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Quantum of area $\Delta A = 8\pi l_P^2$ and a statistical interpretation of black hole entropy

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In contrast to alternative values, the quantum of area $\Delta A = 8\pi l_P^2$ does not follow from the usual statistical interpretation of black hole entropy; on the contrary, a statistical interpretation follows from it. This interpretation is based on the two concepts: nonadditivity of black hole entropy and Landau quantization. Using nonadditivity a microcanonical distribution for a black hole is found and it is shown that the statistical weight of a black hole should be proportional to its area. By analogy with conventional Landau quantization, it is shown that quantization of a black hole is nothing but the Landau quantization. The Landau levels of a black hole and their degeneracy are found. The degree of degeneracy is equal to the number of ways to distribute a patch of area $8\pi l_P^2$ over the horizon. Taking into account these results, it is argued that the black hole entropy should be of the form $S_{\rm bh} = 2\pi \cdot \Delta\Gamma$, where the number of microstates is $\Delta\Gamma = A/8\pi l_P^2$. The nature of the degrees of freedom responsible for black hole entropy is elucidated. The applications of the new interpretation are presented. The effect of noncommuting coordinates is discussed.

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I. INTRODUCTION

The statistical source of the Bekenstein-Hawking black hole entropy

$$S_{\rm bh} = \frac{A}{4l_P^2} \tag{1}$$

is still a central problem in black hole physics. Quantization of the black hole area can be one of the keys to understanding of it. According to Bekenstein [1], quantization of the black hole area means that the area spectrum of a black hole is of the form

$$A_n = \Delta A \cdot n, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where ΔA is the quantum of the black hole area. Despite this classical result there is still no general agreement on the precise value of ΔA; in the literature (see, for example, [2] and references therein), two alternative values are mainly considered:

$$\Delta A = 4\ln(k) l_P^2, \tag{3}$$

where k is an positive integer, and

$$\Delta A = 8\pi l_P^2. \tag{4}$$

The specific value of ΔA is important for a statistical definition of black hole entropy. According to statistical mechanics the entropy of an ordinary object is the logarithm of the number of microstates accessible to it, $\Delta\Gamma$, that is,

$$S = \ln\Delta\Gamma. \tag{5}$$

Since we assume that the entropy of a black hole should also have the form (5), it follows from (1)–(4) that the number of microstates accessible to a black hole is

$$\Delta\Gamma = \begin{cases} k^n, & \text{in the case where } \Delta A = 4\ln(k) l_P^2, \\ \exp(2\pi n), & \text{in the case where } \Delta A = 8\pi l_P^2. \end{cases} \tag{6}$$

The number of microstates is intrinsically an integer. The value $\Delta A = 4\ln(k) l_P^2$ is consistent with this condition, but the value $\Delta A = 8\pi l_P^2$ is not. Since the value $\Delta A = 8\pi l_P^2$, as is well known from the literature, is not restricted only to the semiclassical regime, this inconsistency seems to compound a problem. It is little discussed in the literature. Medved [2] was the first to consider it. Medved suggested that if the Bekenstein-Hawking entropy does not have the strict statistical interpretation of the form (5), then the two values (3) and (4) can be of comparable merit. In this case there is no a problem of $\Delta A = 8\pi l_P^2$.

In this paper I suggest an alternative solution of the problem. Namely I suggest that the black hole entropy is really associated with the number of microstates but, in contrast to ordinary matter (5), without the logarithm, that is,

$$S_{\rm bh} = 2\pi \cdot \Delta\Gamma, \tag{7}$$

where the number of microstates for a given area is

$$\Delta\Gamma \equiv n = \frac{A}{8\pi l_P^2}. \tag{8}$$

As is well known, a number of other entropy calculations have also been proposed to explain black hole statistical mechanics (see, for example, [3] and references therein).

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But they all use the usual expression (5) with the logarithm. The point is that every such ''calculation'' is not a calculation in the ordinary sense, but rather a new *definition* of the black hole entropy, which only may be made precise by referring to the (still missing) quantum theory of gravity. Moreover, none is yet very convincing.

The organization of this paper is as follows. In Sec. II we begin with nonadditive properties of a black hole and define a microcanonical distribution with allowance for nonadditivity. In Sec. III we show that quantization of a black hole is nothing but Landau quantization. We calculate Landau levels of a black hole and their degeneracy. The effect of noncommuting coordinates is discussed. The new definition of black hole entropy is proposed in Sec. IV. There the nature of the degrees of freedom responsible for black hole entropy is elucidated. The applications of the new interpretation are also presented. Finally, in Sec. V we consider the holographic principle and suggest an explanation for the area scaling $S_{\rm bh} \sim A$ in the case where the degrees of freedom do not reside on the horizon but are distributed in a spatial volume.

II. BLACK HOLES AND NONADDITIVITY

A. Motivation

We begin with definitions. The essential reason for taking the logarithm in (5) is to make the entropy an *additive* quantity, for the statistical independent systems. If we can subdivide a system into n, for example, separate subsystems and each subsystem has k states available to it, then the statistical independence of these subsystems signifies mathematically that the number of states for the composite system is the product of the number of states for the separate subsystems [4]:

$$\Gamma = k \times k \times k \times \cdots = k^n. \tag{9}$$

Then the additive property of the entropy defined as log of the number of states follows from (9) directly:

$$S = \ln \Delta\Gamma = n \ln k, \tag{10}$$

that is, the total entropy of the system is n times the entropy of a single subsystem. It is these properties that are essentially used in deriving the value $\Delta A = 4\ln(k) l_P^2$. There are several ways to obtain $\Delta A = 4\ln(k) l_P^2$. A typical assumption is that the horizon surface consists of n independent patches of area $\sim l_P^2$ and every patch has k states available to it. Then the total number of states is $\Delta\Gamma = k^n$, which is the same as (9). Now assuming the usual interpretation of the black hole entropy, we obtain $S = \ln\Delta\Gamma = n \ln k$, which is just (10). On the other hand, the entropy of a black hole is related to the area A of its horizon by the Bekenstein-Hawking formula (1). A comparison of these two expressions just gives $\Delta A = 4\ln(k) l_P^2$. So it is not a surprise that $\Delta A = 4\ln(k) l_P^2$ satisfies the condition (6) on the number of states to be an integer. On the contrary, $\Delta A = 8\pi l_P^2$ is sought without any initial assumptions regarding statistical interpretation of the Bekenstein-Hawking entropy; it follows from the periodicity of the Euclidean black hole solutions, underlying the black hole thermodynamics (see, for example, [5]). In this case, as will be shown below, a new statistical interpretation of the Bekenstein-Hawking entropy follows from $\Delta A = 8\pi l_P^2$.

The above derivation of $\Delta A = 4\ln(k) l_P^2$ as well as the classical formula for the entropy itself rely on the additivity properties of ordinary matter and, more fundamentally, on the very possibility of describing a given system as made up of independent subsystems. However the black holes are not conventional systems: they constitute nonadditive thermodynamical systems (for the sake of simplicity, we shall not make a distinction between nonadditive and nonextensive properties of black holes). As is well known, the fact that the gravitational energy is nonadditive appears already in Newtonian gravity. In general relativity a local definition of mass is not possible; the Arnowitt-Deser-Misner and Komar definitions of mass express this very clearly. Moreover the black hole entropy (1) goes as the square of mass M^2 in a sharp contrast with the additive character of entropy in ordinary thermodynamics. As emphasized by Kaburaki [6] and also Arcioni and Lozano-Tellechea [7], one has to consider a single black hole as a whole system; any discussion related to the possibility of dividing it into subsystems or to the additivity property of the black hole entropy simply does not take place. The statistical independence is a postulate in ordinary statistical physics and many of its general results just fail if this property is not assumed. This departure from the conventional systems is closely related to the long-range behavior of gravitational forces. Note that our proposal $\Delta\Gamma = n$ also gives $S \propto n$ as in the case of conventional systems (10). But our proposal is inconsistent with any hypothesis of the statistical independence. We know that if the number of states for a compound system is a product of factors, each of which depends only on quantities describing one part of the system, then the parts concerned are statistically independent, and each factor is proportional to the number of states of the corresponding part. In our approach the number n cannot be represented as such a product. On the other hand, if the black hole constituents were statistically independent, as in deriving $\Delta A = 4\ln(k) l_P^2$, the entropy (7) would be nonadditive.

It is obvious that the above aspect of nonadditivity cannot be ignored in deriving the black hole entropy. Although the study of nonadditive thermodynamics has been worked out to some extent (see, for example, [6–8] and references therein), there is not (with rare exception [9]) a concrete statistical model of the black hole entropy with allowance for nonadditivity. It is clear: we do not yet have a satisfactory quantum theory of gravity whose classical limit is general relativity. But our task is facilitated by the fact that the black hole area is quantized just with the quantum $\Delta A = 8\pi l_P^2$. Thus we can suggest a more concrete statistical interpretation of the black hole entropy.

B. Microcanonical distribution for a black hole with allowance for nonadditivity

To apply statistical mechanics to a black hole we should at first define the distribution function. In statistical mechanics all properties of a system are encoded in its distribution function [4]. For a quantum system the distribution function w_n determines the probability to find the system in a state with energy E_n. The determination of this function is the fundamental problem of statistical physics. The form of the function is usually postulated; its justification lies in the agreement between results derived from it and the thermodynamic properties of a system.

We begin with conventional systems. The standard determination of the distribution function for them is given in detail by Landau and Lifshitz [4]. Following Landau and Lifshitz, consider an ordinary isolated system consisting of quasi-isolated subsystems in thermal equilibrium. According to Liouville's theorem the distribution function of an isolated system is an integral of the motion. Because of the statistical independence of subsystems and, as a consequence, multiplicativity of their distribution functions, the logarithm of the distribution function must be not merely an integral of the motion, but an additive integral of the motion. It can be shown that the statistical state of a system executing a given motion depends only on its energy. Thus we can deduce that the logarithm of the distribution function must be a linear function of its energy of the form

$$\ln w_n^{(a)} = \alpha^{(a)} + \beta E_n^{(a)}, \tag{11}$$

with constant coefficients α and β, of which α is the normalization constant and β must be the same for all subsystems in a given isolated system; the superscript a refers to the subsystem a. Note that assuming another dependence $\ln w$ on E we may not obtain an additive function on the right side of (11); for example, E^2 is already a nonadditive function. Since the values of nonadditive integrals of the motion do not affect the statistical properties of ordinary system, these properties can be described by any function which depends only on the values of the additive integrals of the motion and which satisfies Liouville's theorem. The simplest such function is

$$dw = \text{const} \times \delta(E - E_0)\prod_a d\Gamma_a, \tag{12}$$

where the number of states of the whole system $d\Gamma$ is a product $d\Gamma = \prod_a d\Gamma_a$ of the numbers $d\Gamma_a$ of the subsystems (such that the sum of the energies of the subsystems lies in the interval of energy of the whole system dE). It defines the probability of finding the system in any of the $d\Gamma$ states. The factor const is the normalization constant and $\delta(E - E_0)$ the Dirac delta function. The distribution (12) is called microcanonical. Note that (11) is nothing but the canonical distribution if we identify $\beta = -1/T$, $\alpha = F/T$, F being the free energy, and T the temperature of the system.

As is easily seen, the statistical independence and additivity play a crucial role in deriving the distribution function for the conventional systems. Now consider the black holes. Because of the nonadditivity, the black holes cannot be considered as made up of any independent subsystems. Therefore, if we want to establish the distribution function for the black holes, we should remove the restrictions of the statistical independence and additivity of integrals of the motion for the subsystems of the black hole. The presence of the logarithm in (11) was just required by the statistical independence of the subsystems. So dropping the logarithm and superscript a in (11) we obtain

$$w_n = f(E_n), \tag{13}$$

where $f(E_n)$ represents a nonadditive integral of the motion and is a nonlinear function of the black hole energy. Besides the energy, in an isolated classical system there is another integral of motion—the phase volume occupied by the system $\Delta\Gamma$ (Liouville's theorem). It follows that any function of $\Delta\Gamma$, in particular the entropy, is also an integral of motion [10]. Since the Bekenstein-Hawking entropy is proportional to M^2 and nonadditive, it is reasonably assumed that so is the integral of motion. Thus the simplest function $f(E_n)$ compatible with this assumption is the square of energy, so we can write

$$w_n = \gamma E_n^2, \tag{14}$$

where γ is a constant coefficient. Note that here, as in ordinary statistics, the form of the distribution function must be regarded only as a postulate, to be justified solely on the basis of agreement of its predictions with the thermodynamical properties of black holes. Our considerations are intended to make it plausible, and nothing more. As a result, the (canonical) distribution for a subsystem (11) transforms to the (microcanonical) distribution for the whole system. Similarly dropping the product and subscript a in (12), we obtain

$$dw = \text{const} \times \delta(E - E_0)d\Gamma, \tag{15}$$

or after integration,

$$w = \text{const} \times \Delta\Gamma, \tag{16}$$

where $\Delta\Gamma$ is the number of states accessible to the whole system in a given state. The functions (14) and (16) are obviously the same and satisfy the same normalization condition $\sum_n w_n = 1$. So comparing (14) and (16) we get

$$\Delta\Gamma \propto M^2, \tag{17}$$

where the energy of the black hole is identified with its mass, M. Thus we arrive at the conclusion that the statistical weight of a black hole should be proportional to the area

$$\Delta\Gamma = \frac{A}{\xi l_P^2}, \tag{18}$$

as is evident from dimensionality considerations. Here ξ is a new constant coefficient $\xi > 1$, which cannot be defined exactly from general considerations without assuming some dynamical model for a black hole.

III. QUANTIZATION OF BLACK HOLE AS LANDAU QUANTIZATION

A. Motivation

As appears from the above, one has to consider a black hole as an indivisible fundamental object, for example, as an elementary particle (in agreement with old idea of 't Hooft). But the degeneracy factor of an elementary particle is relatively small [$\Delta\Gamma = (2s + 1)$, where s is the spin of the particle]. Where does the large $\Delta\Gamma$ of the black hole come from? Landau quantization is interesting for the statistical interpretation of black hole entropy mainly because of the macroscopic degeneracy of Landau levels. This large degeneracy follows from the fact that the orbital angular momentum L_z of an elementary particle can be macroscopically large, proportional to the area of a sample. As will be shown later, a black hole really has an intrinsic angular momentum with such a property and the energy levels of a black hole are nothing but the Landau levels. Finally, Landau quantization is important for black hole physics because of quantization of area and noncommutating coordinates.

B. Electron in two dimensions in a magnetic field

Before we start out discussing Landau quantization of a black hole we need to define the conventional Landau quantization proper. For the convenience of the reader we repeat the relevant material from [11] without proofs, thus making our exposition self-contained. As is well known from quantum mechanics, a magnetic field quantizes the energy of an electron confined in two dimensions. This is the basis of the conventional Landau quantization. So we restrict our attention to the motion of a single spinless electron confined to the xy plane in a perpendicular magnetic field $\vec{B} = B\vec{z}$. In classical mechanics, the centrifugal force is balanced by the Lorentz force

$$\frac{m_e v^2}{r} = \frac{e}{c} vB, \tag{19}$$

where all quantities have the standard meaning, so that a magnetic field $\vec{B}$ forces an electron to move on a circular orbit at the cyclotron frequency in the xy plane

$$\omega_c = \frac{v}{R_c}, \tag{20}$$

where R_c is the cyclotron radius, $R_c = \sqrt{2m_e E_k}/(eB)$, and E_k is the kinetic energy of the electron. For completeness we also define the Larmor frequency

$$\omega_L = \frac{v}{2R_c}. \tag{21}$$

Next, introducing the angular momentum $L_z = mvr$ we obtain from (19)

$$\frac{m_e v^2}{2} = \omega_L L_z. \tag{22}$$

In a quantum mechanical treatment, L_z can take only discrete values $m\hbar$, so that

$$\frac{m_e v^2}{2} = \omega_L \hbar m. \tag{23}$$

The mean potential energy is just as large as the mean kinetic energy, and the one-particle energy simply follows as the sum of both:

$$E = \omega_c \hbar m. \tag{24}$$

In the exact calculation, however, the zero-point energy also appears. The restriction to positive components $L_z > 0$ is a result of the chirality built into the problem by the magnetic field (as will be shown later, in the black hole case this property is caused by the Euclideanization of the black hole metric). The energy levels (23) are degenerate; it appears that the degeneracy is proportional to the area of the system. The macroscopically large degeneracy corresponds to the fact that the center of a classical circular orbit can be located anywhere in the xy plane.

In quantum mechanics, the Hamiltonian describing the cyclotron motion of a single electron is

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2. \tag{25}$$

Here $\vec{p} = (p_x, p_y)$ is the momentum operator and $\vec{A}(x, y)$ is the vector potential. For the "symmetric gauge," $\vec{A} = B(-y, x)/2$, the Hamiltonian (25) can be written as

$$H = \frac{p_x^2}{2m_e} + \frac{m_e \omega_L^2 x^2}{2} + \frac{p_y^2}{2m_e} + \frac{m_e \omega_L^2 y^2}{2} + \omega_L L_z. \tag{26}$$

Note that the first two terms in H form the Hamiltonian of an isotropic two-dimensional oscillator. In the polar coordinates defined by $x = r\cos\varphi$ and $y = r\sin\varphi$ the Hamiltonian (26) reads

$$H = -\frac{\hbar^2}{2m_e}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right] + \frac{m_e \omega_L^2 r^2}{2} - \omega_L i\hbar \frac{\partial}{\partial \varphi}. \tag{27}$$

It has eigenvalues

$$E = \hbar\omega_L(2n_r + 1 + m + |m|), \tag{28}$$

where n_r is the radial quantum number, $n_r = 0, 1, 2, \ldots$, and m is the angular momentum quantum number,

$m = 0, \pm 1, \pm 2, \ldots$. The energy levels are labeled by the principal quantum number n, $n = n_r + (m + |m|)/2$, so that

$$E = \hbar\omega_c\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots. \tag{29}$$

These levels are called Landau levels. The lowest energy level has $n_r = 0$, $m = 0, -1, -2, \ldots$, and energy $E = \hbar\omega_c/2$. The first excited level has $n_r = 1$ and $m = 0, -1, -2, \ldots$, or $n_r = 0$ and $m = 1$, etc. The eigenfunctions can be expressed in terms of the associated Laguerre polynomials. Relatively simple are the eigenfunctions for the lowest Landau level,

$$\psi_{0,m} = \frac{1}{\sqrt{2\pi|m|!}\,l_0}\left(\frac{r^2}{2l_0^2}\right)^{|m|/2} e^{im\varphi} e^{-r^2/4l_0^2}, \tag{30}$$

where l_0 is the characteristic length of the theory, the so-called magnetic length,

$$l_0 = \sqrt{\frac{\hbar}{(m_e\omega_c)}}. \tag{31}$$

It follows that

$$\langle\psi_{0,m}|r^2|\psi_{0,m}\rangle = 2(m+1)l_0^2 \tag{32}$$

and

$$\langle\psi_{0,m}|L_z|\psi_{0,m}\rangle = m. \tag{33}$$

For increasing m the wave function is localized along circles of larger and larger radii. The degree of degeneracy can be determined from the requirement that the radius for the largest m should be inside our system, for example, a disk of radius R,

$$2l_0^2(m+1) = R^2. \tag{34}$$

Expressed in terms of the area $A = \pi R^2$, this gives

$$m + 1 = \frac{A}{2\pi l_0^2}. \tag{35}$$

This is also true for higher Landau levels. We now return to the expression for the energy (28). Because of the smallness of $\hbar$, the energy can only be of macroscopic magnitude for reasonable B, if $(2n_r + 1 + m + |m|)$ is very large. So we have two cases: (i) $m < 0$ and (ii) $m > 0$. It appears that in case (i) n_r is large so the wave functions do not satisfy some natural conditions. There is no such problem in case (ii). If $m > 0$, the factor is $(2n_r + 1 + 2m)$, and it can be large, with n_r small, provided that m is large. The energy now is

$$E = \omega_c\hbar m, \tag{36}$$

in agreement with the classical result (24). Note that L_z is positive, as expected. Note also that for large m the degree of degeneracy is

$$m = \frac{A}{2\pi l_0^2}. \tag{37}$$

This also means that the area of electron orbit is quantized (as will be shown later, the degree of degeneracy of a black hole is related to the quantization of the black hole area in a similar way).

C. Black hole in two-dimensional Euclidean space

It appears that quantization of a black hole is nothing but the Landau quantization. The fact is that kinematics of a black hole in two-dimensional Euclidean Rindler space is similar to that of an electron in two dimensions in a magnetic field. So we start with Rindler space. It is well established [3] that in the near-horizon approximation the metric of an arbitrary black hole can be reduced to the Rindler form. In this approximation the first law of black hole thermodynamics for a Schwarzschild black hole takes the form [12]

$$dE_R = T_R dS_{\rm bh}, \tag{38}$$

where E_R is the Rindler energy, $E_R = 2GM^2$, T_R is the Rindler temperature, $T_R = 1/(2\pi)$, and $S_{\rm bh}$ is the Bekenstein-Hawking entropy. These quantities are related by

$$E_R = T_R S_{\rm bh}. \tag{39}$$

In [5] quantization of the black hole area [1] and value $\Delta A = 8\pi l_P^2$ were derived from the quantization of the angular momentum associated with the Euclidean Rindler space of a black hole. In transforming from Schwarzschild to Euclidean Rindler coordinates the Schwarzschild metric becomes

$$ds_E^2 \approx (k\rho)^2 dt^2 + d\rho^2 + \frac{1}{4k^2} d\Omega^2, \tag{40}$$

where ρ is the proper distance from the horizon and the constant k coincides with the surface gravity of a Schwarzschild black hole, $k = 1/4GM$. This metric is the product of the metric on a two-sphere with radius $2GM$ (the last term) and the Euclidean metric

$$ds_E^2 = \rho^2 d(kt)^2 + d\rho^2. \tag{41}$$

The metric (41) has a coordinate singularity at $\rho = 0$ (corresponding to the gravitational radius $R_g = 2GM$). Regularity is obtained if kt is interpreted as an angular coordinate with periodicity 2π

$$\omega = kt = \frac{t}{4GM}. \tag{42}$$

(t itself has then periodicity $8\pi GM$ which, when set equal to $\hbar/T_H$, gives the Hawking temperature T_H). This periodicity plays the same role in quantization of a black hole as a magnetic field in the conventional Landau quantization. On the other hand, according to quantum mechanics the angle

(42) is conjugate to the zth component of the angular momentum. Therefore, as was suggested in [5], the Rindler energy E_R should be reinterpreted as the zth component of an angular momentum operator $i\hbar\partial/\partial\omega$ with eigenvalue

$$L_z = 2GM^2 \tag{43}$$

(that is why we use the different notations, E_R and L_z, for the same value $2GM^2$). Since $L_z = m\hbar$, $m = 0, \pm 1, \pm 2, \ldots$, the value $2GM^2$ is now quantized. The negative integers m correspond to the region $r < R_g$. But the Euclidean Rindler spacetime has no region corresponding to the region $r < R_g$ in the Lorentzian spacetime, so the negative integers can be ruled out. In [5], quantization of L_z was interpreted as quantization of the black hole area,

$$\frac{A}{8\pi l_p^2} = m, \qquad m = 0, 1, 2, \ldots . \tag{44}$$

In [5] it was shown that this conclusion is also valid for a generic Kerr-Newman black hole. A refined version of this approach extended to generic theories of gravity was presented by Medved [13]. The angular momentum (43) can be also written in the usual classical form

$$L_z = M\upsilon r \tag{45}$$

and associated with some intrinsic motion if we identify M with the mass of a body which moves in a circle of the radius $r = R_g$ with the linear velocity $\upsilon \equiv c = 1$. This does not mean however that our system (i.e., a black hole) represents a rigid rotator, rather, as will be shown below, it represents a harmonic oscillator. Since a black hole as a two-sphere has circumference $2\pi R_g$ the period of such a "motion" is $2\pi R_g$ and the angular frequency is $1/R_g$; by analogy with (20) we shall call this frequency the *cyclotron frequency* and it is denoted by ω_c,

$$\omega_c = \frac{1}{R_g}. \tag{46}$$

Since the Rindler time ω is related to the Schwarzschild time t by (42), a field quantum with Rindler frequency ν_R is seen by a distant Schwarzschild observer to have a redshifted frequency $\nu = \nu_R/(4GM)$. From this it follows [12] that the temperature as seen by the distant observer is just the Hawking temperature $T_H = T_R \times 1/(4GM)$. By analogy with (21) we shall call the quantity $1/(4GM)$ the *Larmor frequency* and denote it by ω_L,

$$\omega_L = \frac{1}{2R_g} \tag{47}$$

(it is just half of the cyclotron frequency, $\omega_L = \omega_c/2$). We now return to the expression for the Rindler energy (39). Taking into account (43) and (47), the expression (39) can be rewritten as

$$\omega_L L_z = T_H S_{\rm bh} \tag{48}$$

(the entropy $S_{\rm bh}$ is an invariant and is not red-shifted). Since $M = 2T_H S_{\rm bh}$ and $L_z = m\hbar$, we can write

$$M = 2\omega_L \hbar m, \tag{49}$$

or equivalently

$$M = \omega_c \hbar m. \tag{50}$$

D. Landau levels of a black hole

Continuing our analogy with Landau quantization, we may expect that in the more general quantum mechanical case the Hamiltonian of a black hole has a form similar to (26), except that now it is defined in the two-dimensional Euclidean plane (41) and all quantities relating to the electron are replaced by the corresponding quantities relating to the black hole,

$$H = \frac{P_x^2}{2M_*} + \frac{M_*\omega_L^2 x^2}{2} + \frac{P_y^2}{2M_*} + \frac{M_*\omega_L^2 y^2}{2} + \omega_L L_z, \tag{51}$$

or in polar coordinates $\rho - \omega$,

$$H = -\frac{\hbar^2}{2M_*}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\omega^2}\right] + \frac{M_*\omega_L^2\rho^2}{2} + \omega_L L_z. \tag{52}$$

Here ω_L is the Larmor frequency of the black hole (47) and L_z is the angular momentum operator introduced near (43). Since the total energy of an electron in a magnetic field is twice the kinetic energy $m_e/2$, we replace the electron mass by $M_* = M/2$, M being the mass of black hole. Accordingly, the magnetic length of the electron is replaced by the characteristic length of black hole l_*

$$l_0 \to l_* = \sqrt{\frac{\hbar}{M_*\omega_c}}. \tag{53}$$

It is important to emphasize that in agreement with nonadditivity of a black hole there are no particular Δm_i in (51), only the total M. The Hamiltonian (51) can be postulated from the very beginning. From (51) it follows that a black hole is a two-dimensional isotropic oscillator with an additional interaction $\omega_L L_z$, like an electron in a magnetic field (26). Since in the black hole case $L_z \geq 0$, the eigenvalues of the Hamiltonian are

$$E = \hbar\omega_L(2n_r + 1 + 2m), \tag{54}$$

where n_r is the radial quantum number, $n_r = 0, 1, 2, \ldots$, and m is the angular momentum quantum number, $m = 0, 1, 2, \ldots$. Analogously to (29),

$$E = \hbar\omega_c\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots, \tag{55}$$

where $n = n_r + m$ is the principal quantum number. By analogy with the energy levels of an electron we call (55) the *Landau levels* of a black hole. The lowest level has $n_r = 0$ and $m = 0$. It may seem strange that there is an energy $E_0 = \hbar\omega_c/2$ in a state with $m = 0$ ($M = 0$). But $\hbar\omega_c \to \infty$ when $M = 0$, so a state with the zero-point energy in the absence of a black hole has no real physical meaning. The energy difference between the subsequent Landau levels is $\hbar\omega_c$. This gap decreases with increasing M and is equal in order of magnitude to T_H. It will be discussed in detail later. In the semiclassical limit $m \gg 1$ we obtain from (54)

$$E = \hbar\omega_c m, \tag{56}$$

which is the same as (50).

E. Degree of degeneracy of Landau levels

We know that a black hole has the entropy so that each Landau level should be degenerate. But where does this degeneration come from? As mentioned above, the complete Euclidean Schwarzschild space (40) has the structure $R^2 \times S^2$. The fact is that the energy of a black hole does not depend on a point of the two-sphere at which the Euclidean space (41) can be attached. Since we associate the Euclidean space with internal cyclotron motion, we can also say that the degeneracy corresponds to the fact that the center of the motion can be located anywhere on the two-sphere (in other words, all axes of rotation are physically equivalent). If the accuracy with which this point can be determined coincides with the size of the area quantum $\Delta A = 8\pi l_P^2$, then the degeneracy factor is given by $\Delta\Gamma = A/(8\pi l_P^2)$. This is nothing but the angular momentum number (44). As is well known, the energy levels of a system whose angular momentum is conserved are always degenerate. It is clear that L_z is conserved. Since in the black hole case L_z can take only positive values and zero,

$$\Delta\Gamma = m + 1. \tag{57}$$

For a typical black hole $m \gg 1$ so

$$\Delta\Gamma = m. \tag{58}$$

On the other hand, L_z is associated with a rotation in the Euclidean Rindler space through an angle ω. In the semiclassical description for any rotational degrees of freedom the number of accessible states equals the total accessible phase-space volume divided by the volume of one state $2\pi\hbar$:,

$$\Delta\Gamma = \frac{\int d\omega dL_z}{2\pi\hbar}. \tag{59}$$

Taking into account (44) and the fact that the angular orientation is unconstrained, so that the integral over $d\omega$ gives 2π, we again obtain $\Delta\Gamma = m$. Note that the black hole is degenerate with respect to L_z exactly as the electron in a magnetic field. So we can determine the degree of degeneracy of a black hole from the corresponding formulas for the electron, replacing all quantities relating to the electron by the corresponding quantities related to the black hole. From (53) it follows that for the black hole the characteristic length is $l_* = 2l_P$. Substituting this value in (37) instead of l_0, we obtain $\Delta\Gamma = A/(8\pi l_P^2)$, as expected.

F. Noncommutative geometry

Coordinate noncommutativity is one of the most fascinating effects of the Landau quantization. It appears [14] that in the limit of very large magnetic field B the energy difference between the subsequent Landau levels $\hbar\omega_c \to \infty$ so that an electron is restricted to the lowest Landau level. As a result, the two coordinates of the xy plane obey the same commutation relations as the momentum p_x and the position x in quantum mechanics,

$$[x, y] = il_0^2. \tag{60}$$

Thus, the two-dimensional coordinate space becomes the *phase space* for the system. As mentioned above, the area of one state in phase space is $\triangle p_x \triangle x = 2\pi\hbar$, so that

$$\Delta x \Delta y = 2\pi l_0^2 \tag{61}$$

(here l_0^2 plays the role of $\hbar$). Therefore the physical plane xy can be thought of as divided into the patches of area $2\pi l_0^2$, where the center of the motion can be localized. Note that the phenomenon of the noncommuting plane is not specific to the lowest Landau level but can be obtained by projecting to an arbitrary finite number of Landau levels [15]. Since the large B limit corresponds to small m_e, we can obtain a similar relation for a black hole setting $M \to 0$ and replacing l_0 by l_*. As a result, we get

$$\Delta x \Delta y = 8\pi l_P^2. \tag{62}$$

We can also obtain a similar relation for the r and t coordinates. Since L_z is conjugate to the angle (42), we have

$$\left[L_z, \frac{t}{4GM}\right] = i\hbar \tag{63}$$

or

$$[R_g, t] = i\hbar 4G. \tag{64}$$

From this it follows that

$$\Delta r \Delta t = 8\pi l_P^2, \tag{65}$$

as required.

IV. THE BLACK HOLE ENTROPY

A. Definition

Now we can define the entropy of a black hole. The entropy plays a particularly fundamental role when the microcanonical ensemble is used. According to the standard formula for the entropy we would have to take the logarithm of $\Delta\Gamma$. But in this case the generalized second law of black hole thermodynamics would be violated. The argument involves a well-known example with the collision of black holes: two identical black holes collide, merge, radiating gravitational wave energy, and form a third black hole. According to (5), the initial entropy of the system is

$$S_i = 2\ln\Delta\Gamma_i = 2\ln\left(\frac{A}{8\pi l_P^2}\right). \tag{66}$$

On the one hand, the final entropy is bounded from above by

$$S_f = \ln\Delta\Gamma_f = \ln\left(\frac{4A}{8\pi l_P^2}\right). \tag{67}$$

On the other hand, by virtue of the generalized second law it must be greater than initial entropy. So we have

$$2\ln\left(\frac{A}{8\pi l_P^2}\right) < S < \ln\left(\frac{4A}{8\pi l_P^2}\right). \tag{68}$$

As is easily seen, these inequalities are satisfied only for $A < 32\pi l_P^2$. This means that the standard interpretation of the entropy in terms of the logarithm of $\Delta\Gamma$ violates the generalized second law. Moreover, since m takes not only positive integral values but also zero, the entropy as the logarithm of $\Delta\Gamma$ makes no sense at all. Thus we conclude that the statistical interpretation of the Bekenstein-Hawking entropy is true only if log is deleted from the Boltzmann formula (5), that is,

$$S_{\rm bh} = 2\pi \cdot \Delta\Gamma = 2\pi m. \tag{69}$$

This is nothing but the angular momentum quantization condition on the phase of a wave function: if the eigenfunction of L_z is to be single-valued, it must be periodic in phase, with period 2π. Note that factor 2π was already noticed in the literature in a topological context [16]. In particular Bunster (Teitelboim) and Carlip noted that the overall factor in front of the area, usually quoted as one fourth in units where Newton's constant is unity, is really the Euler class of the two-dimensional disk. A number of proposals were proposed to quantize the entropy. Prominent among others, besides the classical works of Bekenstein [1], are those of Barvinsky and Kunstatter [17]; Padmanabhan and Patel [18]; Romero, Santiago, and Vergara [19]; and also Dolan [20]. It is important to notice here the following. Although all these researchers obtained the required spectrum $S_{\rm bh} = 2\pi m$, there is an important difference between their result and ours: in their spectrum m is simply a non-negative integer; in ours, it is the statistical weight of the black hole $\Delta\Gamma = m$.

B. The nature of the degrees of freedom

As appears from the above, a Schwarzschild black hole is completely described (at least in the semiclassical approximation) by one quantum number—the angular momentum number m. So, by definition, the black hole has *one* degree of freedom. At first sight it may seem that the horizon surface splits into m elementary patches of area $\Delta A = 8\pi l_P^2$. This is not the case; the number m does not mean that the horizon is really divided into m elementary figures with specific shape and localization, as a globe with quadrangles formed by parallels of latitude and meridians of longitude. According to nonadditivity of a black hole, a black hole cannot be thought as made up of any independent constituents; the black hole is an indivisible fundamental object, like the electron. On the other hand, although the black hole energy, like the energy of the electron motion, is the sum of m quanta with energy $\hbar\omega_c$, this does not mean that a black hole (or an electron) consists of m "photons." Thus the number m is not the number of black hole constituents. Instead, it is *the number of distinguishable ways* to distribute a patch of area $8\pi l_P^2$ over the horizon. This is its physical meaning. But the number m can have a deeper nature. The questions then arising, however, have as yet hardly been studied at all.

C. Applications: mean separation between energy levels of black hole and Poincaré recurrence time

The most distinctive feature of our interpretation of black hole entropy is that the statistical weight $\Delta\Gamma \sim S_{\rm bh}$ in contrast to $\Delta\Gamma \sim \exp(S_{\rm bh})$ of the usual interpretation. Here we consider the cases where the difference between the old and new interpretations of black hole entropy can manifest itself most clearly. We begin with the energy spectrum. According to (55) the energy spacing between the subsequent Landau levels is

$$\Delta E = \hbar\omega_c. \tag{70}$$

This agrees with the characteristic value of Hawking radiation $\sim T_H$. This value however does not agree with estimation obtained from the usual definition of entropy. The entropy of an ordinary system (5), by definition, is the logarithm of the number of states $\Delta\Gamma$ with energy between E and $E + \delta E$. The width δE is some energy interval characteristic of the limitation in our ability to specify absolutely precisely the energy of a macroscopic system. Dividing δE by the number of states $\exp(S)$ we obtain the mean separation between energy levels of the system [4],

$$\langle\Delta E\rangle = \delta E \exp(-S). \tag{71}$$

The interval δE is equal in order of magnitude to the mean canonical-ensemble fluctuation of energy of a system. However a Schwarzschild black hole has the negative specific heat C_v, $C_v = -8\pi GM^2$, so that the energy fluctuations calculated in the canonical ensemble have formally negative variance, $\langle(\delta E)^2\rangle = C_v T_H^2 \sim -m_P^2$, where T_H is the Hawking temperature. The situation is quite different if a black hole is placed in a reservoir of radiation and the total energy of the system is fixed [21]. In this case a stable equilibrium configuration can exist if the radiation and black hole temperatures coincide, $T_{\rm rad} = T_H \equiv T$, and

$E_{\rm rad} < M/4$, where $E_{\rm rad}$ is the energy of radiation. The latter condition can be reformulated as the restriction on the volume of reservoir V, $4aVT^5 < 1$, where a is the radiation constant. According to this condition Pavon and Rubï found [22] that the mean square fluctuation of the black hole energy (mass) is given by

$$\langle(\delta E)^2\rangle = (1/8\pi)T^2 Z, \quad (72)$$

Z being the quantity $4aVT^3/(1-4aVT^5)$, $G = c = \hbar = 1$, and the Boltzmann constant $k_B = (8\pi)^{-1}$. It is clear that (restoring G, c, $\hbar$, and k_B)

$$\langle(\delta E)^2\rangle \sim \frac{m_P^4}{M^2} \quad (73)$$

and

$$\langle(\delta E)\rangle \sim \frac{m_P^2}{M}. \quad (74)$$

In the quantum mechanical description, the accuracy with which the energy of a black hole can be defined by a distant observer is limited by the time-energy uncertainty relation as well as by the decrease of the mass of the black hole due to transition from a higher energy level to a lower one. The lifetime of a state E_n is proportional to the inverse of the imaginary part of the effective action [23]; less formally, it is the time needed to emit a single Hawking quantum and this is proportional to the gravitational radius R_g. So $\delta E_q \sim 1/R_g$, where I have added the subscript q to refer to the quantum uncertainty. On the other hand, $\delta E_q \sim T_H$ due to transition from the state n to the state $n-1$. As is easily seen, δE_q is of the same order of magnitude as (74). So we obtain the mean separation between energy levels for a black hole

$$\langle\Delta E\rangle \sim \frac{m_P^2}{M}\exp(-S_{\rm bh}). \quad (75)$$

This value is however exponentially smaller than (70). Thus a problem arises. It has not yet received attention in the literature. The point is that the energy interval δE contains only a single state. But in this case statistics is not applicable; by definition the statistical treatment is possible only if δE contains many quantum states. This means that Eq. (75) is not applicable to a black hole (contrary to what is assumed in many works [24]). Nevertheless we may attempt to define $\langle\Delta E\rangle$ with the help of the new interpretation of black hole entropy. Namely, taking into account the relation $S_{\rm bh} \sim \Delta\Gamma = m$, we get

$$\langle\Delta E\rangle \sim \frac{\Delta E}{\Delta m} \sim \frac{dM}{dS_{\rm bh}} = T_H; \quad (76)$$

this agrees with (70), as it should. Thus the first law of thermodynamics defines the energy spacing of a black hole.

One further remark is required concerning the equipartition theorem; it is closely connected with energy spacing of a black hole. Recently Padmanabhan [25] and Verlinde [26] set up very interesting hypotheses concerning the nature of gravity. One of the crucial ingredients of their hypotheses is the claim that the horizon degrees of freedom/bits satisfy the equipartition theorem. As is well known [4], the theorem is valid only if the thermal energy $k_B T_H$ is considerably larger than the spacing between energy levels of a system. As is easily seen, if the energy spacing of black hole levels were exponentially small in the black hole entropy as (75), the theorem would be valid, since in this case $k_B T_H \gg \Delta E$. But in fact the energy spacing of black hole levels is the same order of magnitude as $k_B T_H$, $\Delta E \sim k_B T_H$. Therefore the equipartition theorem is not valid for the black holes. On the other hand, we have Eq. (56) which seems to support the assumption of Padmanabhan and Verlinde, if we identify m with the number of bits. How can that be? The point is that a black hole has no independent constituents or bits. That is why the classical equipartition theorem is not applied to a black hole. The number m in (56) is not the number of bits but the number of ways to distribute a patch of area $8\pi l_P^2$ over the horizon. Thus Padmanabhan and Verlinde are right only in a sense that we can extract the frequency ω_c (rather than the temperature) from (56).

As mentioned above, a description of a black hole via a thermal ensemble is inappropriate. From the point of view of statistics this could be explained as follows. According to the canonical distribution, the probability p_i of a system being in a state of energy E_i is proportional to the Boltzmann factor,

$$p_i \sim \Delta\Gamma_i \exp\left(-\frac{E_i}{T}\right), \quad (77)$$

where T is the temperature of the system, $T = T_{\rm heat\,bath}$. Assuming the usual interpretation of entropy (5), the statistical weight of a black hole should grow with M as

$$\exp(4\pi GM^2). \quad (78)$$

In that case, the total probability diverges. However, in the case where the entropy is given by (7), the statistical weight grows as

$$2GM^2, \quad (79)$$

and the probability can converge. So, it seems that the usual value of the statistical weight (78) better explains the breakdown of the canonical ensemble for black holes than suggested (79). But this is not the case. First, it is clear that an indefinitely large heat bath is gravitationally unstable. On the other hand, there is always a size of bath at which the interaction energy between the members of ensemble is not negligible. In both cases, the canonical distribution is not applicable. Secondly, a black hole possesses a very special property which singles it out, namely, its size and temperature are not independent parameters. As a result, the temperature of a black hole does not remain constant at the different E_i so that $T_H \neq T_{\text{heat bath}}$, contrary to the definition of the canonical ensemble. This is irrespective of the form of the statistical weight. Note that the ordinary self-gravitating systems, for example stars and galaxies, also have negative heat capacity. And although their statistical weights grow not so fast as (78), they cannot be in a thermal equilibrium with a heat bath [27]. Thus the apparent divergence of canonical distribution, in the case where the statistical weight grows as (78), cannot be an evidence in support of the usual interpretation of black hole entropy.

One more case, where the old and new interpretations of black hole entropy give different answers—the Poincaré recurrence time. According to the Poincaré recurrence theorem [28], any state of an isolated finite system continuously returns arbitrarily close to its initial value in a finite amount of time (the Poincaré recurrence time t_r). For an ordinary system this time is exponentially large in the thermodynamical entropy of the system:

$$t_r \sim t_0 \exp(S), \tag{80}$$

where t_0 is the time required for a fluctuation, once it occurs, to again degrade. To apply the theorem to the black holes one has to place a black hole in a finite reservoir with a fixed total energy. We shall assume that all requirements of the Poincaré recurrence theorem are satisfied and a black hole is in equilibrium with its own radiation in an appropriate reservoir. We also assume that the total entropy is dominated by the entropy of a single black hole. In this case the Poincaré recurrence time for a black hole is given by

$$t_r \sim t_0 S_{\text{bh}}. \tag{81}$$

Then assuming that for a black hole t_0 is not smaller than its lifetime, $\sim t_P (M/m_P)^3$, we obtain

$$t_r \sim t_P \left(\frac{M}{m_P}\right)^5. \tag{82}$$

This time is considerably smaller than (80). There is however an obvious explanation for this behavior: due to the long-range attractive character of gravitational forces, matter is very unstable with respect to the clumping.

V. THE NATURE OF THE AREA SCALING $S \propto A$

This section has a more speculative character and is not related with previous one directly. Early in counting the number of degrees of freedom responsible for black hole entropy we assumed that they all reside on the horizon. In this case the area scaling $S \propto A$ is natural. There is however a hypothesis that the degrees of freedom are distributed in a spatial volume. But in this case most of them are not involved in black hole thermodynamics. An ordinary quantum field theory underlying thermodynamics cannot explain this fact. In [29] it is noted that, in a fundamental fermion theory with a cutoff at the Planck scale, the total number of independent quantum states in a given volume V of space is $\Delta\Gamma \sim k^V$, where k is the number of state of a single fermion. Note that if there are any fundamental bosons, the number of possible states should be infinite. So the entropy is proportional to the volume instead of being proportional to the area. An explanation is that most of the states of field theory are not observable since their energy is so large that they are confined inside their own gravitational radius. In this way gravitation reduces the number of physical degrees of freedom so that the number of states grows exponentially with the area instead of the volume. It is then conjectured that all degrees of freedom are resided on the surface of volume. This is called the holographic hypothesis.

We want to suggest an alternative mechanism for the area scaling $S \propto A$. Our proposal is based on the analogy with electrons in metal [30]. As is well known, according to the principle of equipartition of energy, the conduction electrons in a metal viewed as a classical electron gas should make a contribution $3N/2$ (N is the number of electrons) to the heat capacity of the metal. In reality the electronic contribution to the heat capacity at room temperature is only of the order of 1 percent of the classical value. This means that only a small fraction of electrons participates in thermal equilibrium, not the total number of free electrons. The observation is completely unexplained by classical theory, but is in good agreement with quantum statistics. It turns out that the difficulty disappears if it is taken into account that an electron gas possesses the properties of a highly degenerate Fermi gas. We now proceed to explain the area scaling $S \propto A$ basing our considerations on the analogy with electrons in metal. First we assume that at the very fundamental level of matter there exist the fundamental fermions. Then suppose that N fundamental fermions with spin $1/2$ are uniformly distributed in a spatial region of volume $\sim R_g^3$ with spacing $\sim l_P$, so that $N \sim (R_g/l_P)^3$. At $T = 0$ the first $N/2$ states up to the energy $E_{\max}$ will be "completely" filled, with two fermions with opposite spins per state (in accordance with the Pauli principle), while all states with $E > E_{\max}$ will be empty (the limiting energy $E_{\max}$ is generally referred to as the Fermi energy of the system and is denoted by the symbol ε_F). It is obvious that there is one and only one way of achieving this arrangement with indistinguishable particles. Therefore $S = 0$. Remember that according to Dirac's prequantum field theory picture the vacuum consists of states of positive and negative energies, with the negative energy states completely filled and states of positive energy empty. Note that this picture applies only to fermions. In the spirit of Dirac's picture we assume that all of our fundamental fermions are also unobservable (perhaps they have negative energies due to the effects of gravitation). Suppose now that the energy levels are uniformly distributed so that the energy difference between neighboring levels is $\Delta E = 2\varepsilon_F/N$. If one goes from the temperature $T = 0$ to a temperature $T > 0$ slightly above zero, then some of the fermions will be thermally excited from states just below the Fermi energy to states just above the Fermi energy. Then the number of fermions close to the Fermi surface which increase their energy by an amount $\sim k_B T$ is given approximately by

$$\Delta N \sim \frac{k_B T}{\Delta E} \sim N \frac{k_B T}{\varepsilon_F}. \tag{83}$$

Since the temperature of a black hole $T \sim 1/R_g$, we have

$$\Delta N \sim \left(\frac{R_g}{l_P}\right)^3 \frac{1}{\varepsilon_F R_g} \sim (\varepsilon_F^{-1} l_P^{-3}) R_g^2 \propto A. \tag{84}$$

This means that the volume remains uninfluenced by the rise in temperature. That is why the area, not volume, is relevant in black hole thermodynamics. Since each of the excited fermions receives an additional energy of $\sim k_B T$, the internal energy of a black hole will be

$$E \sim k_B T \times \Delta N \sim (\varepsilon_F^{-1} l_P^{-3}) R_g. \tag{85}$$

This is just the black hole mass if we identify $G^{-1} \sim \varepsilon_F^{-1} l_P^{-3}$ or $\varepsilon_F \sim m_P$. So the heat capacity C_V is given by

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V \sim -(\varepsilon_F^{-1} l_P^{-3}) R_g^2. \tag{86}$$

If we now deduce the entropy from the heat capacity, $C_V = T(\partial S/\partial T)$, we get

$$S = |C_V| \sim \Delta N \propto A, \tag{87}$$

which is what had to be proved.

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