Note on a well-known equation in cosmological perturbation theory which is in error

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We have a well-known equation in cosmological perturbation theory which appeared only by several simple algebraic errors made in many textbooks. There have been attempts to modify Newtonian equations aiming to reproduce that incorrect equation. We clarify why such attempts are wrong, present the correct equation to try in the modification, and explain its own limitation as well. We show that *any* form of density perturbation equation is possible by a suitable gauge condition.

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The well-known equation in the literature we would like to address in this work is this one:

$$\ddot{\delta}_x + 2H\dot{\delta}_x - \left[4\pi G\mu(1+w)(1+3w) + c_s^2\frac{\Delta}{a^2}\right]\delta_x = 0, \tag{1}$$

where $\delta \equiv \delta\mu/\mu$ a relative energy-density fluctuation, a the cosmic scale factor, $H \equiv \dot{a}/a$, $w \equiv p/\mu$ and $c_s^2 \equiv \dot{p}/\dot{\mu}$. The subindex x indicates a certain gauge condition chosen. In order to make our argument simple we consider $w =$ constant, thus $c_s^2 = w$, and a flat background with vanishing cosmological constant, $K = 0 = \Lambda$.

Our main point is that Eq. (1) is a wrong equation for $w \neq 0$. We will show that it is possible to derive Eq. (1) in a certain class of gauge conditions. However, we will show that in relativistic perturbation theory by a suitable gauge choice one can derive *any* form of differential equation for δ_x. In the literature, Eq. (1) was derived merely by algebraic errors. There have been attempts to modify Newton's gravity in an *ad hoc* way in order to include effects of pressure. The modifications, unfortunately, were *designed* to reproduce Eq. (1). The correct equation one should aim in such modifications must be Eq. (4) to be presented later. We decide to clear the issue especially because Eq. (1) is quite popular in cosmology textbooks.

Equation (1) appears in several textbooks. Equation (15.10.57) in [1], Eq. (10.118) in [2], problem (6.10) in [3], Eq. (11.65) in [4], Eq. (15.25) in [5], and Eq. (10.9.2) in [6] are in error. All these errors are due to fallible algebraic mistakes made in the synchronous gauge. In the presence of pressure the equation for δ in that gauge becomes a third-order differential equation even in the large-scale limit. In the synchronous gauge, we have [7,8]

$$\ddot{\delta}_\alpha + 2H\dot{\delta}_\alpha - \left[4\pi G\mu(1+w)(1+3w) + c_s^2\frac{\Delta}{a^2}\right]\delta_\alpha = 3c_s^2(1+w)\frac{\Delta}{a}Hv_\alpha, \tag{2}$$

$$\dot{v}_\alpha + (1-3c_s^2)Hv_\alpha = \frac{1}{a}\frac{c_s^2}{1+w}\delta_\alpha, \tag{3}$$

where δ_α and v_α are the same as δ and v, respectively, in the synchronous gauge ($\alpha \equiv 0$). If we ignore the right-hand side of Eq. (2) we recover Eq. (1). Notice, however, that for $w \neq 0$ the term in the right-hand side cannot be ignored even in the large-scale limit (despite the presence of the Laplacian!). Thus, for $w \neq 0$ we inevitably have a third-order differential equation for δ_α; this is because the synchronous gauge fails to fix the gauge mode completely [9–11].

The truncated second-order equation in [1] picks up a gauge mode instead of the physical decaying (in an expanding phase) solution. The error in [2] is based on imposing the synchronous gauge ($\alpha \equiv 0$) and the comoving gauge ($v \equiv 0$) simultaneously, which is not allowed in the presence of pressure, and happens to end up with Eq. (1). In [4–6] the authors proposed a simple modification of the Newtonian theory which was *designed* to reproduce Eq. (1). Many works have been published based on this wrong equation.

We have addressed this issue in Sec. 4 of [7], and Sec. 5 and Appendix A of [8]. Yet another error arriving at Eq. (1) can be identified in Hawking's covariant approach in Sec. VI of [12]; the error was resolved in Sec. III-b-i of [13]. In the following we will present a remaining issue of whether Eq. (1) is possible in the relativistic theory and its possible physical significance.

The correct expression is available in the comoving gauge as [10,14,15]

$$\ddot{\delta}_v + (2-3w)H\dot{\delta}_v - \left[4\pi G\mu(1-w)(1+3w) + c_s^2\frac{\Delta}{a^2}\right]\delta_v = 0, \tag{4}$$

where δ_v is the same as δ in the comoving gauge ($v \equiv 0$) or a unique gauge-invariant combination between δ and v: $\delta_v \equiv \delta + 3(1+w)aHv$, see Eq. (6) below. Later we will explain why we believe this is the equation for density perturbation in a relativistic context. It can be written in a compact form [8]

$$\frac{1+w}{a^2 H}\left[\frac{H^2}{a(\mu+p)}\left(\frac{a^3\mu}{H}\delta_v\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\delta_v = 0, \quad (5)$$

which is valid for general $w \equiv p(\mu)/\mu$, and in the presence of general K and Λ. For a completely general equation in the presence of stresses, see Eq. (45) in [8].

Although nontrivial, it is possible to derive Eq. (1) in the relativistic cosmological perturbation theory; in fact, we will show that *any* form of differential equation for δ is possible in Einstein's gravity by a suitable gauge choice. One simple way is to use a gauge transformation from the equation in the comoving gauge. Under a gauge transformation $\hat{x}^a = x^a + \tilde{\xi}^a(x^e)$ with $\tilde{\xi}^0 \equiv \xi^t/a$, we have [11,16]

$$\hat{\delta} = \delta + 3(1+w)H\xi^t, \qquad \hat{v} = v - \xi^t/a. \quad (6)$$

Let us take x^a and $\hat{x}^a$ as coordinates in the comoving gauge ($v \equiv 0$) and the yet unknown gauge ($x \equiv 0$) used in Eq. (1), respectively. Thus

$$\delta_x = \delta_v + 3(1+w)H\xi^t, \qquad v_x = -\xi^t/a, \quad (7)$$

where $\xi^t = \xi^t_{v\to x}$. Combining Eqs. (1), (4), and (7) we can derive

$$(H\xi^t)^{\cdot\cdot} + 2H(H\xi^t)^{\cdot} - \left[4\pi G\mu(1+w)(1+3w) + c_s^2\frac{\Delta}{a^2}\right]H\xi^t = -\frac{w}{1+w}\left[H\dot{\delta}_v - \frac{8\pi G}{3}\mu(1+3w)\delta_v\right]. \quad (8)$$

As long as the gauge transformation ξ^t satisfies Eq. (8), we can achieve Eq. (1) in the x-gauge condition. In the large-scale limit, using the background solution $a \propto t^{2/[3(1+w)]}$, Eqs. (1), (4), and (8) have solutions

$$\delta_v = c_g t^{([2(1+3w)]/[3(1+w)])} + c_d t^{-[(1-w)/(1+w)]}, \quad (9)$$

$$H\xi^t = w\mathcal{O}(\Delta c_g) + \frac{1}{3(1+w)}(-c_d t^{-[(1-w)/(1+w)]} + \bar{c}_g t^{([2(1+3w)]/[3(1+w)])} + \bar{c}_d t^{-1}), \quad (10)$$

$$\delta_x = (c_g + \bar{c}_g)t^{([2(1+3w)]/[3(1+w)])} + \bar{c}_d t^{-1}. \quad (11)$$

Equation (8) becomes highly complicated in the case of general w with K and Λ. Apparently, it is a nontrivial matter to explicitly specify the gauge condition x.

Although we showed that Eq. (1) is possible in a certain class of gauge conditions, this does *not* imply that the original errors made in the literature can be accepted. Nor does this imply that the *ad hoc* modifications of Newton's gravity aiming to reproduce Eq. (1) are supported. Our above analysis reveals that, in fact, via gauge transformation we can derive *any* form of linear differential equation for δ with arbitrary coefficients. Let us write Eqs. (1) and (4) using differential operators as $\mathcal{L}_x\delta_x = 0$ and $\mathcal{L}_v\delta_v = 0$. Now, using Eq. (6) we can show that as long as $\xi^t \equiv \xi^t_{v\to y}$ satisfies

$$\mathcal{L}_y[3(1+w)H\xi^t] = -\mathcal{L}_y\delta_v, \quad (12)$$

we have an equation $\mathcal{L}_y\delta_y = 0$ in the y-gauge for an *arbitrary* linear differential operator $\mathcal{L}_y$. For $y = x$ we have Eq. (8). By a suitable gauge condition we could even have $\mathcal{L}_y = 0$, so that we have $\delta_y = 0$ in that gauge condition: in fact, this is the well-known uniform-density gauge introduced in [11]. Therefore, existence of an equation like Eq. (1) does not guarantee its physical significance. The point is that Eq. (1) is nothing other than a wrong equation derived by errors made in the literature.

We would like to note that these popular but common errors found in the synchronous gauge are only due to simple algebraic errors made, and are *not* due to any esoteric aspect of the gauge choice available in the relativistic perturbation theory, see [7,8]. We wish to recall that although one may need to do more algebra in tracing the remnant gauge mode in the synchronous gauge this was done carefully in the original study by Lifshitz [9].

As we mentioned above one should aim to reproduce Eq. (4) in any heuristic pseudo-Newtonian attempt. We have the following reasons: (i) All the original derivations of Eq. (1) were based on algebraic errors made in the synchronous gauge; (ii) using gauge transformation we can derive an arbitrary form of differential equation for δ; (iii) in the zero-pressure case the comoving gauge and the synchronous gauge are the only fundamental gauge conditions used in the literature which reproduce the Newtonian density perturbation equation [8,10,11]; (iv) in the presence of pressure the synchronous gauge fails to produce a second-order differential equation; and (v) in the zero-pressure case, the density perturbation equation in the comoving gauge coincides exactly with the one known in Newton's gravity even to the second-order perturbations, see Sec. VII.C in [17,18].

By modifying the Newtonian hydrodynamic equations to include pressure we can easily reproduce the cosmological background equations; see [19] for an early attempt. According to Sachs and Wolfe in [20], "When these modified equations were perturbed to first order, their solutions did not agree with the relativistic results, even qualitatively." As yet we do not have the modification which reproduces the proper relativistic result in Eq. (4). Moreover, even in the case we succeed to get such a modification valid to the linear order, it is *not* justified to use that equation in nonlinear situations; for the density perturbation equation in the presence of pressure to the second order, see Sec. VI and particularly Eq. (132) in [21]. It is dangerous to rely on pseudo-Newtonian equations without proper confirmation in the relativistic gravity. Then, we do not see the use of such modifications except for the heuristic purpose of understanding the already known results in the relativistic theory. Currently, no such a modification is available even to the linear order in the cosmological perturbation.

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