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# Lemaître-Tolman-Bondi cosmological wormhole

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We present a new analytical solution of the Einstein field equations describing a wormhole shell of zero thickness joining two Lemaître-Tolman-Bondi universes, with no radial accretion. The material on the shell satisfies the energy conditions and, at late times, the shell becomes comoving with the dust-dominated cosmic substratum.

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## I. INTRODUCTION

Static and asymptotically flat wormhole solutions of the Einstein equations have been known for a long time [1]. The study of wormholes developed with the seminal paper by Morris and Thorne [2], after which many solutions were discovered (see [3] for an extensive discussion). The possibility that inflation in the early universe may enlarge a Planck-size wormhole to a macroscopic-size object was contemplated in Ref. [4]. Dynamical wormholes were discovered and studied in Refs. [5,6], and wormholes in cosmological settings were contemplated in various works (see [7,8] and references therein), with particular attention being paid to wormholes with the cosmological constant $\Lambda$, which are asymptotically de Sitter or anti–de Sitter according to the sign of $\Lambda$ [9].

After the 1998 discovery of the present acceleration of the universe [10] and the introduction of dark energy in cosmological theories to account for this cosmic acceleration, there were claims that phantom energy, an extremely exotic form of dark energy with $P < -\rho$ (where $P$ and $\rho$ are the pressure and the energy density, respectively), could cause the universe to end with a big rip singularity at a finite time in the future [11]. There was then a claim in the literature [12] that, if a wormhole accretes phantom energy, it grows to an enormous size faster than the background universe, swallowing the entire cosmos which would then tunnel through the wormhole throat and reappear in a different portion of the multiverse before reaching the big rip singularity. This claim was based on qualitative arguments and was later disproved by two classes of exact solutions of the Einstein equations representing wormholes embedded in a cosmological background dominated by phantom energy [13]. These wormholes accrete phantom energy but, even if their expansion rate differs from that of the cosmic substratum initially, they become comoving with it as the scale factor of the Friedmann-Lemaître-Robertson-Walker (FLRW) universe in which they are embedded grows.

The first class of solutions consists of a zero-thickness shell which carries exotic energy and does not perturb the two copies of the FLRW universe which it joins. The second, more realistic, class is described by a generalized McVittie metric [14] with an imperfect fluid and a radial energy flow, with the mass of the wormhole shell distorting the surrounding FLRW metric [13]. Another, less general, solution of the Einstein equations describing a cosmological wormhole comoving with the background was presented in Ref. [15].

Cosmological wormholes are truly dynamical, and an interest in this kind of solution has developed in parallel with the increasing attention paid to cosmological black holes [16]. Additionally, gravitational lensing by wormholes was studied in [17], and numerical solutions interpreted as wormholes in accelerating FLRW universes were presented in Refs. [18]. Recently, Maeda, Harada, and Carr have given precise definitions for general cosmological wormholes and have found two new exact solutions of this kind [19]. An important result of this work, which echoes a previous result of [7], is that the null energy condition need not be violated in this dynamical situation, although it must be violated for static wormholes to exist [19]. It seems that the study of cosmological wormholes is developing into a promising new area of research.

In this paper we propose a new analytical solution of the Einstein field equations describing a cosmological wormhole shell joining two Lemaître-Tolman-Bondi (LTB) universes. We are led to this solution by the following considerations: the second class of solutions in Ref. [13] is inspired by the McVittie metric, which describes a central inhomogeneity in a FLRW background. However, the McVittie metric needs to be generalized by removing the McVittie "no-accretion" condition $G_{01} = 0$ (in spherical coordinates) which forbids radial energy flow. The goal of Ref. [13] was to describe the effect of the accretion of

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phantom energy onto the wormhole. Here, we begin by noting that inhomogeneities embedded in a FLRW background are usually described by using a LTB metric [20–22], not a McVittie one. The classical LTB metric describes a spherically symmetric inhomogeneity in a dust-dominated FLRW background. The Bondi condition $G_{01} = 0$ parallels the McVittie no-accretion condition and forbids the (radial) flow of cosmic dust onto the inhomogeneity. It would be interesting to obtain a solution describing a wormhole shell joining two identical LTB universes and perturbing its surroundings in the way described by the LTB metric. This is what we do here. We obtain a wormhole shell composed of exotic matter which expands more slowly than the cosmic substratum (which becomes a spatially flat FLRW universe at late times), but eventually becomes comoving with it.

The next section details how to construct the wormhole shell and satisfy the Israel-Darmois-Lichnerowicz junction conditions [23] on this shell. The Einstein equations on the shell provide expressions for the energy density and pressure of the material on the shell. Section III uses the covariant conservation equation to relate the rate of change of the mass of the shell material, the shell area, and the flux of cosmic fluid onto the shell due to the relative velocity between the shell and the cosmic substratum. The metric signature is $-+++$, we use units in which the speed of light and Newton's constant are unity, and we follow the notations of Ref. [24]. Greek indices run from 0 to 3 and Latin indices assume the values 0, 1, and 2, corresponding to the coordinates $(t, \theta, \varphi)$ of the spherical hypersurface $\Sigma$ defined below.

## II. THE LTB WORMHOLE SOLUTION

The spherically symmetric LTB line element for the critically open universe in polar coordinates $(t, r, \vartheta, \varphi)$ is

$$ds^2 = -dt^2 + [R'(t, r)]^2 dr^2 + R^2(t, r) d\Omega^2, \tag{1}$$

where

$$R(t, r) = \left(r^{3/2} + \tfrac{3}{2}\sqrt{m_e(r)}t\right)^{2/3} \tag{2}$$

is an areal radius, $r$ is a comoving radius,

$$m_e(r) = 4\pi \int_0^r dx x^2 \rho_0(x), \tag{3}$$

$\rho_0(r)$ is the energy density on an initial hypersurface, a prime denotes differentiation with respect to $r$, and $d\Omega^2 \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$. The line element (1) describes a spherical inhomogeneity in a dust-dominated universe (see [20–22]; for a recent review see [25]).

Consider now a wormhole shell $\Sigma$ at $r = r_\Sigma(t)$ which joins two identical copies of a LTB spacetime (this shell describes a wormhole created with the universe and not formed as the result of a dynamical process after the big bang). The wormhole shell is dynamical and moves, possibly also relative to the cosmic substratum, and its motion is described by the form of the function $r_\Sigma(t)$. It is convenient to write the equation of this shell as [26]

$$f(t, r) \equiv R - R_\Sigma(t, r_\Sigma(t)) = 0. \tag{4}$$

To find the unit normal to $\Sigma$ we first compute

$$N_\mu \equiv \nabla_\mu f = \nabla_\mu (R - R_\Sigma) = (R_t - \dot{R}_\Sigma, R', 0, 0) \tag{5}$$

and

$$N^\mu = \left(\dot{R}_\Sigma - R_t, \frac{1}{R'}, 0, 0\right), \tag{6}$$

and then normalize according to $n_\mu = \alpha N_\mu$. Here $R_t \equiv \partial R/\partial t$ and an overdot denotes a total derivative with respect to $t$, i.e., $\dot{R} \equiv dR/dt$.

The normalization $n_\mu n^\mu = 1$ yields

$$\alpha = \frac{1}{\sqrt{1 - (R_t - \dot{R}_\Sigma)^2}}. \tag{7}$$

It is convenient to introduce the radial velocity of the wormhole shell relative to the cosmic substratum

$$v \equiv \dot{R}_\Sigma - R_{t|\Sigma}, \tag{8}$$

where $R_{t|\Sigma} \equiv R_t(t, r_\Sigma(t))$. Then

$$\alpha = \frac{1}{\sqrt{1 - v^2}} = \gamma(v) \tag{9}$$

is an (instantaneous [27]) Lorentz factor for the relative motion shell-background. The unit normal is then

$$n_\mu = (-\gamma v, \gamma R', 0, 0), \tag{10}$$

$$n^\mu = \left(\gamma v, \frac{\gamma}{R'}, 0, 0\right). \tag{11}$$

The restriction of the metric to $\Sigma$ is given by

$$ds^2_{|\Sigma} = -dt^2 + R'^2_\Sigma dr^2_{|\Sigma} + R^2_\Sigma d\Omega^2 \tag{12}$$

or, using the fact that $\dot{R}_\Sigma = R_{t|\Sigma} + R'_\Sigma \dot{r}_\Sigma$ on the shell,

$$ds^2_{|\Sigma} = -(1 - v^2)dt^2 + R^2_\Sigma d\Omega^2, \tag{13}$$

which expresses the fact that the proper time $\tau$ of the shell is given by

$$d\tau = \sqrt{1 - v^2}\, dt; \tag{14}$$

i.e., it is Lorentz dilated with respect to the comoving time $t$ of the background.

Using the triad

$$\{e^\alpha_{(t)}, e^\alpha_{(\vartheta)}, e^\alpha_{(\varphi)}\} = \left\{\sqrt{1 - v^2}\,\delta^{\alpha t}, \delta^{\alpha\vartheta}, \delta^{\alpha\varphi}\right\}, \tag{15}$$

the extrinsic curvature of the shell is given by ($a, b, c = t$, $\vartheta$, or $\varphi$)

$$K_{\alpha\beta} = e^{(a)}_{\alpha} e^{(b)}_{\beta} \nabla_a n_b = e^{(a)}_{\alpha} e^{(b)}_{\beta} (\partial_a n_b - \Gamma^c_{ab} n_c), \quad (16)$$

where $\Gamma^c_{ab}$ are the Christoffel symbols of the three-dimensional metric $g_{ab|\Sigma}$. Equation (16) yields

$$\begin{aligned} K_{tt} &= \frac{1}{\gamma^2}(\partial_t n_t - \Gamma^t_{tt} n_t) = -\frac{v_t}{\gamma} - 2\gamma v_t v^2, \\ K_{\vartheta\vartheta} &= -\Gamma^t_{\vartheta\vartheta} n_t = \gamma^3 v R_\Sigma R_{t|\Sigma}, \\ K_{\varphi\varphi} &= K_{\vartheta\vartheta} \sin^2\vartheta. \end{aligned} \quad (17)$$

The mixed components of the extrinsic curvature are

$$K^t{}_t = \gamma v_t (2\gamma^2 v^2 + 1), \quad (18)$$

$$K^\vartheta{}_\vartheta = \gamma^3 v \frac{R_{t|\Sigma}}{R_\Sigma} = K^\varphi{}_\varphi, \quad (19)$$

while the trace is

$$K = K^t{}_t + K^\vartheta{}_\vartheta + K^\varphi{}_\varphi = 2\gamma^3 v \left( \frac{R_{t|\Sigma}}{R_\Sigma} + v_t v \right) + \gamma v_t. \quad (20)$$

Since there are two identical LTB universes joining at the shell with unit normal $n^\mu$ pointing outward, the jumps of these quantities on $\Sigma$ are

$$[K^a{}_b] = 2K^a{}_b, \qquad [K] = 2K. \quad (21)$$

The Einstein equations at the shell $\Sigma$ are [28]

$$[K^a{}_b - \delta^a{}_b K] = -8\pi S^a{}_b, \quad (22)$$

where $S_{ab}$ is the energy-momentum tensor of the material on the shell. We assume that this matter is a perfect fluid, described by

$$S_{ab} = (\sigma + P_\Sigma) u^{(\Sigma)}_a u^{(\Sigma)}_b + P_\Sigma g_{ab|\Sigma}, \quad (23)$$

where $\sigma$ and $P_\Sigma$ are the two-dimensional surface density and pressure, respectively, and $u^\mu_{(\Sigma)}$ is the four-velocity of the shell given by

$$u^\alpha_{(\Sigma)} = \frac{dx^\alpha_{(\Sigma)}}{d\tau} = \frac{\partial x^\alpha_{(\Sigma)}}{\partial x^\mu} \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} \frac{\partial x^\alpha_{(\Sigma)}}{\partial x^\mu}. \quad (24)$$

The coordinates on $\Sigma$ are $x^\mu_\Sigma = (t, r_\Sigma(t), \vartheta, \varphi)$, which yield

$$u^\alpha_{(\Sigma)} = \left( \gamma, \gamma \frac{v}{R'_\Sigma}, 0, 0 \right), \qquad u^{(\Sigma)}_\alpha = (-\gamma, \gamma v R'_\Sigma, 0, 0). \quad (25)$$

As such, it is easy to see that

$$u^\mu_{(\Sigma)} u^{(\Sigma)}_\mu = 1, \qquad u^\mu_{(\Sigma)} n_\mu = 0. \quad (26)$$

The $(t, t)$ component of the Einstein equations (22) at the shell is

$$\sigma = -\gamma^3 \frac{v}{2\pi} \frac{R_{t|\Sigma}}{R_\Sigma}, \quad (27)$$

while the $(\vartheta, \vartheta)$ or the $(\varphi, \varphi)$ component yields

$$\begin{aligned} P_\Sigma &= \frac{\gamma}{4\pi} \left( v_t + 2\gamma^2 v^2 v_t + \gamma^2 v \frac{R_{t|\Sigma}}{R_\Sigma} \right) \\ &= -\frac{\sigma}{2} + \gamma \frac{v_t}{4\pi} \frac{1 + v^2}{1 - v^2}. \end{aligned} \quad (28)$$

Using Eq. (2), one obtains

$$\frac{R_t}{R} = \frac{\sqrt{m_e(r)}}{r^{3/2} + 3\sqrt{m_e(r)} t/2}; \quad (29)$$

this quantity is asymptotic to $\frac{2}{3t}$, the Hubble parameter of the dust-dominated cosmological background, as $t \to +\infty$. It is also

$$\sigma = -\gamma^3 \frac{v}{2\pi} \frac{R_{t|\Sigma}}{R_\Sigma} = -\gamma^3 \frac{v}{2\pi} \frac{\sqrt{m_e(r_\Sigma)}}{r_\Sigma^{3/2} + 3\sqrt{m_e(r_\Sigma)} t/2}, \quad (30)$$

and $\sigma > 0$ is equivalent to $v < 0$. A wormhole shell with positive surface energy density must necessarily expand slower than the cosmic substratum, a fact that is interpreted as the influence of the inhomogeneity slowing down the expansion locally. Since the expansion rate of the background $\frac{2}{3t}$ tends to zero at late times, the shell expansion rate must also tend to zero and the shell becomes comoving. In fact, since $v < 0$, it is $\dot{R}_\Sigma = R_{t|\Sigma} + R'_\Sigma \dot{r}_\Sigma < R_{t|\Sigma}$ and, since $R'_\Sigma > 0$, it is $\dot{r}_\Sigma < 0$. This inequality is consistent, of course, with the relation

$$\dot{r}_\Sigma = \frac{v}{R'_\Sigma}$$

which is easy to derive.

Now, $r_\Sigma > 0$ decreases monotonically, implying that [29], as $t \to +\infty$, either $r_\Sigma$ tends to a horizontal asymptote $r_\infty > 0$, or $\lim_{t\to+\infty} r_\Sigma(t) = 0^+$. If $r_\Sigma \to 0^+$, then

$$R_\Sigma = \left( r_\Sigma^{3/2} + \tfrac{3}{2}\sqrt{m_e(r_\Sigma)} t \right)^{2/3} \to 0 \quad (31)$$

because $m_e(0) = 0$, and the wormhole disappears asymptotically, which does not make sense physically; this possibility is discarded. Hence,

$$R_\Sigma \to \left( r_\infty^{3/2} + \tfrac{3}{2}\sqrt{m_e(r_\infty)} t \right)^{2/3} \quad (32)$$

and the wormhole shell becomes comoving at late times.

We conclude this section with a comment on the energy conditions. The strong energy conditions for a two-dimensional perfect fluid are $\sigma + P_\Sigma \geq 0$ and $\sigma + 2P_\Sigma \geq 0$: for our wormhole, assuming a non-negative surface density $\sigma$ and hence $v < 0$, they are

$$\sigma + P_\Sigma = \frac{\sigma}{2} + \frac{\gamma v_t}{4\pi} \frac{1 + v^2}{1 - v^2} > 0 \quad (33)$$

and

$$\sigma + 2P_\Sigma = \frac{\gamma v_t}{4\pi}\frac{1+v^2}{1-v^2} > 0 \tag{34}$$

because $v < 0$ and $v \to 0^-$ as $t \to \infty$; hence $v_t > 0$. The weak energy condition on the shell corresponds to $\sigma \geq 0$ and $\sigma + P_\Sigma \geq 0$, while the null energy condition is equivalent to $\sigma + P_\Sigma \geq 0$. Therefore, the material on the shell satisfies the weak, strong, and null energy conditions.

## III. THE COVARIANT CONSERVATION EQUATION

We can now solve the covariant conservation equation projected along the four-velocity of the shell $u^a_{(\Sigma)}$ [28],

$$u^a_{(\Sigma)}\nabla_b S_a{}^b = -\Big[u^\alpha_{(\Sigma)} T_\alpha{}^\beta n_\beta\Big]. \tag{35}$$

It is convenient to note that

$$\begin{aligned} u^a_{(\Sigma)}\nabla_b S_a{}^b &= u^a_{(\Sigma)}\nabla_b\Big[(\sigma + P_\Sigma)u^{(\Sigma)}_a u^b_{(\Sigma)}\Big] + u^a_{(\Sigma)}\nabla_a P_\Sigma \\ &= -\nabla_b\Big(\sigma u^b_{(\Sigma)}\Big) - P_\Sigma \nabla_b u^b_{(\Sigma)} \end{aligned} \tag{36}$$

using the normalization $u^a_{(\Sigma)}u^{(\Sigma)}_a = -1$ and its consequence $u^a_{(\Sigma)}\nabla_b u^{(\Sigma)}_a = 0$. We now compute

$$\nabla_b\Big(\sigma u^b_{(\Sigma)}\Big) = \frac{1}{\sqrt{|g_{|\Sigma}|}}\partial_b\Big(\sqrt{|g_{|\Sigma}|}\sigma u^b_{(\Sigma)}\Big), \tag{37}$$

where $g_{|\Sigma} = \gamma^{-2}R^4_\Sigma \sin^2\vartheta$ is the determinant of the three-dimensional metric $g_{ab|\Sigma}$, obtaining

$$\nabla_b\Big(\sigma u^b_{(\Sigma)}\Big) = \frac{\gamma}{R^2_\Sigma}\partial_t(R^2_\Sigma\sigma) = \gamma\frac{\dot{M}}{A_\Sigma}. \tag{38}$$

Here $A_\Sigma \equiv 4\pi R^2_\Sigma$ is the area of the shell and $M \equiv \sigma A_\Sigma$ is the mass of the material located on the shell. Similarly, one obtains

$$\nabla_b u^b_{(\Sigma)} = \gamma\frac{\dot{A}_\Sigma}{A_\Sigma} \tag{39}$$

and

$$\Big[u^\alpha_{(\Sigma)}T_\alpha{}^\beta n_\beta\Big] = 2u^\alpha_{(\Sigma)}T_\alpha{}^\beta n_\beta = -2\gamma^2\rho v. \tag{40}$$

Putting everything together, we obtain the covariant conservation equation in the form

$$\dot{M} + P_\Sigma\dot{A}_\Sigma = -2\gamma\rho v A_\Sigma. \tag{41}$$

This formula is interpreted physically as follows: the quantity $\rho v$ is the flux density of cosmic fluid onto the shell caused by the relative motion of the shell with respect to the background. The quantity $\rho v A_\Sigma$ is the flux of this material; the factor 2 appears because there are two LTB spacetimes joining at the shell. The Lorentz factor $\gamma$ is due to the Lorentz contraction caused by the radial motion of the shell.

Equation (41) expresses the first law of thermodynamics relating changes over a time interval $dt$,

$$dM + dW = dQ_\Sigma, \tag{42}$$

where $dM = \dot{M}dt$ is the variation of internal energy during $dt$, $dW = P_\Sigma\dot{A}_\Sigma dt$ is a work term due to the variation of the area of the shell, and $dQ_\Sigma$ is the energy input due to the influx of cosmic fluid onto the shell.

## IV. DISCUSSION AND CONCLUSIONS

We have obtained, and interpreted physically, an exact solution of the Einstein field equations representing a wormhole shell joining two identical LTB spacetimes which are dust dominated. This solution is similar to the wormhole solution of Ref. [13] obtained by generalizing the McVittie metric, but there are important differences. First, we adopted the no-accretion condition of Bondi [22] which forbids radial flow of energy into the wormhole, while Ref. [13], being interested in the effect of accretion, allows for radial flow with the consequence that an imperfect fluid is needed in order to obtain solutions in [13]. Here, instead, we can consider a perfect fluid, the dust of classical LTB solutions [20–22]. While in [13] the conservation equation analogous to our Eq. (41) has a right-hand side consisting of two terms, one due to the relative motion between shell and cosmic substratum, and another due to accretion, only the first term appears in our case in which there is no radial flux.

An important result of [19] is that, contrary to static wormholes, the null energy condition need not be violated for their cosmological and dynamical wormholes to stay open; here we propose a different cosmological wormhole solution made with material which satisfies the weak, null, and strong energy conditions on the shell. In other words, the ''stuff'' necessary to keep this wormhole throat open does not need to be very exotic. This feature motivates further studies of dynamical wormhole solutions of the Einstein equations.

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