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# Inflation from supersymmetric quantum cosmology

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We derive a special scalar field potential using the anisotropic Bianchi type I cosmological model from canonical quantum cosmology under determined conditions in the evolution to anisotropic variables $\beta_\pm$. In the process, we obtain a family of potentials that has been introduced by hand in the literature to explain cosmological data. Considering supersymmetric quantum cosmology, this family is scanned, fixing the exponential potential as more viable in the inflation scenario $V(\phi) = V_0 e^{-\sqrt{3}\phi}$.

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## I. INTRODUCTION

One of the main problems of inflationary cosmology is to find a mechanism to derive in a natural way the appropriate scalar field potential in order to develop enough $e$-foldings of inflation. By natural, we understand a mechanism from which some theory provides a scalar field potential that offers the convenient features of inflation. In this work, we derive a scalar field potential from supersymmetric quantum cosmology that gives these conditions.

In a previous work, we determined scalar potentials from an exact solution to the Wheeler-DeWitt (WDW) equation in the quantum cosmology scenario [1], using as a toy model a homogenous and isotropic cosmological model. There we focus on solutions that may be relevant for the early universe constructed within the WKB approximation. Recently, these scalars potentials were obtained using a local supersymmetric scheme [2]. Nowadays it is a common issue in cosmology to make use of scalar fields $\phi$ as the responsible agents of some of the most intriguing aspects of our universe [3–11], such as inflation [12,13], dark matter, and dark energy [14]. The natural derivation of a scalar potential is a challenge, posing the following question: What physical processes provide the adequate scalar field potentials that govern the universe in determined epoch? To answer this question, we use the ideas of quantum cosmology to solve the Wheeler-DeWitt equation with a particular ansatz for the Bianchi type I universe wave function. In this scheme, we obtain two possible scenarios, the first one with a scalar exponential potential $V(\phi) = V_0 e^{\lambda\phi}$, and the second one giving a family of potentials, similar to those obtained in our previous work [1]. It is interesting that in the first scenario the $\lambda$ parameter is not fixed by the quantum scheme, remaining as a free parameter of the theory. To fix it, we invoke a supersymmetric scale, using the tools of supersymmetric quantum cosmology in order to find the most viable scalar potential for the inflationary epoch, in this scale. To do this, we applied supersymmetry as a square root of general relativity [15–18], in which the Grassmann variables are only auxiliary and cannot be identified as the supersymmetric partners of the cosmological bosonic variables. Therefore, we construct a family of scalar potentials treating the quantum solutions to anisotropic Bianchi type I cosmological model in the anisotropic variables $\beta_+$ and $\beta_-$. The conditions we use give us a special structure for the scalar potential; by simplifying the Wheeler-DeWitt equation we obtain two cases: one in which both parameters $\beta_\pm$ have hyperbolic trigonometric functions as solutions, and another where $\beta_-$ ($\beta_+$) have a trigonometric (hyperbolic trigonometric) behavior. This potential is also a good candidate, depending on the parameter value, in order to study inflation, dark matter, dark energy, or tachyon models [19]. The transform Wheeler-DeWitt equation can be solved using a particular ansatz in the WKB approximation (Bohmian representation [20]). This method has been used in the literature [21] to solve the cosmological Bianchi class A models, and in a particular, our result in the second case is similar to the one found in Ref. [1] for the isotropic Friedmann-Robertson-Walker (FRW) cosmological model. On the other hand, the best candidates for quantum solutions become those that have a damping behavior with respect to the scale factor, in the sense that we obtain a good classical solution using the WKB approximation in any scenario in the evolution of our universe [22,23]. The supersymmetric scheme has the particularity that is very restrictive because there are more constraints equations applied to the wave function. So, in this work we found that there exists a tendency for supersymmetric vacua to remain close to their semiclassical limits, because the exact solutions found are also the lowest-order WKB approximation, and do not correspond to the full quantum solutions found previously.

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## II. THE WHEELER-DEWITT EQUATION

On the Wheeler-DeWitt equation there are a lot of papers dealing with different problems, for example, Gibbons and Grishchuk [24] asked the question of what a typical wave function for the universe is. In Ref. [25] there appears an excellent summary of a paper on quantum cosmology where the problem of how the universe emerged from big bang singularity can no longer be neglected in the GUT epoch. Also, an important approach to this problem is the wave-function proposal in which the universe would be completely self-contained without any singularities and without any edges. Our goal in this paper deals with the problem to build the appropriate scalar potential in the inflationary scenario.

We start by recalling the canonical formulation of the ADM formalism to the diagonal Bianchi Class A cosmological models. The metrics have the form

$$ds^2 = -(N^2 - N^j N_j)dt^2 + e^{2\Omega(t)} e^{2\beta_{ij}(t)} \omega^i \omega^j, \quad (1)$$

where $N$ and $N_i$ are the lapse and shift functions, respectively, $\Omega(t)$ is a scalar, and $\beta_{ij}(t)$ a $3 \times 3$ diagonal matrix, $\beta_{ij} = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$, $\omega^i$ are one-forms that characterize each cosmological Bianchi type model, and that obey $d\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k$, $C^i_{jk}$ the structure constants of the corresponding invariance group [26]. The metric for the Bianchi type I, takes the form

$$ds_I^2 = -N^2 dt^2 + e^{2\Omega} e^{2\beta_+ + 2\sqrt{3}\beta_-} dx^2 + e^{2\Omega} e^{2\beta_+ - 2\sqrt{3}\beta_-} dy^2 + e^{2\Omega} e^{-4\beta_+} dz^2, \quad (2)$$

the total Lagrangian density function is given by

$$\mathcal{L}_{\mathrm{Total}} = \mathcal{L}_g + \mathcal{L}_\Lambda + \mathcal{L}_{\mathrm{matter},\phi} = \sqrt{-g}(R - 2\Lambda) + \mathcal{L}_{\mathrm{matter},\phi}. \quad (3)$$

We use as a first approximation a perfect fluid and a scalar field as the matter content, in a comoving frame [26],

$$\mathcal{L}_{\mathrm{Total}} = \sqrt{-g}(R - 2\Lambda + 16\pi G\rho + \tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)), \quad (4)$$

and using (2) we have

$$\mathcal{L}_{\mathrm{Total}} = e^{3\Omega}\left[\frac{6}{N}\dot{\Omega}^2 - \frac{6}{N}\dot{\beta}_+{}^2 - \frac{6}{N}\dot{\beta}_-{}^2 - \frac{6}{N}\dot{\varphi}^2 + 16\pi G N\rho - 2\Lambda N + \frac{V(\varphi)N}{2}\right], \quad (5)$$

where we redefined the original scalar field as $\phi = \sqrt{12}\varphi$.

The corresponding momenta are calculated in the usual way

$$\Pi_\Omega = \frac{\partial \mathcal{L}}{\partial \dot{\Omega}} = \frac{12\dot{\Omega}}{N} e^{3\Omega} \rightarrow \dot{\Omega} = \frac{e^{-3\Omega}}{12} N \Pi_\Omega$$

$$\Pi_+ = \frac{\partial \mathcal{L}}{\partial \dot{\beta}_+} = -\frac{12\dot{\beta}_+}{N} e^{3\Omega} \rightarrow \dot{\beta}_+ = -\frac{e^{-3\Omega}}{12} N \Pi_+$$

$$\Pi_- = \frac{\partial \mathcal{L}}{\partial \dot{\beta}-} = -\frac{12\dot{\beta}_-}{N} e^{3\Omega} \rightarrow \dot{\beta}_- = -\frac{e^{-3\Omega}}{12} N \Pi_-$$

$$\Pi_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{12\dot{\varphi}}{N} e^{3\Omega} \rightarrow \dot{\varphi} = -\frac{e^{-3\Omega}}{12} N \Pi_\varphi$$

now writing (5) in canonical form $\mathcal{L}_{\mathrm{can}} = \Pi_q \dot{q} - N\mathcal{H}$ where $\mathcal{H}$ is the Hamiltonian density function,

$$\mathcal{L}_{\mathrm{can}} = \Pi_\Omega \dot{\Omega} + \Pi_+ \dot{\beta}_+ + \Pi_- \dot{\beta}_- + \Pi_\varphi \dot{\varphi} - \frac{N e^{-3\Omega}}{24}(\Pi_\Omega^2 - \Pi_+^2 - \Pi_-^2 - \Pi_\varphi^2 - e^{6\Omega}[384\pi G\rho - 48\Lambda + 12V(\varphi)])$$

we obtain the corresponding Hamiltonian density function

$$\mathcal{H} = \frac{e^{-3\Omega}}{24}(\Pi_\Omega^2 - \Pi_+^2 - \Pi_-^2 - \Pi_\varphi^2 - e^{6\Omega}[384\pi G\rho - 48\Lambda + 12V(\varphi)]) \quad (6)$$

when we include the energy-momentum tensor for a barotropic perfect fluid $\mathrm{p} = \gamma\rho$, we have

$$\mathcal{H} = \frac{e^{-3\Omega}}{24}(\Pi_\Omega^2 - \Pi_+^2 - \Pi_-^2 - \Pi_\varphi^2 + 48\Lambda e^{6\Omega} - 384\pi G M_\gamma e^{-3(\gamma-1)\Omega} - 12 e^{6\Omega} V(\varphi)). \quad (7)$$

Imposing the quantization condition and applying this Hamiltonian to the wave function $\Psi$, we obtain the WDW equation for these models in the minisuperspace by the usual identification $\mathrm{P}_{q^\mu}$ by $-i\partial_{q^\mu}$ in (7), with $q^\mu = (\Omega, \beta_+, \beta_-, \varphi)$, and following Hartle and Hawking [22] we consider a semigeneral factor ordering, which gives

$$\hat{H}\Psi = \left[-\frac{\partial^2}{\partial\Omega^2} + \frac{\partial^2}{\partial\beta_+^2} + \frac{\partial^2}{\partial\beta_-^2} + \frac{\partial^2}{\partial\varphi^2} + Q\frac{\partial}{\partial\Omega} + 48\Lambda e^{6\Omega} - 384\pi G M_\gamma e^{-3(\gamma-1)\Omega} - 12 e^{6\Omega} V(\varphi)\right]\Psi = 0,$$

where $Q$ measures the ambiguity in the factor ordering between the scalar function $\Omega$ and its corresponding momenta. This equation is not easy to solve, first because we do not have the structure of scalar potential, and second, it depends strongly on the class of scenario we analyze with barotropic equation. In the following, for simplicity we shall use the inflationary case, $\gamma = -1$.

Using the following ansatz for the wavefunction $\Psi(\Omega, \varphi, \beta_\pm) = e^{\pm a_1\beta_+ \pm a_2\beta_-} \Xi(\Omega, \varphi)$, we obtain a reduced WDW

$$\left[-\frac{\partial^2}{\partial\Omega^2}+\frac{\partial^2}{\partial\varphi^2}+Q\frac{\partial}{\partial\Omega}+e^{6\Omega}(48\Lambda-384\pi GM_{-1}-12V(\varphi))+c^2\right]\Xi=0, \tag{8}$$

where the constant $c^2=a_1^2+a_2^2$.

Equation (8) can be written in compact form as

$$\Box\Xi+Q\frac{\partial\Xi}{\partial\Omega}-U(\Omega,\varphi,\lambda_{\rm eff})\Xi=0, \tag{9}$$

where the d'Alambertian in two dimensions is redefined as $\Box\equiv-\partial_\Omega^2+\partial_\varphi^2$, $\lambda_{\rm eff}=48\Lambda-384\pi GM_{-1}$ is the effective cosmological constant, and the potential $U(\Omega,\varphi,\Lambda)=e^{6\Omega}[12V(\varphi)-\lambda_{\rm eff}]-c^2$.

To solve (8) we take the ansatz, which is similar to the one used in the Bohmian formalism into quantum mechanics [20]

$$\Xi(\Omega,\varphi)=W(\Omega,\varphi)e^{-S(\Omega,\varphi)}, \tag{10}$$

where $S(\Omega,\varphi)$ is the *superpotential* function. Equation (9) can be written as the following set of partial differential equations

$$(\nabla S)^2-U=0, \tag{11a}$$

$$W\left(\Box S+Q\frac{\partial S}{\partial\Omega}\right)+2\nabla W\cdot\nabla S=0, \tag{11b}$$

$$\Box W+Q\frac{\partial W}{\partial\Omega}=0, \tag{11c}$$

where the first equation is the classical Hamilton-Jacobi equation, which plays an important role in this work. The different terms in this equation are

$$\nabla W\cdot\nabla S\equiv-(\partial_\Omega W)(\partial_\Omega S)+(\partial_\varphi W)(\partial_\varphi S),$$
$$(\nabla)^2\equiv-(\partial_\Omega)^2+(\partial_\varphi)^2.$$

Any exact solution complying with the set of Eqs. (11a)–(11c) will also be an exact solution of the original WDW equation. Following Ref. [1], first we shall choose to solve Eqs. (11a) and (11b), whose solutions at the end will have to fulfill with Eq. (11c), which plays the role of a constraint equation.

Taking the ansatz [27,28]

$$S(\Omega,\varphi)=\frac{1}{\mu}e^{3\Omega}g(\varphi)+c(b_1\Omega+b_2\Delta\varphi), \tag{12}$$

with $\Delta\varphi=\varphi-\varphi_0$, $\varphi_0$ is a constant scalar field, $b_i$ arbitrary constants, Eq. (11a) is transformed as

$$\left[-\frac{9}{\mu^2}g^2+\frac{1}{\mu^2}\left(\frac{dg}{d\varphi}\right)^2-12V(\varphi)+\lambda_{\rm eff}\right]e^{6\Omega}+c^2[1-b_1^2+b_2^2]+\frac{6c}{\mu}\left[\frac{b_2}{3}\frac{dg}{d\varphi}-b_1g\right]e^{3\Omega}=0. \tag{13}$$

This equation is more difficult to solve, at this point we introduce the main idea of the paper to obtain the scalar potential family, which are strongly dependent to solutions for the anisotropic variables $\beta_\pm$. We include two steps to solve Eq. (13):

(1) First, consider that the second and third parenthesis are null, but maintaining that $c\neq0$. The first condition implies that $b_1=\sqrt{1+b_2^2}$, and the constants $a_1$ and $a_2$ are real, the solutions for the anisotropic variables $\beta_\pm$ can be considered as hyperbolic trigonometric functions. The second condition becomes an ordinary differential equation for the unknown function $g(\varphi)$, yielding

$$g(\varphi)=g_0e^{\alpha/2\Delta\varphi}, \tag{14}$$

with $g_0$ an integration constant and $\alpha=\frac{6b_1}{b_2}=\frac{\pm6\sqrt{1+b_2^2}}{b_2}$ with $b_2\neq0$. The scalar potential function become, when we take the first parenthesis in Eq. (13),

$$V(\varphi)=(4\Lambda-32\pi GM_{-1})+V_0e^{\alpha\Delta\varphi}, \tag{15}$$

with $V_0=\frac{3g_0^2}{4b_2^2\mu^2}$. With these results, the superpotential function (12) is

$$S(\Omega,\varphi)=\frac{g_0}{\mu}e^{3\Omega}e^{\alpha/2\Delta\varphi}+c(\pm\sqrt{1+b_2^2}\Omega+b_2\Delta\varphi). \tag{16}$$

Substituting (16) into (11b), the corresponding solutions for the function $W$ in the form $e^{q\Omega+\eta\varphi}$ become $W={\rm Exp}[\frac{1}{2}(Q-3)\Omega-\frac{1}{4}\alpha\Delta\varphi]$, we develop the following wave function

$$\Psi={\rm Exp}[a_1\beta_++a_2\beta_-+\tfrac{1}{2}(Q-3)\Omega-\tfrac{1}{4}\alpha\Delta\varphi]e^{-(e^{3\Omega+(\alpha/2)\Delta\varphi}/\mu)}, \tag{17}$$

with the constraint on the parameter $\alpha\leq\pm6$.

In the literature [29], the scalar potential type $V(\phi)=e^{\lambda\phi}$ gives a power law in the classical scale factor, considering the flat FRW cosmological model when $\lambda<-\sqrt{2}$. If we consider the extreme values for the $\alpha$ parameter ($Q=0$) and the corresponding transformation between $\varphi\rightarrow\phi$, we obtain the special scalar potential $V(\phi)=V_0e^{\pm\sqrt{3}\phi}$. For a standard inflationary model, this class of potential has the advantage that classical analytical solutions can be found and, for appropriate values of the parameters, inflation can be obtained.

(2) In the second step, we consider that the constant $c=0$, implying that $a_1=\pm ia_2$, then the solutions for the wave function for the anisotropic variables $\beta_\pm$ are considered trigonometric functions for the variable $\beta_+$ and hyperbolic trigonometric function for the variable $\beta_-$. Thus the superpotential term (12)

has the simple form

$$S(\Omega,\varphi)=\frac{1}{\mu}e^{3\Omega}g(\varphi), \tag{18}$$

and Eq. (13) becomes an ordinary differential equation for the unknown function $g(\varphi)$ in terms of the scalar potential $\mathcal{V}(\varphi,\lambda_{\rm eff})=V(\varphi)-\frac{\lambda_{\rm eff}}{12}$,

$$\left(\frac{dg}{d\varphi}\right)^2-9g^2(\varphi)=12\mu^2\left(V(\varphi)-\frac{\lambda_{\rm eff}}{12}\right)=12\mu^2\mathcal{V}(\varphi,\lambda_{\rm eff}), \tag{19}$$

this equation is similar to the one obtained in Ref. [1]. It is not surprising that this equation is similar to Eq. (12) in Ref. [1], because the anisotropic Bianchi type I cosmological model is the generalization of the flat FRW model. The last equation has several exact solutions, which can be generated in the following way. Consider that $\mathcal{V}=g^2F(g)$, where $F(g)$ is an arbitrary function of its argument. So, Eq. (19) can be written in quadratures as

$$\Delta\varphi=\pm\frac{1}{2\sqrt{3}}\int\frac{d\ln g}{\sqrt{\frac{3}{4}+\mu^2F(g)}}. \tag{20}$$

In this way, we can solve the $g(\varphi)$ function, and then use the expression for the potential term $\mathcal{V}=g^2F(g)$ back again to find the corresponding scalar potential that leads to an exact solution to the Hamilton-Jacobi equation (11a). Some examples are shown in Table I.
In this way, the superpotential $S(\Omega,\varphi)$ is known.
For solve (11b) we assume

$$W=e^{[z(\Omega)+\omega(\varphi)]}, \tag{21}$$

we arrive to a set of ordinary differential equations for the functions $z(\Omega)$ and $\omega(\varphi)$

$$2\frac{dz}{d\Omega}-Q=k,\rightarrow z(\Omega)=\frac{Q+k}{2}\Omega \tag{22}$$

$$\frac{d^2g}{d\varphi^2}+2\frac{dg}{d\varphi}\frac{d\omega}{d\varphi}=3(k+3)g,\rightarrow\omega(\varphi)=\frac{3k}{2}\int\frac{d\varphi}{\partial_\varphi(\ln g)}-3\mu^2\int\frac{d[\mathcal{V}(\varphi,\lambda_{\rm eff})]}{(\partial_\varphi g)^2}. \tag{23}$$

Thus, the explicit form for the function $W$ becomes

$$W=\exp\left\{\frac{3k}{2}\left[\frac{\Omega}{3}+\int\frac{d\varphi}{\partial_\varphi(\ln g)}\right]+\frac{Q}{2}\Omega-3\mu^2\int\frac{d[\mathcal{V}(\varphi,\lambda_{\rm eff})]}{(\partial_\varphi g)^2}\right\}. \tag{24}$$

The constraint (11c) can be written as

$$\partial_\varphi^2\omega+(\partial_\varphi\omega)^2-\frac{k^2-Q^2}{4}=0, \tag{25}$$

and

$$\partial_\varphi\omega=\frac{3k}{2\partial_\varphi(\ln g)}-3\mu^2\frac{\partial_\varphi[\mathcal{V}(\varphi,\lambda_{\rm eff})]}{(\partial_\varphi g)^2}.$$

Taking into account Table I, we present the corresponding wave function in each case, in Table II.

In a special case, the third line in Table II, has a damping term that corresponds to $e^{-S}$ and a plane wave type, similar to Eq. (17) obtained in the first step. The first line has this behavior, but corresponds to null scalar potential.

In this way, using the quantum formalism in the sector of inflationary scenario, we found that the scalar potential becomes an exponential behavior, however the coupling constant is undetermined. The question is, how can we fix the coupling constant? The answer could be in the supersymmetric quantum cosmology using differential operators to the Grassmann variables, where we present the formalism in the next section.

TABLE I. Some exact solutions to Eq. (19) and their corresponding scalar potentials, where $n$ is any real number and $V_0$ is an arbitrary constant, different to step one. The third line is equivalent to this obtained by the first step.

| $F(g)$ | $g(\varphi)$ | $\mathcal{V}(\varphi,\lambda_{\rm eff})$ |
|---|---|---|
| $0$ | $\mathrm{Exp}[\pm3\Delta\varphi]$ | $0$ |
| $V_0g^{-2}$ | $\sqrt{\frac{4\mu^2V_0}{3}}\sinh(\pm3\Delta\varphi)$ | $V_0$ |
| $V_0$ | $e^{\alpha/2\Delta\varphi}$ | $V_0\exp(\alpha\Delta\varphi)$, $\alpha=\pm4\sqrt{3}\sqrt{\frac{3}{4}+\mu^2V_0}$ |
| $V_0g^{-n}$ $(n\neq2)$ | $[\frac{e^{\eta\Delta\varphi}-4\mu^2V_0e^{-\eta\Delta\varphi}}{2\sqrt{3}}]^{2/n}$ | $[\frac{e^{\eta\Delta\varphi}-4\mu^2V_0e^{-\eta\Delta\varphi}}{2\sqrt{3}}]^{2(2-n)/n}$, $\eta=\frac{3n}{2}$ |
| $\ln g$ | $e^{u(\varphi)}$ | $ue^{2u}$, $u=(\mu\sqrt{3}\Delta\varphi)^2-\frac{3}{4\mu^2}$ |
| $(\ln g)^2$ | $e^{r(\varphi)}$ | $r^2e^{2r}$, $r=\frac{1}{2}[e^{u(\varphi)}-\frac{3}{4\mu^2}e^{-u(\varphi)}]$, $u=2\sqrt{3}\mu\Delta\varphi$ |

TABLE II. Wave function corresponding to Table I.

| $g(\varphi)$ | wave function $\Psi$ |
|---|---|
| $\mathrm{Exp}[\pm 3\Delta\varphi]$ | $\exp\{a_2(\pm i\beta_+ + \beta_-) + (\frac{k+Q}{2})\Omega \pm \frac{3k}{2}\Delta\varphi\}e^{-(e^{3\Omega\pm 3\Delta\varphi}/\mu)}$ |
| $\sqrt{\frac{4\mu^2 V_0}{3}}\sinh(\pm 3\Delta\varphi)$ | $\cosh^{k/2}(\pm 3\Delta\varphi)\exp\{a_2(\pm i\beta_+ + \beta_-) + (\frac{k+Q}{2})\Omega\}e^{-(e^{3\Omega}\sqrt{\frac{4\mu^2 V_0}{3}}\sinh(\pm 3\Delta\varphi)/\mu)}$ |
| $e^{\alpha/2\Delta\varphi}$ | $\exp\{a_2(\pm i\beta_+ + \beta_-) + (\frac{k+Q}{2})\Omega + (\frac{3k-12\mu^2}{\alpha})\Delta\varphi\}e^{-(e^{3\Omega+\frac{\alpha}{2}\Delta\varphi}/\mu)}$ |
| $[\frac{e^{\eta\Delta\varphi}-4\mu^2 V_0 e^{-\eta\Delta\varphi}}{2\sqrt{3}}]^{2/n}$ | $\exp\{a_2(\pm i\beta_+ + \beta_-) + (\frac{k+Q}{2})\Omega + \omega(\varphi)\}e^{-(e^{3\Omega+g(\varphi)}/\mu)}$ |
| | $\omega(\varphi) = \frac{k}{2}[-\Delta\varphi + \frac{2}{3n}\mathrm{Ln}(4\mu^2 V_0 + e^{2\eta\Delta\varphi})] - \frac{\mu^2(2-n)}{3nV_0}\mathrm{Arctanh}(\frac{1}{4\mu^2 V_0}e^{2\eta\Delta\varphi})$ |
| $e^{u(\varphi)}$ | $\exp\{a_2(\pm i\beta_+ + \beta_-) + (\frac{k+Q}{2})\Omega + \frac{k+3-2\mu^2}{4\mu^2}\mathrm{Ln}\Delta\varphi - \frac{3\mu^2}{2}\Delta\varphi^2\}e^{-(e^{3\Omega+u(\varphi)}/\mu)}$ |
| $e^{r(\varphi)}$ | $\exp\{a_2(\pm i\beta_+ + \beta_-) + (\frac{k+Q}{2})\Omega + \omega(\varphi)\}e^{-(e^{3\Omega+r(\varphi)}/\mu)}$ |
| | $\omega(\varphi) = \frac{\sqrt{3k}}{6\mu}\mathrm{Arctan}(\frac{2\mu}{\sqrt{3}}e^{u(\varphi)}) + 6\mu^3\Delta\varphi - \frac{\mu}{2}\mathrm{Ln}(\frac{3}{4\mu^2} + e^{2u(\varphi)})$ |
| | $+ \frac{\mu}{4}[\frac{3}{4\mu^2}e^{u(\varphi)} - e^{-u(\varphi)}] - \frac{\sqrt{3}}{8}\mathrm{Arctan}(\frac{\sqrt{3}}{2\mu}e^{-u(\varphi)})$ |

## III. SUPERSYMMETRIC QUANTUM MECHANICS

In the following, we shall apply the supersymmetric quantum formalism at the quantum structure obtained in the previous section, to obtain a closed value to the parameters that optimize the inflation scenario, i.e., we do an analysis to the family obtained in Table I for the function $g(\varphi)$; or in other words, which is the constraint on the superpotential function that appears in the quantum level, Eq. (18), based in Tables I or II?. In this order of ideas, we found one integrability condition on the $g(\varphi)$ function, which fixes the coupling parameter of our problem.

To obtain these results, in this section we consider only a reduced supersymmetry in two bosonic variables $(\Omega, \varphi)$, without considering the anisotropic parameters, because the full problem does not contemplate the initial condition on our original problem. For instance, the decomposition of the wave function in the full expansion has 16 components, or $16 \times 16$ matrix components, and the solution is very complicated. In this sense, to solve our problem we use a reduced bosonic Hamiltonian, Eq. (8), and in consequence, one reduced supersymmetry.

The idea of Witten [30] is to find the supersymmetric supercharges operators $Q$, $\bar{Q}$ that produce a super-Hamiltonian $\mathcal{H}_{\rm susy}$, that satisfies the closed superalgebra

$$\mathcal{H}_{\rm susy} = \tfrac{1}{2}[Q, \bar{Q}], \qquad [\bar{Q}, \bar{Q}] = [Q, Q] = 0, \qquad (26)$$

where the super-Hamiltonian $\mathcal{H}_{\rm susy}$ has the following form

$$\mathcal{H}_{\rm susy} = \mathcal{H}_b + \frac{\partial^2 S}{\partial q^\mu \partial q^\nu}[\bar{\psi}^\mu, \psi^\nu], \qquad (27)$$

with $\mathcal{H}_b$ is the bosonic Hamiltonian (8) taking the constant $c = 0$, then the supersymmetric approach will only be applied to the reduced Hamiltonian, and $S$ is the corresponding superpotential function that is related with the potential term that appears in the bosonic Hamiltonian, i.e., has the same structure that in the quantum level, proposed in the last section. This idea was applied in Ref. [18] for all Bianchi Class A models without matter content, and in [31] to the FRW cosmological model. For example, in Ref. [32] it is explained that, in particular, we can formulate a *particle dynamics* in a potential $V(q^\mu)$ on a curved manifold and supersymmetry requires that the potential $V(q^\mu)$ is derivable from a globally defined superpotential $S(q^\mu)$ via $V(q^\mu) = \frac{1}{2}G^{\mu\nu}(q)\frac{\partial S(q)}{\partial q^\mu}\frac{\partial S(q)}{\partial q^\nu}$, where $G^{\mu\nu}(q^\mu)$ is the metric in the curved space. This equation is represented in the quantum level by Eq. (11a).

In this approach, a supersymmetric state with $Q|\psi\rangle = 0$ is automatically a zero energy ground state, in a similar way that it is in the quantum regime. This simplifies the problem of finding a supersymmetric ground state because the energy is known *a priori* and the factorization of $\mathcal{H}_{\rm susy}|\psi\rangle = 0$ into $Q|\psi\rangle = 0$, $\bar{Q}|\psi\rangle = 0$ often provides a simpler first-order equation for the ground state wave function. The simplicity of this factorization is related to the solubility of certain bosonic Hamiltonians. In this work, as in others, we find for the empty $(+)$ and filled $(-)$ sector of the expansion of the wave function in this approach, in the sector of the fermion Fock space zero energy solutions $|\mathcal{A}_\pm\rangle = e^{\pm S}|\pm\rangle$ where $\mathcal{A}_\pm$ are the corresponding components for the empty and filled fermionic sector.

The corresponding supercharges that satisfy the superalgebra, when we consider the bosonic Hamiltonian given by Eq. (8) become

$$Q = \psi^\mu\left[\frac{\partial}{\partial q^\mu} + \frac{\partial S}{\partial q^\mu}\right], \qquad \bar{Q} = \bar{\psi}^\nu\left[\frac{\partial}{\partial q^\nu} - \frac{\partial S}{\partial q^\nu}\right]. \qquad (28)$$

We consider the following algebra for the fermionic variables [18]

$$\{\psi^\mu, \bar{\psi}^\nu\} = \eta^{\mu\nu}, \qquad \{\bar{\psi}^\mu, \bar{\psi}^\nu\} = \{\psi^\mu, \psi^\nu\} = 0, \qquad (29)$$

and the corresponding representation

$$\bar{\psi}^{\nu} = \theta^{\nu}, \qquad \psi^{\mu} = \eta^{\mu\nu}\frac{\partial}{\partial\theta^{\nu}}. \tag{30}$$

Equation (28) are

$$\begin{aligned} Q &= -\left[\frac{\partial}{\partial q^0} + \frac{\partial S}{\partial q^0}\right]\frac{\partial}{\partial\theta^0} + \left[\frac{\partial}{\partial q^1} + \frac{\partial S}{\partial q^1}\right]\frac{\partial}{\partial\theta^1},\\ \bar{Q} &= \theta^0\left[\frac{\partial}{\partial q^0} - \frac{\partial S}{\partial q^0}\right] + \theta^1\left[\frac{\partial}{\partial q^1} - \frac{\partial S}{\partial q^1}\right]. \end{aligned} \tag{31}$$

The decomposition of the wave function becomes

$$\Xi(\Omega, \varphi) = \mathcal{A}_+ + \mathcal{B}_0\theta^0 + \mathcal{B}_1\theta^1 + \mathcal{A}_-\theta^0\theta^1, \tag{32}$$

where the coordinates fields are $q^\mu = (q^0, q^1) = (\Omega, \varphi)$, $\mathcal{A}_\pm$, $\mathcal{B}_0$, $\mathcal{B}_1$ are the bosonic and fermionic contributions to the wave function.

The supersymmetric equations $Q|\Xi> = 0$, $\bar{Q}|\Xi> = 0$ are

$$\begin{aligned} Q\Xi = &-\left[\frac{\partial}{\partial q^0} + \frac{\partial S}{\partial q^0}\right]\frac{\partial}{\partial\theta^0}[\mathcal{A}_+ + \mathcal{B}_0\theta^0 + \mathcal{B}_1\theta^1\\ &+ \mathcal{A}_-\theta^0\theta^1] + \left[\frac{\partial}{\partial q^1} + \frac{\partial S}{\partial q^1}\right]\frac{\partial}{\partial\theta^1}[\mathcal{A}_+ + \mathcal{B}_0\theta^0\\ &+ \mathcal{B}_1\theta^1 + \mathcal{A}_-\theta^0\theta^1], \end{aligned} \tag{33}$$

$$\begin{aligned} \bar{Q}\Xi = &\,\theta^0\left[\frac{\partial}{\partial q^0} - \frac{\partial S}{\partial q^0}\right][\mathcal{A}_+ + \mathcal{B}_0\theta^0 + \mathcal{B}_1\theta^1 + \mathcal{A}_-\theta^0\theta^1]\\ &+ \theta^1\left[\frac{\partial}{\partial q^1} - \frac{\partial S}{\partial q^1}\right][\mathcal{A}_+ + \mathcal{B}_0\theta^0 + \mathcal{B}_1\theta^1\\ &+ \mathcal{A}_-\theta^0\theta^1]. \end{aligned} \tag{34}$$

Then (34) gives the following set of differential equations

$$\theta^0\colon \left[\frac{\partial\mathcal{A}_+}{\partial q^0} - \mathcal{A}_+\frac{\partial S}{\partial q^0}\right] = 0, \tag{35}$$

$$\theta^1\colon \left[\frac{\partial\mathcal{A}_+}{\partial q^1} - \mathcal{A}_+\frac{\partial S}{\partial q^1}\right] = 0, \tag{36}$$

$$\theta^0\theta^1\colon \left[\frac{\partial\mathcal{B}_1}{\partial q^0} - \mathcal{B}_1\frac{\partial S}{\partial q^0}\right] - \left[\frac{\partial\mathcal{B}_0}{\partial q^1} - \mathcal{B}_0\frac{\partial S}{\partial q^1}\right] = 0, \tag{37}$$

whose solutions to Eqs. (35) and (36) are

$$\mathcal{A}_+ = a_+e^S, \tag{38}$$

On the other hand, Eq. (33) gives

$$\text{free term:}\ -\left[\frac{\partial\mathcal{B}_0}{\partial q^0} + \mathcal{B}_0\frac{\partial S}{\partial q^0}\right] + \left[\frac{\partial\mathcal{B}_1}{\partial q^1} + \mathcal{B}_1\frac{\partial S}{\partial q^1}\right] = 0, \tag{39}$$

$$\theta^1\colon \left[\frac{\partial\mathcal{A}_-}{\partial q^0} + \mathcal{A}_-\frac{\partial S}{\partial q^0}\right] = 0, \tag{40}$$

$$\theta^0\colon \left[\frac{\partial\mathcal{A}_-}{\partial q^1} + \mathcal{A}_-\frac{\partial S}{\partial q^1}\right] = 0, \tag{41}$$

where (39) can be written as

$$\eta^{\mu\nu}(\partial_\mu\mathcal{B}_\nu + \mathcal{B}_\nu\partial_\mu S) = 0, \tag{42}$$

considering the following ansatz for the fields $\mathcal{B}_\mu$

$$\mathcal{B}_\nu = e^S\partial_\nu f_+, \tag{43}$$

(37) is satisfied identically, and (42) is

$$\begin{aligned} &\eta^{\mu\nu}(\partial_\mu\partial_\nu f_+ + \partial_\nu f_+\partial_\mu S + \partial_\nu f_+\partial_\mu S)\\ &\quad = \eta^{\mu\nu}(\partial_\mu\partial_\nu f_+ + 2\partial_\nu f_+\partial_\mu S) = 0, \end{aligned} \tag{44}$$

where a possible solution is $f_+ = h(\Omega \pm \varphi)$, and $h$ is any function dependent to the argument, given the following constraint on the superpotential function

$$\frac{\partial S}{\partial\Omega} = \pm\frac{\partial S}{\partial\varphi}, \tag{45}$$

and considering the structure of the superpotential (18) we find one condition on integrability over the function $g(\varphi)$, given

$$g(\varphi) = g_0e^{\pm3\Delta\varphi}. \tag{46}$$

and taking into account Tables I or II, we obtain the following constraint in the parameter $\alpha$ of the models when Eq. (45) is satisfied

$$\frac{\alpha}{2} = \pm3,$$

so, only the exponential scalar potential can survive in Table I, and the coupling constant become $\alpha = \pm6$, given the scalar potential $V(\phi) = V_0e^{-\sqrt{3}\Delta\phi}$. In this way, supersymmetric quantum mechanics fix the values for the $\alpha$ parameter, being valid the argument introduced in the quantum scheme.

Equations (40) and (41) can be written as

$$\begin{aligned} &\frac{\partial\mathcal{A}_-}{\partial q^\mu} + \mathcal{A}_-\frac{\partial S}{\partial q^\mu} = 0,\\ &\frac{1}{\mathcal{A}_-}\frac{\partial\mathcal{A}_-}{\partial q^\mu} = -\frac{\partial S}{\partial q^\mu} \rightarrow \frac{\partial \mathrm{Ln}\mathcal{A}_-}{\partial q^\mu} = -\frac{\partial S}{\partial q^\mu}, \end{aligned} \tag{47}$$

with solution

$$\mathcal{A}_- = a_-e^{-S}, \tag{48}$$

then, the set of contributions for the supersymmetric wave functions are found to be

$$\mathcal{A}_\pm = a_\pm e^{\pm S} \qquad \mathcal{B}_0 = e^S\partial_0(f_+) \qquad \mathcal{B}_1 = e^S\partial_1(f_+)$$

It is interesting to note that supersymmetry is very restrictive because there exist more constraints equations applied to the wave function. In this sense, we observe a tendency for supersymmetric vacua to remain close to their semiclassical limits, because the exact solutions found are also the lowest-order WKB approximation.

## IV. CONCLUSIONS

Using the quantum formalism in the inflationary scenario, we find that the scalar potential has an exponential behavior as a good candidate. However, the coupling constant is undetermined. The question was, how can we fix the value of the coupling constant? The answer was in the supersymmetric quantum cosmology using differential operators to the Grassmann variables, where the coupling constant is found under one condition of integrability on the function $g(\varphi) = g_0 e^{\pm 3\Delta\varphi}$, and taking into account Tables I or II, $\frac{\alpha}{2} = \pm 3$. So, the main goal in this paper was to fix the value for the coupling constant to the inflationary scenario $\lambda = \frac{\alpha}{2} = \pm 3$ using the supersymmetric approach, when the quantum approach only gives the general structure for the scalar potential. Also we find exact solutions in both regimes. In the quantum level, we found that the possible solutions become the contributions to the empty ( + ) and filled ( − ) sector of decomposition to the wave function in the supersymmetric approach.

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