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Possibility of hyperbolic tunneling

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Traversable wormholes are primarily useful as ''gedanken experiments'' and as a theoretician's probe of the foundations of general relativity. In this work, we analyze the possibility of having tunnels in a hyperbolic spacetime. We obtain exact solutions of static and *pseudo*-spherically symmetric spacetime tunnels by adding exotic matter to a vacuum solution referred to as a degenerate solution of class A. The physical properties and characteristics of these intriguing solutions are explored, and through the mathematics of embedding it is shown that particular constraints are placed on the shape function, that differ significantly from the Morris-Thorne wormhole. In particular, it is shown that the energy density is always negative, and the radial pressure is positive, at the throat, contrary to the Morris-Thorne counterpart. Specific solutions are also presented by considering several equations of state, and by imposing restricted choices for the shape function or the redshift function.

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I. INTRODUCTION

Wormholes are hypothetical tunnels in spacetime, possibly through which observers may freely traverse. However, it is important to emphasize that these solutions are primarily useful as ''gedanken experiments'' and as a theoretician's probe of the foundations of general relativity (GR). In classical general relativity, wormholes are supported by exotic matter, which involves a stress-energy tensor that violates the null energy condition (NEC) [1,2]. Note that the NEC is given by $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, where k^{μ} is *any* null vector. Thus, it is an important and intriguing challenge in wormhole physics to find a realistic matter source that will support these exotic spacetimes. Several candidates have been proposed in the literature, among which we refer to solutions in higher dimensions, for instance, in Einstein-Gauss-Bonnet theory [3,4], wormholes on the brane [5,6]; solutions in Brans-Dicke theory [7]; wormhole solutions in semiclassical gravity (see Ref. [8] and references therein); exact wormhole solutions using a more systematic geometric approach were found [9]; geometries supported by equations of state responsible for the cosmic acceleration [10–12], and solutions in conformal Weyl gravity were also found [13], etc. (see Refs. [14,15] for more details and [15] for a recent review).

In this work, instead of further exploring wormholes in some extension of GR, we instead analyze the possibility of having tunneling solutions in general relativistic solutions with negatively curved spatial surfaces which might be considered the analogue of wormholes. Indeed, we consider a largely ignored metric which belongs to a class of vacuum solutions referred as degenerate solutions of class A [16] by Ehlers and Kundt [17,18], which are axisymmetric solutions and thus also belong to Weyl's class [19], given by

$$ds^2 = -e^{\mu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(du^2 + \sinh^2 u\, dv^2), \quad (1)$$

where the usual $2-d$ spheres are replaced by pseudospheres, $d\sigma^2 = du^2 + \sinh^2 u\, dv^2$, hence by surfaces of negative, constant curvature. These are still surfaces of revolution around an axis, and v represents the corresponding rotation angle. The specific case of

$$e^{\mu(r)} = e^{-\lambda(r)} = \left(\frac{2\mu}{r} - 1\right) \quad (2)$$

is of particular interest, where μ is a constant [16,20,21].

In our opinion this metric can be seen as an anti-Schwarzschild in the same way the de Sitter model with negative curvature is an anti-de Sitter model. We immediately see that the static solution holds for $r < 2\mu$ and that there is a coordinate singularity at $r = 2\mu$ (note that $|g|$ neither vanishes nor becomes ∞ at $r = 2\mu$) [22]. This is the complementary domain of the exterior Schwarzschild solution in the region outside the Schwarzschild horizon. In the domain $r > 2\mu$, as with the latter solution, the g_{tt} and g_{rr} metric coefficients swap signs. Defining $\tau = r$ and $\rho = t$, we obtain $ds^2 = -d\tilde{\tau}^2 + A^2(\tilde{\tau})d\rho^2 + B^2(\tilde{\tau})(du^2 + \sinh^2 u\, dv^2)$, with the following parametric definitions $\tilde{\tau} = -\tau + 2\mu \ln|\tau - 2\mu|$, $A^2 = 2\mu/\tau - 1$ and $B^2(\tau) = \tau^2$, which is a particular case of a Bianchi III axisymmetric universe.

Using pseudospherical coordinates $\{x = r\sinh u \cos v, y = r \sinh u \sin v, z = r\cosh u, w = b(r)\}$, the spatial part of the metric (1) can be related to the hyperboloid $w^2 + x^2 + y^2 - z^2 = (b^2/r^2 - 1)r^2$ embedded in a four-dimensional flat space. We then have

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$$dw^2 + dx^2 + dy^2 - dz^2 = [(b'(r))^2 - 1]dr^2 + r^2(du^2 + \sinh^2 u dv^2), \quad (3)$$

where the prime stands for differentiation with respect to r, and $b(r) = \mp 2\sqrt{2\mu}\sqrt{2\mu - r}$. We can recast metric (1) into the following:

$$ds^2 = -\tan^2[\ln(\bar{r})^{\mp 1}]d\tau^2 + \left(\frac{2\mu}{\bar{r}}\right)^2 \cos^4[\ln(\bar{r})^{\mp 1}] \times [d\bar{r}^2 + \bar{r}^2(du^2 + \sinh^2 u dv^2)], \quad (4)$$

which is the analogue of the isotropic form of the Schwarzschild solution. In the neighborhood of $u = 0$, i.e., for $u \ll 1$, we can cast the metric of the two-dimensional hyperbolic solid angle as

$$d\sigma^2 \simeq du^2 + u^2 dv^2 \quad (5)$$

so that it confounds itself with the tangent space to the spherically symmetric S^2 surfaces in the neighborhood of the poles. The apparent arbitrariness of the locus $u = 0$, is overcome simply by transforming it to another location by means of a hyperbolic rotation, as in the case of the spherically symmetries case where the poles are defined up to a spherical rotation [SO(3) group]. Thus, the spatial surfaces are conformally flat. However, we cannot recover the usual Newtonian weak-field limit for large r, because of the change of signature that takes place at $r = 2\mu$.

Analyzing the "radial" motion of test particles, we have the following equation:

$$\dot{r}^2 + \left(\frac{2\mu}{r} - 1\right)\left(1 + \frac{h^2}{r^2 \sinh^2 u_*}\right) = \epsilon, \quad (6)$$

where ϵ and h are constants of motion defined by $\epsilon = (2\mu/r - 1)\dot{t} = \text{const}_t$ and $h^2 = r^2 \sinh^2 u_* \dot{v} = \text{const}_v$, for fixed $u = u_*$. The former and latter constants represent the energy and angular momentum per unit mass, respectively. We thus define the potential $2V(r) = (\frac{2\mu}{r} - 1) \times (1 + \frac{h^2}{r^2 \sinh^2 u_*})$. This potential is manifestly repulsive, crosses the r axis at $r = 2\mu$, and for sufficiently high values of h it has a minimum at $r_\pm = (h^2 \mp \sqrt{h^4 - 12\mu^2 h^2})/(2\mu)$. However, this minimum falls outside the $r = 2\mu$ divide. So a test particle is subject to a repulsive potential forcing it to inevitably cross the event horizon at $r = 2\mu$ attracted either by some mass at the minimum or by masses at infinity. In [20] it is hinted that the nonexistence of a clear Newtonian analogue is related to the existence of mass sources at ∞, but no definite conclusions were drawn [23].

A more detailed analysis of the physical properties and characteristics of this intriguing solution is presently underway [24], as well as an application to the cosmological features of the model, namely, the study of negatively curved spacetimes in order to understand the ultimate stages of underdensities [25].

Here we shall study the extension of the solution (2) which arises from adding exotic matter to analyze the possibility of tunneling in hyperbolic spacetimes. These would add to other nonspherically symmetric wormholes which have already been considered in the literature. For instance, extending the spherically symmetric Morris-Thorne wormholes [1] and motivated by the aim of minimizing the violation of the energy conditions, polyhedral solutions and, in particular, cubic wormholes were constructed in Ref. [26]. In Ref. [27], the static spherically symmetric traversable wormhole solution was generalized to that of a (nonplanar) toruslike topology [27], denoted as a ringhole. In Ref. [28], solutions of plane symmetric wormholes in the presence of a negative cosmological constant by matching an interior spacetime to the exterior anti-de Sitter vacuum solution were constructed. It is interesting to note that the construction of these plane symmetric wormholes does not alter the topology of the background spacetime (i.e., spacetime is not multiply connected), so that these solutions can instead be considered domain walls. The dynamic stability analysis of plane symmetric wormholes was further analyzed in Ref. [29].

Thus, it is the purpose of this paper to study static and *pseudospherically* symmetric counterparts to the usual wormholes by adding exotic matter to the vacuum degenerate solution of class A, given by (1). The physical properties and characteristics of these intriguing solutions are explored, and through the mathematics of embedding it is shown that particular constraints are placed on the shape function, that differ radically from the Morris-Thorne wormhole. In particular, it is shown that the energy density is always negative and the radial pressure is positive, at the throat, contrary to its Morris-Thorne counterpart. Specific solutions are also presented by considering various equations of state, and by imposing restricted choices for the shape function or the redshift function.

This paper is organized in the following manner: In Sec. II, the spacetime metric, the field equations and the mathematics of embedding of these pseudospherically symmetric geometries are analyzed in detail. In Sec. III, specific solutions are found by considering several equations of state. We conclude in Sec. IV.

II. TRAVERSABLE TUNNELS IN PSEUDOSPHERICAL SYMMETRY

A. Spacetime metric and field equations

Consider the following pseudospherically symmetric and static solution:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{b(r)/r - 1} + r^2(du^2 + \sinh^2 u dv^2), \quad (7)$$

where the coordinates u and v have the following range: $-\infty < u < +\infty$ and $0 \leq v \leq 2\pi$. The r coordinate yields the curvature radius of the two-dimensional pseudospherical surfaces that thread the spacetime: $(2)R = -r^{-2}$, and, hence, is a generalized radial coordinate. $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate r. As in the Morris-Thorne wormhole [1], we denote $\Phi(r)$ the redshift function, for it is related to the gravitational redshift, and $b(r)$ the shape function, as will be shown below by embedding diagrams, determines the shape of the tunnel (in analogy to the analysis considered in [1]). The coordinate r is nonmonotonic in that it decreases from a constant value C to a minimum value r_0, representing the location of the throat of the wormhole, where $b(r_0) = r_0$, and then it increases from r_0 back to the value C. Note that the condition $(b/r - 1) \geq 0$ imposes that $b(r) \geq r$, contrary to its Morris-Thorne counterpart.

The Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, provides the following stress-energy scenario:

$$\rho(r) = -\frac{1}{8\pi}\frac{b'}{r^2}, \tag{8}$$

$$p_r(r) = \frac{1}{8\pi}\left[\frac{b}{r^3} + 2\left(\frac{b}{r} - 1\right)\frac{\Phi'}{r}\right], \tag{9}$$

$$p_t(r) = \frac{1}{8\pi}\left(\frac{b}{r} - 1\right)\left[\Phi'' + (\Phi')^2 + \frac{b'r + b - r}{2r(b - r)}\Phi' + \frac{b'r - b}{2r^2(b - r)}\right], \tag{10}$$

in which $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, $p_t(r)$ is the pressure measured in the tangential directions, orthogonal to the radial direction.

Assuming that the redshift function is finite $\forall r$, note that the radial pressure is always positive at the throat, i.e., $p_r = 1/(8\pi r_0^2)$, contrary to the Morris-Thorne wormhole, where a radial tension at the throat is needed to sustain the wormhole. In addition to this, we also show below that $b'(r_0) > 1$ at the throat, which implies a negative energy density at the throat. This condition is another significant difference to the Morris-Thorne wormhole, where the existence of negative energy densities at the throat is not a necessary condition.

By taking the derivative with respect to the radial coordinate r, of Eq. (9), and eliminating b' and Φ'', given in Eq. (8) and (10), respectively, we obtain the following equation:

$$p_r' = -(\rho + p_r)\Phi' + \frac{2}{r}(p_t - p_r). \tag{11}$$

Equation (11) is the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the hyperbolic spacetime tunnel.

We now have a system of three equations, namely, Eqs. (8)–(10), with five unknown functions of r, i.e., the stress-energy components, $\rho(r)$, $p_r(r)$ and $p_t(r)$, and the metric fields, $b(r)$ and $\Phi(r)$. To construct specific solutions, we may adopt several approaches, and in this work we shall mainly use the strategy of considering a specific equation of state given by $p_r = p_r(\rho)$, and restricted choices for $b(r)$ or $\Phi(r)$. One may also impose a specific form for the stress-energy components and through the field equations and the equation of state, $p_r = p_r(\rho)$, determine $b(r)$ and $\Phi(r)$. We will show below, through the embedding analysis, that $b'(r_0) > 1$, so that throughout this paper we only consider the cases of a negative energy density at the throat, $\rho|_{r_0} < 0$.

B. Mathematics of embedding

The embedding diagrams are useful to represent the geometry of the tunnelling solution and extract some useful information for the choice of the shape function, $b(r)$ [30]. Because of the pseudospherically symmetric nature of the problem, without a significant loss of generality, consider a slice with the specific value of $u = u_0$ which imposes $\sinh(u_0) = 1$. The respective line element, considering a fixed moment of time, $t = \text{const}$, is given by

$$ds^2 = \frac{dr^2}{b(r)/r - 1} + r^2 dv^2. \tag{12}$$

To visualize this slice, one embeds this metric into three-dimensional Euclidean space [30], in which the metric can be written in cylindrical coordinates, (r, v, z), as

$$ds^2 = dz^2 + dr^2 + r^2 dv^2. \tag{13}$$

Now, in the three-dimensional Euclidean space the embedded surface has equation $z = z(r)$, and thus the metric of the surface can be written as

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right]dr^2 + r^2 dv^2. \tag{14}$$

Comparing Eq. (14) with (12), we have the equation for the embedding surface, given by

$$\frac{dz}{dr} = \pm\left[\frac{2r - b(r)}{b(r) - r}\right]^{1/2}. \tag{15}$$

To be a solution of a spacetime tunnel, the geometry has a minimum radius, $r = b(r) = r_0$, denoted as the throat, at which the embedded surface is vertical, i.e., $dz/dr \to \infty$, see Fig. 1. Note also that contrary to the Morris-Thorne traversable wormhole, the shape function is constrained in the present case. More specifically, taking into account that $b(r) \geq r$, then the embedding surface (15) also imposes the condition $b(r) \leq 2r$. Thus, the shape function is restricted to lie within the following range: $r \leq b(r) \leq 2r$.

In addition to this, to be a solution of a tunnel in spacetime, one needs to impose that the throat flares out, as in Fig. 1. Mathematically, this flaring-out condition entails that the inverse of the embedding function $r(z)$, must satisfy $d^2r/dz^2 > 0$ at or near the throat r_0. Differentiating $dr/dz = \pm(r/b(r) - 1)^{1/2}$ with respect to

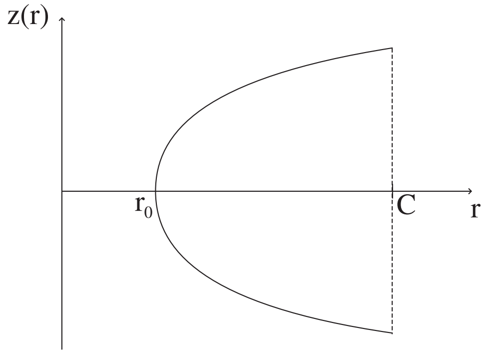

FIG. 1. The embedding diagram of a two-dimensional section along the slice ($t = \text{const}$, $u = u_0 = \sinh^{-1}(1)$) of a tunnel in spacetime.

z, we have

$$\frac{d^2 r}{dz^2} = \frac{b'r - b}{2(2r - b)^2} > 0. \tag{16}$$

At the throat we verify that the shape function satisfies the condition $b'(r_0) > 1$, also contrary to its Morris-Thorne counterpart. This condition plays a fundamental role in the analysis of the violation of the energy conditions.

In particular, considering the NEC, i.e., $p_r + \rho \geq 0$, the term $p_r + \rho$, taking into account the field Eqs. (8) and (9), is given by

$$\rho + p_r = \frac{1}{8\pi}\left[\frac{b - rb'}{r^3} + 2\left(\frac{b}{r} - 1\right)\frac{\Phi'}{r}\right]. \tag{17}$$

At the throat, we have

$$(\rho + p_r)|_{r_0} = \frac{1 - b'(r_0)}{8\pi r_0^2} < 0, \tag{18}$$

which implies the important condition $b'(r_0) > 1$ at the throat, contrary to its Morris-Thorne counterpart. As mentioned above, this implies that the energy density, at the throat, is always negative for these exotic hyperbolic, tunneling geometries.

III. SPECIFIC SOLUTIONS

In this Section, we find exact solutions by considering several appropriate equations of state, $p_r = p_r(\rho)$, and by imposing specific shape functions or specific redshift functions. To this effect, an adequate shape function is the following case:

$$b(r) = r_0\left(\frac{r}{r_0}\right)^{\delta}. \tag{19}$$

Taking the radial derivative, we have

$$b'(r) = \delta\left(\frac{r}{r_0}\right)^{\delta - 1}, \tag{20}$$

which at the throat implies $b'(r_0) = \delta > 1$. From the condition $b(r) < 2r$, one has $r < 2^{1/(\delta-1)} r_0$. Thus, we have the following conditions:

$$\text{If } \delta \to \infty, \quad \text{then } r \to r_0 \tag{21}$$

$$\text{If } \delta \to 1, \quad \text{then } r \to \infty. \tag{22}$$

We verify that one may have an arbitrary large tunnel by imposing the condition $\delta \to 1$.

A. Linear equation of state

1. Specific shape function

Consider the linear equation of state $p_r(r) = \omega\rho(r)$, in which wormhole solutions were extensively analyzed in Ref. [10], and consequently denoted as the "phantom wormhole." Now, using the above-mentioned equation of state and taking into account Eqs. (8) and (9), one deduces the following expression:

$$\Phi'(r) = -\frac{b(r) + \omega b'(r) r}{2r[b(r) - r]}, \tag{23}$$

which is the relationship governing what may be considered the hyperbolic analogue of the "phantom wormhole." Note that the NEC violation, and taking into account $\rho|_{r_0} < 0$, imposes that $\omega > -1$, contrary to its spherically symmetric counterpart.

By imposing a specific redshift function, one finds the general solution for the shape function

$$b(r) = r^{-(1/\omega)} e^{-(2/\omega)\Phi}\left[\frac{2}{\omega}\int \Phi' r^{1+(1/\omega)} e^{(2/\omega)\Phi} dr + r_0^{1+1/\omega} e^{(2/\omega)\Phi(r_0)}\right]. \tag{24}$$

However, one also deduces an adequate solution by imposing the shape function given by Eq. (19). Using relationship (23), the redshift function is provided by

$$\Phi(r) = -\frac{1 + \delta\omega}{2(\alpha - 1)} \ln\left[\left(\frac{r}{r_0}\right)^{\delta - 1} - 1\right] + C_1, \tag{25}$$

where C_1 is an integration constant. Now, to avoid an event horizon at r_0, one needs to impose that $\omega = -1/\delta$, so that the redshift function simplifies to $\Phi(r) = C_1$.

One may also further restrict the parameters by considering certain traversability conditions in analogy to the Morris-Thorne wormhole [1]. In particular, it is important that an observer traversing through the hyperbolic tunnel should not be ripped apart by enormous tidal forces. Thus, the tidal traversability condition requires that the tidal accelerations felt by the traveller should not exceed, for instance, the Earth's gravitational acceleration, $g_\oplus$, which is translated by the following inequalities:

$$2\left|\left(\frac{b}{r} - 1\right)\left[\Phi'' + (\Phi')^2 + \frac{b'r - b}{2r(r - b)}\Phi'\right]\right| \leq g_\oplus, \tag{26}$$

$$\left|\frac{\gamma^2}{r^2}\left[v^2\left(\frac{b}{r}-b'\right)+2(b-r)\Phi'\right]\right| \le g_\oplus. \tag{27}$$

We refer the reader to [1] for details in deducing these relationships. The radial tidal constraint, Eq. (26), constrains the redshift function, and the lateral tidal constraint, Eq. (27), constrains the velocity with which observers traverse the tunnel. These inequalities are particularly simple at the throat, r_0,

$$|\Phi'(r_0)| \le \frac{g_\oplus r_0}{(b'-1)}, \tag{28}$$

$$\gamma^2 v^2 \le \frac{g_\oplus r_0^2}{(b'-1)}. \tag{29}$$

In particular, considering nonrelativistic velocities, i.e., $v \ll 1$ and $\gamma \sim 1$, using the shape function given by (19) and $\Phi(r) = C_1$, one readily verifies that the radial tidal constraint is satisfied. The lateral tidal constraint provides the condition $v \le g_\oplus r_0^2/(\delta - 1)$, which essentially depends on the values taken for the parameter δ. Alternatively, by specifying the traversal velocity, we find a restriction placed on δ given by $\delta \le 1 + g_\oplus r_0^2/v$.

2. *Specific redshift function*

It is also adequate to find solutions by considering a specific redshift function. Consider the redshift function given by $\Phi(r) = r_0/r$. Using expression (23), one deduces the shape function given by

$$b(r) = \frac{4r_0^2}{\omega}\left(-\frac{2r_0}{\omega r}\right)^{1-\omega/\omega} e^{-(2r_0/\omega r)}\mathcal{F}\left(\frac{\omega-1}{\omega}, -\frac{2r_0}{\omega r}\right) - 2r_0 + e^{-(2r_0/\omega r)} r^{-(1/\omega)} C_2, \tag{30}$$

where C_2 is an integration constant and, for notational simplicity, the function $\mathcal{F}$ is defined as

$$\mathcal{F}\left(\frac{\omega-1}{\omega}, -\frac{2r_0}{\omega r}\right) = \Gamma\left(\frac{\omega-1}{\omega}\right) - \Gamma\left(\frac{\omega-1}{\omega}, -\frac{2r_0}{\omega r}\right), \tag{31}$$

and $\Gamma(x)$ and $\Gamma(x, z)$ are the Gamma and the incomplete Gamma functions, respectively. Note that the constant of integration C_2 may be found by taking into account the condition $b(r_0) = r_0$, and is given by the following relationship:

$$C_2 = r_0^{1+(1/\omega)}\left[3e^{2/\omega} + 2\left(-\frac{2}{\omega}\right)^{1/\omega}\mathcal{F}\left(\frac{\omega-1}{\omega}, -\frac{2r_0}{\omega r}\right)\right]. \tag{32}$$

B. "Generalized Chaplygin gas" equation of state

In cosmology, the equation of state representing the generalized Chaplygin gas (GCG) is given by $p_{\rm ch} = -A/\rho_{\rm ch}^{\alpha}$, where A and α are positive constants, and the latter lies in the range $0 < \alpha \le 1$ [31,32]. The particular case of $\alpha = 1$ corresponds to the Chaplygin gas. An attractive feature of this model is that at early times the energy density behaves as matter, $\rho_{\rm ch} \sim a^{-3}$, where a is the scale factor, and as a cosmological constant at a later stage, i.e., $\rho_{\rm ch} = {\rm const}$. In a cosmological context, at a late stage dominated by an accelerated expansion of the Universe, the cosmological constant may be given by $8\pi A^{1/(1+\alpha)}$. This dual behavior is responsible for the interpretation that the GCG model is a candidate of a unified model of dark matter and dark energy [33].

It was noted in Ref. [34] that the GCG equation of state is that of a polytropic gas with a negative polytropic index, and thus suggested that one could analyze astrophysical implications of the model. In this context, the construction of traversable wormholes, possibly arising from a density fluctuation in the GCG cosmological background was explored in [11]. These latter solutions were denoted Chaplygin wormholes. As in Ref. [11], it was considered that the pressure in the GCG equation of state is a radial pressure, and the tangential pressure can be determined from the Einstein equations, in particular, Eq. (10).

To compare to the spherically symmetric Chaplygin wormholes, we construct their pseudospherically symmetric duplicates. Thus, taking into account the GCG equation of state in the form $p_r = -A/\rho^{\alpha}$, and using Eqs. (8) and (9), we have the following condition:

$$\Phi'(r) = \left[A(8\pi)^{1+\alpha}\frac{r^{2\alpha+1}}{2(b')^{\alpha}} - \frac{b}{2r^2}\right]\Big/\left(\frac{b}{r}-1\right). \tag{33}$$

Solutions of the metric (7), satisfying Eq. (33) may be considered as the analogues of "Chaplygin wormholes" analyzed in [11].

As shown above, to be a spacetime tunnel, the condition $b'(r_0) > 1$ is imposed. Now, using the GCG equation of state, evaluated at the throat, and taking into account Eq. (9), we verify that the energy density at r_0 is given by $\rho(r_0) = -(8\pi r_0^2 A)^{1/\alpha}$. Finally, using Eq. (8), and the condition $b'(r_0) > 1$, we verify that for these solutions, the following condition is imposed $A > (8\pi r_0^2)^{-(1+\alpha)}$, contrary to its spherically symmetric counterpart [11].

We consider next a specific example, by taking into account the shape function given by Eq. (19). Thus, the differential Eq. (33) provides the following redshift function:

$$\begin{aligned}\Phi(r) = \frac{1}{2}\Bigg\{&\left(\frac{r}{r_0}\right)^{\delta} r_0\left[\sum_{k=0}^{-1}\left(\frac{1}{1+k}\right)\left(\frac{r}{r_0}\right)^{k(\delta-1)}\right] \\ &- A(8\pi)^{1+\alpha}\left[\delta\left(\frac{r}{r_0}\right)^{\delta-1}\right] r^{2\alpha+3} \\ &\times \text{Lerch Phi}\left[\left(\frac{r}{r_0}\right)^{\delta-1}, 1, \frac{3\alpha+2-\alpha\delta}{\delta-1}\right] \\ &- r\ln\left[\left(\frac{r}{r_0}\right)^{\delta-1} - 1\right]\Bigg\}\Big/[(\delta-1)r] + C_3,\end{aligned} \tag{34}$$

where C_3 is a constant of integration; and Lerch Phi is the general Lerch Phi function, defined as $\mathrm{Lerch\,Phi}(z, a, v) = \sum_{n=0}^{\infty} z^n/(v+n)^a$.

It is interesting to consider the specific case of $\delta = 2$ and $\alpha = 1$ (this latter value corresponds to the Chaplygin gas equation of state), which yields the following simpler solution:

$$\Phi(r) = 8\pi^2 A r_0^3 r\left(2 + \frac{r}{r_0}\right) + \frac{1}{2}(8\pi A r_0^2 - 1)\ln\left(\frac{r}{r_0} - 1\right) + C_3, \tag{35}$$

Note the existence of an event horizon at the throat, rendering the hyperbolic tunnel nontraversable. Thus, to avoid this we impose the condition $A = 1/(8\pi r_0^2)$, so that the above solution simplifies to

$$\Phi(r) = 8\pi^2 A r_0^3 r\left(2 + \frac{r}{r_0}\right) + C_3, \tag{36}$$

yielding a traversable geometry.

C. Van der Waals equation of state

Another case that lends itself to our analysis, is that of the van der Waals (VDW) quintessence equation of state, which seems to provide a solution to the puzzle of dark energy, without the presence of exotic fluids or modifications of the Friedmann equations. In Ref. [12], the construction of inhomogeneous compact spheres supported by a VDW equation of state was explored. These relativistic stellar configurations were denoted as *van der Waals quintessence stars*. Despite the fact that, in a cosmological context, the VDW fluid is considered homogeneous, inhomogeneities may arise through gravitational instabilities. Thus, these solutions may possibly originate from density fluctuations in the cosmological background. Exact solutions were found, and their respective characteristics and physical properties were further explored in Ref. [12].

The van der Waals equation of state is given by

$$p = \frac{\gamma\rho}{1 - \beta\rho} - \alpha\rho^2, \tag{37}$$

where ρ is the energy density and p the pressure of the VDW fluid. The accelerated and decelerated periods depend on the parameters, α, β and γ of the equation of state, and in the limiting case $\alpha, \beta \to 0$, one recovers the dark energy equation of state, with $\gamma = p/\rho < -1/3$ [35].

As in the previous exact solutions, we may also consider the counterpart of the static and spherically symmetric van der Waals wormhole. Thus, taking into account the specific shape function given by Eq. (19), with the parameter $\delta = 2$, and considering that p is a radial pressure (see [12] for details), we have the following solution:

$$\Phi(r) = -\frac{1}{4\pi r_0^2 \bar{\beta}}\left[(\bar{\alpha}\,\bar{\beta} + 16\pi^2 r_0^4 \gamma)\ln\left(1 - \frac{r_0}{r}\right) + 4\pi r_0^2 \beta\gamma \ln\left(1 + \frac{4\pi r_0 r}{\beta}\right)\right] + C_4, \tag{38}$$

where C_4 is a constant of integration, and we have considered the following definitions for notational simplicity:

$$\bar{\alpha} = \alpha + 2\pi r_0^2, \qquad \bar{\beta} = \beta + 4\pi r_0^2. \tag{39}$$

Note the presence of an event horizon, due to the term $\ln(1 - r_0/r)$. Thus, to avoid this, we impose the following condition:

$$\gamma = -\frac{\bar{\alpha}\,\bar{\beta}}{16\pi^2 r_0^4} = -\frac{(\alpha + 2\pi r_0^2)(\beta + 4\pi r_0^2)}{16\pi^2 r_0^4}, \tag{40}$$

so the redshift function reduces to

$$\Phi(r) = \frac{(\alpha + 2\pi r_0^2)\beta}{16\pi^2 r_0^4}\ln\left(1 + \frac{4\pi r_0 r}{\beta}\right) + C_4. \tag{41}$$

Using the simplified relationship (41), one may find a restriction from the lateral tidal acceleration constraint (29), provided by

$$|\bar{\alpha}| \leq 4\pi r_0^4 g_{\oplus} |\bar{\beta}/\beta|. \tag{42}$$

IV. SUMMARY, DISCUSSION AND FUTURE OUTLOOK

Traversable wormholes possess a peculiar property, namely, "exotic matter," involving a stress-energy tensor that violates the null energy condition. In this work, we have constructed exact solutions of static and pseudospherically symmetric spacetime tunnels by adding exotic matter to a vacuum solution referred to as a degenerate solution of class A. The usual two-dimensional spheres are replaced by pseudospheres, which are still surfaces of revolution around an axis, but now consist of a negative and constant curvature. The physical properties and characteristics of these intriguing, hyperbolic solutions were further explored, and through the mathematics of embedding it was shown that particular constraints are placed on the shape function, that differ radically from the Morris-Thorne wormhole. In particular, it was shown that the energy density is always negative and the radial pressure is positive, at the throat, contrary to the Morris-Thorne counterpart. Specific solutions were also presented by considering several different equations of state, and by imposing restricted choices for the shape function or the redshift function.

Standard definitions of a wormhole usually require either asymptotic flatness with a localized bridge to another asymptotically flat region, or the ability to identify a compact region that can usefully be thought of as a throat. The geometries considered in the current article also exhibit asymptotic flatness and share the possibility of having localized tunnels, with throats, bridging different spacetime regions. However, the spatial hypersurfaces of the asymptotic flat limit of the hyperbolic spacetime under consideration here is peculiar in that it is not Euclidean, but rather pseudo-Euclidean (it is a three-dimensional Lorentzian space). Hence, rather than denote the exotic geometric objects analyzed in this work as wormholes, we prefer the term hyperbolic tunnels. They should indeed be understood as a theoretician's probe of the foundations of GR, on the same grounds as in the case of the Morris-Thorne wormhole.

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