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Inhomogeneous loop quantum cosmology: Hybrid quantization of the Gowdy model

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The Gowdy cosmologies provide a suitable arena to further develop loop quantum cosmology, allowing the presence of inhomogeneities. For the particular case of Gowdy spacetimes with the spatial topology of a three-torus and a content of linearly polarized gravitational waves, we detail a hybrid quantum theory in which we combine a loop quantization of the degrees of freedom that parametrize the subfamily of homogeneous solutions, which represent Bianchi I spacetimes, and a Fock quantization of the inhomogeneities. Two different theories are constructed and compared, corresponding to two different schemes for the quantization of the Bianchi I model within the *improved dynamics* formalism of loop quantum cosmology. One of these schemes has been recently put forward by Ashtekar and Wilson-Ewing. We address several issues, including the quantum resolution of the cosmological singularity, the structure of the superselection sectors in the quantum system, or the construction of the Hilbert space of physical states.

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I. INTRODUCTION

In the absence of a full theory of canonical quantum gravity, it is most instructive to study symmetry-reduced systems in order to test and develop mathematical techniques that can help in achieving the quantization of general relativity, as well as to progress in the understanding of the physical phenomena emerging in quantum gravity. The symmetry reduction makes these systems more manageable for a complete quantization and facilitates the extraction of physical predictions. In particular, mimicking the techniques applied in loop quantum gravity (LQG) [1], a quantization program for homogeneous models, known as loop quantum cosmology (LQC) [2], has been developed in recent years (see, e.g., [3–13]). The so-called *polymeric* quantization applied to these systems has interesting physical consequences. Remarkably, the analog of the classical cosmological singularity is not present in the quantum theory and is therefore resolved.

In view of the success of LQC, it is compelling to extend the application of its techniques to the quantization of inhomogeneous cosmological systems. Several reasons motivate this extension. On the one hand, this kind of models preserves one of the most characteristic features of the full theory, namely, the fieldlike nature of the degrees of freedom. Thus, one would expect that their quantization gives trustable insights about some of the open questions in LQG which are related to the presence of an infinite number of degrees of freedom. On the other hand, it is essential to investigate the role played by inhomogeneities in the quantum theory in order to understand the physical laws that explain the origin and evolution of the Universe, and reach in this way realistic predictions that can be confronted with cosmological observations. Furthermore, the analysis of the inhomogeneities would allow us to check the robustness of the results obtained in homogeneous LQC, in particular, those concerning the quantum resolution of classical cosmological singularities.

With this aim, we have carried out a complete quantization of the linearly polarized Gowdy T^3 model [14]. We have chosen this model because it is the simplest cosmological system *in vacuo* which contains inhomogeneities. Indeed, not only its classical solutions are well known [15,16], but also their quantization (by more conventional methods) has deserved much attention since the 70s [17]. Actually, a complete Fock quantization of the deparametrized system has been achieved [18], which has been shown to be essentially unique [19]. Unlike these previous analyses, which involve standard nonpolymeric techniques, we will discuss here a hybrid quantization that combines the polymeric procedures of LQC—applied to the homogeneous sector of the system, namely, the set of degrees of freedom that describe the subfamily of homogeneous solutions of the Gowdy model—with the Fock quantization of the inhomogeneities. Since the full polymeric quantization of the model is an extremely complicated task, we have chosen a more conservative approach, which explores the effects of the quantum discrete geometry underlying LQC only on the homogeneous gravitational sector. A natural treatment for the inhomogeneities is then a Fock quantization. This approach assumes that there exists a regime in which the most relevant phenomena emerging from quantum geometry are those affecting the homogeneous subsystem, whereas such effects are small and can be ignored for the inhomogeneities, even if the latter may still present a standard (Fock) quantum

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behavior. Although our quantization approach is not completely derived from LQG, it retains interesting features associated with the discrete and polymeric nature of the geometry, declaring a certain type of perturbative hierarchy on their relevance for different subsectors of the cosmological system.

As we have commented, the homogeneous sector of the Gowdy model coincides with the phase space of the Bianchi I model. It turns out that there exist two different quantizations of this spacetime in the literature of LQC, developed in Refs. [11,12], which correspond to two different schemes of the so-called *improved dynamics* prescription and that, for brevity, we will denote as schemes A and B, respectively. In Refs. [20,21], we presented our hybrid quantization making use of scheme A in the representation of the homogeneous sector, i.e., applying the quantization of Ref. [11] (see also Ref. [9]). The main aim of this paper is to revisit and develop our hybrid quantization by incorporating the alternative scheme B, put forward by Ashtekar and Wilson-Ewing, in the construction of the Bianchi I representation for the homogeneous sector. We will review the quantization of Ref. [12], improve and complete some aspects of the quantization procedure, extend our hybrid framework to adapt it to the alternative scheme and, finally, compare the results obtained with the two hybrid approaches.

With this objective, the constraints present in the model will be represented by well-defined operators on a (hybrid) kinematical Hilbert space. These constraints are a global diffeomorphism constraint, which generates translations in the circle, and a global Hamiltonian constraint which couples the homogeneous and inhomogeneous sectors in a complicated form. As it is typical in the context of LQC, in both schemes the Hamiltonian constraint operator *superselects* the kinematical Hilbert space—which as a whole is nonseparable—in separable sectors. In particular, we will be able to determine the structure of these superselection sectors for scheme B, which are highly nontrivial and had not been identified previously (see Ref. [12]). Furthermore, our Hamiltonian constraint leads in fact to a difference equation in an internal, strictly positive parameter with support on semilattices of points with constant separation. In scheme A, this parameter is the classical global time used in the Fock quantization to deparametrize the system [18]. In Refs. [20,21], we showed that the solutions to the Hamiltonian constraint are determined by the data on the *initial* section, namely, the section where that parameter takes its lowest allowed value. We will see that, in scheme B, the analogous discrete variable is (up to a constant factor) the volume of the Bianchi I spacetime associated with the Gowdy universe, and we will analyze the resulting structure of the solutions.

Actually, one of the motivations of our work is the expectation that the quantum field theory should remain valid on the loop quantized Bianchi I background after imposing the quantum constraints, therefore validating the standard Fock description for the inhomogeneities in the approximation adopted here. In the hybrid approach for scheme A, we demonstrated in Refs. [20,21] that the physical Hilbert space has the expected tensor product structure: the physical Hilbert space of the Bianchi I model times a Fock space for the inhomogeneous sector, which is equivalent to that obtained in Ref. [18]. Hence, we indeed recover the standard quantum field theory for the inhomogeneities. We will discuss whether this continues to be the case in the new hybrid approach, obtained with the alternative quantization scheme for Bianchi I.

Another important motivation, as we have commented, is the study of the fate of the cosmological singularities in the quantum theory. We will see that the polymeric quantization of the homogeneous sector, irrespective of the adopted improved dynamics scheme, is enough to *cure* the singularity. This fact endorses the previous results of singularity resolution obtained in homogeneous LQC.

The rest of the paper is organized as follows: In Sec. II, we summarize the basic features of the classical Gowdy T^3 model. In Sec. III, we construct the kinematical Hilbert space, and represent the elementary operators in both schemes A and B. In addition, we represent the diffeomorphism constraint in the quantum theory. We construct the Hamiltonian constraint operator in Sec. IV for the two studied schemes. In Sec. V, we review the structure of the physical Hilbert space for scheme A. The physical sector for case B is investigated in Sec. VI. In Sec. VII, we discuss the main results of this work, and conclude with some additional remarks. Finally, we include four appendices, which contain some supplementary technical details.

II. THE REDUCED MODEL

Gowdy models describe globally hyperbolic spacetimes *in vacuo* which possess two spacelike commuting Killing vector fields and whose spatial sections are compact [14]. The simplest case is the Gowdy T^3 model (i.e., the spatial topology is that of a three-torus). We will focus our discussion on the subsystem with linearly polarized gravitational waves. This model satisfies the additional restriction that the Killing fields be hypersurface orthogonal. Hence, they can be chosen mutually orthogonal everywhere. Let ∂_σ and ∂_δ be these two axial Killing fields. We may then describe the spacetime (globally) with coordinates adapted to these symmetries: $\{t, \theta, \sigma, \delta\}$, with θ, σ, $\delta \in S^1$. The metric components depend only on t and θ. Since they are periodic in θ, they can be expanded in a Fourier series.

We use the symmetries of the system to fix completely the gauge freedom associated with the diffeomorphism constraints in σ and δ, the directions defined by the Killing fields, as explained in Ref. [22]. We further fix all the gauge freedom associated with the inhomogeneous (nonzero) modes of the θ-momentum constraint and of the Hamiltonian constraint. The details of this gauge fixing

can be found in Ref. [21]. At the end of the day, two global constraints remain in the model: a generator of translations in the circle, C_θ, and a Hamiltonian constraint, $C_{\rm G}$. Except for these constraints, all the gauge freedom is fixed. The reduced phase space contains three pairs of "point-particle" degrees of freedom, which parameterize the sector of (spatially) homogenous spacetimes and is therefore called the homogeneous sector, and all the nonzero modes of a field and its conjugate momentum, which form the so-called inhomogeneous sector.

In order to quantize the homogeneous sector with polymeric techniques according to LQC, we need to describe it in terms of Ashtekar variables. The subfamily of homogeneous solutions of the Gowdy model, encoded in this sector as we have mentioned, corresponds to empty Bianchi I spacetimes with three-torus topology. In a diagonal gauge, the nontrivial components of the $SU(2)$ gravitational connection and of the densitized triad are $c_i/(2\pi)$ and $p_i/(4\pi^2)$, respectively, with $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$ and $i, j = \theta, \sigma, \delta$ (see, e.g., [10–12]). Here G is the Newton constant and γ the Immirzi parameter. For the inhomogeneous sector we will carry out the Fock quantization presented in Ref. [18]. With this aim we adopt as basic variables the creation and annihilation variables which would be naturally associated with a free massless scalar field, (a_m, a_m^*), where m can be any nonzero integer. In Appendix A, we provide the form of the metric in terms of the chosen variables. The constraints are given by [21]

$$C_\theta = \sum_{m=1}^{\infty} m(a_m^* a_m - a_{-m}^* a_{-m}) = 0, \tag{2.1}$$

$$C_{\rm G} = \frac{\mathcal{C}_{\rm G}}{\sqrt{|p_\theta p_\sigma p_\delta|}} = 0, \tag{2.2}$$

$$\mathcal{C}_{\rm G} = -\frac{2}{\gamma^2}[c_\theta p_\theta c_\sigma p_\sigma + c_\theta p_\theta c_\delta p_\delta + c_\sigma p_\sigma c_\delta p_\delta] + G\left[\frac{(c_\sigma p_\sigma + c_\delta p_\delta)^2}{\gamma^2 |p_\theta|} H_{\rm int}^\xi + 32\pi^2 |p_\theta| H_0^\xi\right], \tag{2.3}$$

$$H_{\rm int}^\xi = \sum_{m\neq 0} \frac{1}{2|m|}[2a_m^* a_m + a_m a_{-m} + a_m^* a_{-m}^*], \tag{2.4}$$

$$H_0^\xi = \sum_{m\neq 0} |m| a_m^* a_m. \tag{2.5}$$

The first line of $\mathcal{C}_{\rm G}$ is just the standard form of the densitized Hamiltonian constraint of the Bianchi I model in Ashtekar variables (see, e.g., [10]). The inhomogeneities are present in the term H_0^ξ, which is the Hamiltonian corresponding to a free massless scalar field, and in the term $H_{\rm int}^\xi$, which represents an interaction term quadratic in the field. Note that the inhomogeneities are coupled with the homogeneous sector in a complicated way, what makes the hybrid quantization of this system far from straightforward.

III. THE KINEMATICAL HILBERT SPACE

The kinematical Hilbert space of this reduced Gowdy model is the tensor product of the kinematical Hilbert spaces of the two sectors.

A. The homogeneous sector

In order to construct the kinematical Hilbert space of the homogeneous sector, we consider the two different polymeric quantizations of the Bianchi I model developed in Refs. [11,12]. The main distinction between them lies in the way in which the quantization prescription known as improved dynamics (introduced in Ref. [4] with successful results in isotropic cosmologies) is adapted to the anisotropic case. Although the quantization of Ref. [11] is well defined in the considered model with compact spatial slices, it suffers from some drawbacks in noncompact situations [10]. This fact motivated the consideration of an alternative prescription, whose corresponding quantization has been recently developed by Ashtekar and Wilson-Ewing in Ref. [12]. In the following, we will call scheme A the prescription adopted in Ref. [11], and scheme B that of Ref. [12].

Let us summarize the main characteristics of both quantizations. The elementary configuration variables are holonomies of connections computed along edges of oriented coordinate length $2\pi\mu_i$ in the direction i, where μ_i is any real number. On the other hand, the elementary momentum variables are triad fluxes through rectangles orthogonal to those directions. The configuration algebra is the algebra of almost periodic functions generated by the matrix elements of the holonomies, namely, $\mathcal{N}_{\mu_j}(c_j) = \exp(i\mu_j c_j/2)$ (no Einstein summation convention is adopted). We call $\mathrm{Cyl}_{\rm S} = \otimes_i \mathrm{Cyl}_{{\rm S},i}$ the corresponding vector space, and employ the ket notation $|\mu_i\rangle$ to denote the states $\mathcal{N}_{\mu_i}(c_i)$ in the triad representation. The kinematical Hilbert space, $\mathcal{H}_{\rm Kin} = \otimes_i \mathcal{H}_{{\rm Kin},i}$, is then the completion of the space $\mathrm{Cyl}_{\rm S}$ with respect to the discrete inner product $\langle\mu_i|\mu_i'\rangle = \delta_{\mu_i\mu_i'}$ for each direction. Hence, the states $|\mu_i\rangle$, which are eigenstates of the operator $\hat{p}_i$ associated with fluxes, provide an *orthonormal* basis of $\mathcal{H}_{{\rm Kin},i}$. In turn, the operators $\hat{\mathcal{N}}_{\mu_i'}$ associated with holonomies produce a shift equal to μ_i' in the label μ_i of this basis of states (see Refs. [3,9,11]). We assume that the action of any operator defined on $\mathrm{Cyl}_{\rm S}$ is the identity when acting on the kinematical Hilbert space of the inhomogeneous sector, which will be introduced below.

1. Improved dynamics: Schemes A and B

The so-called improved dynamics prescription is based on the assertion that, because of the existence of a mini-

mum gap Δ in the spectrum of the physical area operator in LQG [4,23], fiducial surfaces cannot be as small as one wants, but there is a minimum fiducial surface, $S_{\min}$, whose corresponding physical area is equal to Δ. As a consequence, the coordinate length of the holonomy along each edge turns out to exhibit a minimum nonzero value $2\pi\bar{\mu}_i$. Prescription A is derived assuming that $S_{\min}$, lying, e.g., in the $j-k$ plane, is a fiducial square with a minimum fiducial side determined by $2\pi\bar{\mu}_i$, whereas according to prescription B it must be instead a rectangle with minimum fiducial sides given by $2\pi\bar{\mu}_j$ and $2\pi\bar{\mu}_k$. Here, and in what follows, whenever the three indices i, j, and k appear together, we assume $\epsilon_{ijk} \neq 0$.

As a result one gets two different expressions for $\bar{\mu}_i$, one for each prescription, given by [9,10,12]

$$\text{A}: \frac{1}{\bar{\mu}_i} = \frac{\sqrt{|p_i|}}{\sqrt{\Delta}}, \qquad \text{B}: \frac{1}{\bar{\mu}_i} = \frac{1}{\sqrt{\Delta}}\sqrt{\left|\frac{p_j p_k}{p_i}\right|}. \tag{3.1}$$

Although these arguments can be considered heuristic inasmuch as the true relation between full LQG and LQC is still to be understood, there is a feature of prescription B (deduced in Ref. [12] following a procedure of this type) which distinguishes it and has not been realized until now in the literature. Provided that the $\bar{\mu}_i$'s depend only on fluxes [i.e., $\bar{\mu}_i = \bar{\mu}_i(p_j)$], because so does the physical area, prescription B is uniquely determined by the requirement that, for all directions i, the exponents $\bar{\mu}_i c_i$ of the holonomy elements $\mathcal{N}_{\bar{\mu}_i}(c_i)$ have a fixed *constant* Poisson bracket with the variable

$$v := \mathrm{sgn}(p_\theta p_\sigma p_\delta)\frac{\sqrt{|p_\theta p_\sigma p_\delta|}}{2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}}, \tag{3.2}$$

up to a sign depending on the orientations of the triad components. Here, $l_{\mathrm{Pl}} = \sqrt{G\hbar}$ is the Planck length. Note that v is proportional to the volume of the associated Bianchi I universe. Then, at least at the level of the Poisson bracket algebra, the introduced requirement can be understood as the condition that the holonomies produce a constant shift in the volume.

For any of the prescriptions, and owing to the dependence of $\bar{\mu}_i$ on the coefficients of the densitized triad, the elementary operator $\hat{\mathcal{N}}_{\bar{\mu}_i}$ generates in fact a state-dependent nonlinear transformation on the basis of states $|\mu_\theta, \mu_\sigma, \mu_\delta\rangle = \otimes_i|\mu_i\rangle$. It is possible to relabel this basis with affine parameters instead of the labels μ_i, so that the transformation generated by $\hat{\mathcal{N}}_{\bar{\mu}_i}$ is simply a shift in the labels. Each prescription requires a different affine reparametrization.

2. Quantum representation in case A

In scheme A, since $\bar{\mu}_i$ only depends on p_i, the three fiducial directions are not mixed, and we can calculate a new parameter $v_i(\mu_i)$ for each fiducial direction to rename the states in our basis, so that the operator $\hat{\mathcal{N}}_{\bar{\mu}_i}$ generates a constant shift in the new label v_i, as was done in Refs. [4,9]. The action of the basic operators on the relabeled states $|v_\theta, v_\sigma, v_\delta\rangle$ is given by [9,11]

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_i}|v_i\rangle = |v_i \pm 1\rangle, \tag{3.3}$$

$$\hat{p}_i|v_i\rangle = (6\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta})^{2/3}\mathrm{sgn}(v_i)|v_i|^{2/3}|v_i\rangle. \tag{3.4}$$

3. Quantum representation in case B

Similarly, one can introduce new parameters λ_i for the three fiducial directions such that the action of the operator $\hat{\mathcal{N}}_{\bar{\mu}_i}$ has only a nontrivial effect on the label λ_i, whereas it does not change the other two labels of the state, λ_j and λ_k [12]. Remarkably, this effect is not a constant shift anymore but depends on the values of those two labels. Nonetheless, as we have commented, there exists a variable v, given in terms of the λ_i's by $v = 2\lambda_\theta\lambda_\sigma\lambda_\delta$, such that all the operators $\hat{\mathcal{N}}_{\bar{\mu}_i}$ produce a constant shift on it (for fixed orientation of the triad components). Therefore, it is convenient to work, e.g., with the relabeled states $|v, \lambda_\sigma, \lambda_\delta\rangle$. The two λ's are variables which measure the degree of anisotropy. The representation of the basic operators is determined by [12]

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_\theta}|v, \lambda_\sigma, \lambda_\delta\rangle = |v \pm \mathrm{sgn}(\lambda_\sigma\lambda_\delta), \lambda_\sigma, \lambda_\delta\rangle, \tag{3.5}$$

$$\hat{p}_\theta|v, \lambda_\sigma, \lambda_\delta\rangle = (4\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta})^{2/3}\mathrm{sgn}\left(\frac{v}{\lambda_\sigma\lambda_\delta}\right)\frac{v^2}{4\lambda_\sigma^2\lambda_\delta^2} \times |v, \lambda_\sigma, \lambda_\delta\rangle, \tag{3.6}$$

$$\hat{\mathcal{N}}_{\pm\bar{\mu}_\sigma}|v, \lambda_\sigma, \lambda_\delta\rangle = \left|v \pm \mathrm{sgn}(\lambda_\sigma v), \lambda_\sigma \pm \left|\frac{\lambda_\sigma}{v}\right|, \lambda_\delta\right\rangle, \tag{3.7}$$

$$\hat{p}_\sigma|v, \lambda_\sigma, \lambda_\delta\rangle = (4\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta})^{2/3}\mathrm{sgn}(\lambda_\sigma)\lambda_\sigma^2|v, \lambda_\sigma, \lambda_\delta\rangle. \tag{3.8}$$

The actions of $\hat{\mathcal{N}}_{\pm\bar{\mu}_\delta}$ and $\hat{p}_\delta$ can be obtained from Eqs. (3.7) and (3.8) by interchanging λ_σ and λ_δ [24]. When we construct the Hamiltonian constraint in Sec. IV, we will see that states with $v = 0$—and *a fortiori* with vanishing λ_σ or λ_δ—are removed from our kinematical Hilbert space, and therefore the above representation is well defined. Unlike in case A, note that all directions are mixed now, and the operators $\hat{\mathcal{N}}_{\bar{\mu}_i}$ and $\hat{\mathcal{N}}_{\bar{\mu}_j}$ $(i \neq j)$ do not commute.

B. The inhomogeneous sector

For this sector we employ a Fock quantization, promoting the variables a_m and a_m^* to annihilation and creation

operators $\hat{a}_m$ and $\hat{a}_m^\dagger$, respectively, such that $[\hat{a}_m, \hat{a}_{\tilde{m}}^\dagger] = \delta_{m\tilde{m}}$. As before, we assume that these operators are the identity acting on the homogeneous sector. We call $\mathcal{S}$ the vector space whose elements are finite linear combinations of n-particle states,

$$|\mathfrak{n}\rangle := |\ldots, n_{-2}, n_{-1}, n_1, n_2, \ldots\rangle, \tag{3.9}$$

with $\sum_m n_m < \infty$, $n_m \in \mathbb{N}$ being the occupation number (or number of particles) of the m-th mode. Then, the symmetric Fock space $\mathcal{F}$ is the completion of the space $\mathcal{S}$ with respect to the Fock inner product $\langle \mathfrak{n}'|\mathfrak{n}\rangle = \delta_{\mathfrak{n}'\mathfrak{n}}$. Therefore, the n-particle states provide an orthonormal basis of the Fock space $\mathcal{F}$.

In the totally deparametrized system, this is the unique Fock quantization in which the field dynamics is unitarily implemented and that also provides a natural unitary implementation of the gauge group of S^1 translations [19].

C. Quantum representation of the S^1 symmetry

Since the generator of translations in the circle, given in Eq. (2.1), only affects the inhomogeneities, it is the same for the Gowdy model in both schemes A and B. Employing the above Fock representation and taking normal ordering, we obtain its quantum counterpart

$$\hat{C}_\theta = \sum_{m>0}^{\infty} m\hat{X}_m, \qquad \hat{X}_m = \hat{a}_m^\dagger \hat{a}_m - \hat{a}_{-m}^\dagger \hat{a}_{-m}. \tag{3.10}$$

The n-particle states annihilated by this operator are those which satisfy the condition

$$\sum_{m>0}^{\infty} mX_m = 0, \qquad X_m = n_m - n_{-m}. \tag{3.11}$$

They form a dense set of a proper subspace of the Fock space $\mathcal{F}$ that we denote by $\mathcal{F}_p$. It is (unitarily equivalent to) the physical Hilbert space of Ref. [18].

IV. THE HAMILTONIAN CONSTRAINT OPERATOR

Physical states must also be annihilated by the operator that represents the (nondensitized) Hamiltonian constraint [25]. We will carry out a process of densitization that will allow us to give an equivalent (and more convenient) description in which physical states will be annihilated by the densitized version of the Hamiltonian constraint. Actually, this procedure is also adopted in the quantizations of Refs. [5,11].

A. Densitization of the Hamiltonian constraint

We define the subspace of zero homogeneous volume states as the kernel of the homogeneous volume operator $\hat{V} = \otimes_i \widehat{\sqrt{|p_i|}}$ which represents the physical volume of the (coordinate cell in the) Bianchi I spacetime associated with the Gowdy cosmology. Let then $\hat{C}_{\rm G} = \hat{C}_{\rm BI} + \hat{C}_\xi$ be the operator that represents the (nondensitized) Hamiltonian constraint for the Gowdy model, where $\hat{C}_{\rm BI}$ denotes the Hamiltonian constraint for the Bianchi I model and $\hat{C}_\xi$ is the term that involves the inhomogeneities. One first constructs the operator $\hat{C}_{\rm BI}$ following LQG procedures. When symmetrizing it, one can always adopt a suitable factor ordering such that $\hat{C}_{\rm BI}$ annihilates the subspace of states with zero homogeneous volume and leaves its orthogonal complement invariant. On the other hand, one can construct the other operator $\hat{C}_\xi$ so that it inherits the same properties. Therefore, the subspace of states with vanishing homogeneous volume decouples under the action of the full Hamiltonian constraint $\hat{C}_{\rm G}$, and one can then remove it from the kinematical Hilbert space and restrict the study to its complement. This complement will be denoted by $\tilde{\mathcal{H}}_{\rm Kin} \otimes \mathcal{F}$, where $\tilde{\mathcal{H}}_{\rm Kin}$ is the completion of

$$\begin{aligned}\widetilde{\rm Cyl}_{\rm S} &= {\rm span}\{|v_\theta, v_\sigma, v_\delta\rangle; v_\theta v_\sigma v_\delta \neq 0\} \\ &= {\rm span}\{|v, \lambda_\sigma, \lambda_\delta\rangle; v\lambda_\sigma\lambda_\delta \neq 0\}.\end{aligned} \tag{4.1}$$

In the last formula, we have made explicit that only nonzero values of λ_σ and λ_δ are allowed, even if this is implicit in the nonvanishing of $v = 2\lambda_\theta\lambda_\sigma\lambda_\delta$.

In general, nontrivial physical states, which are annihilated by $\hat{C}_{\rm G}$, are not normalizable in $\tilde{\mathcal{H}}_{\rm Kin} \otimes \mathcal{F}$. In principle, they should belong to a larger space, typically the algebraic dual of a suitable dense subspace of this kinematical Hilbert space, e.g., the tensor product of $\widetilde{\rm Cyl}_{\rm S}$ and a suitable dense subspace of $\mathcal{F}$. We will denote these states by $(\tilde{\psi}|$. Actually, they can be transformed into other states $(\psi|$ by the map

$$(\tilde{\psi}| \to (\psi| = (\tilde{\psi}|\widehat{\left[\frac{1}{V}\right]}^{1/2}, \tag{4.2}$$

which is a bijection in the considered kind of algebraic dual spaces. Here, the operator $\widehat{[1/V]}$ represents the inverse of the homogeneous volume and is well defined in $\widetilde{\rm Cyl}_{\rm S} \otimes \mathcal{F}$. In Appendix B, we provide its explicit form for each of the schemes A and B. In both cases, the resulting operator is diagonal in our basis of states, annihilates the zero homogeneous volume states and is bounded. Therefore it can be extended uniquely to the kinematical Hilbert space. Moreover, its inverse $\widehat{[1/V]}^{-1}$ is also well defined via the spectral theorem once we have restricted the discussion to the kinematical Hilbert space $\tilde{\mathcal{H}}_{\rm Kin} \otimes \mathcal{F}$, because the discrete spectrum of $\widehat{[1/V]}$ in this space does not contain the zero anymore.

The transformed physical states $(\psi|$ are then annihilated by the (adjoint of the) symmetric operator

$$\hat{\mathcal{C}}_{\rm G} = \widehat{\left[\frac{1}{V}\right]}^{-1/2} \hat{C}_{\rm G} \widehat{\left[\frac{1}{V}\right]}^{-1/2} = \hat{C}_{\rm BI} + \hat{C}_\xi. \tag{4.3}$$

It is worth noting that the relation between the Hamiltonian constraint of the Gowdy model and its densitized version does not involve the volume of the Gowdy spacetime, but the volume of the associated Bianchi I spacetime. Hence, the above operator $\hat{\mathcal{C}}_{\rm G}$ is in fact the quantum counterpart of the constraint $\mathcal{C}_{\rm G}$ given in Eq. (2.3).

B. Quantum representation of the densitized Hamiltonian constraint

For both schemes, we start with the (nondensitized) Hamiltonian constraint of the Bianchi I model $\hat{C}_{\rm BI}$. We take advantage of the freedom in the factor ordering to get a representation as convenient as possible. More specifically, as mentioned above, we adopt a symmetric factor ordering which has the two following features: i) the same powers of $\widehat{|p_i|}$ appear on the left and right of every term, in a way such that $\hat{C}_{\rm BI}$ decouples the zero homogeneous volume states; ii) the factors of the type $\sin(\widehat{\bar{\mu}_i c_i})\,\widehat{\rm sgn}(p_i)$ [where $\sin(\widehat{\bar{\mu}_j c_j}) = i(\hat{\mathcal{N}}_{-2\bar{\mu}_j} - \hat{\mathcal{N}}_{2\bar{\mu}_j})/2$], are symmetrized in the form

$$\tfrac{1}{2}[\,\sin(\widehat{\bar{\mu}_i c_i})\widehat{\rm sgn}(p_i) + \widehat{\rm sgn}(p_i)\sin(\widehat{\bar{\mu}_i c_i})]. \tag{4.4}$$

As a consequence, our operator $\hat{C}_{\rm BI}$ also decouples states with different orientations of the densitized triad components. Both properties have relevant consequences, as was discussed in detail in Ref. [5], where the flat Friedmann-Robertson-Walker model coupled to a scalar field was quantized adopting the same procedure. As we will see later on, the procedure leads to simple superselection sectors with neat physical properties. In particular, this fact allowed us to solve explicitly the Hamiltonian constraint and determine the physical Hilbert space of the hybrid Gowdy model for scheme A in Refs. [20,21]. Therefore, it seems most reasonable to apply the same kind of symmetrization in case B as well.

By densitizing $\hat{C}_{\rm BI}$ we obtain then the densitized Bianchi I term $\hat{\mathcal{C}}_{\rm BI}$. The contribution of the inhomogeneities is contained in the term $\hat{\mathcal{C}}_{\xi}$, which is constructed by promoting the second line of Eq. (2.3) to a symmetric operator. In particular, the comparison between $\hat{\mathcal{C}}_{\rm BI}$ and its classical counterpart [first line of Eq. (2.3)] gives us a natural quantum prescription to represent the term $(c_\sigma p_\sigma + c_\delta p_\delta)^2$. In addition, we also know how to represent the term $1/|p_\theta|$, as explained in Appendix B. These terms, acting on the homogeneous sector, have a different representation in schemes A and B, and we will analyze them for each case separately. But let us deal first with the terms that have a nontrivial action in the inhomogeneous sector, which are the same for both cases.

1. Inhomogeneities

Choosing normal ordering, the quantum counterparts of the free Hamiltonian and the interaction term are

$$\hat{H}^{\xi}_{0} = \sum_{m>0}^{\infty} m\hat{N}_m, \qquad \hat{N}_m = \hat{a}^{\dagger}_m\hat{a}_m + \hat{a}^{\dagger}_{-m}\hat{a}_{-m}, \tag{4.5}$$

$$\hat{H}^{\xi}_{\rm int} = \sum_{m>0}^{\infty} \frac{\hat{N}_m + \hat{Y}_m}{m}, \qquad \hat{Y}_m = \hat{a}_m\hat{a}_{-m} + \hat{a}^{\dagger}_m\hat{a}^{\dagger}_{-m}. \tag{4.6}$$

The inhomogeneous sector of the kinematical Hilbert space, i.e., the Fock space $\mathcal{F}$, can be written as a direct sum of dynamically invariant Fock subspaces. Indeed, the operator $\hat{Y}_m$, which is the only operator in the Hamiltonian that does not act diagonally on the basis of states $|\mathfrak{n}\rangle$ of $\mathcal{F}$, annihilates and creates pairs of particles in modes with the same wave number (i.e., the modes with wave vectors m and $-m$). Therefore the quantities X_m, defined in Eq. (3.11), are conserved under the action of the Hamiltonian constraint $\hat{\mathcal{C}}_{\rm G}$. Hence, it is convenient to relabel the basis of n-particle states with the quantum numbers X_m, for all positive integers m, together, e.g., with the eigenvalues $N_m = n_m + n_{-m}$ of the operators $\hat{N}_m$, defined in Eq. (4.5). That is, we rewrite the states in our basis as

$$|X_1, X_2, \ldots; N_1, N_2, \ldots\rangle := |\mathfrak{X}; \mathfrak{N}\rangle. \tag{4.7}$$

Here, the numbers X_m can take any integer value, whereas $N_m \in \{|X_m| + 2k, k \in \mathbb{N}\}$.

On these states, the action of the relevant operators is

$$\hat{X}_m|\mathfrak{X}; \mathfrak{N}\rangle = X_m|\mathfrak{X}; \mathfrak{N}\rangle, \tag{4.8}$$

$$\hat{N}_m|\mathfrak{X}; \mathfrak{N}\rangle = N_m|\mathfrak{X}; \mathfrak{N}\rangle, \tag{4.9}$$

$$\hat{Y}_m|\mathfrak{X}; \mathfrak{N}\rangle = \frac{\sqrt{N_m^2 - X_m^2}}{2}|\mathfrak{X}; \ldots, N_m - 2, \ldots\rangle + \frac{\sqrt{(N_m+2)^2 - X_m^2}}{2}|\mathfrak{X}; \ldots, N_m + 2, \ldots\rangle, \tag{4.10}$$

and thus the sequence $\mathfrak{X} = \{X_1, X_2, \ldots\}$ is not affected, as we pointed out above. The first contribution in Eq. (4.10) vanishes if $N_m < 2$.

In addition, we will denote by $\mathcal{S}_{\mathfrak{X}}$ the subspace of $\mathcal{S}$ spanned by the n-particle states with fixed sequence $\mathfrak{X}$, and by $\mathcal{F}_{\mathfrak{X}}$ the respective completion. Then

$$\mathcal{F} = \oplus_{\mathfrak{X}}\mathcal{F}_{\mathfrak{X}}. \tag{4.11}$$

In practice, as far as the action of the Hamiltonian constraint operator is concerned, we can restrict the study of the inhomogeneous sector to any specific subspace $\mathcal{F}_{\mathfrak{X}}$. In what follows, to simplify the notation, we will denote the

n-particle states by $|\mathfrak{N}\rangle$ and obviate their dependence in the fixed sequence $\mathfrak{X}$.

Obviously the operator $\hat{H}^{\xi}_{0}$ with domain $\mathcal{S}_{\mathfrak{X}}$ is well defined in $\mathcal{F}_{\mathfrak{X}}$, because it acts diagonally on the n-particle states and maps $\mathcal{S}_{\mathfrak{X}}$ into itself. On the other hand, the interaction term $\hat{H}^{\xi}_{\rm int}$ creates infinite pairs of particles, and thus it does not leave invariant the domain $\mathcal{S}_{\mathfrak{X}}$, which only contains states with a finite number of them. From Eqs. (4.6), (4.9), and (4.10), we have

$$\|\hat{H}^{\xi}_{\rm int}|\mathfrak{N}\rangle\|^2 = \Big(\sum_{m>0}^{\infty}\frac{N_m}{|m|}\Big)^2 + \sum_{m>0}^{\infty}\frac{N_m^2 - X_m^2 + 2N_m}{2m^2} + \sum_{m>0}^{\infty}\frac{1}{m^2}. \tag{4.12}$$

Since in the n-particle states only a finite set of occupation numbers differ from zero, among the above three sums only the last one involves an infinite number of nonvanishing terms. Actually this sum converges, and hence the norm of $\hat{H}^{\xi}_{\rm int}|\mathfrak{N}\rangle$ is finite. In conclusion, $\hat{H}^{\xi}_{\rm int}|\mathfrak{N}\rangle \in \mathcal{F}_{\mathfrak{X}}$ for all $|\mathfrak{N}\rangle \in \mathcal{S}_{\mathfrak{X}}$, and therefore $\hat{H}^{\xi}_{\rm int}$ with domain $\mathcal{S}_{\mathfrak{X}}$ is also well defined.

2. Hamiltonian constraint in scheme A

Following the procedure sketched in the beginning of this subsection, for the densitized Hamiltonian constraint and in scheme A we get an operator of the form (see Ref. [21] for the details of the construction):

$$\hat{\mathcal{C}}^{\rm A}_{\rm G} = -\frac{2}{\gamma^2}[\hat{\Theta}_\theta\hat{\Theta}_\delta + \hat{\Theta}_\theta\hat{\Theta}_\sigma + \hat{\Theta}_\sigma\hat{\Theta}_\delta] + l_{\rm Pl}^2\Big\{\frac{(\hat{\Theta}_\sigma + \hat{\Theta}_\delta)^2}{\gamma^2}\Big[\widehat{\frac{1}{\sqrt{|p_\theta|}}}\Big]^2\hat{H}^{\xi}_{\rm int} + 32\pi^2\widehat{|p_\theta|}\hat{H}^{\xi}_{0}\Big\}, \tag{4.13}$$

where $\hat{H}^{\xi}_{0}$ and $\hat{H}^{\xi}_{\rm int}$ are given in Eqs. (4.5) and (4.6), respectively, $\widehat{|p_\theta|}$ is constructed from Eq. (3.4), and $[1/\widehat{\sqrt{|p_\theta|}}]$ is defined in Eq. (B2). The symmetric operator $\hat{\Theta}_i$ represents the classical quantity $c_i p_i$, and its action on the basis of states $|\upsilon_i\rangle$ of the homogeneous sector, as determined by the quantization procedure explained above, has the form [11]

$$\hat{\Theta}_i|\upsilon_i\rangle = -i\pi\gamma l_{\rm Pl}^2[f_+(\upsilon_i)|\upsilon_i + 2\rangle - f_-(\upsilon_i)|\upsilon_i - 2\rangle]. \tag{4.14}$$

Here, $f_\pm(\upsilon_i)$ are two positive functions which satisfy that $f_-(\upsilon_i) = f_+(\upsilon_i - 2)$ and, remarkably, that $f_+(\upsilon_i)$ vanishes in the whole interval $\upsilon_i \in [-2, 0)$. Their explicit expressions are provided in Appendix C.

It is worth noticing that, in scheme A, the homogeneous sector is completely factorized in three independent directional subsectors.

3. Hamiltonian constraint in scheme B

Motivated by our previous analysis, carried out for the Bianchi I model [11] as well as for the hybrid Gowdy model in scheme A [20,21], we follow in case B the very same densitization and symmetrization procedure.

Let us focus first on the Bianchi I term. Taking into account Eq. (3.1) for prescription B, one can see that the densitized term $\hat{\mathcal{C}}^{\rm B}_{\rm BI}$ corresponds to the symmetrization of

$$-\frac{2}{\gamma^2\Delta}\sum_{j<k}\hat{V}^2\widehat{\rm sgn}(p_j)\,\widehat{\sin}(\bar{\mu}_j c_j)\,\widehat{\rm sgn}(p_k)\,\widehat{\sin}(\bar{\mu}_k c_k). \tag{4.15}$$

This is precisely the gravitational part of the constraint represented in Ref. [12], up to a constant multiplicative factor [26]. Nonetheless our symmetrization is different. Adopting a symmetrization similar to that introduced in previous subsections, we get

$$\hat{\mathcal{C}}^{\rm B}_{\rm BI} = \hat{\mathcal{C}}^{(\theta)} + \hat{\mathcal{C}}^{(\sigma)} + \hat{\mathcal{C}}^{(\delta)}, \tag{4.16}$$

$$\hat{\mathcal{C}}^{(i)} = -\frac{1}{4\gamma^2\Delta}\widehat{\sqrt{V}}[\hat{F}_j\hat{V}\hat{F}_k + \hat{F}_k\hat{V}\hat{F}_j]\widehat{\sqrt{V}}, \tag{4.17}$$

$$\hat{F}_i = \widehat{\sin}(\bar{\mu}_i c_i)\,\widehat{\rm sgn}(p_i) + \widehat{\rm sgn}(p_i)\,\widehat{\sin}(\bar{\mu}_i c_i). \tag{4.18}$$

The action of the operators $\hat{F}_i$ is displayed in Appendix D 1, while the action of $\hat{V} = \otimes_i\widehat{|p_i|}$ is obtained from Eqs. (3.6) and (3.8).

The final result for the densitized Hamiltonian constraint in scheme B is

$$\hat{\mathcal{C}}^{\rm B}_{\rm G} = \hat{\mathcal{C}}^{(\theta)} + \hat{\mathcal{C}}^{(\sigma)} + \hat{\mathcal{C}}^{(\delta)} + l_{\rm Pl}^2\Big\{\Big[\widehat{\frac{1}{|p_\theta|^{1/4}}}\Big]^2\hat{G}\Big[\widehat{\frac{1}{|p_\theta|^{1/4}}}\Big]^2\hat{H}^{\xi}_{\rm int} + 32\pi^2\widehat{|p_\theta|}\hat{H}^{\xi}_{0}\Big\}, \tag{4.19}$$

where $\hat{H}^{\xi}_{0}$ and $\hat{H}^{\xi}_{\rm int}$ are again given in Eqs. (4.5) and (4.6), $[1/\widehat{|p_\theta|^{1/4}}]$ can be found in Eq. (B3), $\widehat{|p_\theta|}$ is constructed from Eq. (3.6), and we choose the symmetric operator

$$\hat{G} = \frac{1}{4\gamma^2\Delta}\widehat{\sqrt{V}}[\hat{F}_\sigma\hat{V}\hat{F}_\sigma + \hat{F}_\delta\hat{V}\hat{F}_\delta]\widehat{\sqrt{V}} - \hat{\mathcal{C}}^{(\theta)} \tag{4.20}$$

to be the quantum counterpart of $[(c_\sigma p_\sigma + c_\delta p_\delta)/\gamma]^2$.

In contrast with scheme A, the homogeneous sector is not factorized anymore in three independent directional subsectors, since the operators $\hat{F}_i$ and $\hat{F}_j$ $(i \neq j)$ do not commute.

C. Superselection in the homogeneous sector

1. Superselection in scheme A

As discussed in Refs. [20,21], $\hat{\mathcal{C}}^{\rm A}_{\rm G}$ leaves invariant the Hilbert subspaces $\mathcal{H}^{\pm}_{\varepsilon_i}$, defined as the Cauchy completion (with respect to the discrete inner product) of the subspaces

$$\mathrm{Cyl}^{\pm}_{\mathrm{S},\varepsilon_i} = \mathrm{span}\{|v_i\rangle; v_i \in \mathcal{L}^{\pm}_{\varepsilon_i}\}, \tag{4.21}$$

where $\mathcal{L}^{\pm}_{\varepsilon_i}$ denotes the semilattice of step 2 defined by

$$\mathcal{L}^{\pm}_{\varepsilon_i} = \{\pm(\varepsilon_i + 2k); k \in \mathbb{N}\}, \qquad \varepsilon_i \in (0, 2]. \tag{4.22}$$

Therefore, for each ε_i, the kinematical Hilbert space

$$\mathcal{H}^{+}_{\vec{\varepsilon}} \otimes \mathcal{F}, \quad \text{with} \quad \mathcal{H}^{+}_{\vec{\varepsilon}} = \otimes_i \mathcal{H}^{+}_{\varepsilon_i}, \tag{4.23}$$

provides a superselection sector, and the Hamiltonian constraint operator is well defined in any of its subspaces $\mathcal{H}^{+}_{\vec{\varepsilon}} \otimes \mathcal{F}_{\mathfrak{X}}$, with dense domain $\otimes_i \mathrm{Cyl}^{+}_{\mathrm{S},\varepsilon_i} \otimes \mathcal{S}_{\mathfrak{X}}$. Here, we are assuming that physical observables can distinguish between different modes and thus they do not superselect $\mathcal{F}_{\mathfrak{X}}$.

In conclusion, in case A we can restrict the study to a kinematical Hilbert space that is separable, and in which the quantum numbers representing the homogeneous degrees of freedom are strictly positive, with minimum values ε_i, and distributed on cubic lattices of step 2.

2. Superselection in scheme B

Let us analyze the action of the Hamiltonian constraint operator $\hat{\mathcal{C}}^{\mathrm{B}}_{\mathrm{G}}$ on the homogeneous sector component of any state $|v^\star, \lambda^\star_\sigma, \lambda^\star_\delta\rangle \otimes |\mathfrak{N}\rangle$. The explicit expression of this operator is given in Appendix D 2. First, concerning just the homogeneous sector, it is straightforward to see that the action of the constraint leaves invariant the subspace of positive densitized triad coefficients, given by

$$\mathrm{Cyl}^{+}_{\mathrm{S}} = \mathrm{span}\{|v, \lambda_\sigma, \lambda_\delta\rangle; v, \lambda_\sigma, \lambda_\delta > 0\}. \tag{4.24}$$

We restrict our discussion to this subspace from now on.

It is worth commenting that, whereas Ref. [12] adopts the same symmetrization for the powers of $\hat{V}$ that we have proposed (see, e.g., Refs. [20,21]), so that the states with vanishing (homogeneous) volume are indeed decoupled, the symmetrization chosen for the signs of the triad components differs from ours, and therefore states with different orientations of those densitized triad components are not decoupled. Moreover, although the discussion in Ref. [12] is also restricted to the space $\mathrm{Cyl}^{+}_{\mathrm{S}}$, that restriction is incorporated there on the basis of the symmetry under parity, but the (gravitational part of the) Hamiltonian constraint operator does not leave invariant this domain, since it mixes states with different orientations. In other words, it is only on the subspace of parity symmetric (or antisymmetric) states that the restriction of the Hamiltonian constraint to the sector of positive orientations can be defined, because the symmetry under parity allows one to identify states with negative orientations with a counterpart in this sector.

The Hamiltonian constraint produces shifts on the variable $v^\star$ which are equal to 4 or -4. Nonetheless, in the considered sector, a shift by -4 is possible only if $v^\star > 4$. Then, if we define

$$\mathcal{L}_{\varepsilon} = \{\varepsilon + 4k; k \in \mathbb{N}\}, \qquad \varepsilon \in (0, 4], \tag{4.25}$$

the action of $\hat{\mathcal{C}}^{\mathrm{B}}_{\mathrm{G}}$ never mixes states with values of v in different semilattices of this kind. Furthermore, acting on a state $|v^\star, \lambda^\star_\sigma, \lambda^\star_\delta\rangle \otimes |\mathfrak{N}\rangle$, this operator produces new states with the following quantum numbers v and (λ_a, λ_b), where the two identifications $(a, b) = (\sigma, \delta)$ and $(a, b) = (\delta, \sigma)$ are allowed:

(i) $v = v^\star - 4 > 0$

$$\left(\lambda^\star_a, \frac{v^\star - 4}{v^\star - 2}\lambda^\star_b\right), \qquad \left(\lambda^\star_a, \frac{v^\star - 2}{v^\star}\lambda^\star_b\right),$$
$$\left(\frac{v^\star - 2}{v^\star}\lambda^\star_a, \frac{v^\star - 4}{v^\star - 2}\lambda^\star_b\right), \qquad \left(\lambda^\star_a, \frac{v^\star - 4}{v^\star}\lambda^\star_b\right),$$

(ii) $v = v^\star$

$$\left(\lambda^\star_a, \frac{v^\star}{v^\star + 2}\lambda^\star_b\right), \qquad \left(\lambda^\star_a, \frac{v^\star + 2}{v^\star}\lambda^\star_b\right),$$
$$\left(\frac{v^\star + 2}{v^\star}\lambda^\star_a, \frac{v^\star}{v^\star + 2}\lambda^\star_b\right), \qquad (\lambda^\star_a, \lambda^\star_b),$$

and, if $v^\star > 2$, also

$$\left(\lambda^\star_a, \frac{v^\star - 2}{v^\star}\lambda^\star_b\right), \qquad \left(\lambda^\star_a, \frac{v^\star}{v^\star - 2}\lambda^\star_b\right),$$
$$\left(\frac{v^\star - 2}{v^\star}\lambda^\star_a, \frac{v^\star}{v^\star - 2}\lambda^\star_b\right),$$

(iii) $v = v^\star + 4$

$$\left(\lambda^\star_a, \frac{v^\star + 4}{v^\star + 2}\lambda^\star_b\right), \qquad \left(\lambda^\star_a, \frac{v^\star + 2}{v^\star}\lambda^\star_b\right),$$
$$\left(\frac{v^\star + 2}{v^\star}\lambda^\star_a, \frac{v^\star + 4}{v^\star + 2}\lambda^\star_b\right), \qquad \left(\lambda^\star_a, \frac{v^\star + 4}{v^\star}\lambda^\star_b\right).$$

We see that the effect caused on the λ labels does not depend on the reference quantum numbers $\lambda^\star_\sigma$ and $\lambda^\star_\delta$, but only on the value of $v^\star = \varepsilon + 4k^\star$. This dependence is through fractional factors whose denominator is two or four units bigger or smaller than the numerator. Therefore, it is possible to see that, starting with $|v^\star, \lambda^\star_\sigma, \lambda^\star_\delta\rangle \otimes |\mathfrak{N}\rangle$ and restricting the consideration to the given value $v^\star$ of the label v, the iterative action of the constraint operator leads only to states whose quantum numbers λ_a are of the form $\lambda_a = \omega_\varepsilon \lambda^\star_a$ $(a = \sigma, \delta)$, with ω_ε belonging to the set

$$\mathcal{W}_{\varepsilon} = \left\{\left(\frac{\varepsilon - 2}{\varepsilon}\right)^z \prod_{m,n \in \mathbb{N}} \left(\frac{\varepsilon + 2m}{\varepsilon + 2n}\right)^{k^m_n}\right\}, \tag{4.26}$$

where $k^m_n \in \mathbb{N}$, and $z \in \mathbb{Z}$ if $\varepsilon > 2$, while z vanishes when $\varepsilon \leq 2$. The discrete set $\mathcal{W}_\varepsilon$ is countably infinite and turns out to be dense in the positive real line. The proof of this last statement can be found in Appendix D 3. Therefore,

whereas the variable υ has support on simple semilattices of constant step, the variables λ_a take values in much more complicated sets. Nonetheless, they are also superselected in separable sectors, a fact which had not been realized in previous literature. As a particular case, we can see that if $\varepsilon = 2$ and $\lambda_a^\star$ is a fraction, then λ_a can take any value in the set of positive rational numbers.

In conclusion, $\hat{\mathcal{C}}_{\rm G}^{\rm B}$ leaves invariant the Hilbert subspaces $\mathcal{H}_{\varepsilon,\lambda_\sigma^\star,\lambda_\delta^\star}$ defined as the Cauchy completion with respect to the discrete inner product of

$$\mathrm{Cyl}_{\mathrm{S},\varepsilon,\lambda_\sigma^\star,\lambda_\delta^\star} = \mathrm{span}\{|\upsilon, \lambda_\sigma, \lambda_\delta\rangle; \upsilon \in \mathcal{L}_\varepsilon, \lambda_a = \omega_\varepsilon \lambda_a^\star, \omega_\varepsilon \in \mathcal{W}_\varepsilon, \lambda_a^\star \in \mathbb{R}^+\}. \tag{4.27}$$

As a consequence, the Hilbert subspaces

$$\mathcal{H}_{\varepsilon,\lambda_\sigma^\star,\lambda_\delta^\star} \otimes \mathcal{F} \tag{4.28}$$

provide superselection sectors. Moreover, the Hamiltonian constraint operator $\hat{\mathcal{C}}_{\rm G}^{\rm B}$ has well-defined restrictions on any of the subspaces $\mathcal{H}_{\varepsilon,\lambda_\sigma^\star,\lambda_\delta^\star} \otimes \mathcal{F}_{\mathfrak{X}}$, with corresponding dense domain given by $\mathrm{Cyl}_{\mathrm{S},\varepsilon,\lambda_\sigma^\star,\lambda_\delta^\star} \otimes \mathcal{S}_{\mathfrak{X}}$.

V. PHYSICAL SECTOR IN CASE A

A. Imposition of the Hamiltonian constraint

In Ref. [21] we already determined the solutions $(\psi|$ of the Hamiltonian constraint $\hat{\mathcal{C}}_{\rm G}^{\rm A}$, which were given by an expansion of the form

$$(\psi| = \sum_{\upsilon_\theta \in \mathcal{L}_{\varepsilon\theta}^+} \int_{\mathbb{R}^2} d\omega_\sigma d\omega_\delta \langle \upsilon_\theta| \otimes \langle \omega_\sigma| \otimes \langle \omega_\delta| \otimes (\psi_{\omega_\sigma,\omega_\delta}(\upsilon_\theta)|. \tag{5.1}$$

Here, $|\omega_a\rangle$ $(a = \sigma, \delta)$ denotes the generalized eigenstates (with real generalized eigenvalue ω_a) of $\hat{\Theta}_a$. These operators are constants of motion (they commute with $\hat{\mathcal{C}}_{\rm G}^{\rm A}$), and therefore the states $|\omega_a\rangle$ are stable under the action of $\hat{\mathcal{C}}_{\rm G}^{\rm A}$. The concrete expression of these states can be found in Ref. [11]. On the other hand, remarkably, the projections $(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta + 2k)|$ of the solution on the sections with constant value of $\upsilon_\theta = \varepsilon_\theta + 2k$ for every $k \in \mathbb{N}^+$, are all formally determined by the initial data $(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta)|$ through the action of a complicated operator, which involves the iterative action of $\hat{H}_0^\xi$ and $\hat{H}_{\rm int}^\xi$. Again, the explicit expression is provided in Ref. [21].

In addition, the physical solutions must be annihilated by the generator of translations in S^1. This S^1 symmetry is preserved by the dynamics, since the quantities X_m are constants of motion; therefore we can ignore it at this stage and impose it after dealing with the Hamiltonian constraint.

B. Physical Hilbert space

The resulting form of the general solutions $(\psi|$ of the Hamiltonian constraint is only formal, in the sense that they do not belong to the dual of the domain of definition of $\hat{\mathcal{C}}_{\rm G}^{\rm A}$. Namely, some of the coefficients $(\psi_{\omega_\sigma,\omega_\delta}(\upsilon_\theta)|\mathfrak{N}\rangle$ of the solutions diverge, for example, when $\upsilon_\theta \geq \varepsilon_\theta + 4$. Indeed, it is not difficult to see that the space $\mathcal{S}_{\mathfrak{X}}$ is not in the domain of the operator $\hat{H}_0^\xi \hat{H}_{\rm int}^\xi$, because this contains the term $\sum_m \hat{N}_m \hat{Y}_m$. The action of this term on a generic state of $\mathcal{S}_{\mathfrak{X}}$ does not lead to a normalizable state of the Fock space $\mathcal{F}_{\mathfrak{X}}$, not even in the (conventional) generalized sense, since, in particular, it involves the creation of pairs in an infinite number of modes and then a sum over the number of all of those particles.

This problem can be traced back to the choice of domain for the Hamiltonian constraint, since the present one is not invariant under the action of $\hat{\mathcal{C}}_{\rm G}^{\rm A}$. Indeed, it is worth pointing out that the determination of an invariant domain (and such that the Hamiltonian constraint be an essentially self-adjoint operator) would allow one to resort to group averaging techniques [27] in order to construct the space of physical solutions. However, the selection of an alternative domain that remains invariant is an extremely difficult task, given the complexity of our model, and no satisfactory choice is actually at hand. In this sense, the kinematical structure of our quantization is not well adapted to the physical one, and their relation is not straightforward *a priori*.

Nonetheless, we can still complete the quantization program: we just need to make sense of the solutions and provide them with a Hilbert structure. As we have already said, the solutions to the densitized Hamiltonian constraint are completely determined, at least formally, by a single piece of initial data $(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta)|$, and then we can identify these solutions with the corresponding data. The determination of a complete set of real classical observables acting on these initial data, together with the condition that they be represented as self-adjoint operators, determines a unique inner product [28] that characterizes the Hilbert structure. Moreover, taking into account the additional S^1 symmetry, implemented by condition (3.11), we conclude [20,21] that the initial data $(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta)|$ must belong to the Hilbert space

$$L^2(\mathbb{R}^2, d\omega_\sigma d\omega_\delta) \otimes \mathcal{F}_p, \tag{5.2}$$

which we identify as the physical Hilbert space.

Alternatively, we can argue that this is the physical Hilbert space following a different line of reasoning, based on the idea that we can regularize the theory by means of a cutoff for the wave number $m > 0$ and analyze the limit of arbitrarily large values of the cutoff. Note that we are

allowed to carry out this further reduction of the phase space because, in the Gowdy model, the field modes with different wave numbers are not mixed dynamically.

Regularized model

Let $\mathcal{S}_M$ be the subspace of $\mathcal{S}$ spanned by the n-particle states which satisfy $N_m = 0$ for all $m > M > 0$, and let $\mathcal{S}_M^\perp$ be its orthogonal complement. In addition, let $\hat{P}_M$ and $\hat{P}_M^\perp$ be the projectors on the subspaces $\mathcal{S}_M$ and $\mathcal{S}_M^\perp$, respectively. We are now interested in finding the physical Hilbert space of the truncation of the Gowdy model in which all the field modes with wave number bigger than the cutoff M vanish. Obviously, this truncated system is governed by the constraints

$$(\psi_{\omega_\sigma,\omega_\delta}(v_\theta)|\hat{P}_M^\perp = 0, \tag{5.3}$$

$$(\psi_{\omega_\sigma,\omega_\delta}(v_\theta)|\hat{P}_M\hat{\mathcal{C}}_{\mathrm{G}}^{\mathrm{A}}\hat{P}_M = 0, \tag{5.4}$$

$$(\psi_{\omega_\sigma,\omega_\delta}(v_\theta)|\hat{P}_M\hat{\mathcal{C}}_\theta\hat{P}_M = 0. \tag{5.5}$$

Simultaneous solutions to the first two equations have the same form as the solutions to the Hamiltonian constraint in the full Gowdy model, but containing exclusively wave numbers m equal or smaller than M, so that they now possess only a finite number of terms. As a consequence, the divergences caused by the infinite production of pairs of particles disappear, and the coefficients $(\psi_{\omega_\sigma,\omega_\delta}(v_\theta)|\mathfrak{N}\rangle$ of the solutions to Eqs. (5.3) and (5.4) are now finite for all v_θ. Therefore, the solutions (for fixed v_θ, ω_σ, and ω_δ) live in the dual $\mathcal{S}_M^*$ of the space spanned by the n-particle states with the cutoff. Similarly to what we did in the previous subsection, we can now endow the data at $v_\theta = \varepsilon_\theta$ with a Hilbert structure. We construct the same complete set of observables [21], with the difference that now we do not have an infinite number of them in the inhomogeneous sector, but only $4M$ (corresponding to 2 degrees of freedom on phase space for each of the $2M$ wave vectors). The resulting observables act on the solutions $(\psi_{\omega_\sigma,\omega_\delta}(\varepsilon_\theta)|$ and are self-adjoint in $L^2(\mathbb{R}^2, d\omega_\sigma d\omega_\delta) \otimes \mathcal{F}_M$, where $\mathcal{F}_M$ is the Fock space obtained by completing $\mathcal{S}_M$. Taking into account the remaining constraint (5.5), which implements the S^1 symmetry, the physically admissible states have finally the Hilbert structure

$$L^2(\mathbb{R}^2, d\omega_\sigma d\omega_\delta) \otimes (\mathcal{F}_p)_M, \tag{5.6}$$

where $(\mathcal{F}_p)_M$ is the subspace of $\mathcal{F}_M$ spanned by the n-particle states that verify the condition $\sum_{m>0}^M mX_m = 0$.

As the cutoff M increases, we get closer to the nontruncated theory for the Gowdy model. This indicates that, in the limit $M \to \infty$, we should recover the full Fock space: $(\mathcal{F}_p)_M \to \mathcal{F}_p$. This supports our previous statement that the physical Hilbert space of the nontruncated Gowdy model is in fact the space (5.2).

The introduction of the cutoff in the wave number also makes manageable the numerical study of the effect that the inhomogeneities have on the dynamical behavior of the system and, in particular, of the changes that occur in the dynamics of the Bianchi I background when the inhomogeneities grow. Actually, this kind of analysis has already been carried out at the effective level and is included in the study of Ref. [29], where the classical Hamiltonian has been modified in an effective way in order to take into account the corrections that arise from the quantum theory—although the results obtained there are extended to the full model without the cutoff.

VI. PHYSICAL SECTOR IN CASE B

A. Imposition of the Hamiltonian constraint

Unlike in case A, the homogeneous sector of the kinematical Hilbert space, $\mathcal{H}_{\varepsilon,\lambda_\sigma^\star,\lambda_\delta^\star}$, is not factorized in three independent directional subsectors, and none of the operators $\widehat{\sqrt{V}}\hat{F}_i\widehat{\sqrt{V}}$ appearing in the expression of the constraint $\hat{\mathcal{C}}_{\mathrm{G}}^{\mathrm{B}}$ (i.e., the counterpart of $\hat{\Theta}_i$ in case A) is now a constant of motion [30]. Therefore, in this case we cannot simplify the action of the Hamiltonian constraint operator by diagonalizing it in the directional subsectors labeled by σ and δ, as we did in scheme A.

Then, in order to solve the Hamiltonian constraint represented by $\hat{\mathcal{C}}_{\mathrm{G}}^{\mathrm{B}}$, we expand the solutions $(\psi|$ using the basis of states $|v, \lambda_\sigma, \lambda_\delta\rangle$ for the homogeneous sector. Namely,

$$(\psi| = \sum_{v\in\mathcal{L}_\varepsilon}\sum_{\omega_\varepsilon\in\mathcal{W}_\varepsilon}\sum_{\bar{\omega}_\varepsilon\in\mathcal{W}_\varepsilon} \langle v, \omega_\varepsilon\lambda_\sigma^\star, \bar{\omega}_\varepsilon\lambda_\delta^\star| \otimes (\psi(v, \omega_\varepsilon\lambda_\sigma^\star, \bar{\omega}_\varepsilon\lambda_\delta^\star)|. \tag{6.1}$$

Inserting this expansion into the (dual of the) constraint equation, projecting on the homogeneous sector, and taking into account the action of $\hat{\mathcal{C}}_{\mathrm{G}}^{\mathrm{B}}$ given in Appendix D 2, we get that the projections

$$(\psi(v, \lambda_\sigma, \lambda_\delta)| = (\psi(v, \omega_\varepsilon\lambda_\sigma^\star, \bar{\omega}_\varepsilon\lambda_\delta^\star)|$$

satisfy a series of relations that can be interpreted as difference equations in v. Introducing the projections of $(\psi|$ on the combinations of states defined in Eqs. (D5)–(D8) of Appendix D 2, which we call

$$(\psi_\pm(v \pm 4, \lambda_\sigma, \lambda_\delta)| = (\psi|v \pm 4, \lambda_\sigma, \lambda_\delta\rangle_\pm$$

and in a similar way for the rest of projections, the relation obtained is

$$
\begin{aligned}
&(\psi_+(v+4,\lambda_\sigma,\lambda_\delta)| - \beta b_\theta^2(v,\lambda_\sigma,\lambda_\delta) b_\theta^2(v+4,\lambda_\sigma,\lambda_\delta)\frac{v+4}{v}(\psi'_+(v+4,\lambda_\sigma,\lambda_\delta)|\hat{H}^\xi_{\rm int} \\
&= -\frac{1}{\beta}\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}(\psi(v,\lambda_\sigma,\lambda_\delta)|\hat{H}^\xi_0 + \frac{x_0^-(v)}{x_+(v)}(\psi_{0^-}(v,\lambda_\sigma,\lambda_\delta)| + \frac{x_0^+(v)}{x_+(v)}(\psi_{0^+}(v,\lambda_\sigma,\lambda_\delta)| \\
&\quad - \frac{x_-(v)}{x_+(v)}(\psi_-(v-4,\lambda_\sigma,\lambda_\delta)| + \beta b_\theta^2(v,\lambda_\sigma,\lambda_\delta)\Big\{b_\theta^2(v-4,\lambda_\sigma,\lambda_\delta)\frac{v-4}{v}\frac{x_-(v)}{x_+(v)}(\psi'_-(v-4,\lambda_\sigma,\lambda_\delta)| \\
&\quad - b_\theta^2(v,\lambda_\sigma,\lambda_\delta)\Big[\frac{x_0^-(v)}{x_+(v)}(\psi'_{0^-}(v,\lambda_\sigma,\lambda_\delta)| + \frac{x_0^+(v)}{x_+(v)}(\psi'_{0^+}(v,\lambda_\sigma,\lambda_\delta)|\Big]\Big\}\hat{H}^\xi_{\rm int}.
\end{aligned}
\tag{6.2}
$$

Here, b_θ, $x_\pm$, and $x_0^\pm$ are the functions defined in Appendices B 2 and D 2.

B. Analysis of the solutions

We want to prove now that, as in case A, the solution is completely determined by the data on the first section $v = \varepsilon$ (at least formally). Specifically, we want to show that, given a set of initial data $(\psi(\varepsilon,\omega_\varepsilon\lambda_\sigma^\star,\bar{\omega}_\varepsilon\lambda_\delta^\star)|$ (belonging to $\mathcal{S}^*_{\hat{\mathcal{X}}}$ for all ω_ε, $\bar{\omega}_\varepsilon \in \mathcal{W}_\varepsilon$), it is possible to determine each term $(\psi(v,\lambda_\sigma,\lambda_\delta)|$ of the solution, for every $v > \varepsilon$ in $\mathcal{L}_\varepsilon$.

The presence of the interaction term on the left hand side of Eq. (6.2) complicates a direct proof of the above statement. However, it is possible to attain the result by means of an asymptotic analysis of the solutions. Remarkably, our theory involves a dimensionless parameter, β, introduced in Eq. (D4), and recurring to it we can naturally adopt an asymptotic approach without the need to introduce any external parameter by hand. Note that, since the area gap Δ is proportional to $\gamma l_{\rm Pl}^2$, β is proportional to the inverse of the Immirzi parameter γ. Thus, in the limit $\beta \to 0$, we expand the solutions in asymptotic series of the form:

$$
(\psi(\varepsilon+4k,\lambda_\sigma,\lambda_\delta)| = \sum_{n\in\mathbb{N}} \beta^{n-k}({}^n\psi(\varepsilon+4k,\lambda_\sigma,\lambda_\delta)|, \quad \forall k \in \mathbb{N}^+. \tag{6.3}
$$

Note that the linear combinations introduced in Eq. (6.2), like, e.g., $(\psi_\pm(v\pm4,\lambda_\sigma,\lambda_\delta)|$, adopt then similar expansions, and we will denote their terms using an obvious notation, for instance, $({}^n\psi_\pm(v\pm4,\lambda_\sigma,\lambda_\delta)|$. Substituting the expansion (6.3) in the constraint (6.2), and considering powers of β order by order, we obtain an expression for every term $({}^n\psi_+(v+4,\lambda_\sigma,\lambda_\delta)|$ (for generic v) provided that the data at v and $v-4$ are known. The explicit result is the following (to simplify the notation, we obviate the dependence of the states and of the function b_θ on the λ's):

(i) Leading term:

$$
({}^0\psi_+(v+4)| = -\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}({}^0\psi(v)|\hat{H}^\xi_0, \tag{6.4}
$$

(ii) first order correction:

$$
\begin{aligned}
({}^1\psi_+(v+4)| = &-\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}({}^1\psi(v)|\hat{H}^\xi_0 + \frac{x_0^-(v)}{x_+(v)}({}^0\psi_{0^-}(v)| + \frac{x_0^+(v)}{x_+(v)}({}^0\psi_{0^+}(v)| \\
&+ b_\theta^2(v) b_\theta^2(v+4)\frac{v+4}{v}({}^0\psi'_+(v+4)|\hat{H}^\xi_{\rm int},
\end{aligned}
\tag{6.5}
$$

(iii) second order correction:

$$
\begin{aligned}
({}^2\psi_+(v+4)| = &-\frac{32v^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(v)}({}^2\psi(v)|\hat{H}^\xi_0 + \frac{x_0^-(v)}{x_+(v)}({}^1\psi_{0^-}(v)| + \frac{x_0^+(v)}{x_+(v)}({}^1\psi_{0^+}(v)| - \frac{x_-(v)}{x_+(v)}({}^0\psi_-(v-4)| \\
&+ b_\theta^2(v)\Big\{-b_\theta^2(v)\Big[\frac{x_0^-(v)}{x_+(v)}({}^0\psi'_{0^-}(v)| + \frac{x_0^+(v)}{x_+(v)}({}^0\psi'_{0^+}(v)|\Big] + b_\theta^2(v+4)\frac{v+4}{v}({}^1\psi'_+(v+4)|\Big\}\hat{H}^\xi_{\rm int},
\end{aligned}
\tag{6.6}
$$

(iv) n-th order correction ($n \geq 3$):

$$({}^n\psi_+(\upsilon+4)| = -\frac{32\upsilon^2}{\lambda_\sigma^2\lambda_\delta^2 x_+(\upsilon)}({}^n\psi(\upsilon)|\hat{H}_0^\xi + \frac{x_0^-(\upsilon)}{x_+(\upsilon)}({}^{n-1}\psi_{0^-}(\upsilon)| + \frac{x_0^+(\upsilon)}{x_+(\upsilon)}({}^{n-1}\psi_{0^+}(\upsilon)|$$
$$- \frac{x_-(\upsilon)}{x_+(\upsilon)}({}^{n-2}\psi_-(\upsilon-4)| + b_\theta^2(\upsilon)\Big\{-b_\theta^2(\upsilon)\Big[\frac{x_0^-(\upsilon)}{x_+(\upsilon)}({}^{n-2}\psi'_{0^-}(\upsilon)| + \frac{x_0^+(\upsilon)}{x_+(\upsilon)}({}^{n-2}\psi'_{0^+}(\upsilon)|\Big]$$
$$+ b_\theta^2(\upsilon+4)\frac{\upsilon+4}{\upsilon}({}^{n-1}\psi'_+(\upsilon+4)| + b_\theta^2(\upsilon-4)\frac{\upsilon-4}{\upsilon}({}^{n-3}\psi'_-(\upsilon-4)|\Big\}\hat{H}_{\rm int}^\xi. \tag{6.7}$$

The above expressions simplify considerably in the cases $\upsilon = \varepsilon$ and $\upsilon = \varepsilon + 4$. This is due to the fact that, on the one hand, the data on the very initial section are not given by asymptotic series, that is, in principle $(\psi(\varepsilon)| = ({}^0\psi(\varepsilon)|$ and hence $({}^n\psi(\varepsilon)| = 0$ for all $n \geq 1$. And, on the other hand, $x_-(\varepsilon) = 0$ for all $\varepsilon \in (0, 4]$, whereas $x_0^-(\varepsilon) = 0$ if $\varepsilon \leq 2$ [see Eqs. (D9) and (D11)].

In view of these equations, we see that, for $\upsilon > 4$, the knowledge of the solution on the sections $\upsilon - 4$ and υ, together with the terms of asymptotic order $n - 1$ on the section $\upsilon + 4$, determine on this last section the terms of the solution at order n via the following linear combinations:

$$({}^n\psi_+(\upsilon+4, \lambda_\sigma, \lambda_\delta)| = \Big({}^n\psi\Big(\upsilon+4, \lambda_\sigma, \frac{\upsilon+4}{\upsilon+2}\lambda_\delta\Big)\Big| + \Big({}^n\psi\Big(\upsilon+4, \lambda_\sigma, \frac{\upsilon+2}{\upsilon}\lambda_\delta\Big)\Big| + \Big({}^n\psi\Big(\upsilon+4, \frac{\upsilon+4}{\upsilon+2}\lambda_\sigma, \frac{\upsilon+2}{\upsilon}\lambda_\delta\Big)\Big|$$
$$+ \Big({}^n\psi\Big(\upsilon+4, \frac{\upsilon+4}{\upsilon+2}\lambda_\sigma, \lambda_\delta\Big)\Big| + \Big({}^n\psi\Big(\upsilon+4, \frac{\upsilon+2}{\upsilon}\lambda_\sigma, \frac{\upsilon+4}{\upsilon+2}\lambda_\delta\Big)\Big|$$
$$+ \Big({}^n\psi\Big(\upsilon+4, \frac{\upsilon+2}{\upsilon}\lambda_\sigma, \lambda_\delta\Big)\Big|. \tag{6.8}$$

On the other hand, for $0 < \upsilon < 4$, the data on the section $\upsilon - 4$ are spurious, and their knowledge its not required to determine the terms of the solution for $\upsilon + 4$.

Actually, a similar structure appears also in the solutions to the Hamiltonian constraint for the Bianchi I model, inasmuch as the solution on the section $\upsilon + 4$ is determined in terms of the same kind of linear combinations. Furthermore, it has been recently shown [31] that, for all $\upsilon > 0$, the set of linear combinations

$$\{(\psi_+(\upsilon+4, \omega_\varepsilon\lambda_\sigma^\star, \bar{\omega}_\varepsilon\lambda_\delta^\star)|; \omega_\varepsilon, \bar{\omega}_\varepsilon \in \mathcal{W}_\varepsilon,\}$$

determines the set of individual terms

$$\{(\psi(\upsilon+4, \omega_\varepsilon\lambda_\sigma^\star, \bar{\omega}_\varepsilon\lambda_\delta^\star)|; \omega_\varepsilon, \bar{\omega}_\varepsilon \in \mathcal{W}_\varepsilon\}$$

through a one-to-one map. This result can be applied here as well as for Bianchi I cosmology. Therefore, starting with the initial data, we can obtain, step by step, the terms of the solution—up to the desired asymptotic order—on all of the consecutive υ sections.

In conclusion, the initial data $(\psi(\varepsilon, \lambda_\sigma, \lambda_\delta)|$ (with λ_σ and λ_δ taking all possible values in the corresponding superselection sectors) completely determines the solution, as we wanted to show. The solutions, constructed in this way, are formal, as happens to be the case in scheme A, in the sense that the objects $({}^n\psi(\upsilon+4, \lambda_\sigma, \lambda_\delta)|$ do not belong in general to the dual space of the domain chosen to define the Hamiltonian constraint operator, owing to the presence of the operator $\hat{H}_{\rm int}^\xi$ in their expressions.

C. Physical Hilbert space

In order to provide the solutions with a Hilbert structure, we proceed in the very same way as in case A. Once we have shown that the set of initial data

$$\{(\psi(\varepsilon, \omega_\varepsilon\lambda_\sigma^\star, \bar{\omega}_\varepsilon\lambda_\delta^\star)|; \omega_\varepsilon, \bar{\omega}_\varepsilon \in \mathcal{W}_\varepsilon\}$$

characterizes the solution, we can identify solutions with their corresponding data, and the physical Hilbert space with a Hilbert space of such initial data.

The requirement that a complete set of observables acting on these initial data be self-adjoint operators determines again uniquely the inner product that provides the Hilbert structure. Such observables are given, for instance, by a complete set of observables for the Bianchi I model *in vacuo* and by the observables introduced in scheme A for the inhomogeneities. Imposing the remaining S^1 symmetry on the resulting Hilbert space, we finally get the same structure found in case A for the physical Hilbert space, namely, the tensor product of the Fock subspace $\mathcal{F}_p$ and the physical Hilbert space of the Bianchi I model *in vacuo*, though now in scheme B:

$$\mathcal{H}_{\rm phys}^{\rm B} = \mathcal{H}_{\rm phys,BI}^{\rm B} \otimes \mathcal{F}_p. \tag{6.9}$$

The explicit form of $\mathcal{H}_{\rm phys,BI}^{\rm B}$ is analyzed in Ref. [31].

VII. SUMMARY AND CONCLUSION

A. Recovery of the standard quantum field theory

As we have commented in the Introduction, one of the motivations of this work is to investigate the plausibility of the recovery of standard quantum field theory in the framework of loop quantization. In particular we wanted to show that, in the Gowdy model, one attains a Fock description of the inhomogeneities over a polymerically quantized Bianchi I background in the space of physical states, starting with a hybrid quantization in the kinematical setting. Indeed, we have proved that this is the case, since the physical Hilbert space obtained in both schemes has the structure of the tensor product of the physical Hilbert space of the Bianchi I model and a Fock space, which turns out to be equivalent to the space obtained in the standard Fock quantization [18,19]. This result supports the validity of the hybrid quantization, because the latter should lead to the standard quantization of the system in the limit in which the effects arising from the discreteness of the geometry become negligible. Let us remark that the result is nontrivial, inasmuch as the hybrid approach is introduced in the kinematical arena and the relation between the kinematical and physical Hilbert structures cannot be anticipated before completing the quantization, even more if one takes into account the fieldlike complexity of the model.

B. Resolution of the cosmological singularity

The classical solutions of the linearly polarized Gowdy T^3 model present generically a cosmological singularity. In Ref. [15], e.g., a curvature invariant was explicitly calculated and proven to diverge almost everywhere at initial time. In terms of the variables that we have employed, the cosmological singularity corresponds to vanishing values for the components p_i of the densitized triad. Actually, as we can see in Eq. (A1), the metric is ill defined if any of the p_i's is zero.

In our quantum theory, the polymeric quantization performed in the homogeneous sector succeeds in eliminating the singularity. More explicitly, we have been able to remove the kernel of all the operators $\hat{p}_i$ and, as a consequence, an analog of the classical cosmological singularity does not exist any more quantum mechanically. This resolution of the singularity is achieved at a kinematical level. Of course, it persists in the physical Hilbert space, since physical states do not have projection onto the zero eigenspaces of the operators $\hat{p}_i$. Furthermore, they only have support on a sector with fixed orientation of the triad components and, then, they do not cross the singularity to another branch of the universe corresponding to a different orientation.

On the other hand, in addition to this kinematical resolution of the cosmological singularity, it is worth commenting that, at least for scheme A and in the framework of the effective description corresponding to the hybrid quantization put forward here, the numerical simulations performed so far for the Gowdy model show the presence of a bounce which replaces the singularity and which emerges owing to the quantum geometry corrections to Einstein's theory [29]. Similar numerical calculations are being developed currently for scheme B [32], in order to validate this stronger result about the resolution of the singularity.

Let us emphasize that the standard (nonpolymeric) quantum methods do not succeed in resolving the cosmological singularity. On the one hand, in the Fock quantization of the deparametrized system, which has been discussed in the literature [17] and which has been successfully accomplished till completion in Refs. [18,19], a classical time parameter is present explicitly in the quantum description, and the curvature invariant calculated in Ref. [15] depends on its inverse in such a way that the invariant still blows up at initial time. On the other hand, if one does not deparametrize the system, following our gauge reduction, and quantizes the homogeneous sector in a standard (nonpolymeric) way, as in the Wheeler-De Witt approach, then the zero eigenvalue would be included in the continuous spectrum of the triad operator, instead of the discrete spectrum, and there would not exist a proper subspace associated with this eigenvalue which could be decoupled and removed.

C. Concluding remarks

In conclusion, we have rigorously constructed a hybrid quantization of the Gowdy model with three-torus topology and linearly polarized gravitational waves. The homogeneous sector of the phase space, which coincides with the phase space of a Bianchi I model, has been polymerically quantized, whereas we have applied a (distinguished) Fock quantization to represent the inhomogeneous sector. In the LQC literature, there exist two different schemes for the polymeric quantization of the Bianchi I universes, denoted in this paper as schemes A and B. We had already analyzed the hybrid quantization of the Gowdy model adopting scheme A in Refs. [20,21]. Here, we have revisited that quantization and extended our hybrid approach to the alternative case B, in which the homogeneous sector has a different representation.

In both schemes, the quantum Hamiltonian constraint has been densitized, in order to deal with a simpler constraint, and then has been promoted to an operator, well defined in some dense domain of the kinematical Hilbert space. This is truly a nontrivial result, because our system possesses an infinite number of degrees of freedom and the two sectors, on which the constraint operator acts, are coupled and quantized with entirely different methods.

As we have seen, the kinematical structure over which we have defined the theory and the choice of domain for our quantum operators do not suffice to make sense of the formal solutions to the Hamiltonian constraint. Nonetheless, we have found a procedure to overcome the prob-

lem and complete the quantization. Indeed, in our hybrid approach, the Hamiltonian constraint provides a difference equation in an internal discrete parameter (v_θ in scheme A, v in scheme B) which has a strictly positive minimum value, and the solutions to the Hamiltonian constraint are completely determined by the data provided on the initial section of such a discrete parameter. One can say that the solutions follow a no-boundary prescription, in the sense that they arise in a single section without the need to impose any particular boundary condition. This behavior has two important consequences. On the one hand, this immediately resolves the classical singularity in the quantum theory at a kinematical level. On the other hand, it allows one to deal with these solutions by identifying them with initial data. In this way, we have been able to characterize the physical Hilbert space in both schemes A and B. Remarkably, as we have pointed out, this procedure leads to the recovery of the standard quantum field theory for the inhomogeneities.

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APPENDIX A: CLASSICAL METRIC

To derive the form of the classical metric of the linearly polarized Gowdy T^3 model, one can start with its expression in the (field) parametrization of Ref. [18], apply the gauge fixing procedure, and perform a canonical transformation from the elementary variables chosen for the homogeneous sector in that parametrization to the corresponding Ashtekar variables $\{c_i, p_i\}$. A careful calculation shows that, in our variables, the nonvanishing components of the induced three-metric are

$$\begin{aligned} q_{\theta\theta} &= \frac{1}{4\pi^2}\left|\frac{p_\sigma p_\delta}{p_\theta}\right| \exp\left\{\frac{2\pi}{\sqrt{|p_\theta|}}\left(2\frac{c_\delta p_\delta}{c_\sigma p_\sigma + c_\delta p_\delta} - 1\right)\tilde{\xi}(\theta)\right. \\ &\quad \left. - \frac{\pi^2}{|p_\theta|}[\tilde{\xi}(\theta)]^2 - \frac{8\pi G\gamma}{c_\sigma p_\sigma + c_\delta p_\delta}\zeta(\theta)\right\}, \\ q_{\sigma\sigma} &= \frac{1}{4\pi^2}\left|\frac{p_\theta p_\delta}{p_\sigma}\right| \exp\left\{-\frac{2\pi}{\sqrt{|p_\theta|}}\tilde{\xi}(\theta)\right\}, \\ q_{\delta\delta} &= \frac{1}{4\pi^2}\left|\frac{p_\theta p_\sigma}{p_\delta}\right| \exp\left\{\frac{2\pi}{\sqrt{|p_\theta|}}\tilde{\xi}(\theta)\right\}, \end{aligned} \tag{A1}$$

where

$$\tilde{\xi}(\theta) = \frac{1}{\pi}\sum_{m\neq 0}\sqrt{\frac{G}{|m|}}(a_m + a^*_{-m})e^{im\theta},$$

$$\begin{aligned} \zeta(\theta) &= i\sum_{m\neq 0}\sum_{\tilde{m}\neq 0}\mathrm{sgn}(m+\tilde{m})\frac{\sqrt{|m+\tilde{m}||\tilde{m}|}}{|m|} \\ &\quad \times (a_{-\tilde{m}} - a^*_{\tilde{m}})(a_{m+\tilde{m}} + a^*_{-(m+\tilde{m})})e^{im\theta}. \end{aligned}$$

Besides, owing to the homogeneity of the shift function N^θ, required by the gauge fixing, we can reabsorb it by means of the following redefinition of the coordinate θ:

$$\theta + \int_{t_i}^{t} dt' N^\theta(t') \to \theta,$$

where t_i is any initial time. Then, the spacetime metric becomes

$$\begin{aligned} ds^2 &= -q_{\theta\theta}\left(\frac{|p_\theta|}{4\pi^2}\right)^2 \underline{N}^2 dt^2 + q_{\theta\theta}d\theta^2 + q_{\sigma\sigma}d\sigma^2 \\ &\quad + q_{\delta\delta}d\delta^2. \end{aligned}$$

Here, $\underline{N}$ is the densitized lapse function, which in our gauge is spatially homogenous. The metric of the Bianchi I spacetime is the result of ignoring the inhomogeneities in Eq. (A1), setting $\tilde{\xi} = 0 = \zeta$.

APPENDIX B: INVERSE HOMOGENEOUS VOLUME OPERATOR

Following the procedures of LQG, the classical expression of $1/(|p_i|^{1-r})$ in LQC, for $r > 0$, is represented by the regularized operator (see, e.g., Ref. [3])

$$\begin{aligned} \widehat{\left[\frac{1}{|p_i|^{1-r}}\right]} &= \frac{\widehat{\mathrm{sgn}(p_i)}}{8\pi\gamma l_{\mathrm{Pl}}^2 r}\widehat{\left[\frac{1}{\bar{\mu}_i}\right]}[\hat{\mathcal{N}}_{-\bar{\mu}_i}\widehat{|p_i|}^r\hat{\mathcal{N}}_{\bar{\mu}_i} \\ &\quad - \hat{\mathcal{N}}_{\bar{\mu}_i}\widehat{|p_i|}^r\hat{\mathcal{N}}_{-\bar{\mu}_i}], \end{aligned} \tag{B1}$$

where $\widehat{[1/\bar{\mu}_i]}$ is the quantum counterpart of the expression given in Eq. (3.1) for each of the considered schemes. The choice of r is arbitrary.

1. Scheme A

In case A, and for $r = 1/2$, one obtains [9,11]

$$\begin{aligned} \widehat{\left[\frac{1}{V}\right]} &= \otimes_i\widehat{\left[\frac{1}{\sqrt{|p_i|}}\right]}, \\ \widehat{\left[\frac{1}{\sqrt{|p_i|}}\right]}|v_i\rangle &= b(v_i)|v_i\rangle, \\ b(v_i) &= \frac{1}{2(2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta})^{1/3}}|v_i|^{1/3}||v_i+1|^{1/3} \\ &\quad - |v_i - 1|^{1/3}|. \end{aligned} \tag{B2}$$

2. Scheme B

In case B, choosing $r = 1/4$ and combining all the powers of $|p_\theta|$, we obtain the expression

$$\Big[\widehat{\frac{1}{|p_\theta|^{1/4}}}\Big] = \frac{\widehat{\mathrm{sgn}(p_\theta)}}{2\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta}}\sqrt{\widehat{|p_\sigma p_\delta|}}[\hat{\mathcal{N}}_{-\bar{\mu}_\theta}\widehat{|p_\theta|}^{1/4}\hat{\mathcal{N}}_{\bar{\mu}_\theta} - \hat{\mathcal{N}}_{\bar{\mu}_\theta}\widehat{|p_\theta|}^{1/4}\hat{\mathcal{N}}_{-\bar{\mu}_\theta}], \tag{B3}$$

and similarly for $[1/\widehat{|p_\sigma|^{1/4}}]$ and $[1/\widehat{|p_\delta|^{1/4}}]$. Their action on our basis of states turns out to be

$$\Big[\widehat{\frac{1}{|p_i|^{1/4}}}\Big]|v, \lambda_\sigma, \lambda_\delta\rangle = \frac{b_i(v, \lambda_\sigma, \lambda_\delta)}{(4\pi\gamma l_{\mathrm{Pl}}^2\sqrt{\Delta})^{1/6}}|v, \lambda_\sigma, \lambda_\delta\rangle, \tag{B4}$$

where $(a = \sigma, \delta)$

$$\begin{aligned} b_\theta(v, \lambda_\sigma, \lambda_\delta) &= \sqrt{2|\lambda_\sigma\lambda_\delta|}|\sqrt{|v+1|} - \sqrt{|v-1|}|, \\ b_a(v, \lambda_\sigma, \lambda_\delta) &= \sqrt{\Big|\frac{v}{\lambda_a}\Big|}|\sqrt{|v+1|} - \sqrt{|v-1|}|. \end{aligned} \tag{B5}$$

The inverse homogeneous volume operator can then be represented as the regularized operator

$$\Big[\widehat{\frac{1}{V}}\Big] = \otimes_i\Big[\widehat{\frac{1}{|p_i|^{1/4}}}\Big]^2. \tag{B6}$$

APPENDIX C: DETAILS OF THE QUANTUM MODEL FOR SCHEME A

For completeness in the presentation, we include here some details about the operator $\hat{\Theta}_i$ that appears in the expression (4.13) of the densitized Hamiltonian constraint operator $\hat{C}_{\mathrm{G}}^{\mathrm{A}}$ for scheme A. The definition of $\hat{\Theta}_i$ in terms of holonomy and fluxes operators is [11,21]:

$$\begin{aligned} \hat{\Theta}_i = -\frac{i}{4\sqrt{\Delta}}\Big[\widehat{\frac{1}{\sqrt{|p_i|}}}\Big]^{-1/2}\widehat{\sqrt{|p_i|}}[(\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i})\widehat{\mathrm{sgn}(p_i)} \\ + \widehat{\mathrm{sgn}(p_i)}(\hat{\mathcal{N}}_{2\bar{\mu}_i} - \hat{\mathcal{N}}_{-2\bar{\mu}_i})]\widehat{\sqrt{|p_i|}}\Big[\widehat{\frac{1}{\sqrt{|p_i|}}}\Big]^{-1/2}. \end{aligned} \tag{C1}$$

Its action on the states $|v_i\rangle$ is given in Eq. (4.14), where the functions $f_\pm(v_i)$ are:

$$f_\pm(v_i) = g(v_i \pm 2)s_\pm(v_i)g(v_i), \tag{C2}$$

with

$$\begin{aligned} s_\pm(v_i) &= \mathrm{sgn}(v_i \pm 2) + \mathrm{sgn}(v_i), \\ g(v_i) &= \Big|\Big|1 + \frac{1}{v_i}\Big|^{1/3} - \Big|1 - \frac{1}{v_i}\Big|^{1/3}\Big|^{-1/2}, \quad \text{if } v_i \neq 0, \\ g(0) &= 0. \end{aligned} \tag{C3}$$

APPENDIX D: DETAILS OF THE QUANTUM MODEL FOR SCHEME B

1. The operators $\hat{F}_i$

Expressing each operator $\hat{F}_i$ in terms of $\hat{\mathcal{N}}_{\pm\bar{\mu}_i}$, a straightforward calculation shows that

$$\begin{aligned} \hat{F}_\theta|v, \lambda_\sigma, \lambda_\delta\rangle = i\frac{\mathrm{sgn}(\lambda_\sigma\lambda_\delta)}{2}\sum_{l=+1,-1} l[\mathrm{sgn}(v) \\ + \mathrm{sgn}\{v + 2l\mathrm{sgn}(\lambda_\sigma\lambda_\delta)\}] \\ \times |v + 2l\mathrm{sgn}(\lambda_\sigma\lambda_\delta), \lambda_\sigma, \lambda_\delta\rangle, \end{aligned} \tag{D1}$$

$$\hat{F}_\sigma|v, \lambda_\sigma, \lambda_\delta\rangle = i\frac{\mathrm{sgn}(\lambda_\sigma)}{2}\sum_{l=+1,-1} l[\mathrm{sgn}\{|v| + 2l\mathrm{sgn}(\lambda_\sigma)\} + 1]\Big|v + 2l\mathrm{sgn}(v\lambda_\sigma), \lambda_\sigma + 2l\Big|\frac{\lambda_\sigma}{v}\Big|, \lambda_\delta\Big\rangle. \tag{D2}$$

The action of $\hat{F}_\delta$ is similar to that of $\hat{F}_\sigma$, interchanging the roles of λ_σ and λ_δ.

2. The operator $\hat{C}_{\mathrm{G}}^{\mathrm{B}}$

The action of the constraint operator on the states $|v, \lambda_\sigma, \lambda_\delta\rangle \otimes |\mathfrak{N}\rangle$ of our basis, with v, λ_σ, and λ_δ being all positive, is the following:

$$\begin{aligned}\widehat{\mathcal{C}}^{\mathrm{B}}_{\mathrm{G}}|v,\lambda_\sigma,\lambda_\delta\rangle\otimes|\mathfrak{R}\rangle = \frac{(\pi l_{\mathrm{Pl}}^2)^2}{4}\Big\{&x_-(v)|v-4,\lambda_\sigma,\lambda_\delta\rangle_- - x_0^-(v)|v,\lambda_\sigma,\lambda_\delta\rangle_{0^-}\\ &- x_0^+(v)|v,\lambda_\sigma,\lambda_\delta\rangle_{0^+} + x_+(v)|v+4,\lambda_\sigma,\lambda_\delta\rangle_+ + \frac{32}{\beta}\frac{v^2}{\lambda_\sigma^2\lambda_\delta^2}\widehat{H}_0^\xi|v,\lambda_\sigma,\lambda_\delta\rangle - \beta b_\theta^2(v,\lambda_\sigma,\lambda_\delta)\widehat{H}^\xi_{\mathrm{int}}\\ &\times\Big[b_\theta^2(v-4,\lambda_\sigma,\lambda_\delta)\frac{v-4}{v}x_-(v)|v-4,\lambda_\sigma,\lambda_\delta\rangle'_- - b_\theta^2(v,\lambda_\sigma,\lambda_\delta)\\ &\times[x_0^-(v)|v,\lambda_\sigma,\lambda_\delta\rangle'_{0^-} + x_0^+(v)|v,\lambda_\sigma,\lambda_\delta\rangle'_{0^+}] + b_\theta^2(v+4,\lambda_\sigma,\lambda_\delta)\frac{v+4}{v}x_+(v)|v+4,\lambda_\sigma,\lambda_\delta\rangle'_+\Big]\Big\}\\ &\otimes|\mathfrak{R}\rangle,\end{aligned} \tag{D3}$$

where

$$\beta = \left(\frac{l_{\mathrm{Pl}}}{4\pi\gamma\sqrt{\Delta}}\right)^{2/3} \tag{D4}$$

and we have introduced the following notation:

$$\begin{aligned}|v\pm4,\lambda_\sigma,\lambda_\delta\rangle_\pm = &\left|v\pm4,\lambda_\sigma,\frac{v\pm4}{v\pm2}\lambda_\delta\right\rangle + \left|v\pm4,\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle + \left|v\pm4,\frac{v\pm4}{v\pm2}\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle\\ &+ \left|v\pm4,\frac{v\pm4}{v\pm2}\lambda_\sigma,\lambda_\delta\right\rangle + \left|v\pm4,\frac{v\pm2}{v}\lambda_\sigma,\frac{v\pm4}{v\pm2}\lambda_\delta\right\rangle + \left|v\pm4,\frac{v\pm2}{v}\lambda_\sigma,\lambda_\delta\right\rangle,\end{aligned} \tag{D5}$$

$$\begin{aligned}|v,\lambda_\sigma,\lambda_\delta\rangle_{0^\pm} = &\left|v,\lambda_\sigma,\frac{v}{v\pm2}\lambda_\delta\right\rangle + \left|v,\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle + \left|v,\frac{v}{v\pm2}\lambda_\sigma,\lambda_\delta\right\rangle + \left|v,\frac{v}{v\pm2}\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle\\ &+ \left|v,\frac{v\pm2}{v}\lambda_\sigma,\lambda_\delta\right\rangle + \left|v,\frac{v\pm2}{v}\lambda_\sigma,\frac{v}{v\pm2}\lambda_\delta\right\rangle,\end{aligned} \tag{D6}$$

$$\begin{aligned}|v\pm4,\lambda_\sigma,\lambda_\delta\rangle'_\pm = &\left|v\pm4,\lambda_\sigma,\frac{v\pm4}{v}\lambda_\delta\right\rangle + \left|v\pm4,\frac{v\pm4}{v\pm2}\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle + \left|v\pm4,\frac{v\pm4}{v}\lambda_\sigma,\lambda_\delta\right\rangle\\ &+ \left|v\pm4,\frac{v\pm2}{v}\lambda_\sigma,\frac{v\pm4}{v\pm2}\lambda_\delta\right\rangle,\end{aligned} \tag{D7}$$

$$|v,\lambda_\sigma,\lambda_\delta\rangle'_{0^\pm} = 2|v,\lambda_\sigma,\lambda_\delta\rangle + \left|v,\frac{v\pm2}{v}\lambda_\sigma,\frac{v}{v\pm2}\lambda_\delta\right\rangle + \left|v,\frac{v}{v\pm2}\lambda_\sigma,\frac{v\pm2}{v}\lambda_\delta\right\rangle, \tag{D8}$$

and

$$x_-(v) = 2\sqrt{v}(v-2)\sqrt{v-4}[1+\mathrm{sgn}(v-4)], \tag{D9}$$

$$x_+(v) = x_-(v+4), \tag{D10}$$

$$x_0^-(v) = 2(v-2)v[1+\mathrm{sgn}(v-2)], \tag{D11}$$

$$x_0^+(v) = x_0^-(v+2). \tag{D12}$$

3. Support of the anisotropies

We want to prove that the set $\mathcal{W}_\varepsilon$ is dense in $\mathbb{R}^+$. For this, we will show that its subset U_ε, defined as

$$\mathcal{W}_\varepsilon \supset U_\varepsilon = \left\{\frac{\varepsilon+4m}{\varepsilon+4n}; m,n\in\mathbb{N}\right\}, \tag{D13}$$

is already dense in the positive real line.

Let a and b be any two positive real numbers, such that $b-a>0$. Besides, we define the set

$$V_\varepsilon = \left\{\frac{\varepsilon}{4}+n; n\in\mathbb{N}\right\}. \tag{D14}$$

Then there always exists a number $s=\varepsilon/4+n_1\in V_\varepsilon$ such that $1<s(b-a)$, or equivalently

$$sa+1<sb. \tag{D15}$$

Let us denote by $t=\varepsilon/4+m_1$ the largest number in V_ε which is smaller or equal than $sa+1$, that is

$$sa<\frac{\varepsilon}{4}+m_1\le sa+1. \tag{D16}$$

Equations (D15) and (D16) imply that $sa<\varepsilon/4+m_1<sb$. These inequalities can be written equivalently as

$$a < \frac{\varepsilon + 4m_1}{\varepsilon + 4n_1} < b. \tag{D17}$$

In conclusion, given any two positive numbers a and b with $a < b$, there always exists a number $u \in U_\varepsilon$ such that $a < u < b$. As a consequence U_ε, and $\mathcal{W}_\varepsilon$ *a fortiori*, are dense in the positive real axis, as we wanted to prove.

[1] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2007); C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004); A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).

[2] M. Bojowald, Living Rev. Relativity **11**, 4 (2008).

[3] A. Ashtekar, M. Bojowald, and J. Lewandowski, Adv. Theor. Math. Phys. **7**, 233 (2003).

[4] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. D **74**, 084003 (2006).

[5] M. Martín-Benito, G. A. Mena Marugán, and J. Olmedo, Phys. Rev. D **80**, 104015 (2009).

[6] A. Ashtekar, T. Pawlowski, P. Singh, and K. Vandersloot, Phys. Rev. D **75**, 024035 (2007).

[7] L. Szulc, W. Kaminski, and J. Lewandowski, Classical Quantum Gravity **24**, 2621 (2007); K. Vandersloot, Phys. Rev. D **75**, 023523 (2007).

[8] E. Bentivegna and T. Pawlowski, Phys. Rev. D **77**, 124025 (2008).

[9] D. W. Chiou, Phys. Rev. D **75**, 024029 (2007).

[10] D. W. Chiou, Phys. Rev. D **76**, 124037 (2007).

[11] M. Martín-Benito, G. A. Mena Marugán, and T. Pawlowski, Phys. Rev. D **78**, 064008 (2008).

[12] A. Ashtekar and E. Wilson-Ewing, Phys. Rev. D **79**, 083535 (2009).

[13] A. Ashtekar and E. Wilson-Ewing, Phys. Rev. D **80**, 123532 (2009).

[14] R. H. Gowdy, Ann. Phys. (N.Y.) **83**, 203 (1974).

[15] V. Moncrief, Phys. Rev. D **23**, 312 (1981).

[16] J. Isenberg and V. Moncrief, Ann. Phys. (N.Y.) **199**, 84 (1990).

[17] See, e.g., C. W. Misner, Phys. Rev. D **8**, 3271 (1973); B. K. Berger, Ann. Phys. (N.Y.) **83**, 458 (1974); Phys. Rev. D **11**, 2770 (1975); Ann. Phys. (N.Y.) **156**, 155 (1984); G. A. Mena Marugán, Phys. Rev. D **56**, 908 (1997); M. Pierri, Int. J. Mod. Phys. D **11**, 135 (2002).

[18] A. Corichi, J. Cortez, and G. A. Mena Marugán, Phys. Rev. D **73**, 041502 (2006); **73**, 084020 (2006).

[19] A. Corichi, J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Classical Quantum Gravity **23**, 6301 (2006); J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Phys. Rev. D **75**, 084027 (2007).

[20] M. Martín-Benito, L. J. Garay, and G. A. Mena Marugán, Phys. Rev. D **78**, 083516 (2008).

[21] G. A. Mena Marugán and M. Martín-Benito, Int. J. Mod. Phys. A **24**, 2820 (2009).

[22] G. A. Mena Marugán and M. Montejo, Phys. Rev. D **58**, 104017 (1998).

[23] Here, we call physical area that measured by the area operator defined on the kinematical Hilbert space, to distinguish it from the fiducial area.

[24] There is a discrepancy in signs between this representation and that of Ref. [12] because, in the latter, γ is considered to change sign under internal parity transformations, namely, $\gamma = |\gamma|\mathrm{sgn}(v)$, while we treat γ just as a positive free parameter, which therefore is unaffected by transformations of the dynamical variables.

[25] We understand the *nondensitized* Hamiltonian constraint as referring to the scalar constraint with the same densitization as in LQG.

[26] This difference is due to the factor $1/(16\pi G)$ accompanying the classical action, which is not included in the definition of the constraint in this paper.

[27] D. Marolf, arXiv:gr-qc/9508015; Classical Quantum Gravity **12**, 1199 (1995); **12**, 1441 (1995); **12**, 2469 (1995).

[28] A. D. Rendall, Classical Quantum Gravity **10**, 2261 (1993); arXiv:gr-qc/9403001.

[29] D. Brizuela, G. A. Mena Marugán, and T. Pawlowski, Classical Quantum Gravity **27**, 052001 (2010).

[30] Nonetheless, we note that $\widehat{\sqrt{V}}(\hat{F}_i - \hat{F}_j)\widehat{\sqrt{V}}$ is still a constant of motion. We thank T. Pawlowski for pointing out this fact.

[31] M. Martín-Benito, G. A. Mena Marugán, and E. Wilson-Ewing, arXiv:1006.2369.

[32] G. A. Mena Marugán, T. Pawlowski, and E. Wilson-Ewing "An Effective Analysis of the Hybrid Quantization of the Gowdy Model" (unpublished).