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Toroidal spiral Nambu-Goto strings around higher-dimensional black holes

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We present solutions of the Nambu-Goto equation for test strings in a shape of toroidal spiral in five-dimensional spacetimes. In particular, we show that stationary toroidal spirals exist around the five-dimensional Myers-Perry black holes. We also show the existence of innermost stationary toroidal spirals around the five-dimensional black holes like geodesic particles orbiting around four-dimensional black holes.

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I. INTRODUCTION

A lot of attention has been paid to higher-dimensional spacetimes, which are inspired by unified theories. In particular, many studies are devoted to the understanding of physical properties of higher-dimensional black holes. It has been clarified that the black holes in higher dimensions have rich geometrical structures that have no analogue in four dimensions (see [1] for a review).

The motion of a test particle is one of useful probes of geometry around the black holes. It is remarkable that a higher-dimensional generalization of rotating black hole, the Myers-Perry metric [2], allows separation of variables in the geodesic Hamilton-Jacobi equation [3,4]. In addition, it was shown that the Myers-Perry black hole also admits separation of variables in the Nambu-Goto equation for stationary strings along the Killing time [5–7].

In this article, we study a Nambu-Goto test string that has a geometrical symmetry in the target space described by the Myers-Perry metric in five dimensions. The metric admits an isometry group containing two commutable rotations. We assume a Killing vector, which generates a combination of the two rotations, is tangent to a world sheet of a string. Then, on a time slice, the string has a spiral shape along a circle. We call it a "*toroidal spiral string*." The strings in the same configuration were considered in flat spacetime in Ref. [8].

Generally, if a Killing vector of a target space is tangent to a world sheet of a string, the string is called a cohomogeneity-one string [9,10]. Both the stationary strings, which are associated with a timelike Killing vector, and the toroidal spiral strings, which are defined above, are members of the cohomogeneity-one strings. The Nambu-Goto action for a cohomogeneity-one string associated with a Killing vector, say ξ, is reduced to the geodesic action [9,10]

$$S = \int \sqrt{(\xi \cdot \xi) h_{MN} dx^M dx^N} \tag{1}$$

in the orbit space of ξ. Here, h_{MN} is the projection metric with respect to ξ, which is defined by

$$h_{MN} = g_{MN} - \frac{\xi_M \xi_N}{\xi \cdot \xi}, \tag{2}$$

where g_{MN} is a metric of a target space which admits the Killing vector ξ, and $\xi \cdot \xi$ is the norm of ξ. There are a lot of works on the cohomogeneity-one strings in a variety of contexts [9–13]. We consider here the cohomogeneity-one string as a probe of the geometry of higher-dimensional black holes.

A gravitational field around a compact object allows stable circular orbits of particles in four dimensions. This is easily understood by a balance of the gravitational force and the centrifugal force. In general relativity, strong gravity of a black hole forbids the existence of stable circular orbit inside a critical radius, the so-called *innermost stable circular orbit*. In contrast, there is no stable circular orbit of the geodesic particle at all around a higher-dimensional asymptotically flat black hole [4].[1] This is because the gravitational force cannot be in balance with the centrifugal force in the higher dimensions.

We investigate the system of toroidal spiral strings in the Myers-Perry black holes as a potential problem. We show that there exist stationary toroidal spiral strings, the world sheets are tangent to a timelike Killing vector, around the black holes which are achieved by the balance of centrifugal force and string tension in five-dimensional cases. The radii of toroidal spiral strings can oscillate around the stationary solutions. Furthermore, we show the appearance of innermost stationary toroidal spiral strings around five-dimensional black holes by the effect of gravity.

II. TOROIDAL SPIRAL STRINGS IN THE FIVE-DIMENSIONAL FLAT SPACETIME

For understanding the toroidal spiral strings, it is helpful that we see the toroidal spiral string solutions in flat spacetime before considering black hole spacetimes. The solu-

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[1]There exist stable circular orbits of particles around five-dimensional squashed Kaluza-Klein black holes [14].

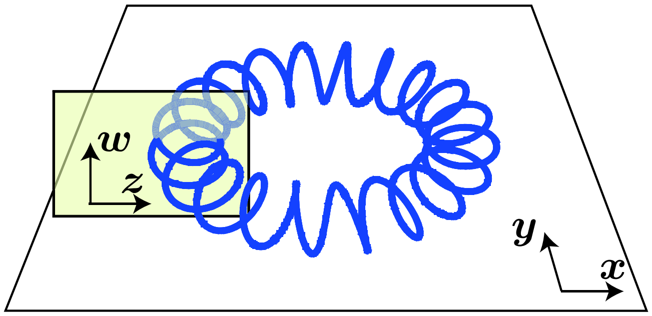

FIG. 1 (color online). Snap shot of a toroidal spiral string. The string coils around a torus embedded in the four-dimensional Euclid space $(x, y, z, w) = (\rho\cos\phi, \rho\sin\phi, \zeta\cos\psi, \zeta\sin\psi)$.

tions for the Nambu-Goto string equation are presented in [8] by the Cartesian coordinates and the conformal gauge in the form

$$t = \tilde{\tau}, \qquad x = \rho_0 \cos\frac{\tilde{\sigma} - \tilde{\tau}}{2\rho_0}, \qquad y = \rho_0 \sin\frac{\tilde{\sigma} - \tilde{\tau}}{2\rho_0}, \qquad z = \zeta_0 \cos\frac{\tilde{\sigma} + \tilde{\tau}}{2\zeta_0}, \qquad w = \zeta_0 \sin\frac{\tilde{\sigma} + \tilde{\tau}}{2\zeta_0}, \tag{3}$$

where ρ_0, and ζ_0 are constants, and $\tilde{\tau}$ and $\tilde{\sigma}$ are parameters on the string world sheet. A typical shape of the string described by (3) is expressed in Fig. 1. It is easy to see that a Killing vector of the flat target space is tangent to the world sheet of the string. For convenience of treating the strings in black hole spacetimes, we reproduce the toroidal spiral string solutions in terms of the Killing vectors of the target space as discussed in Sec. I.

Let us consider a cohomogeneity-one string in the five-dimensional Minkowski spacetime of the metric

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + d\zeta^2 + \zeta^2 d\psi^2. \tag{4}$$

We concentrate on a toroidal spiral string, namely, we assume that the Killing vector of the metric

$$\xi = \partial_\phi + \alpha\partial_\psi \tag{5}$$

is tangent to the world sheet of the string, where α is an arbitrary constant. We use the parameter σ on the world sheet such that the Killing vector is written as $\xi = \partial_\sigma$ on the world sheet.

The dynamical system of the toroidal spiral strings is reduced to the system of a free particle in the metric

$$\begin{aligned} ds_4^2 &= (\xi \cdot \xi) h_{\mu\nu} dx^\mu dx^\nu \\ &= (\rho^2 + \alpha^2\zeta^2)(-dt^2 + d\rho^2 + d\zeta^2) + \rho^2\zeta^2 d\bar{\varphi}^2, \end{aligned} \tag{6}$$

where we have used a new variable $\bar{\varphi} := \psi - \alpha\phi$. The action (1) in this metric is equivalent to

$$S = \int \left[\frac{1}{2N}((\rho^2 + \alpha^2\zeta^2)(-\dot{t}^2 + \dot{\rho}^2 + \dot{\zeta}^2) + \rho^2\zeta^2\dot{\bar{\varphi}}^2) - \frac{N}{2} \right] d\tau, \tag{7}$$

where the overdot denotes differentiation with respect to a parameter τ, and N is the Lagrange multiplier.

The action is represented as

$$S = \int [p_\mu \dot{x}^\mu - H] d\tau, \tag{8}$$

by the Hamiltonian for the particle

$$H = \frac{N}{2}\left(\frac{-p_t^2 + p_\rho^2 + p_\zeta^2}{\rho^2 + \alpha^2\zeta^2} + \frac{p_{\bar{\varphi}}^2}{\rho^2\zeta^2} + 1 \right), \tag{9}$$

where p_μ is the canonical momentum conjugate to x^μ. Using the constants of motion $p_t = -E$ and $p_{\bar{\varphi}} = L$, we have the reduced Hamiltonian in two-dimensions

$$H = \frac{N}{2}\left(\frac{p_\rho^2 + p_\zeta^2}{\rho^2 + \alpha^2\zeta^2} + V_{\rm eff}(\rho, \zeta) \right), \tag{10}$$

where

$$V_{\rm eff}(\rho, \zeta) = -\frac{E^2}{\rho^2 + \alpha^2\zeta^2} + \frac{L^2}{\rho^2\zeta^2} + 1. \tag{11}$$

By variation of the action (10) with N, we have the constraint equation $H \approx 0$.

We easily find stationary solutions $\rho = \rho_0 =$ const and $\zeta = \zeta_0 =$ const at the global minimum of $V_{\rm eff}$. The constant radii and E are given by

$$\rho_0 = \sqrt{\alpha L}, \qquad \zeta_0 = \sqrt{\frac{L}{\alpha}}, \qquad \text{and} \qquad E^2 = 4\alpha L. \tag{12}$$

In the parameter choice $\tau = t$, i.e., $N = \sqrt{\alpha L}$, and $\sigma = \phi$, we have

$$\psi = \alpha\sigma + \sqrt{\frac{\alpha}{L}}\tau. \tag{13}$$

These solutions are equivalent to the solutions (3). They are represented in different gauges.

On a $t =$ const surface, the string has a shape of toroidal spiral which lies on the two-dimensional torus

$$ds^2_{S^1\times S^1} = \rho_0^2 d\phi^2 + \zeta_0^2 d\psi^2 \tag{14}$$

embedded in the four-dimensional Euclidean space (See Fig. 1). If α is a rational number n_2/n_1, where n_1 and n_2

are relatively prime integers, the string coils around the torus n_1 times in ϕ direction while n_2 times in ψ direction, then the string is closed. The tangent vector to the world sheet

$$\eta = \partial_\tau - \frac{1}{2\sqrt{\alpha L}}\partial_\sigma = \partial_t - \frac{1}{2}\left(\frac{1}{\rho_0}\partial_\phi - \frac{1}{\zeta_0}\partial_\psi\right) \quad (15)$$

is a timelike Killing vector then the string is stationary. Since the world sheet is spanned by two commutable Killing vectors ξ and η, the two-dimensional surface is intrinsically flat. The angular momenta in ϕ and ψ directions, and the total energy of the string are in proportion to $-\alpha L$, L, and E, respectively.

III. TOROIDAL SPIRAL STRINGS IN FIVE-DIMENSIONAL BLACK HOLE SPACETIMES

Next, we consider the toroidal spiral strings in the five-dimensional Myers-Perry metric

$$\begin{aligned} ds^2 = &-dt^2 + \frac{2M}{\Sigma}(dt - a\sin^2\theta d\phi - b\cos^2\theta d\psi)^2 \\ &+ \frac{\Sigma dr^2}{\Delta} + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \\ &+ (r^2+b^2)\cos^2\theta d\psi^2, \end{aligned} \quad (16)$$

where

$$\Delta = \frac{(r^2+a^2)(r^2+b^2)}{r^2} - 2M, \quad (17)$$

$$\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \quad (18)$$

and a, b, and M are parameters related to two independent rotations and mass, respectively.

The metric of the orbit space with respect to the Killing vector (5) is

$$\begin{aligned} ds_4^2 &= (\xi\cdot\xi)h_{\mu\nu}dx^\mu dx^\nu \\ &= -\left[(\xi\cdot\xi) - \frac{2M}{\Sigma}((r^2+a^2)\sin^2\theta + \alpha^2(r^2+b^2)\cos^2\theta)\right]dt^2 - \frac{2M\cos^2\theta\sin^2\theta}{\Sigma}[b(r^2+a^2) - \alpha a(r^2+b^2)]dt d\bar{\varphi} \\ &\quad + \frac{(\xi\cdot\xi)\Sigma}{\Delta}dr^2 + (\xi\cdot\xi)\Sigma d\theta^2 + \frac{\cos^2\theta\sin^2\theta}{\Sigma}[2M(r^2(a^2\sin^2\theta + b^2\cos^2\theta) + a^2b^2) + \Sigma(r^2+a^2)(r^2+b^2)]d\bar{\varphi}^2, \end{aligned} \quad (19)$$

where $\bar{\varphi} = \psi - \alpha\phi$, and the norm of ξ is given by

$$\xi\cdot\xi = (r^2+a^2)\sin^2\theta + \alpha^2(r^2+b^2)\cos^2\theta + \frac{2M}{\Sigma}(a\sin^2\theta + \alpha b\cos^2\theta)^2. \quad (20)$$

Since the metric (19) has the Killing vectors ∂_t and $\partial_{\bar{\varphi}}$, the corresponding canonical momenta are constants of motion. Setting $p_t = -E = \text{const}$ and $p_{\bar{\varphi}} = L = \text{const}$, we get the reduced Hamiltonian as

$$H = \frac{N}{2}[(\xi\cdot\xi)^{-1}(h^{rr}p_r^2 + h^{\theta\theta}p_\theta^2) + V_{\text{eff}}(r,\theta)], \quad (21)$$

where $h^{rr} = \Delta/\Sigma$ and $h^{\theta\theta} = 1/\Sigma$, and

$$\begin{aligned} V_{\text{eff}}(r,\theta) = &-\frac{1}{\Sigma}\left((r^2+2M) + \frac{4M^2}{\Delta} + (a^2\cos^2\theta + b^2\sin^2\theta)\right)\frac{E^2}{\xi\cdot\xi} \\ &+ \frac{1}{\Sigma}\left(\frac{(a^2-b^2)(r^2(1-\alpha^2) + (a^2-\alpha^2b^2)) - 2M(a-\alpha b)^2}{r^2\Delta} + \frac{1}{\cos^2\theta} + \frac{\alpha^2}{\sin^2\theta}\right)\frac{L^2}{\xi\cdot\xi} \\ &+ \frac{4M(-b(r^2+a^2) + \alpha a(r^2+b^2))}{r^2\Delta\Sigma}\frac{EL}{\xi\cdot\xi} + 1. \end{aligned} \quad (22)$$

The dynamical system of the toroidal spiral strings in the Myers-Perry metric reduces to the two-dimensional particle system specified by the Hamiltonian (21) with (22). A typical potential shape is drawn as a contour plot in Fig. 2. There exists a stationary solution $r = r_0 = \text{const}$ and $\theta = \theta_0 = \text{const}$ at the local minimum of $V_{\text{eff}}(r,\theta)$.

To inspect conditions for existence of the stationary solution, we restrict ourselves to the nonrotational case $a = b = 0$, for simplicity. By use of the variables $\rho := r\sin\theta$ and $\zeta := r\cos\theta$ which are used in Minkowski case, the effective potential (22) becomes

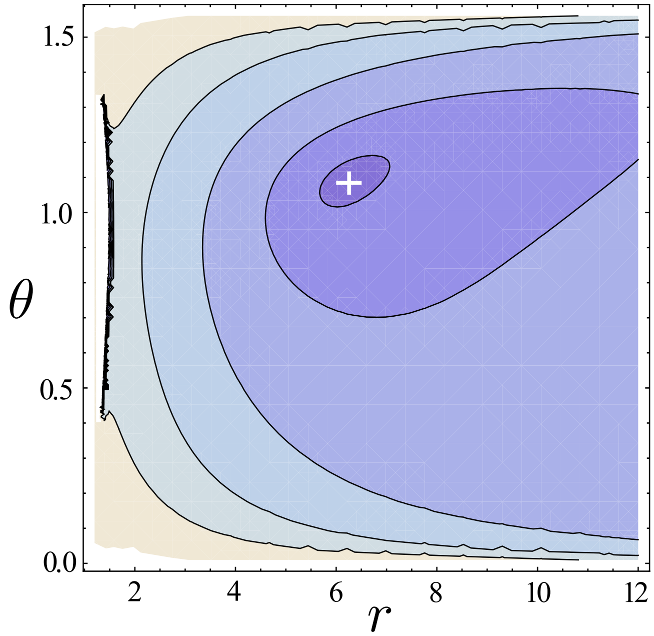

FIG. 2 (color online). Contour plot of typical shape of the effective potential in the Myers-Perry metric. Parameters of toroidal spiral string are chosen as $\alpha = 2$, $L/E = 3/2$, and black hole parameters are $2M = 1$, $a = 1/4$, $b = 1/5$, for example. The local minimum of the potential exists at the mark "+" in the figure.

$$V_{\rm eff}(\rho, \zeta) = \frac{E^2}{\rho^2 + \alpha^2\zeta^2}\left(-1 - \frac{2M}{\rho^2 + \zeta^2 - 2M}\right) + \frac{L^2}{\rho^2 + \alpha^2\zeta^2}\left(\frac{\alpha^2}{\rho^2} + \frac{1}{\zeta^2}\right) + 1. \tag{23}$$

We can interpret the two terms in the first parenthesis as attractive terms by string tension and gravity, and two terms in the second parenthesis as centrifugal repulsion by ϕ and ψ rotations, respectively.

The potential shapes are shown in Fig. 3. As same as in the Minkowski case, we can find stationary solutions $\rho = \rho_0$ and $\zeta = \zeta_0$ at the minimum of $V_{\rm eff}(\rho, \zeta)$. The explicit values of ρ_0 and ζ_0 are given by solving the coupled algebraic equations

$$\partial_\rho V_{\rm eff}(\rho, \zeta) = 0, \quad \partial_\zeta V_{\rm eff}(\rho, \zeta) = 0, \quad \text{and} \quad V_{\rm eff}(\rho, \zeta) = 0. \tag{24}$$

On a time slice $t =$ const, the stationary toroidal spiral string solution coils around a two-dimensional torus $S^1 \times S^1$ of radii $\rho = \rho_0$ and $\zeta = \zeta_0$ that lies on S^3 of the radius $r_0^2 = \rho_0^2 + \zeta_0^2$ which is surrounding the black hole.

In contrast to the flat case, because of the gravitational term the minimum of $V_{\rm eff}(\rho, \zeta)$ is a local minimum in the black hole case. The local minimum in the ρ-ζ plane moves with the value of L/E, and disappears if L/E becomes less than a critical value for each α.

We see from (24) that the local minimum points (ρ_0, ζ_0) for fixed α stay on the curve

$$(\rho_0^2 + \zeta_0^2)^2(\rho_0^2 - \alpha^2\zeta_0^2) + 4M(1 - \alpha^2)\rho_0^2\zeta_0^2 = 0 \tag{25}$$

in the ρ-ζ plane. In the case of $\alpha = 1$, ρ_0 is simply equal to ζ_0. The cross section of $V_{\rm eff}(\rho, \zeta)$ in $\alpha = 1$ case along the line $\zeta = \rho$ becomes

$$V_{\rm eff}^{\alpha=1}|_{\zeta=\rho}(\rho) = -\frac{E^2}{2\rho^2}\left(1 + \frac{M}{\rho^2 - M}\right) + \frac{L^2}{\rho^4} + 1. \tag{26}$$

We easily find that the innermost radii are $\rho_0 = \zeta_0 = \sqrt{3M}$ (see Fig. 4). For $\alpha \neq 1$ cases the curves (25) and innermost radii of stationary toroidal spirals are shown in Fig. 5.

We mention that a toroidal spiral string circles around a black hole and rotates about two axes of the black hole. The center of mass of the toroidal spiral does not orbit the

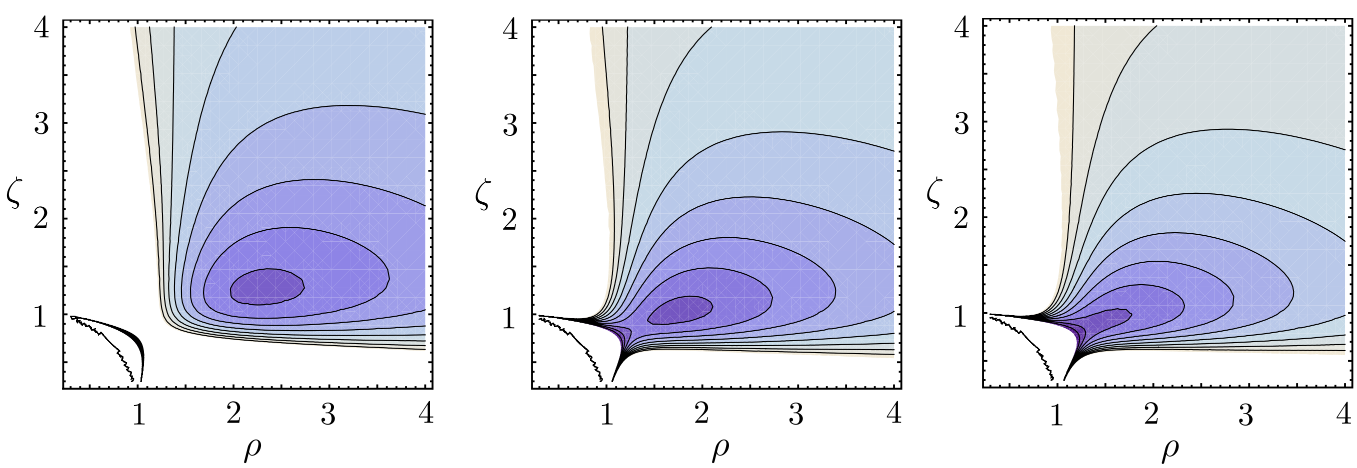

FIG. 3 (color online). Contour plot of typical shapes of effective potential in the Schwarzschild-Tangherlini metric with $2M = 1$. Parameters of toroidal spiral string are chosen as $\alpha = 2$, $L/E = 0.7$ (left), $L/E = 0.62$ (center), and $L/E = 0.6$ (right). Local minima of the effective potential appear in the left and center panels, while the local minimum disappears in the right panel.

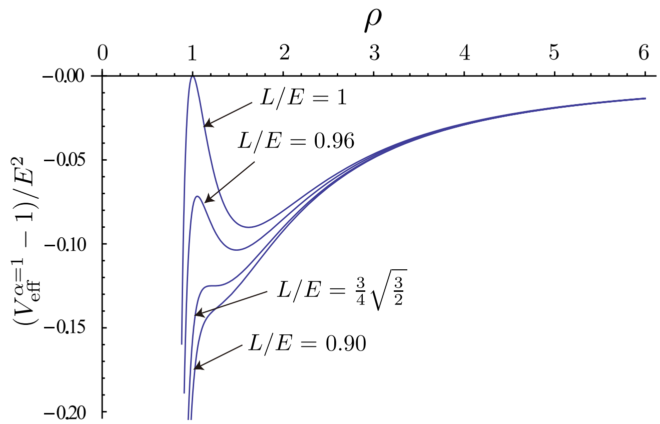

FIG. 4 (color online). Cross sections of effective potential for $\alpha = 1$ and various values of L/E in the Schwarzschild-Tangherlini metric with $2M = 1$. The local minimum disappears if $L/E = \frac{3}{4}\sqrt{\frac{3}{2}}$.

black hole. However, the appearance of the innermost radii of the toroidal spiral strings around higher-dimensional black holes is analogous to the geodesic particles around four-dimensional black holes. The toroidal spiral strings would play a role of probes to reveal geometrical properties of higher-dimensional black holes. The toroidal spirals could be captured by a black hole and be settled into stationary states if they lose center-of-mass energy by any processes. Because strings in higher dimensions hardly intersect with others [15], toroidal spirals could accumulate around a black hole like a cloud.

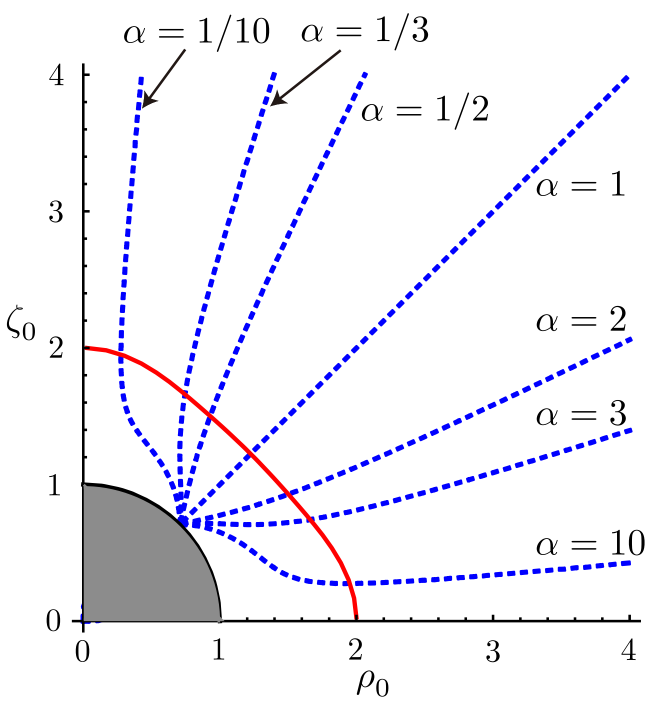

FIG. 5 (color online). The radii (ρ_0, ζ_0) of innermost toroidal spiral strings are plotted (solid curve). The radii for stationary toroidal spirals stay on broken curves for fixed α. Shaded region denotes inside the black hole horizon.

IV. SUMMARY AND DISCUSSION

We have presented stationary solutions of the Nambu-Goto string in a shape of toroidal spiral in the five-dimensional Myers-Perry black hole spacetimes. The toroidal spirals are characterized by a linear combination of commuting Killing vectors of which integral curves are closed in the target space. The stationary toroidal spirals are realized essentially by the balance of the string tension and the centrifugal force. By the effect of gravity of the black holes, there exist the critical radii of toroidal spirals inside which there is no stationary solution.

It is an interesting problem to study the gravitational field around the toroidal spirals. Investigation of gravitational wave emission from the toroidal spirals would be important. In four dimensions, it is well known that cusps commonly appear in the evolution of closed strings [16]. However, even if strings are closed in five dimensions, as has been shown in this paper, they can be the stationary toroidal spirals without cusp. The difference comes from the dimensionality of the target space of the strings. It would be natural that the stationary toroidal spirals emit gravitational waves almost constantly in a long duration with periodic waveforms [17] in the higher-dimensional universe.

It is straightforward to extend the solution in target spaces with more than five dimensions. The toroidal spirals are extended sources of gravity which have two independent rotations. They would mimic gravitational fields of black rings in far regions. In six or more dimensions, the toroidal spirals would give an important insight into construction of black ring solutions [18]. It is also possible to consider more complicated spiraling strings generated by more numbers of Killing vectors.

Though we mainly focus on the stationary solution of toroidal spirals in this article, investigation of the dynamics of toroidal spirals is reported in a separate paper [19]. The integrability of toroidal spiral strings are shown there.

ACKNOWLEDGMENTS

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