PHYSICAL REVIEW D **82,** 043528 (2010)

Dynamical decompactification and three large dimensions

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(Received 29 June 2010; published 25 August 2010)

We study string gas dynamics in the early universe and seek to realize the Brandenberger-Vafa mechanism—a goal that has eluded earlier works—that singles out three or fewer spatial dimensions as the number that grows large cosmologically. Considering wound string interactions in an impact parameter picture, we show that a strong exponential suppression in the interaction rates for $d > 3$ spatial dimensions reflects the classical argument that string world sheets generically intersect in at most four spacetime dimensions. This description is appropriate in the early universe if wound strings are heavy—wrapping long cycles—and *diluted*. We consider the dynamics of a string gas coupled to dilaton gravity and find that (a) for any number of dimensions the universe generically stays trapped in the Hagedorn regime and (b) if the universe fluctuates to a radiation regime any residual winding modes are diluted enough so that they freeze-out in $d > 3$ large dimensions while they generically annihilate for $d = 3$. In this sense the Brandenberger-Vafa mechanism is operative.

DOI: 10.1103/PhysRevD.82.043528 PACS numbers: 98.80.Cq

I. INTRODUCTION

One of the few mechanisms aiming to explain the hierarchy between three large and six small spatial dimensions within superstring theory is due to a suggestion, some two decades ago, by Brandenberger and Vafa [1]; see [2] for a review. In this scenario the early universe consists of a hot string gas in thermal equilibrium near the Hagedorn temperature. The topology of space has nontrivial cycles supporting winding modes in the gas. The background metric and string coupling evolve with the low-energy effective dilaton-gravity equations of motion according to which the winding modes resist the expansion of the spatial directions they wrap. If due to a thermal fluctuation a number of dimensions starts growing, then eventually the equilibrium number of winding modes will drop to zero. The winding modes have the capacity to relax to equilibrium through annihilations with antiwinding modes; if these interactions are efficient, then at large volumes the winding numbers will vanish, allowing the corresponding dimensions to grow. The Brandenberger-Vafa (BV) mechanism relied on a simple dimension-counting argument that wound strings generically intersect in at most three spatial dimensions, singling this out as the maximum number of dimensions in which winding numbers have the capacity to track their equilibrium values, thereby dropping to zero and allowing the dimensions to grow large. In [3] this argument was supported using numerical simulations of a network of classical strings, though gravitational dynamics was not taken into account.

Over time it became clear that strings are not the only fundamental degrees of freedom of string theory and that higher dimensional objects (membranes) are also fundamental states of the theory; superstring theory was shown to result from compactification of a higher dimensional theory, M theory. In a paper by Alexander, Brandenberger and Easson [4] the setup of [1] was extended to include p-branes for $p = 0, 1, 2, 4, 5, 6, 8$ in the weak-coupling limit of M theory with one small dimension compactified on S^1 (type IIA string theory). The other spatial dimensions were compactified on a 9-torus. The authors argued that fundamental string winding modes are still the decisive objects regarding decompactification and that the conclusions of [1] still hold. They also pointed out that a further hierarchy between dimensions could arise. Past the string scale, as the universe grows, more and more energy is needed to support wound branes of highest p; hence highest p branes would tend to decay first. As two p-branes can intersect in at most $2p + 1$ spatial dimensions, there is no obstacle for the disappearance of p-branes for $p > 2$. But 2-branes can allow for a five-dimensional subspace to grow first. Further, within this subspace, 1-branes will only allow for a three-dimensional space to continue expanding, as in [1]; hence one is left with a 3-2-4 dimensional hierarchy.

These claims relied on heuristic thermodynamic and topological arguments. Aiming to carry out a more rigorous investigation, Easther *et al.* [5] considered the full equations of motion for 11D supergravity on a homogeneous but anisotropic toroidal background, coupled to a gas of branes and supergravity particles. Focusing on the late time behavior of the system, they justifiably ignored excitations on the branes and included only M2-branes, since M5-branes (the other fundamental states of M theory) would annihilate efficiently in the full 11-dimensional

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(11D) spacetime. Motivated by the BV mechanism and the arguments of [4], the authors of [5] chose initial states resulting from fluctuations that would leave three dimensions unwrapped, some number of dimensions partially wrapped, and some fully wrapped. The conclusion was that indeed the dynamics leverage the topological reasoning and a hierarchy among dimensions is established. This conclusion was supported further by [6], where in addition nontrivial fluxes were included (the 3-form gauge field of 11D supergravity). In fact, the presence of fluxes seemed to enlarge the possible space of initial conditions that lead to three large dimensions at late times. Specifically, for the case of six initially unwrapped dimensions, the dynamics of fluxes introduced a new hierarchy suppressing the growth of three out of the six unwrapped dimensions.

An apparent limitation of the BV argument is that it seems to depend crucially on noncontractible spatial cycles and their associated topologically stable winding modes. Phenomenologically viable compactifications of string theory, however, may not have such cycles. Nonetheless the authors of [7] surmised that these more general spaces might still support ‘‘pseudowound’’ modes, long strings that extend around a dimension but are contractible. If these strings are stable over time scales larger than the cosmological Hubble scale, then as far as the dynamics are concerned they play the same role as stable wound strings. In [7], using numerical simulations for string networks on toroidal orbifolds with trivial fundamental group, the authors showed that pseudowound strings generically do persist for many Hubble times, suggesting that the requirement of noncontractible cycles can be relaxed.

The results up to this point seemed promising, but it remained to actually test the heart of the argument: whether at early times, thermal fluctuations near the Hagedorn era and string (or brane) interactions really lead to annihilation of winding modes in a three-dimensional subspace. An early attempt to investigate this was carried out in [8]. The authors considered a gas of 2-branes and supergravity particles, along with excitations on the branes that lead to a limiting Hagedorn temperature. This setup was within the low-energy limit of M theory compactified on a 10-torus, with an anisotropic and homogeneous metric evolving according to 11-dimensional Einstein gravity. The winding numbers of 2-branes evolved according to Boltzmann equations. The authors assumed initial conditions in which the total volume of the torus was fixed but otherwise assumed that all states were equally likely. By numerically solving the coupled Boltzmann-gravity equations the authors concluded that the number of unwrapped dimensions at late times depended crucially on the initial volume of the torus. Typically a large (and monotonically increasing) overall volume would decrease the interaction cross section of branes too quickly, eventually leading to brane number freeze-out. If the initial volume was constrained according to holographic arguments, the initial winding numbers proved so small that all dimensions would decompactify early on. Three dimensions was not found to be singled out by the dynamics.

Similar all-or-nothing behavior was found in [9,10] for IIA theory compactified on T^9. In these papers the dilaton-gravity equations for the background were coupled to Boltzmann equations for winding modes and radiation. Even though this behavior was attributed to the rolling of the dimensionally reduced coupling to weaker values, we emphasize here a more decisive phenomenon that prevents the annihilation of the winding modes and yields the all-or-nothing behavior. If the initial energy density of the universe is large, the system is found in the Hagedorn phase with a significant amount of winding present in thermal abundance but in a regime that resembles a matter-dominated universe with vanishing pressure. This, along with ‘‘friction’’ due to the dilaton’s velocity, results in an insignificant growth of the wrapped dimensions (even over an infinite amount of time). With the total energy nearly constant, the equilibrium number of winding modes generically does not fall to zero. In this sense the system stays ‘‘trapped’’ in the Hagedorn phase. It is very likely, however, that this problem is particular to the approximation of treating the background with the lowest order dilaton-gravity dynamics. Corrections to these [11], or a different treatment of the metric degrees of freedom, could alleviate it. An alternative approach would be to keep the background dynamics to lowest order and still consider a high density initial phase—a fairly natural assumption—but consider large volume fluctuations that could yield an exit. This is the approach we adopt here.

Finally, and most importantly for our purposes, the aforementioned problems were independent of the number of dimensions growing large. The reason was that the rate at which wound strings annihilated only fell off like the inverse volume of the transverse dimensions. This failed to single out three large dimensions as special, suggesting that the Brandenberger-Vafa argument might not be supported by the dynamics underlying string/M theory.

In this paper, we reexamine this conclusion and suggest a possible way in which string dynamics may indeed favor three large dimensions. Our basic approach is this: According to the Brandenberger-Vafa dimension-counting argument, one expects that string interaction rates should be dramatically suppressed when the number of large spatial dimensions is bigger than three. Moreover, as the dimension-counting argument is purely classical, one expects it to be valid in a regime where the wound strings behave nearly classically and can be regarded as one-dimensional extended objects tracing a two-dimensional world volume. In such a regime the quantum thickness of the strings should be small compared to their length along the dimension they wrap and also small compared to the size of the transverse space. This suggests, in contrast to our previous work, that we hope for a *dilute* gas of winding strings. Furthermore, as we discuss, in a dilute regime we are led to work in an impact parameter representation of the string scattering amplitude. As we will see, this makes manifest the distinction between three and more large spatial dimensions regarding the interactions of winding modes. Our main observation is that if the universe fluctuates out of an initial dense Hagedorn regime—something that we believe is generically necessary in order to match to a realistic expanding cosmology in the string gas scenario independently of the decompactification mechanism—then any residual winding modes that were thermally excited in the Hagedorn phase are indeed diluted enough so that they freeze-out in $d > 3$ spatial dimensions. Further, for $d = 3$, the enhancement in interaction rates due to the length of wound strings generically overcomes the suppression due to the weak coupling and winding modes may annihilate efficiently. While a number of important issues remain, this appears to be the first demonstration of dynamical string theory decompactification that generically yields three large spatial dimensions.

By way of outline we begin with a discussion of the impact parameter representation, proceed to set up our model for the string gas, and finish with a numerical simulation, along the lines of [9], that will allow us to identify the regions of phase space in which three or more spatial dimensions decompactify.

II. INTERACTION AMPLITUDES AND IMPACT PARAMETER PICTURE

In this section we derive interaction rates for wound strings in a semiclassical impact parameter picture. We will show that when long strings interact at impact parameters larger than their thickness, there is an exponential suppression in the interaction rates for $d > 3$.

The starting point is the Virasoro-Shapiro amplitude for wound strings in $d = D - 1$ large dimensions given by [12,13]

$$A(s, t) = -\kappa_{D-2}^2 \frac{s^2}{t}(\alpha' s/4)^{\alpha' t/2} e^{-i\pi\alpha' t/4} \tag{1}$$

with s computed either from the right-moving or left-moving momenta of the closed string, $s \approx 4R^2/\alpha'^2$ with R the radius of the dimension that the strings wrap. The imaginary part of the amplitude as $t \to 0$ is

$$\mathrm{Im}\, A(s, t = 0) = \frac{\alpha' \pi}{4} \kappa_{D-2}^2 s^2. \tag{2}$$

Here $\kappa_{D-2}^2 = \kappa^2/V$ is the gravitational coupling in $D - 2$ dimensions, where V is the transverse compactification volume times the area of the torus wrapped by the strings [12]. By the optical theorem $\frac{1}{s}\,\mathrm{Im}(A(s,t))|_{t=0} \sim \alpha' \kappa_{D-2}^2 s$ controls string interactions. It is crucial to observe that for $D = 4$ this quantity is dimensionless and gives the probability for two colliding winding strings to interconnect and unwind (to leading order), while for $D > 4$ it has units of $(\mathrm{length})^{D-4}$ and represents a cross section in the $D - 4$ dimensions transverse to the moving strings. This reflects the fact that long strings generically intersect in $D = 4$, like point particles moving on a line, while they generically miss in $D > 4$ and the relevant quantity becomes a cross section.

One can consider the interaction probability in an impact parameter picture. As discussed above, long wound strings have an effective impact parameter in the $D - 4$ directions transverse to the motion of both strings. The impact parameter b is the conjugate variable to the transverse momentum $q = \sqrt{-t}$, and the amplitude in this representation is obtained by the following transform in the transverse directions:

$$A(s, b) = \int \frac{d^{D-4}q}{(2\pi)^{D-4}} e^{-iqb} \frac{A(s, t)}{s}. \tag{3}$$

For the Virasoro-Shapiro amplitude (1), using $q^{-2} = \alpha' \int_0^1 dx x^{\alpha' q^2 - 1}$, this gives

$$\begin{aligned} A(s, b) &= \alpha' \kappa_{D-2}^2 s \int_0^1 \frac{dx}{x} \\ &\quad \times \int \frac{d^{D-4}q}{(2\pi)^{D-4}} e^{-(Y - i(\pi/4) - \log(x))\alpha' q^2 - ibq} \\ &= \frac{\kappa_{D-2}^2 s}{4\pi^{(D/2)-2}} b^{6-D} \gamma\left(\frac{D}{2} - 3, \frac{b^2/(4\alpha')}{Y - i\frac{\pi}{4}}\right), \end{aligned} \tag{4}$$

where $Y = \log(\frac{\alpha' s}{4})$ and $\gamma(a, x)$ is the lower incomplete gamma function. The imaginary part of the above amplitude in the limit $b^2 \gg Y\alpha'$ is

$$\mathrm{Im}\, A(s, b) \to \frac{\pi \alpha' \kappa_{D-2}^2 s}{4(4\pi Y \alpha')^{D/2-2}} e^{-(b^2/4Y\alpha')}. \tag{5}$$

These results are similar to those found in [14], the difference being that the authors of [14] consider graviton scattering and take the number of transverse directions to be $D - 2$. In fact, the interpretation of b as a classical impact parameter in (3) can be justified along the lines of [14]. In the high energy limit ($s \to \infty$ which for wound strings is $R \to \infty$—precisely our limit of interest) the authors of [14] sum up the amplitude to all loop orders to a unitary eikonal form. The large R or large energy limit localizes strings in the transverse directions and reveals classical behavior, much as the eikonal treatment in quantum mechanics (or optics) reveals semiclassical particle (or ray) behavior.

Note that $A(s, b)$ is dimensionless for any D. It determines the annihilation probability $P(b)$ via

$$P(b) = \frac{1}{v} \mathrm{Im}(A(s, b)) \tag{6}$$

with $\mathrm{Im}A(s, b)$ as in Eq. (5) and v the velocity of the colliding strings in their center of mass frame. This prescription can be shown to satisfy the usual unitarity conditions in the large s limit [15–17].

The quantity $\Delta x^2 \equiv 4Y\alpha' = 4\alpha' \log(R^2/\alpha')$ appearing in (5) is interpreted as the quantum thickness of the string. It measures the fluctuations about the classical straight string configuration. The fact that it increases logarithmically with the string's length reflects that it is energetically less costly to excite oscillators on a long string. Similar string spreading effects occur in high energy collisions and for strings falling into black holes [18]. Note that this string spreading does not include the effect of real (as opposed to virtual) oscillator excitations as would be appropriate in the Hagedorn phase of a string gas. In the Hagedorn phase wound strings are highly excited and their spread in the transverse directions is comparable to the length of the dimension they wrap [19].[1] These wiggly strings are very likely to intersect, leading to rapid interactions which keep the strings in equilibrium. But as the universe expands and cools down, the equilibrium phase becomes one of pure radiation. Then the oscillator excitations decay away and the spread of the wound strings approaches Δx. This justifies our use of the amplitude (5) if $b > \Delta x$.

It is useful to contrast the impact parameter picture to the more standard method of obtaining a scattering probability. Typically one derives a cross section σ, and the collision probability is simply $n\sigma$ where n is the number of targets per transverse volume. If one has a collision probability in impact parameter space, $P(b)$, then the scattering cross section is obtained via [20,21]

$$\sigma = \frac{1}{n}\int d^{\perp} b n P(b). \tag{7}$$

In other words, $n\sigma$ is an averaged probability in impact parameter space. Most frequently it is assumed that the targets are uniformly distributed ($n = \text{const}$) and one obtains

$$\sigma = \int d^{\perp} b P(b). \tag{8}$$

In the case of the optical theorem, for example, one can immediately derive $\sigma = \frac{1}{vs}\,\mathrm{Im}A(s, t = 0)$ using Eqs. (8), (6), and (3). It is the $n = \text{const}$ assumption that we are willing to relax here. There are two ways in which it can be justified. First, if the targets are dense and uniform as in collider experiments, a test particle in that case will interact with targets at all impact parameters so one can integrate as in (8). Second, if the time between collisions is much smaller than the total time over which collisions take place, then the test particle is given enough time to interact with targets at all impact parameters (assuming each collision is at a random impact parameter) and the averaging over impact parameters is essentially a time average.

But if the winding modes in a string gas are dilute, with a mean separation much larger than their thickness, the dense target assumption above does not apply. It could still be that, since the strings move in a compact space, they collide repeatedly with each other and a time average is appropriate. It then becomes a matter of time scales. We need to compare the mean time between collisions with the recollapse time, the time required for winding modes to pull the universe back to a small-radius regime where winding modes are no longer dilute. An additional effect which must be taken into account is that the string coupling is time dependent. This could also invalidate the use of a time-averaged cross section.

We thus have to develop a model for the distribution of interactions over impact parameters. We will return to this in the next section after we set up the rest of the dynamics.

III. EQUATIONS OF MOTION

In this section we write down coupled dilaton-gravity and Boltzmann equations for the matter degrees of freedom. For further details on the thermodynamic phases and energy conservation see [9,11] and references therein.

The general setup is similar to [9], except that instead of an anisotropic universe we consider d growing dimensions all with the same radius, and hold the remaining $9 - d$ dimensions frozen at the self-dual radius. By removing the randomness in the choice of initial radii present in [9] we can see more clearly the dependence of the winding annihilations on the number of growing dimensions.

We consider type IIA string theory with a flat Friedmann-Robertson-Walker metric on a torus for the $d = D - 1$ growing dimensions,

$$ds^2 = -dt^2 + \alpha' e^{2\lambda(t)} \sum_i dx_i^2, \qquad 0 \le x_i \le 2\pi, \tag{9}$$

and a homogeneous shifted dilaton $\varphi(t)$. From now on we set $\alpha' = 1$. When the metric and dilaton are coupled to matter, the equations of motion are

$$\ddot{\varphi} = \frac{1}{2}(\dot{\varphi}^2 + d\dot{\lambda}^2), \qquad \ddot{\lambda} = \dot{\varphi}\,\dot{\lambda} + \frac{1}{8\pi^2} e^{\varphi} P, \tag{10}$$

and the Hamiltonian constraint (Friedmann equation) is

$$E = (2\pi)^2 e^{-\varphi}(\dot{\varphi}^2 - d\dot{\lambda}^2). \tag{11}$$

Here E is the total energy in the string gas and P is the pressure (times the volume) in d dimensions.

A. Matter content and Boltzmann equations

The background equations of motion are coupled to phenomenological Boltzmann equations that govern the evolution of matter. We model matter with three species:

[1] In the Hagedorn phase strings perform a random walk in all directions. As their energy scales with their length, their mean extent in all directions scales as $\sqrt{E}$. This is the dependence of the winding number on energy as we will see in the section on thermodynamics.

(i) Winding modes that evolve according to

$$\dot{W} = -\Gamma_W(W^2 - \langle W\rangle^2). \quad (12)$$

We specify the interaction rates Γ and the thermal equilibrium values $\langle\cdot\rangle$ below. The total energy in winding and antiwinding modes is $E_W = 2dWe^{\lambda}$ and their contribution to the pressure is $P_W = -2We^{\lambda}$.

(ii) Radiation, or pure Kaluza-Klein (KK) modes, evolve according to

$$\dot{K} = -\Gamma_K(K^2 - \langle K\rangle^2). \quad (13)$$

The energy in KK and anti-KK modes is $E_K = 2dKe^{-\lambda}$, and their pressure is $P_K = 2Ke^{-\lambda}$.

(iii) Finally we include string oscillators, or massive string modes, as pressureless matter. The oscillator modes fill up the energy budget via $E_{\text{osc}} = E - (E_W + E_k)$. We do not need a Boltzmann equation for these modes since the dilaton-gravity equations of motion automatically conserve energy, $dE = -PdV$.

B. Thermodynamic phases and interaction rates

Near the self-dual radius the gas of strings is in a high density Hagedorn phase. The thermodynamics of this phase has been studied in [22,23]. The quantities of interest here are the equilibrium values of the winding and KK numbers[2]

$$\langle W\rangle = \frac{1}{12}\sqrt{\frac{E}{\pi}}e^{-\lambda}, \qquad \langle K\rangle = \frac{1}{12}\sqrt{\frac{E}{\pi}}e^{\lambda}. \quad (14)$$

Since $E \gg 1$ most of the energy in the Hagedorn phase resides in oscillator modes ($E_{\text{osc}} \simeq E - \sqrt{E}$).

As the volume of the universe grows and the energy density drops, the equilibrium state should be one with only radiation.[3] As the condition for that transition we set

$$\frac{E}{V_d} \leq c_d T_H^{d+1} \quad (15)$$

with T_H the Hagedorn temperature, $V_d = (2\pi)^d e^{d\lambda}$ the volume of the large dimensions, and c_d a Stefan-Boltzmann constant appropriate to the IIA gas of 128 massless Bose and Fermi degrees of freedom in d dimensions,

$$c_d = 128\frac{2d!\zeta(d+1)}{(4\pi)^{d/2}\Gamma(d/2)}(2 - 1/2^d). \quad (16)$$

In this phase the equilibrium values are

$$\langle W\rangle = 0, \qquad \langle K\rangle = \frac{1}{2d}Ee^{\lambda}. \quad (17)$$

That is, at equilibrium all the energy is in radiation (KK and anti-KK modes).

Now we need to specify the interaction rates entering the Boltzmann equations. Recall that for winding modes, with an impact parameter b in $D-4$ dimensions, the interaction probability is $P(b) = \frac{1}{v}\,\text{Im}A(s, b)$ with $\text{Im}A(s, b)$ given in (5). For two wound strings moving in the x_1 direction and with opposite winding along x_2, the right-moving momenta are $p_{R1,2} = (E, \pm Ev, \pm R/\alpha')$ so $s_R = -(p_{R1} + p_{R2})^2 = 4E^2 \simeq (2R/\alpha')^2$ for slowly moving strings. Putting things together, the interaction probability per unit time (per winding mode in the direction of motion) can be written as

$$\Gamma_W = \Gamma_0 \times \Gamma_b \equiv \frac{\pi\alpha'}{4}\frac{\kappa_{10}^2}{V}\left(\frac{2R}{\alpha'}\right)^2\left[\left(\frac{2\pi R}{(\pi\Delta x^2)^{1/2}}\right)^{D-4} e^{-(b^2/\Delta x^2)}\right] \quad (18)$$

with V the total spatial volume of the 9-torus. In terms of our variables, and with $\alpha' = 1$, we have $\frac{\kappa_{10}^2}{V} = \frac{1}{2(2\pi)}e^{\varphi}$ and $R = e^{\lambda}$. Note that Γ_0 is the interaction rate used in [9].

As explained earlier, an impact parameter representation is appropriate only in the radiation phase, when the separation between winding modes r is larger than the string thickness Δx. From the thermodynamic distributions of [23] we can estimate how the mean velocity $\bar{v}$ of a single winding mode depends on R and E (see Appendix A).[4] The mean time between collisions, or recollision time, is then $t_r \simeq \frac{r}{\bar{v}}$. In practice, as we numerically integrate the equations of motion, once we are in the dilute regime we randomly choose an impact parameter b on every recollision time. The impact parameter is chosen at random, from a uniform distribution in the transverse $D-4$ dimensions, up to the maximum value $b = r$. In effect, under the assumption of isotropy and even distribution of winding modes, we treat their interactions as d-many copies of a single interaction over a periodic lattice r^d ($d = D-1$). We have now enough information to determine Γ_b.

Another concern, raised earlier, is that in the dilute regime the winding strings might not have time to collide before the universe recollapses to a dense Hagedorn phase. In principle this could happen even in $D = 4$. We test for this as follows. Upon entering the dilute regime we estimate the recollision time t_r and turn off interactions, i.e. set $\Gamma = 0$. If the negative pressure from the frozen winding

[2]These values are derived under the assumption that the microcanonical energy is split equally amongst all dimensions; see [9].

[3]See [24,25] for a detailed treatment of the conditions for equilibrium between massive and massless modes in a string gas.

[4]Even though we are working off-equilibrium we consider the equilibrium velocities to be a good approximation. In other words, we are assuming kinetic equilibrium and explore the possibility of chemical equilibrium. To be precise, note that $\bar{v} = v_{\text{rms}} = \sqrt{\langle v^2\rangle}$ is the root-mean-squared velocity.

modes recollapses the universe in a time smaller than t_r, it means that freezing the interactions was consistent; that is, the winding modes truly had no time to collide. On the other hand, if after time t_r we are still in the dilute regime, then string interactions must be taken into account.

Thus (18) provides our description of string interactions in the dilute regime. In the Hagedorn phase the strings have highly excited oscillator modes which enhance the interaction rates since more string is available. This was studied in [9], and it amounts to inserting an overall factor of $\frac{16}{9}E$ in the Boltzmann equations.

We also need to specify the interaction rate for KK modes. Since the wavelength of these modes grows with R, a semiclassical impact parameter picture at large R is not appropriate. Instead we should average over impact parameters. Since we already know the averaged interaction rate Γ_0 for winding modes, by T-duality we can take $\Gamma_K = \Gamma_0|_{\lambda\to-\lambda}$.

C. Initial conditions

We need to integrate the coupled Eqs. (10), (12), and (13) subject to the constraint (11). We need six initial conditions: φ_0, $\dot\varphi_0$, λ_0, $\dot\lambda_0$, W_0, and K_0. In this section we describe our method for sampling from the space of initial conditions. The basic idea is to fix the value of $\dot\varphi$, scan over allowed values of λ_0 and φ_0, and average over the values of $\dot\lambda_0$, W_0, and K_0 using a suitable probability distribution.

For effective supergravity to be valid and to ensure weak coupling we take $\dot\varphi_0 = -1$ and $\varphi_0 \lesssim -1$. Recall that the dilaton evolves monotonically to weaker coupling with the absolute value of its velocity decreasing [9].

With $\dot\varphi_0$ and φ_0 fixed we consider the universe at equilibrium with $\dot\lambda = 0$ and $\lambda = 0$ (at the self-dual radius). Calculating the energy $E_{\rm eq} = E|_{\dot\lambda=0}$ determines the equilibrium number of winding and KK modes per dimension the universe would have at the self-dual radius,

$$\langle W\rangle_{\rm sd} = \langle K\rangle_{\rm sd} = \frac{1}{12}\sqrt{\frac{E_{\rm eq}}{\pi}}.$$

We will use these values to set upper (lower) bounds for choosing W_0 (K_0) below, once the volume fluctuates to larger values.

Next we choose the scale factor λ_0 of the d-torus.[5] We take $\lambda_0 > 0$ since by T-duality we need not consider smaller volumes. Having already determined $E_{\rm eq}$, choosing λ_0 fixes the energy density and thus the equilibrium thermodynamic phase. The choice of λ_0 can be thought of as merely a choice of initial condition, but since we choose the winding and KK modes with respect to their values at $\lambda = 0$ (see below) it is more appropriate to interpret it as a fluctuation from $\lambda = 0$ to a larger volume. Since we are choosing λ_0 at random, we are then assuming the thermal distribution to be flat. This is not an arbitrary choice. The entropy to leading order in the Hagedorn phase is $S_0 \simeq E/T_H$, and thus it costs no entropy or energy for such a fluctuation. To next order the dependence of the entropy on the radii $R = e^\lambda$ for d spatial dimensions is [23,26]

$$S_1 \simeq \log[1 - \Gamma(2d)^{-1}(\eta E)^{2d-1}e^{-\eta E}]$$

with $\eta \sim 1/R^2$. Even though this contribution is very small for the ranges of energies and radii we consider, it would be interesting to study more precisely the effect that these corrections give to our scenario.

The Hubble rate $\dot\lambda$ can also fluctuate away from zero, which is the value that maximizes the entropy. In the Hagedorn phase the entropy is given to a good approximation by $S = E/T_H$. Thus $\dot\lambda_0$ is chosen randomly from the Gaussian distribution

$$e^S \propto e^{-\dot\lambda_0^2/(2\sigma_H^2)} \tag{19}$$

with $\sigma_H^2 = \frac{T_H e^{\varphi_0}}{2(2\pi)^2 d}$. In the radiation phase the entropy is $S = \frac{d+1}{d} c_d V_d (\frac{E}{c_d V_d})^{d/(d+1)}$. Using (11), to leading order in $\dot\lambda_0$ we have the distribution

$$e^S \propto e^{-\dot\lambda_0^2/(2\sigma_r^2)} \tag{20}$$

with

$$\sigma_r^2 = \left(\frac{\rho}{\rho_H}\right)^{1/(d+1)}\sigma_H^2, \qquad \rho = \frac{E|_{\dot\lambda=0}}{V_d}, \qquad \rho_H \equiv c_d T_H^{d+1}. \tag{21}$$

It remains to choose the initial winding and KK numbers. Depending on whether we are in the radiation or Hagedorn phase, the equilibrium number of winding modes could be zero or not. Since we do not want to begin with zero winding (we would not be testing the BV mechanism in that case) the lowest value of W we may pick is 0.5, our chosen threshold between zero and nonzero winding. The furthest we can fluctuate from equilibrium (the largest winding) is $W_{\rm sd}$. However, it is possible that the volume is so large that there is not enough energy to support that much winding. This occurs if $W_{\rm sd} > Ee^{-\lambda}/(2d)$. Putting everything together, the initial winding number is chosen randomly in the range

$$\left(\mathrm{Max}\{0.5, \langle W\rangle\}, \mathrm{Min}\left\{\langle W\rangle_{\rm sd}, \frac{E}{2d}e^{-\lambda}\right\}\right). \tag{22}$$

In the Hagedorn phase the KK number can fluctuate between $\langle K\rangle_{\rm sd}$ and the equilibrium value at the given λ_0, so we choose a value randomly in this range. In the radiation phase, given W_0, we compute the energy in winding $E_W = 2dW_0e^\lambda$. The rest of the energy should be available to

[5]We consider the dynamics of d dimensions, while the remaining $9 - d$ are kept frozen at the self-dual radius, $\lambda = 0$ with $\dot\lambda = 0$ for all times.

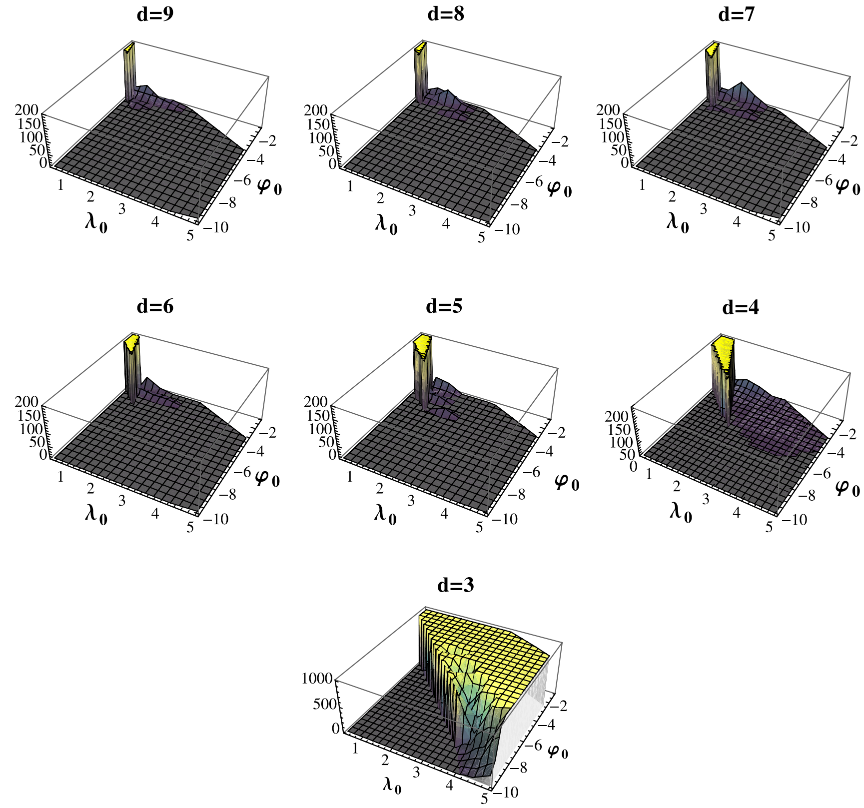

FIG. 1 (color online). Number of cases decompactifying as a function of φ_0 and λ_0 for different choices of growing dimensions d. For $d = 4, \ldots, 9$ the z axis is clipped at 200 to make the fewer decompactifying cases visible.

radiation with the maximum KK number being $K_{\max} = (E - E_W)e^{\lambda}/(2d)$. Therefore K_0 is chosen randomly in the range $(\langle K \rangle_{\rm sd}, K_{\max})$. If $K_{\max} < \langle K \rangle_{\rm sd}$, we set $K_0 = K_{\max}$.[6]

Once initial conditions are fixed, we integrate the equations of motion until either the winding modes annihilate ($W < 0.5$) or the interactions freeze out, which we define as $\Gamma_0 W < 0.1H$. We use the maximum rate Γ_0 instead of the total Γ_W to allow for the possibility that, depending on the randomly chosen value for b, strings could interact even for $D > 4$.

IV. RESULTS

With $\dot{\varphi}_0$ fixed at -1 we can scan initial conditions over a two-dimensional lattice of points (λ_0, φ_0). For each lattice point we do 1000 runs to average over different choices of $\dot{\lambda}_0$, W_0, and K_0.

The results for values of d ranging from 9 to 3 are contrasted in Figs. 1 and 2. For all values of d we find that equilibrium is maintained during the Hagedorn phase. A consequence of this is that for a large range of initial conditions, as long as the system starts out in the Hagedorn phase, it will remain forever trapped there (see Fig. 2). This is related to the limiting value of $\lambda(t)$ as $t \to \infty$ in the solution to the equations of motion (10) in thermal equilibrium [11], and it is also the same behavior found in [26] at exact equilibrium. When the volume is large enough such that $\langle W \rangle \to 0$[7] yet the system is still in the Hagedorn phase, then the universe decompactifies for any d. This region gets narrower as d increases as seen in Fig. 2. But for $d > 3$ this is essentially the only region in parameter space where decompactification occurs. If the system gets to the radiation era, as the oscillators decay to massless modes and the long strings start diluting, hardly any choice of initial conditions leads to decompactification. For $d > 3$ and outside the Hagedorn phase there are few cases (of order 1%) in which the universe decompactifies. Those rare cases have a very small winding number and happen to have collisions at small impact parameters.

By contrast, for the $d = 3$ case shown in Fig. 1, we see that even in the radiation phase long strings are able to annihilate. It is interesting to note that the effects of large λ enter in two competing ways. First, because of the factor of $s \sim R^2 = e^{2\lambda}$ in the amplitude, long strings interact more efficiently, even at weak coupling. But at large λ the effect

[6]When integrating the equations of motion we need to be careful not to produce more radiation than energy conservation allows. If at some point $E = E_K + E_W$, that is, all the oscillators decay, we set $\langle K \rangle = (E - E_W)e^{\lambda}/(2d)$.

[7]In practice this would be $\langle W \rangle < 0.5$. We round W to zero and not $\langle W \rangle$.

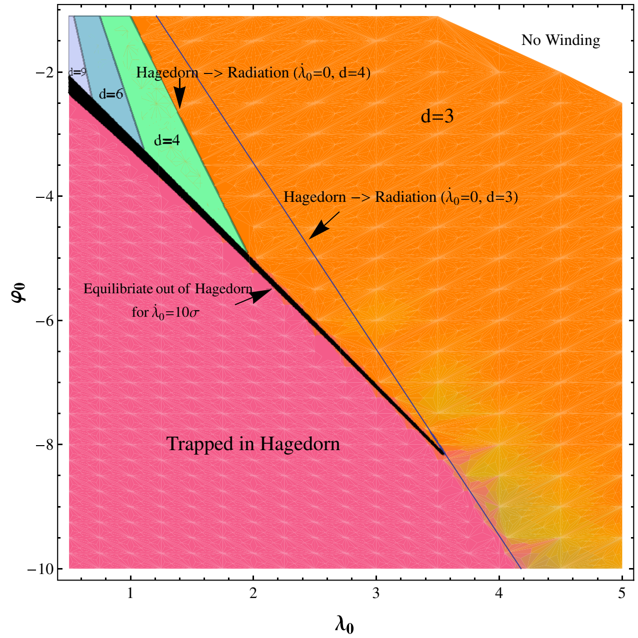

FIG. 2 (color online). A plot of the (λ_0, φ_0) plane contrasting the cases $d = 3, 4, 6, 9$. If the initial equilibrium winding number in the Hagedorn phase is nonzero, the system typically stays trapped in the Hagedorn phase, unless the initial Hubble rate is large and the initial winding number is small (thin dark region labeled "Equilibriate out of Hagedorn for $\dot{\lambda}_0 = 10\sigma$"). If the initial equilibrium winding number is zero in the Hagedorn phase, then the universe typically decompactifies in any number of dimensions (regions in the upper left corner, labeled by dimension). But if the universe begins in a radiation phase with a dilute gas of winding strings, then only $d = 3$ will decompactify (orange region to the right of the grey line).

of dilution is more dramatic and strongly (exponentially) suppresses interactions for $d > 3$.

A comment on the choice $\dot{\varphi}_0 = -1$. We could have considered smaller (absolute) values for $\dot{\varphi}_0$, still valid within supergravity. This can be compensated by a small (logarithmic) shift in the initial dilaton to smaller values such that the initial energy remains the same. The qualitative results should be unaltered.

V. SUMMARY AND DISCUSSION

The Brandenberger-Vafa mechanism relies on a classical dimension-counting argument, namely, that the world volumes of one-dimensional objects will generically intersect in at most three spatial dimensions. To see the mechanism at work one needs to be in a regime where strings behave as semiclassical one-dimensional objects. That is, one needs their length to be much larger than their effective quantum thickness, and also their thickness to be much smaller than the size of the transverse directions (the space is compact). We realized these conditions in a simple isotropic setup where the length of the string R was the same as the size of the compactification manifold. With oscillators excited, as in the Hagedorn phase, strings have a significant spread in all directions and the classical picture fails. But when oscillators decay, as in the radiation phase, the thickness of strings grows as $\Delta x \sim \sqrt{\log R}$. Thus strings begin to behave classically as $R/\sqrt{\log R}$ grows. To model string interactions in this regime we developed an impact parameter representation of the string scattering amplitude. This allowed us to show that in this regime the BV mechanism indeed operates and favors decompactification of three spatial dimensions.

To enter this regime we had to consider departures from equilibrium, often large. Clearly in the radiation phase this is necessary since the equilibrium number of winding strings is zero. In the Hagedorn phase strings rapidly come to equilibrium and the pressure vanishes. This means the universe tends to remain stuck in the Hagedorn phase, and for some number of dimensions to decompactify a large fluctuation is needed, either in the Hubble rate or in the initial volume, to send the system to a regime where the equilibrium winding number is zero. As the distribution (19) typically allows for only small fluctuations in the Hubble rate, we had to consider large fluctuations in the volume to realize the BV mechanism. An important next step would be to understand the likelihood of such a fluctuation taking place in the early universe.

One shortcoming of our framework was that, even though we considered a dilute gas of winding strings, we modeled the resulting pressure as a homogeneous term in the gravity equations of motion. This led us to consider their backreaction on spacetime in an all-or-nothing manner, in which any amount of winding would oppose expansion while zero winding would not. As far as testing the interactions and eventual annihilation of strings, which was our focus, this should not be a concern. But a more detailed investigation of string gas cosmology should address the issue of spatial inhomogeneity. Finally it would be interesting to extend the analysis of this paper to the more general context of M theory, taking into account the effects of the full p-brane spectrum.

ACKNOWLEDGMENTS

D. K. is supported by U.S. National Science Foundation Grant No. PHY-0855582 and PSC-CUNY Grant No. 60038-39-40, and is grateful to the Aspen Center for Physics where this work was completed. S. M. is supported by an A. G. Leventis Foundation grant.

APPENDIX: ROOT-MEAN-SQUARE VELOCITY OF WINDING MODES

Thermodynamic quantities can be calculated using string distributions derived by Deo, Jain, and Tan [22,23] in the microcanonical ensemble. They show that the average number of strings with winding charge vector $\mathbf{w}$, Kaluza-Klein charge vector $\mathbf{k}$, and energy ϵ on a D-torus with total energy E is given by

$$D(\epsilon, \mathbf{w}, \mathbf{k}, E) = \frac{N}{\epsilon} u^D e^{-u\mathbf{q}^T A^{-1}\mathbf{q}/4}, \tag{A1}$$

where

$$u = \frac{E}{\epsilon(E-\epsilon)}, \qquad \mathbf{q} = (\mathbf{w}, \mathbf{k}), \qquad N = \frac{(2\sqrt{\pi})^{-2D}}{\sqrt{\det A}},$$

$$A = \begin{pmatrix} \frac{1}{4\pi^2 R_i^2}\delta_{ij} & 0 \\ 0 & \frac{R_i^2}{4\pi^2}\delta_{ij} \end{pmatrix}.$$

One can consider a unit winding mode, $\mathbf{w}_1 = (1, 0, \ldots)$, along one of the d large dimensions ($R_i = R$, $i = 1, \ldots, d$ and $R_j = 1$, $j = d+1, \ldots, D$) and calculate the mean momentum squared of the string.

$$\langle k^2 \rangle = \frac{\int_0^E d\epsilon \int d^D\mathbf{k} k^2 D(\epsilon, \mathbf{w} = \mathbf{w}_1, \mathbf{k}, E)}{\int_0^E d\epsilon \int d^D\mathbf{k} D(\epsilon, \mathbf{w} = \mathbf{w}_1, \mathbf{k}, E)} = \frac{dR^2}{2\pi^2} \frac{\int_0^E \frac{d\epsilon}{\epsilon} u^{D/2-1} e^{-u\pi R^2}}{\int_0^E \frac{d\epsilon}{\epsilon} u^{D/2} e^{-u\pi R^2}}. \tag{A2}$$

Given the total energy E and the radius R, the two integrals above can be evaluated with saddle point methods or numerically. For a heavy winding mode (large R) we have $\langle v^2 \rangle \simeq \frac{\langle k^2 \rangle}{R^4}$ and $v_{\rm rms} = \sqrt{\langle v^2 \rangle}$.

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