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Gravothermal catastrophe

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This work discusses *gravothermal catastrophe* in astrophysical systems and provides an analytic collapse solution which exhibits many of the catastrophe properties. The system collapses into a trapped surface with outgoing energy radiated to a future boundary and provides an example of catastrophic collapse.

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I. INTRODUCTION

Since 1968, Lynden-Bell [1–3] has illuminated the concept of *negative* heat capacity in astrophysical systems. He explained Antonov's theorem [4] which states (roughly) that a spherical collection of self-gravitating point masses has no global entropy maximum. Explanation was needed since the most probable state for such a system is at the maximum of the Boltzmann entropy. Lynden-Bell and Wood considered a self-gravitating gas sphere. They calculated the energy E and heat capacity C_V of the isothermal sphere. A graph of the sphere's binding energy vs density contrast showed a local maximum near an inflection point, and the plot of specific heat vs density contrast showed a stable branch for negative specific heat.

Negative C_V can be understood as follows: the virial theorem for *inverse square* forces of bounded systems relates the average kinetic energy and average potential energy as

$$\langle K.E.\rangle = -\tfrac{1}{2}\langle P.E.\rangle.$$

With total energy $E = \langle K.E.\rangle + \langle P.E.\rangle$ one has

$$E = -\langle K.E.\rangle \quad \text{negative},$$

but for moving particles $K.E. = \frac{3}{2}Nk_BT$. It follows that the specific heat is negative

$$C_V = \frac{dE}{dT} = -\frac{3}{2}Nk_B.$$

A negative C_V system in contact with a large thermal reservoir will have fluctuations that add energy and make its transient temperature lower, causing inward heat flow which will drive it to even lower temperatures. Thus negative C_V systems cannot reach thermal equilibrium.

Lynden-Bell coined the name ''gravothermal catastrophe'' for stellar systems undergoing such collapse. To describe the collapse, we quote [5]: ''Conductive transfer of heat from the central region will raise the high central temperature faster than it raises the lower temperature of the outer parts. No equilibrium is possible; the center continues to contract and get hotter, sending out heat to the outer parts.''

Astrophysical systems where gravothermal catastrophe may occur [6] are older globular clusters with compact cores, and bright elliptical galaxies with high central density profiles. Lynden-Bell and Eggleton studied the core collapse of globular clusters. In particular, they calculated self-similar collapse features of a gravitating gas sphere. One of the things they learned from their study was that to model the formation of a central black hole in a globular cluster, they needed to go beyond self-similarity and study more dissipative collapse.

In this work, we present an analytic solution [7] with shear and radial heat flow (SRH) which models many of the features of catastrophic collapse. The system collapses into a trapped surface with outgoing energy radiated to a future boundary.

In the following section, the SRH collapse metric with pressure and density is developed. The results are given in Sec. III and summarized in Sec. IV. This is followed by two appendixes. Appendix A contains the original SRH solution with arbitrary functions, and Appendix B has mass and trapped surface equations.

II. COLLAPSING SRH FLUID

The SRH spacetime [7] is divided into an exterior region covered by the Vaidya metric and a collapsing interior region with spherically symmetric metric

$$g^{\rm SRH}_{\mu\nu}dx^\mu dx^\nu = A^2dt^2 - B^2dr^2 - R^2d\Omega^2, \qquad (1)$$

where $A = A(t, r)$, $B = B(t, r)$, $R = R(t, r)$, and $d\Omega^2$ is the metric of the unit sphere. The stress-energy tensor is given by ($G = c = 1$)

$$T^{\mu\nu} = w\hat{u}^\mu\hat{u}^\nu - p\gamma^{\mu\nu} + q^\mu\hat{u}^\nu + \hat{u}^\mu q^\nu. \qquad (2)$$

p is the isotropic pressure, w is the mass-energy density, $\gamma^{\mu\nu} = g^{\mu\nu} - \hat{u}^\mu\hat{u}^\nu$, and q^μ is the radial heat flow vector orthogonal to $\hat{u}^\mu$. Time is comoving with $\hat{u}^\mu\partial_\mu = A^{-1}\partial_t$, $\hat{u}^\mu\hat{u}_\mu = 1$. We use the notation of Taub [8] for the mass-energy density w. Taub's purpose was to distinguish mass-energy density from proper mass density ρ, where $w = \rho(1 + \epsilon)$ and ϵ is specific internal energy.

The original SRH solution, given in Appendix A, provides the following metric functions and physical scalars (overdots denote $\partial/\partial t$ and primes denote $\partial/\partial r$):

$$R(t,r) = e^{t/t_0}[(\beta_0^2/2)e^{-4t/t_0} + (r+r_0)^2]^{1/2}, \quad (3a)$$

$$A(t,r) = \alpha_0 \dot{R}, \quad (3b)$$

$$B(t,r) = \beta_0/R. \quad (3c)$$

Metric functions A and B should be dimensionless and R should have dimensions of length. We write $A = \alpha_0(\dot{R}/c)$ so it is clear that α_0 is dimensionless. The other constants have the following dimensions: $\dim(t_0) = \text{time}$, $\dim(\beta_0) = \text{length}$, and $\dim(r_0) = \text{length}$.

The heat flow vector and scalar are given by $4\pi q_\mu dx^\mu = Q dr$. The heat flow scalar is

$$Q = -\frac{1}{\alpha_0}\left(\frac{r+r_0}{R^3}\right)e^{2t/t_0}. \quad (4)$$

The pressure is

$$8\pi p = \frac{(r+r_0)^2 e^{4t/t_0}}{\beta_0^2 R^2}\left[\frac{R^2 + \beta_0^2 e^{-2t/t_0}}{R^2 - \beta_0^2 e^{-2t/t_0}}\right] - \frac{(1+1/\alpha_0^2)}{R^2}, \quad (5)$$

and the mass-energy density is

$$8\pi w = \frac{1}{R^2}\left[1 - \frac{1}{\alpha_0^2} - \frac{(r+r_0)^2}{\beta_0^2}e^{4t/t_0}\right] - \frac{2}{\beta_0^2}e^{2t/t_0}. \quad (6)$$

w is negative when

$$\left[\alpha_0^2 - 1 - \frac{\alpha_0^2(r+r_0)^2}{\beta_0^2}e^{4t/t_0}\right] - \frac{2\alpha_0^2 R^2}{\beta_0^2}e^{2t/t_0} < 0,$$

$$\alpha_0^2\beta_0^2 - \beta_0^2 - \alpha_0^2(r+r_0)^2 e^{4t/t_0} - 2\alpha_0^2 R^2 e^{2t/t_0} < 0,$$

$$\text{or} \quad \beta_0^2 + 3\alpha_0^2(r+r_0)^2 e^{4t/t_0} > 0.$$

This inequality always holds, and so w is negative.

III. COLLAPSE RESULTS

A. End stage of collapse

For small β_0, R goes as

$$R \simeq e^{t/t_0}(r+r_0). \quad (7)$$

The pressure and mass-energy density, at late times and for small β_0, go as

$$8\pi p \simeq \frac{e^{2t/t_0}}{\beta_0^2}, \quad (8)$$

$$8\pi w \simeq -3\frac{e^{2t/t_0}}{\beta_0^2}. \quad (9)$$

At late times, the magnitude of the mass-energy density increases exponentially, and the fluid approaches a photon gas with equation of state

$$p = \tfrac{1}{3}\,|\,w\,|\,. \quad (10)$$

Negative mass-energy density w is linked to the *gravothermal catastrophe*. The collapsing fluid has a negative specific heat which derives from negative internal energy ϵ. While both w and ϵ are negative, the proper mass density, $\rho = w/(1+\epsilon)$, remains positive. To separately compute ϵ would require a complete thermodynamic analysis. One would need a causal relativistic description of thermodynamics such as the Israel-Stewart "second order" type theory [9,10]. This has been left for future work.

As a model of gravitational dyamics, Chavanis *et al.* [11] studied the collapse of a gas of self-gravitating Brownian particles in a closed sphere. For catastrophic collapse in the microcanonical ensemble, they found the mass density to go as

$$\rho \simeq \rho_0(r_0/r)^{2.21} \quad (11)$$

and further, in a stability study, they found the perturbation $\delta\rho/\rho$ has a "core-halo" structure (hinting at central black hole formation). Without developing the thermodynamics of a collapsing SRH fluid, the density approximation in Eq. (11) can be used to compute internal energy from $\epsilon = w/\rho - 1$. With w at late times given in Eq. (9), we find

$$\epsilon \simeq -\text{const}\frac{e^{2t/t_0}}{\rho_0\beta_0^2}(r/r_0)^{2.21} - 1. \quad (12)$$

When the density is given a temperature profile (in the canonical ensemble $\rho \sim T^{-1/2}$), then $\epsilon = \epsilon(T)$ yields negative specific heat $c_V = d\epsilon/dT$.

B. Trapped surface

From Eq. (B2) the Misner-Sharp mass within a $t = \text{const}$, $r = \text{const}$ 2-surface is

$$2m = R\left[1 + \frac{1}{\alpha_0^2} - \frac{(r+r_0)^2}{\beta_0^2}e^{4t/t_0}\right]. \quad (13)$$

Null rays entering and leaving the 2-surface are described by the expansions of their respective generators. Equations (B6) and (B7) show that both null generators have nonnegative expansions and so a trapped surface exists at $R = 2m$. At the trapped surface, Eq. (13) sets a value for constants α_0 and β_0.

$$\frac{\beta_0^2}{\alpha_0^2} = (r_{\text{trap}} + r_0)^2 e^{4t_{\text{trap}}/t_0}. \quad (14)$$

The expression for R, Eq. (3a), yields

$$R_{\text{trap}}^2 = (r_{\text{trap}} + r_0)\beta_0\frac{(2+\alpha_0^2)}{2\alpha_0}. \quad (15)$$

We see from Eq. (4) that, at late times, the heat flow scalar goes to zero

$$Q \simeq -\frac{1}{\alpha_0}\frac{1}{(r+r_0)^2 e^{t/t_0}} \to 0. \quad (16)$$

The heat flow shuts off as the fluid collapses into the trapped surface.

C. Rate of collapse

The rate of collapse scalar is $\Theta = \nabla_\mu \hat{u}^\mu = -1/(\alpha_0 R)$. We again quote Lynden-Bell: "During the gravothermal catastrophe… the center continues to constrict and get hotter, giving out heat to the outer parts, but the temperature difference increases and drives the collapse onwards still faster.'' At distances just beyond the trapped surface

$$\Theta \simeq -\frac{e^{t/t_0}}{\alpha_0\beta_0}. \tag{17}$$

At late times the rate of collapse increases exponentially. This is a necessary component for a model of catastrophic collapse.

IV. SUMMARY

An analytic solution of Einstein's equations for dissipative collapse has been presented. The original SRH solution contains arbitrary functions of time which have been chosen here to provide an explicit solution. The system collapses into a trapped surface with outgoing energy radiated to a future asymptotic boundary. The collapsing fluid has negative mass energy, which has been related to negative specific heat. Lynden-Bell has linked collapse with negative heat capacity to gravothermal catastrophe. The SRH collapse has many features that model the gravothermal catastrophe.

APPENDIX A: ORIGINAL SRH SOLUTION

This solution is given in Eqs. (14), (15), and (16) of [7] [there is an error in the first term of Eq. (16). $\beta_0^2 h_1$ should be $(\beta_0^2/2)h_1$]. The metric components are, with arbitrary functions $h_1(t)$ and $h_2(t)$

$$A(t, r) = \alpha_0\dot{R}, \tag{A1a}$$

$$B(t, r) = \beta_0/R, \tag{A1b}$$

$$R(t, r) = [(\beta_0^2/2)h_1 + h_1^{-1}(r + h_2)^2]^{1/2}. \tag{A1c}$$

The match of g^{SRH} to exterior Vaidya was done in [7].

The pressure, with parameters and arbitrary functions unchosen, is

$$\begin{aligned} 8\pi p &= \frac{R^2}{\beta_0^2}\left[\left(\frac{R'}{R}\right)^2 + 2\frac{\dot{R}'}{\dot{R}}\frac{R'}{R}\right] - \frac{1}{\alpha_0^2\dot{R}^2}\left[\left(\frac{\dot{R}}{R}\right)^2\right] - \frac{1}{R^2} \\ &= \frac{1}{\beta_0^2}\left[(R')^2 + (R^2)'\frac{\dot{R}'}{\dot{R}}\right] - \frac{1}{R^2}\left(1 + \frac{1}{\alpha_0^2}\right) \\ &= -\frac{(\frac{r+h_2}{h_1})^2}{\beta_0^2 R^2} + \frac{2(\frac{r+h_2}{h_1})[(\frac{r+h_2}{h_1}) - \frac{\dot{h}_2}{\dot{h}_1}]}{\beta_0^2[R^2 - \beta_0^2 h_1 - 2(\dot{h}_2/\dot{h}_1)(r + h_2)]} \\ &\quad - \frac{(1 + 1/\alpha_0^2)}{R^2} \end{aligned} \tag{A2}$$

with mass-energy density

$$\begin{aligned} 8\pi w &= \frac{1}{\alpha_0^2\dot{R}^2}\left[-\left(\frac{\dot{R}}{R}\right)^2\right] - \frac{R^2}{\beta_0^2}\left[2\frac{R''}{R} + 3\left(\frac{R'}{R}\right)^2\right] + \frac{1}{R^2} \\ &= \frac{1}{\alpha_0^2 R^2}\left[\alpha_0^2 - 1 - \frac{\alpha_0^2(r + h_2)^2}{\beta_0^2 h_1^2}\right] - \frac{2}{\beta_0^2 h_1}. \end{aligned} \tag{A3}$$

The heat flow vector and scalar are given by $4\pi q_\mu dx^\mu = Q dr$,

$$Q = -\frac{1}{\alpha_0}\frac{R'}{R^2} = -\frac{1}{\alpha_0 R^2}\left[\frac{r + h_2}{h_1}\right]. \tag{A4}$$

For the particular case above, we choose

$$h_1 = e^{-2t/t_0}, \qquad h_2 = r_0. \tag{A5}$$

APPENDIX B: MASS AND TRAPPED SURFACE

The collapse metric is spanned by the tetrad

$$\hat{u}_\mu dx^\mu = A dt, \qquad \hat{r}_\mu dx^\mu = B dr, \qquad \hat{\vartheta}_\mu dx^\mu = R d\vartheta, \qquad \hat{\varphi}_\mu dx^\mu = R\sin\vartheta d\varphi,$$

such that

$$g_{\mu\nu} = \hat{u}_\mu\hat{u}_\nu - \hat{r}_\mu\hat{r}_\nu - \hat{\vartheta}_\mu\hat{\vartheta}_\nu - \hat{\varphi}_\mu\hat{\varphi}_\nu. \tag{B1}$$

The unique mass [12] m within $t = $ const, $r = $ const 2-surfaces is

$$\begin{aligned} 2m &= R^3 R_{\alpha\beta\mu\nu}\hat{\vartheta}^\alpha\hat{\varphi}^\beta\hat{\vartheta}^\mu\hat{\varphi}^\nu \\ &= R[1 + \dot{R}^2/A^2 - (R')^2/B^2]. \end{aligned} \tag{B2}$$

The metric is Petrov type **D** and the only nonzero Weyl tensor component Ψ_2 is invariantly expressed as $48(\Psi_2)^2 = C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$. The two principal null vectors of metric (1), normal to (ϑ, φ) 2-surfaces, are

$$l^\mu\partial_\mu = A^{-1}\partial_t + B^{-1}\partial_r, \tag{B3}$$

$$n^\mu\partial_\mu = A^{-1}\partial_t - B^{-1}\partial_r, \tag{B4}$$

with respective expansions

$$l^\mu_{;\mu} = \frac{\sqrt{2}}{R}\left(\frac{\dot{R}}{A} + \frac{R'}{B}\right), \qquad n^\mu_{;\mu} = \frac{\sqrt{2}}{R}\left(\frac{\dot{R}}{A} - \frac{R'}{B}\right). \tag{B5}$$

Consider a 2-surface S generated by l^μ and n^μ. If both null generators converge, i.e. expansions $l^\mu_{;\mu}$ and $n^\mu_{;\mu}$ have the same positive sign, then the 2-surface is trapped [13]. When $R = 2m$, Eq. (B2) provides

$$\frac{\dot{R}}{A} = \frac{R'}{B},$$

which implies

$$n^\mu_{;\mu} = 0, \qquad l^\mu_{;\mu} = \frac{2\sqrt{2}\dot{R}}{RA}. \tag{B6}$$

Using $A = \alpha_0\dot{R}$, the expansion of l^μ is

$$l^\mu_{;\mu} = \frac{2\sqrt{2}}{\alpha_0 R}. \tag{B7}$$

When $\alpha_0 > 0$ both expansions are non-negative signifying a trapped surface at $R = 2m$.

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