J = \begin{bmatrix} c & s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{A7}$$

where $c = \cosh(\alpha(x))$ and $s = \sinh(\alpha(x))$. It is immediate to see that under such transformation the connection above transforms to

$$\begin{aligned} \underset{\leftarrow}{\tilde{\omega}}^{01} &= -\alpha'(x)dx, & \underset{\leftarrow}{\tilde{\omega}}^{12} &= -\lambda(x)d\theta, \\ \underset{\leftarrow}{\tilde{\omega}}^{02} &= -\lambda(x)d\theta, & \underset{\leftarrow}{\tilde{\omega}}^{13} &= -\lambda(x)\sin(\theta)d\phi, \\ \underset{\leftarrow}{\tilde{\omega}}^{03} &= -\lambda(x)\sin(\theta)d\phi, & \underset{\leftarrow}{\tilde{\omega}}^{23} &= -\cos(\theta)d\phi, \end{aligned} \tag{A8}$$

where the arrows below the components denote the pull-back of the one-forms to Δ, and $\lambda(x) = \frac{2x}{F'\Omega}\exp(\alpha(x))$. We can obviously chose this Lorentz transformation in order for $\lambda(x) = \lambda_0$ with λ_0 an arbitrary constant. We have

$$\lambda_0 = \frac{2x}{F'\Omega}\exp(\alpha_0(x)). \tag{A9}$$

This can be made compatible with the time gauge by changing the space-time foliation just at the intersection with the horizon Δ so that $\tilde{e}^0 = (\Lambda \cdot e)^0$ is the new normal.[9] Now we are ready to write the quantities we were looking for:

$$\begin{aligned} \underset{\Leftarrow}{K}^1 &= 0, & \underset{\Leftarrow}{\Gamma}^3 &= \lambda_0 d\theta, \\ \underset{\Leftarrow}{K}^2 &= -\lambda_0 d\theta, & \underset{\Leftarrow}{\Gamma}^2 &= -\lambda_0\sin(\theta)d\phi, \\ \underset{\Leftarrow}{K}^3 &= -\lambda_0\sin(\theta)d\phi, & \underset{\Leftarrow}{\Gamma}^1 &= \cos(\theta)d\phi, \end{aligned} \tag{A10}$$

where $\Gamma^i = -\frac{1}{2}\epsilon^{ijk}\omega_{jk}$ and $K^i = \omega^{0i}$. The self-dual connection $A^i_+ \equiv \Gamma^i + iK^i$ and the Ashtekar-Barbero connection become

$$\begin{aligned} \underset{\Leftarrow}{A}{}^3_+ &= \lambda_0(-i\sin(\theta)d\phi + d\theta), & \underset{\Leftarrow}{A}{}^3_\beta &= \lambda_0(-\beta\sin(\theta)d\phi + d\theta), \\ \underset{\Leftarrow}{A}{}^2_+ &= \lambda_0(-\sin(\theta)d\phi - id\theta), & \underset{\Leftarrow}{A}{}^2_\beta &= \lambda_0(-\sin(\theta)d\phi - \beta d\theta), \\ \underset{\Leftarrow}{A}{}^1_+ &= \cos(\theta)d\phi, & \underset{\Leftarrow}{A}{}^1_\beta &= \cos(\theta)d\phi. \end{aligned} \tag{A11}$$

[9] Recently, a similar analysis as the one presented here—and also in [43]—has been done [44]. In that reference the authors derive a result which is compatible with the above equations in the *singular* vanishing extrinsic curvature slicing $\lambda_0 = 0$. Such (null) slicing is however inconsistent with the canonical formulation that is necessary for the LQG quantization of the bulk degrees of freedom.

The curvature of the self-dual and Ashtekar-Barbero connections is (when pulled back to the cross sections H)

$$\begin{aligned} \underset{\Leftarrow}{F}{}^3_+ &= 0, & \underset{\Leftarrow}{F}{}^3_\beta &= 0, \\ \underset{\Leftarrow}{F}{}^2_+ &= 0, & \underset{\Leftarrow}{F}{}^2_\beta &= 0, \\ \underset{\Leftarrow}{F}{}^1_+ &= -\sin(\theta)d\theta\wedge d\phi, & \underset{\Leftarrow}{F}{}^1_\beta &= -(1-\lambda_0^2[1+\beta^2])\sin(\theta)d\theta\wedge d\phi. \end{aligned} \tag{A12}$$

Using that $a_H = 4\pi r^2$ we can write the previous equations as

$$\underset{\Leftarrow}{F}{}^i_+ = -\frac{2\pi}{a_H}\underset{\Leftarrow}{\Sigma}{}^i \tag{A13}$$

and

$$\underset{\Leftarrow}{F}{}^i_\beta = (1-\lambda_0^2(1+\beta^2))\underset{\Leftarrow}{F}{}^i_+ = -\frac{2\pi(1-\lambda_0^2(1+\beta^2))}{a_H}\underset{\Leftarrow}{\Sigma}{}^i. \tag{A14}$$

In the following section we will show that $\lambda_0 = -1/\sqrt{2}$ defines the frame where the IH surface gravity matches the stationary black hole one. With this value of λ_0, the previous two equations and Eq. (A10) imply Eqs. (3), (5), and (6), respectively. For completeness we write the components of Σ^{IJ}

$$\begin{aligned} \underset{\leftarrow}{\Sigma}{}^{01} &= 0, \\ \underset{\leftarrow}{\Sigma}{}^{02} &= r\Omega\exp(\alpha)dx\wedge d\theta, \\ \underset{\leftarrow}{\Sigma}{}^{03} &= r\Omega\exp(\alpha)\sin(\theta)dx\wedge d\phi, \\ \underset{\leftarrow}{\Sigma}{}^{12} &= r\Omega\exp(\alpha)dx\wedge d\theta, \\ \underset{\leftarrow}{\Sigma}{}^{13} &= r\Omega\exp(\alpha)\sin(\theta)dx\wedge d\phi, \\ \underset{\leftarrow}{\Sigma}{}^{23} &= r^2\sin(\theta)d\theta\wedge d\phi, \\ \underset{\leftarrow}{\Sigma}{}^3_+ &= r\Omega\exp(\alpha)dx\wedge d\theta + ir\Omega\exp(\alpha)\sin(\theta)dx\wedge d\phi, \\ \underset{\leftarrow}{\Sigma}{}^2_+ &= -\exp(\alpha)\Omega r\sin(\theta)dx\wedge d\phi + ir\Omega\exp(\alpha)dx\wedge d\theta, \\ \underset{\leftarrow}{\Sigma}{}^1_+ &= r^2\sin(\theta)d\theta\wedge d\phi, \end{aligned} \tag{A15}$$

where on the right we have written the corresponding self-dual components.

2. Surface gravity and the value of λ_0

For stationary black holes, the surface gravity κ_H is defined by the equation

$$\ell^a\nabla_a\ell^b = \kappa_H\ell^b, \tag{A16}$$

where ℓ^a is the Killing vector field tangent to the horizon. For isolated horizons there is no unique notion of ℓ^a. We shall define ℓ_a in terms of the tetrad in the usual way with $\ell_a \equiv (e_a^1 - e_a^0)/\sqrt{2}$.[10] However, this definition still allows the freedom associated with the Lorentz transformations (A7) which send $\ell^a \to \exp(-\alpha(x))\ell^a$. We can fix this freedom by demanding the surface gravity to match that of a Reissner-Nordstrom black hole with mass M and charge Q for which

$$\kappa_H = \frac{\sqrt{(M^2-Q^2)}}{2M[M+\sqrt{(M^2-Q^2)}]-Q^2}. \tag{A17}$$

Indeed this choice is the one that makes the zero, and first law of IH look just as the corresponding laws of stationary black hole mechanics.

This choice is then physically motivated. In turn this will fix the value of λ_0 in (A14). If we define $n_a \equiv -(e_a^0 + e_a^1)/\sqrt{2}$ then we have that (A16) implies

$$\begin{aligned} \ell^a n_b\nabla_a\ell^b &= -\kappa_H, \\ -\tfrac{1}{2}\ell^a(e_b^0+e_b^1)\nabla_a(e^{1b}-e^{0b}) &= -\kappa_H, \\ \ell^a\omega_a^{01} &= \kappa_H. \end{aligned} \tag{A18}$$

Notice that after the Lorentz transformation (A7) we have

$$\begin{aligned} \kappa_H &= \ell^a\omega_a^{01} = -\alpha'\ell^a dx_a = -\alpha' g^{ab}\ell_a dx_b \\ &= -\exp(-\alpha)\frac{\Omega}{\sqrt{2}}\alpha' g^{ax}(dt_a - dx_a) \\ &= \exp(-\alpha)\frac{\Omega}{\sqrt{2}}\alpha' g^{xx} = -(\exp(-\alpha))'\frac{\Omega}{\sqrt{2}}g^{xx} \\ &= -(\exp(-\alpha))'\frac{1}{\sqrt{2}\Omega}. \end{aligned} \tag{A19}$$

Now we can fix $\alpha(x) = \alpha_0(x)$ so that κ_H takes the Reissner-Nordstrom value. Recalling Eq. (A9) and using the above equations, a simple calculation shows that this happens for

$$\lambda_0 = -\frac{1}{\sqrt{2}}, \tag{A20}$$

which implies the desired result

$$F^i_\beta = \tfrac{1}{2}(1-\beta^2)F^i_+. \tag{A21}$$

Notice that

$$\nabla_a\ell_b = \omega_a^{01}\ell_b, \tag{A22}$$

and that [according to (A8)] we also have

$$d\omega^{01} = 0. \tag{A23}$$

[10]The future pointing null generators of the horizon ℓ^a are such that $\ell^a \propto (\partial/\partial x)^a + (\partial/\partial t)^a$. This implies that $\ell_a \propto dx_a - dt_a$ from which we get $\ell_a = (e_a^1 - e_a^0)/\sqrt{2}$ and $n_a = -(e_a^1 + e_a^0)/\sqrt{2}$ so that $n\cdot\ell = -1$.

All this implies that $\mathcal{L}_\ell \omega^{01} = d(\ell \lrcorner \omega^{01}) + \ell \lrcorner d\omega^{01} = d\kappa_H = 0$ as expected from $[\mathcal{L}_\ell, D] = 0$ (general proof in Lemma 2). In other words, the ℓ we have chosen by means of fixing the boost freedom $\ell \to \exp(-\alpha(x))\ell$ is a member of the equivalence class $[\ell]$ in Definition II.

3. Solving Cartan's equation

For this we first compute de, namely:

$$
\begin{aligned}
de^0 &= \Omega'(x,t)dr \wedge dt|_\Delta = 2\Omega' \frac{x}{F'} dx \wedge dt,\\
de^1|_\Delta &= 2\Omega' \frac{x}{F'} dx \wedge dt,\\
de^2|_\Delta &= \frac{2x}{F'}(dx \wedge d\theta - dt \wedge d\theta),\\
de^3|_\Delta &= -r\cos(\theta) d\theta \wedge d\phi\\
&\quad + \frac{2x}{F'}\sin(\theta)(dx \wedge d\phi - dt \wedge d\phi).
\end{aligned} \tag{A24}
$$

Now we are ready to explicitly write Cartan's first structure equations. They are

$$
\begin{aligned}
0|_\Delta &= 2\Omega' \frac{x}{F'} dx \wedge dt + \Omega \omega^{01} \wedge dx + r\omega^{02} \wedge d\theta\\
&\quad + r\sin(\theta)\omega^{03} \wedge d\phi,\\
0|_\Delta &= 2\Omega' \frac{x}{F'} dx \wedge dt + \Omega \omega^{01} \wedge dt + r\omega^{12} \wedge d\theta\\
&\quad + r\sin(\theta)\omega^{13} \wedge d\phi,\\
0|_\Delta &= \frac{2x}{F'}(dx \wedge d\theta - dt \wedge d\theta) + \Omega\omega^{02} \wedge dt\\
&\quad + \Omega\omega^{21} \wedge dx + r\sin(\theta)\omega^{23} \wedge d\phi,\\
0|_\Delta &= -r\cos(\theta)d\theta \wedge d\phi\\
&\quad + \frac{2x}{F'}\sin(\theta)(dx \wedge d\phi - dt \wedge d\phi)\\
&\quad + \Omega\omega^{03} \wedge dt + \Omega\omega^{31} \wedge dx + r\omega^{32} \wedge d\theta.
\end{aligned} \tag{A25}
$$

Let us now study the previous equation individually. The six components of the first, Eq. (A25), become

$$
\begin{aligned}
0|_\Delta &= dx \wedge dt\left(2\Omega' \frac{x}{F'} - \omega_t^{01}\Omega\right),\\
0|_\Delta &= dx \wedge d\theta(-\Omega\omega_\theta^{01} + \omega_x^{02} r),\\
0|_\Delta &= dx \wedge d\phi(-\Omega\omega_\phi^{01} + \omega_x^{03} r\sin(\theta)),\\
0|_\Delta &= dt \wedge d\theta(r\omega_t^{02}),\\
0|_\Delta &= dt \wedge d\phi(\omega_t^{03} r\sin(\theta)),\\
0|_\Delta &= d\theta \wedge d\phi(-r\omega_\phi^{02} + \omega_\theta^{03} r\sin(\theta)).
\end{aligned} \tag{A26}
$$

The six components of the second, Eq. (A25), become

$$
\begin{aligned}
0|_\Delta &= dx \wedge dt\left(2\Omega' \frac{x}{F'} + \omega_x^{01}\Omega\right),\\
0|_\Delta &= dx \wedge d\theta(\omega_x^{12} r),\\
0|_\Delta &= dx \wedge d\phi(\omega_x^{13} r\sin(\theta)),\\
0|_\Delta &= dt \wedge d\theta(-\Omega\omega_\theta^{01} + r\omega_t^{12}),\\
0|_\Delta &= dt \wedge d\phi(-\Omega\omega_\phi^{01} + r\sin(\theta)\omega_t^{13}),\\
0|_\Delta &= d\theta \wedge d\phi(-r\omega_\phi^{12} + \omega_\theta^{13} r\sin(\theta)).
\end{aligned} \tag{A27}
$$

The six components of the third, Eq. (A25), become

$$
\begin{aligned}
0|_\Delta &= dx \wedge dt(\omega_x^{02}\Omega + \omega_t^{21}\Omega),\\
0|_\Delta &= dx \wedge d\theta\left(2\frac{x}{F'} - \omega_\theta^{21}\Omega\right),\\
0|_\Delta &= dx \wedge d\phi(-\omega_\phi^{21}\Omega + \omega_x^{23} r\sin(\theta)),\\
0|_\Delta &= dt \wedge d\theta\left(-2\frac{x}{F'} - \Omega\omega_\theta^{02}\right),\\
0|_\Delta &= dt \wedge d\phi(-\Omega\omega_\phi^{02} + r\sin(\theta)\omega_t^{23}),\\
0|_\Delta &= d\theta \wedge d\phi(\omega_\theta^{23} r\sin(\theta)).
\end{aligned} \tag{A28}
$$

Finally, the six components of the fourth, Eq. (A25), become

$$
\begin{aligned}
0|_\Delta &= dx \wedge dt(\omega_x^{03}\Omega - \omega_t^{31}\Omega),\\
0|_\Delta &= dx \wedge d\theta(-\omega_\theta^{31}\Omega + \omega_x^{32} r),\\
0|_\Delta &= dx \wedge d\phi\left(2\frac{x}{F'}\sin(\theta) - \omega_\phi^{31}\Omega\right),\\
0|_\Delta &= dt \wedge d\theta(-\omega_\theta^{03}\Omega + r\omega_t^{32}),\\
0|_\Delta &= dt \wedge d\phi\left(-2\frac{x}{F'}\sin(\theta) - \Omega\omega_\phi^{03}\right),\\
0|_\Delta &= d\theta \wedge d\phi(-r\cos(\theta) - \omega_\phi^{32} r).
\end{aligned} \tag{A29}
$$

At this point we make the following ansatz $\omega_{01}^\theta = 0$, $\omega_{01}^\phi = 0$, $\omega_{23}^x = 0$, and $\omega_{23}^t = 0$, from which we get the solution (A6).

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