Inflation in models with conformally coupled scalar fields: An application to the noncommutative spectral action

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Slow-roll inflation is studied in theories where the inflaton field is conformally coupled to the Ricci scalar. In particular, the case of Higgs field inflation in the context of the noncommutative spectral action is analyzed. It is shown that while the Higgs potential can lead to the slow-roll conditions being satisfied once the running of the self-coupling at two-loops is included, the constraints imposed from the CMB data make the predictions of such a scenario incompatible with the measured value of the top quark mass. We also analyze the role of an additional conformally coupled massless scalar field, which arises naturally in the context of noncommutative geometry, for inflationary scenarios.

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I. INTRODUCTION

Cosmological inflation is the most widely accepted mechanism to resolve the shortcomings of the standard hot big bang model. This mechanism, leading to a phase of exponential expansion in the very early universe, is deeply rooted in the fundamental principles of general relativity and field theory, and once combined with the principles of quantum mechanics, it can account for the origin of the observed large scale structures and the measured temperature anisotropies of the cosmic microwave background (CMB). However, despite its success, cosmological inflation remains a paradigm in search of a model which should be motivated by a fundamental theory. The strength of the inflationary mechanism is based on the assumption that its onset is generically independent of the initial conditions. Nevertheless, even this issue is under debate [1–6] given the lack of a complete theory of quantum gravity.

The inflaton field (usually a scalar field) is assumed to dominate the evolution of the Universe at early times, but its origin and the form of its effective potential both remain unknown; for this reason it would be attractive if the one scalar field that is commonly thought to exist, namely, the Higgs field, also doubled as the long searched for inflaton. Unfortunately, it seems that if the Higgs field is minimally coupled to gravity this cannot be achieved, which has led some authors to consider large nonminimal couplings of the Higgs field to gravity where inflation might be achieved [7].

It is commonly assumed/chosen that there is no coupling (i.e., minimal coupling) between the inflaton field and the background geometry (the Ricci curvature). However, this assumption/choice seems to lack a solid justification. A first (and merely aesthetic) motivation comes from the observation that in the early Universe (where masses are negligible), the equations of motion for spinors and gauge bosons have a natural conformal invariance in four space-time dimensions, while the same is true for scalar fields only when they couple to the Ricci scalar in a specific way. More compelling is the fact that even if classically the coupling between the scalar field and the Ricci curvature could be set equal to zero, a nonminimal coupling will be induced once quantum corrections in the classical field theory are considered. Moreover, a nonminimal coupling seems to be needed in order to renormalize the scalar field theory in a curved space-time. The precise value of the coupling constant (denoted by ξ) then depends on the choice of the theory of gravity and the scalar field [8]. It has also been argued that in all metric theories of gravity, including general relativity, in which the scalar field is not part of the gravitational sector (e.g., when the scalar field is the Higgs field), the coupling constant should be conformal in order for the short distance propagators of the theory to match those found in a Minkowski space-time—a requirement of the strong equivalence principle [8,9] (in our notation, conformal coupling means $\xi = 1/12$). Finally, in the context of finite theories at one-loop level, it was shown [10,11] that the nonminimal coupling ξ tends either to its conformal value or increases exponentially in modulus, depending on the specific structure of the theory.

In what follows, we will investigate whether scalar fields, and, in particular, the Higgs field, could play the role of the inflaton in the presence of a small positive nonminimal coupling between the scalar field and the background geometry. The coupling constant ξ is not a free parameter which could be tuned to achieve a successful inflationary scenario avoiding severe fine-tuning of inflationary parameters (e.g., the self-coupling of the inflaton field), ξ should instead be dictated by the underlying theory. For negative values of ξ, exponential expansion is more easily achieved than in the minimal case, and it can in fact lead to inflation consistent with observational data in the strong coupling limit [7,12]. In fact, the slow-roll

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parameters for large $|\xi|$ are independent of ξ and only depend on the number of e-folds. However, exponential expansion is less favored for positive values (in our conventions) such as conformal coupling [13]. In light of the motivations for a small positive ξ outlined above, we will investigate whether quantum corrections to the Higgs potential can lead to a slow-roll inflationary era and if so, whether the constraints imposed from the CMB temperature anisotropies are satisfied.

We will apply this analysis to the spectral action of noncommutative geometry (NCG). This theory leads naturally to a Lagrangian with a conformal coupling between the Higgs field and the background geometry, in the form of a boundary condition at high energies $E \geq \Lambda$, where Λ is a characteristic scale of the model. NCG provides an elegant way of accounting for the standard model (SM) of particle physics and its phenomenology [14]. Our motivation is to investigate cosmological consequences of the NCG spectral action and, in particular, to test whether slow-roll inflation driven by one of the scalar fields arising naturally within NCG could be realized in agreement with experimental data and astrophysical measurements.

In a previous study, we (one of us and a collaborator) studied [15] the conditions on the couplings so that the Higgs field could play the role of the inflaton in the context of the NCG. Since, however, the running of couplings with the cutoff scale had been only analyzed [14] neglecting the nonminimal coupling between the Higgs field and the curvature, we were not able to reach a definite conclusion. In this respect, the study below is a follow-up of Ref. [15]. Moreover, it has been argued [16] that inflation with a conformally coupled Higgs boson could be realized in the context of NCG due to the running of the effective gravitational constant. In what follows, we will also analyze the validity of this statement. Finally, the NCG spectral action provides, in addition to the Higgs field, another conformally coupled (massless) scalar field, which exhibits no coupling to the matter sector [17]. One may *a priori* wish/expect that this field could be another candidate for the inflaton; we will examine this scenario as well.

Concluding, we analyze slow-roll inflation within models that exhibit a conformal coupling between the Higgs field and the Ricci curvature. Our motivation is to investigate whether any of the two scalar fields arising naturally within the NCG spectral action could be identified as the inflaton. As we will explicitly show, our analysis leads us to the conclusion that unfortunately such a slow-roll inflationary scenario fails to remain in agreement with current data from high energy physics experiments and astrophysical measurements.

This paper is organized as follows: In Sec. II, we study the issue of the realization of slow-roll inflation within theories with a nonminimal coupling between the scalar field and the Ricci curvature, classically. The analysis is first performed in the Jordan frame and then in the Einstein frame. In Sec. III, we consider corrections to the Higgs potential through a two-loop renormalization group analysis of the minimally coupled standard model; we then enlarge this study in the case of a conformal coupling. We focus on the gravitational and Higgs field sector of the Lagrangian density, obtained within the noncommutative spectral action, which has a conformal coupling, in Sec. IV. We find that even though we can accommodate an era of slow-roll inflation, it seems difficult to reach an agreement with the CMB data. This conclusion holds not only for the Higgs field but also for the other scalar field which appears generically in the theory. We then examine in Sec. V, whether the running of the gravitational constant could modify our conclusions with regards to the realization of a successful inflationary scenario driven through one of the scalar fields in the NCG theory. We round up our conclusions in Sec. VI.

Our signature convention is $(- + + +)$; the Riemann and Ricci tensors are defined as

$$R^{\sigma}{}_{\mu\nu\rho} = \Gamma^{\sigma}_{\mu\rho,\nu} - \Gamma^{\sigma}_{\nu\rho,\mu} + \Gamma^{\tau}_{\mu\rho}\Gamma^{\sigma}_{\tau\nu} - \Gamma^{\tau}_{\nu\rho}\Gamma^{\sigma}_{\tau\mu},$$
$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu},$$

respectively. Note that within our definition of ξ, conformal coupling means $\xi = 1/12$.

II. SLOW-ROLL INFLATION WITH NONMINIMALLY COUPLED SCALAR FIELDS

In this section, we will study whether slow-roll inflationary scenarios can be realized within models with an implicit nonminimal coupling between the inflaton field and the scalar curvature. We will first work in the Jordan frame and then we will perform the analysis in the Einstein frame. The Jordan frame is natural (physical) and offers some useful insights on the effect of conformal coupling, while the Einstein frame is mathematically more convenient, especially when including more complicated corrections to the potential.

A. Analysis in the Jordan frame

Let us consider the action of a Higgs boson (or any other scalar field ϕ) nonminimally coupled to gravity:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}f(\phi)R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right\}, \quad (1)$$

where

$$f(\phi) = 1 - 2\kappa^2\xi\phi^2,$$

with $\kappa \equiv \sqrt{8\pi G} = m_{\rm Pl}^{-1}$ and g being the determinant of the metric tensor. The scalar potential of ϕ is

$$V(\phi) = \lambda\phi^4 - \mu^2\phi^2. \quad (2)$$

The term $-\xi\phi^2 R$ in the action encodes the explicit nonminimal coupling of the scalar field ϕ to the Ricci curvature R.

The background geometry during inflation is of the Fridemann-Lemaître-Robertson-Walker (FLRW) form:

$$ds^2 = dt^2 - a^2(t)d\Sigma, \tag{3}$$

where t stands for cosmological time, $a(t)$ is the scale factor and $d\Sigma$ describes spatial sections of constant curvature.

Einstein's equations read

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R = \kappa^2 f(\phi)^{-1}T^{\mu\nu}(\phi), \tag{4}$$

where the energy-momentum tensor, obtained by varying the action with respect to the metric, is [18,19]

$$T^{\mu\nu}(\phi) = (1-4\xi)\nabla^\mu\phi\nabla^\nu\phi + 4\xi\phi(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu)\phi + g^{\mu\nu}\left[-\left(\frac{1}{2}-4\xi\right)\nabla_\rho\phi\nabla^\rho\phi - V(\phi)\right]. \tag{5}$$

Here $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplace-Beltrami operator and Greek and Latin indices take values 0, 1, 2, 3 and 1, 2, 3, respectively.[1] The equation of motion (Klein-Gordon equation) of the Higgs field reads

$$\Box\phi - 2\xi R\phi - \frac{dV}{d\phi} = 0. \tag{6}$$

For vanishing and quartic potentials, Eq. (6) is invariant under conformal transformations $g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu}$ and $\phi \to \Omega(x)^{-1}\phi$ at conformal coupling $\xi = 1/12$.

For a FLRW background and a spatially homogeneous ϕ, Eqs. (4) and (6) combine to

$$H^2 = \frac{\kappa^2}{3f(\phi)}\left[\frac{1}{2}\dot\phi^2 + V(\phi) + 12\xi H\phi\dot\phi\right], \tag{7}$$

$$0 = \ddot\phi + 3H\dot\phi - \frac{2\xi(1-12\xi)\kappa^2\phi\dot\phi^2}{1-2\xi(1-12\xi)\kappa^2\phi^2} + \frac{8\xi\kappa^2\phi V(\phi) + f(\phi)V'(\phi)}{1-2\xi(1-12\xi)\kappa^2\phi^2}, \tag{8}$$

where overdots denote time derivatives and primes stand for derivatives with respect to the argument [e.g., $V'(\phi) \equiv dV/d\phi$]. Note that $2\xi(1-12\xi)$ is zero at both, minimal (i.e., $\xi = 0$) and conformal (i.e., $\xi = 1/12$) couplings.

Inflationary models are usually built upon the slow-roll approximation, consisting of neglecting the most slowly varying terms in the equation of motion for the inflaton field. However, in the case of positive nonminimal coupling (i.e., $\xi \neq 0$), it is more difficult to achieve the slow-rolling of the inflaton field. More precisely, the nonminimal coupling term in the action, $-\xi\phi^2 R$, plays the role of an effective mass term for the scalar field, distorting the flatness of the scalar potential. Thus, in the case of a nonminimal coupling, inflationary requirements such as $-\dot H < H^2$ (where H denotes the Hubble parameter) do not translate in an equally straight-forward manner to relations on the inflaton fields and their scalar potentials. Indeed, there is no common choice of conditions (see, e.g., Refs. [18,20,21]), and no analog of slow-roll parameters in terms of which quantities such as the number of e-folds of expansion or perturbation amplitudes are evaluated.

With a tentative choice of conditions [18]

$$\left|\frac{\ddot\phi}{\dot\phi}\right| \ll H, \qquad \left|\frac{\dot\phi}{\phi}\right| \ll H, \quad \text{and} \quad \frac{1}{2}\dot\phi^2 \ll V(\phi), \tag{9}$$

and a negligible mass term in the potential at high energies, the energy constraint, Eq. (7), and field equation, Eq. (8), reduce to

$$H^2 \approx \frac{\lambda\kappa^2\phi^4}{3f(\phi)}\left[1 - \frac{16\xi}{1-2\xi(1-12\xi)\kappa^2\phi^2}\right], \tag{10}$$

$$3H\dot\phi \approx -\frac{4\lambda\phi^3}{1-2\xi(1-12\xi)\kappa^2\phi^2}, \tag{11}$$

respectively. These equations determine the background solution, given by

$$a(\phi) = (1-2\xi\kappa^2\phi^2)^{(1/4)}\exp\left[-\frac{1-12\xi}{8}\kappa^2\phi^2\right]. \tag{12}$$

It is the second factor, in Eq. (12) above, which has the potential to generate sufficient number of e-folds, as the first one will only lead to logarithmic corrections. For $\xi \neq 1/12$ (i.e., nonconformal coupling), a large enough change in $\sqrt{|\xi|}\kappa\phi$ can lead to sufficient inflation to resolve the horizon problem. This leaves some room to play with the coupling and the field values, and it has indeed been shown [12,22] in recent literature that inflation can be achieved in a manner consistent with CMB data for large negative $|\xi| \sim 10^4$.

At conformal coupling ($\xi = 1/12$) however, the argument in the exponential vanishes identically. For this particular value the smallness of $\sqrt{|\xi|}$ can thus not be compensated by a larger value of ϕ during inflation to generate the required expansion.

What about quantum corrections to ξ? For values close to conformal coupling, $\delta\xi = \xi - 1/12$, the number of e-folds is approximately

$$N(\phi) = \tfrac{3}{2}\delta\xi\kappa^2(\phi^2 - \phi_e^2), \tag{13}$$

(ϕ_e denotes the value of ϕ at the end of the inflationary era) which requires a minimum initial Higgs field of the order of $\phi \approx \sqrt{N/|\delta\xi|}$. Renormalization group analysis shows that $\delta\xi$ (as a function of the energy scale) is small in the inflationary region, namely, less than $\mathcal{O}(\xi)$ [10,11]. The initial Higgs amplitude required for sufficient number of e-

[1]Note that it is really the tensor $\bar T^{\mu\nu}(\phi) = f(\phi)^{-1}T^{\mu\nu}(\phi)$ which is covariantly conserved rather than $T^{\mu\nu}(\phi)$ [13], but this ambiguity in the choice of the energy-momentum tensor will not be relevant in our analysis.

folds with such values of $\delta\xi$ generally lies above the Planck scale. Whether this implies energies above the Planck mass relies in turn on the value of the parameter λ. Note, however, that the same renormalization group analysis of the nonminimally coupled standard model suggests that there are no quantum corrections to ξ, if it is exactly conformal at some energy scale [10,11]. This is based on the observation that there are no nonconformal values for the coupling ξ for which there is a renormalization group flow towards the conformal value as one runs the standard model parameters up in the energy scale. It thus indicates that if one expects an exactly conformal coupling for the Higgs field at some specific scale, it will be exactly conformal at all scales, hence $\delta\xi = 0$.

The fact that conformal coupling destroys the accelerated expansion has been noted previously [13]. How can conformal invariance be connected to the conditions for inflation? The implications of conformal invariance on the stress-energy tensor are well known: if the matter sector of the theory is invariant under the conformal transformation

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \qquad \phi \to \Omega^{-1}\phi, \tag{14}$$

then the trace of $T^{\mu\nu}$ vanishes covariantly, and hence the scalar curvature R is zero. However, for a FLRW universe the scalar curvature reads

$$R = 6(\dot{H} + 2H^2), \tag{15}$$

and therefore $R = 0$ implies

$$-\frac{\dot{H}}{H^2} = 2, \tag{16}$$

which is, for example, satisfied during the radiation-dominated period of the evolution of a universe in the context of general relativity. However, it rules out inflationary solutions which require[2] $-\dot{H}/H^2 < 1$. Indeed, taking $T^{\mu\nu}(\phi)$ from Eq. (5), its trace evaluates to

$$T^{\mu}_{\mu}(\phi) = -[1 - 12\xi]\nabla_\rho\phi\nabla^\rho\phi + [12\xi\phi V'(\phi) - 4V(\phi)] + 24\xi^2 R\phi^2, \tag{17}$$

having used the equation of motion for the scalar, Eq. (6). However, from Eq. (4), the trace of the energy-momentum tensor of ϕ reads

$$T^{\mu}_{\mu}(\phi) = -\kappa^{-2} f(\phi) R = -\kappa^{-2}(1 - 2\xi\kappa^2\phi^2)R. \tag{18}$$

Thus, Eqs. (17) and (18) imply

$$-[1 - 12\xi]\nabla_\rho\phi\nabla^\rho\phi + [12\xi\phi V'(\phi) - 4V(\phi)] + 24\xi^2 R\phi^2 = -\kappa^{-2}(1 - 2\xi\kappa^2\phi^2)R. \tag{19}$$

Let us analyze Eq. (19): For vanishing ($V = 0$) or quartic ($V = \lambda\phi^4$) potential, conformal invariance ($\xi = 1/12$) implies that the terms in square brackets vanish and the last term on the left-hand side cancels with the last term on the right-hand side, leading to zero scalar curvature R and thus zero trace of the energy-momentum tensor. However, when conformal invariance is broken, due, for example, to a nonzero mass term for the inflaton field (i.e., $\mu \neq 0$), the induced corrections to the scalar curvature are

$$\delta R = 2\mu^2\kappa^2\phi^2. \tag{20}$$

In this case, the inflationary condition $-\dot{H}/H^2 < 1$ requires that $\mu\kappa\phi > \sqrt{3}|H|$, which is not satisfied by a light scalar inflaton.

For a $V(\phi) = \lambda\phi^4$ potential, classical analysis therefore seems to exclude an inflationary regime. However, it is worth investigating whether quantum corrections to the quartic self-coupling λ can induce potential terms that break conformal invariance, and whether this can have a sufficiently strong effect as to enable inflationary solutions. This can happen if these corrections are drastic enough to generate terms in the effective potential which alter the *local* profile of the potential, i.e., $V(\phi) \to V_{\rm eff} = V(\phi) + \alpha\delta\phi$ with $\mathcal{O}((\delta\phi)') \sim \mathcal{O}(V')$. Then the slow-roll parameters will have a different form and may allow inflation.

For slow-roll analysis with more complex potentials, it is convenient to perform a transformation to the Einstein frame, where the action is formulated in terms of a rescaled metric and a new scalar field with a minimal coupling to the curvature scalar of the new metric. Any meaningful conclusions should of course be independent of the choice of conformal frame used during the calculation.

B. Analysis in the Einstein frame

Performing a suitable Weyl transformation, the action, Eq. (48), can be recast in terms of a new metric

$$\hat{g}_{\mu\nu} = f(\phi) g_{\mu\nu} = (1 - 2\xi\kappa^2\phi^2) g_{\mu\nu}, \tag{21}$$

and a canonical scalar field $\chi(\phi)$ that is minimally coupled and related to the Higgs field by

$$\frac{d\chi}{d\phi} = \frac{\sqrt{1 - 2\xi(1 - 12\xi)\kappa^2\phi^2}}{f(\phi)}. \tag{22}$$

It should be noted that the transformation is singular for $\phi_s = 1/(\kappa\sqrt{2\xi})$. In fact, solving for the canonical field χ, one can show that it covers only the range $|\phi| \leq \phi_s$, implying that the analysis in the Einstein frame is valid only for this restricted domain of the original scalar. The value ϕ_s also has special status in the Jordan frame itself. At $\xi = 1/12$ in particular, it was shown that although the scalar field evolves smoothly through ϕ_s in isotropic background cosmologies, its anisotropic shear diverges. We will safely stay below this point in the Einstein frame analysis, still having access to Higgs field values all the way up to the Planck scale, as long as $\xi \leq 1$.

[2]Note that conformal invariance is considered here solely in the matter sector. The Einstein-Hilbert term is not conformally invariant.

The Weyl transformation is not a diffeomorphism and the space-time coordinates are left unchanged, $\hat{x}^\mu = x^\mu$. Now $a(\hat{t}) = a(t)$ is not the FRWL scale factor of the Universe described by the Einstein frame variables. However, by defining a new time coordinate

$$d\hat{\tau} = a(\hat{t})d\hat{t} = a(t)dt, \tag{23}$$

the metric takes the FRWL form in the Einstein frame with a scale factor

$$\hat{a}(\tau) = \sqrt{f(\phi)}a(t), \tag{24}$$

and the Hubble parameter can be defined as

$$\hat{H} = \frac{1}{\hat{a}}\frac{d\hat{a}}{d\hat{\tau}}. \tag{25}$$

This leaves us with an Einstein frame action

$$S_{\rm E} = \int d^4x\sqrt{-\hat{g}}\left\{\frac{1}{2\kappa^2}\hat{R} - \frac{1}{2}(\hat{\nabla}\chi)^2 - \hat{V}(\chi)\right\}, \tag{26}$$

and a scalar potential

$$\hat{V}(\chi) = \frac{V(\phi(\chi))}{[f(\phi(\chi))]^2} = \frac{\lambda[\phi(\chi)]^4 - \mu^2[\phi(\chi)]^2}{[f(\phi(\chi))]^2}. \tag{27}$$

The expression for $\phi(\chi)$ is obtained from Eq. (22) and can be solved analytically for any ξ [23]. In this study however we shall express any functions (e.g., slow-roll parameters) of the Einstein frame in terms of ϕ, the physical degree of freedom, so we leave the new potential in terms of ϕ. Of course, our interpretation of the Einstein frame as *unphysical but mathematically convenient* presupposes that the "observables" computed therein have no immediate physical meaning. We will come back to this point, and particularly the translation from Einstein frame observables to physical Jordan frame observables, later.

It is now possible to look for an inflationary regime within the Einstein frame cosmology. We shall neglect the mass term in the following analysis[3] since we consider energy scales $E \gg \mu$. In terms of the Higgs field ϕ, the canonical first and second slow-roll parameters are given by the formulas:

$$\hat{\epsilon}(\phi) = \frac{1}{2\kappa^2}\frac{1}{\hat{V}^2}\left(\frac{d\hat{V}}{d\phi}\right)^2\left(\frac{d\chi}{d\phi}\right)^{-2}, \tag{28}$$

$$\hat{\eta}(\phi) = \frac{1}{\kappa^2}\frac{1}{\hat{V}}\left[\left(\frac{d\chi}{d\phi}\right)^{-2}\frac{d^2\hat{V}}{d\phi^2} - \frac{d^2\chi}{d\phi^2}\left(\frac{d\chi}{d\phi}\right)^{-3}\frac{d\hat{V}}{d\phi}\right]. \tag{29}$$

The number of Einstein frame e-folds is

[3]It is worth noting that the potential takes a particularly simple form at $\xi = 1/12$ when the (conformal invariance breaking) mass term is neglected: $V(\chi) = 36\lambda\kappa^{-4}\sinh^2(\kappa\chi/\sqrt{6})$.

$$\hat{N} = \int_t^{t_{\rm end}} \hat{H} d\hat{\tau} = \kappa\int_{\chi_{\rm end}}^{\chi}\frac{1}{\sqrt{2\hat{\epsilon}(\chi)}}d\chi = \kappa\int_{\phi_{\rm end}}^{\phi}\frac{1}{\sqrt{2\hat{\epsilon}(\phi)}}\frac{d\chi}{d\phi}d\phi, \tag{30}$$

and is related to the true number of e-folds in the Jordan frame by

$$N = \hat{N} + \frac{1}{2}\ln\left[\frac{f(\phi)}{f(\phi_{\rm end})}\right]. \tag{31}$$

Classically, we have $\hat{V}(\phi) = \lambda\phi^4/[f(\phi)]^2$, which gives

$$\hat{\epsilon}(\phi) = \frac{8}{\kappa^2\phi^2[1 - 2\xi(1 - 12\xi)\kappa^2\phi^2]} \overset{\rm CC}{\rightarrow} \frac{8}{\kappa^2\phi^2}, \tag{32}$$

$$\hat{\eta}(\phi) = 4\left[\frac{3 + 24\xi^2\kappa^2\phi^2 - 2\xi(1 - 12\xi)\kappa^2(4\xi\kappa\phi^2 + 1)\phi^2}{\kappa^2\phi^2[1 - 2\xi(1 - 12\xi)\kappa^2\phi^2]}\right] \overset{\rm CC}{\rightarrow} \frac{4}{3} + \frac{12}{\kappa^2\phi^2}, \tag{33}$$

where CC denotes the conformal coupling limit. It thus emerges that the slow-roll parameters admit no slow-roll region at all at conformal coupling. This can also be seen from the total number of e-folds:

$$\hat{N}(\phi) = \frac{(1 - 12\xi)\kappa^2}{8}(\phi^2 - \phi_{\rm end}^2) - \frac{3}{4}\ln\left[\frac{f(\phi)}{f(\phi_{\rm end})}\right], \tag{34}$$

which lacks the first, exponential expansion generating, term when $\xi = 1/12$.

Comparing the number of e-folds in the Jordan frame, obtained in the Einstein frame analysis, namely, from Eq. (31), with the scalar factor $a(t)$ given from Eq. (12), one can confirm that it indeed agrees with the previous result obtained within the Jordan frame. This shows that the canonical slow-roll conditions in the Einstein frame and the ones chosen in the Jordan frame produce agreeing results, at least at the level of the observed expansion. Of course this does not imply the equivalence of other quantities such as the perturbation amplitudes in the two frames. As it has been explicitly shown in Ref. [24], the scalar two-point correlation functions evaluated in the Jordan frame are different than those calculated after the field redefinitions in the Einstein frame. Therefore, one should keep in mind that there is a number of ambiguities when quantum fluctuations of the scalar fields are studied in different frames in the context of generalized Einstein theories. Primordial spectral indices are calculated to second order in slow-roll parameters in Ref. [20] for different inflationary models, in the context of theories with a nonminimal coupling between the inflaton field and the Ricci curvature scalar. It has been shown that there are inflationary models (e.g., new inflation) for which there are discrepancies between the values of the spectral index $n_{\rm s}$ calculated in the Einstein and the Jordan frame, while for others (e.g., chaotic inflation) there are not. Finally, the reader should keep in mind that while the realization of slow-roll inflation in the (physical) Jordan frame, in which the inflaton is nonminimally coupled to the Ricci curvature, implies slow-roll inflation in the (unphysical) Einstein frame, the *vive versa* does not hold [13].

III. FLAT POTENTIAL THROUGH QUANTUM CORRECTIONS

The Higgs potential takes the classical form

$$V(\phi) = \lambda\phi^4 - \mu^2\phi^2, \tag{35}$$

however, both μ and λ are subject to radiative corrections as a function of energy. For very large values of the field ϕ one therefore needs to calculate the renormalized value of these parameters at the energy scale $\mu \sim \phi$. The running of the top Yukawa coupling and the gauge couplings cannot be neglected and must be evolved simultaneously. We follow the analysis of Ref. [25], which relies upon the β functions and improved effective potential presented in Refs. [26–28]. This involves taking the measured values of the gauge couplings at low energy and evolving them upwards in energy, taking into account the thresholds where quark species come into the running. It is necessary to simultaneously evolve all three gauge couplings and the top quark Yukawa coupling in order to accurately predict the full effect upon the Higgs self-coupling. Care must be taken to use the correct relationship between the pole masses and the parameters used in the running [25].

At high energies the mass term is subdominant and one can write the effective potential as

$$V(\phi) = \lambda(\phi)\phi^4. \tag{36}$$

Then for a given mass m_t of the top quark, a smaller value of the Higgs mass will result in the quartic coupling being driven down at large values of ϕ, such that it may develop a metastable or true vacuum at expectation values of ϕ, far in excess of that observed from standard model physics $\langle\phi\rangle = 246$ GeV. For typical values of m_t, if this false vacuum appears at all, it will show up relatively close to the Planck scale. When calculating the running of λ it is in fact necessary to go to two-loop accuracy since at one-loop this second minimum develops at scales typically far in excess of the Planck scale, where we would really expect higher order nonrenormalizable contributions to the potential to become important.

For each value of m_t there is therefore a value of the Higgs mass, m_h, where the effective potential is on the verge of developing a metastable minimum at large values of ϕ and the Higgs potential is locally flattened. This is illustrated in Fig. 1. Since the region where the potential becomes flat is narrow, slow-roll must be very slow (i.e., the slow-roll parameters very small), in order to provide a sufficiently long period of quasiexponential expansion.

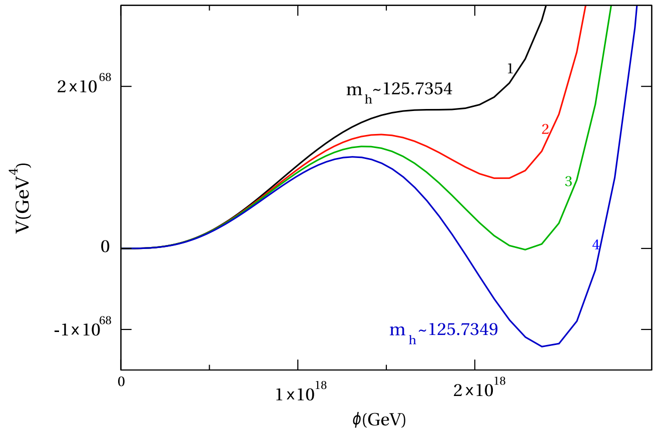

FIG. 1 (color online). Sub-Planckian flattening of the Higgs potential due to two-loop corrections in the standard mdel ($\xi = 0$). We analyze slow-roll for profiles just above the top (black) curve, which feature no metastable vacua.

The slow-roll parameters for the top (black curve) potential profile of Fig. 1 are shown in Fig. 2, and one can see that the region where ϵ is extremely small takes the form of a narrow dip. It is there that the integral $N \sim \int \epsilon^{-(1/2)} d\phi$ can generate the required number of e-folds.

It was noted in Ref. [29] that in the minimally coupled model, slow-roll through this flat region will not match the observed amplitude of density perturbations $\Delta^2_{\mathcal{R}}$ in the cosmic microwave background. Inflation predicts the latter to be related to the potential and first slow-roll parameter at horizon crossing (labeled by stars). Its value as measured by WMAP7 [30] imposes the constraint

$$\left(\frac{V_*}{\epsilon_*}\right)^{(1/4)} = 2\sqrt{3\pi} m_{\rm Pl} \Delta_{\mathcal{R}}^{(1/2)} = (2.75 \pm 0.30) \times 10^{-2} m_{\rm Pl}, \tag{37}$$

where $\epsilon_* \leq 1$. The mismatch arises because ϵ needs to be

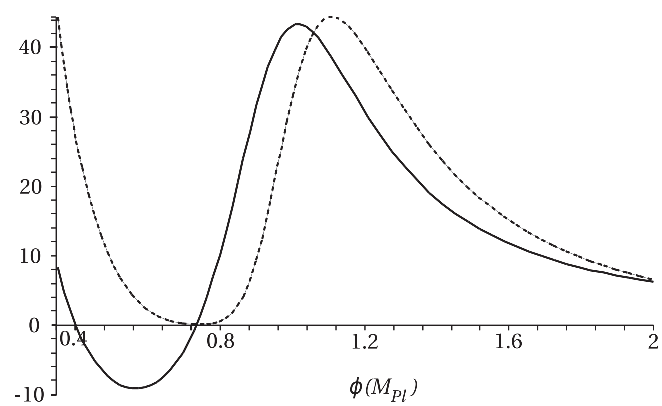

FIG. 2. Typical profiles of ϵ (dotted line) and η (solid line) with a small sub-Planckian region of slow-roll, plotted here for $m_t = 172$ GeV and $\delta = 0$. There is a narrow region in which both are very small.

extremely small in order to allow for sufficient e-folds and the potential energy is then too large to fit the condition. However, even in the minimally coupled model, there remains the possibility that horizon crossing occurs close to the beginning of inflation, where ϵ is not yet so small, provided the flat region occurs at low enough energy. Since $\epsilon_* \leq 1$, the maximum potential energy at horizon crossing is $5.7 \times 10^{-7} m_{\rm Pl}^4$. We shall see in the renormalization group analysis that there exist values of the top quark mass for which the flattening does happen at energies below this value. Furthermore, the presence of nonminimal coupling has additional effects since it changes the potential felt by the Higgs field.

When the nonminimal coupling ξ of the Higgs boson to gravity is included in the standard model, it has a β function induced by the coupling between the Higgs field and the matter sector whose behavior has been analyzed to one-loop [10,11]. As previously stated, we take $\beta_\xi = 0$, since the presence of a boundary value $\xi = 1/12$ at some energy scale suggests that $\xi = 1/12$ at all scales. The β function of the quartic Higgs self-coupling changes as well due to the $-\xi R \phi^2$ term, and this can have significant effects on the remaining standard model parameters when ξ is large [31]. We have worked out how large ξ needs to be to impact the normal standard model running by considering the two cases $\xi = 1$ and $\xi = -1$ at low energies and running these up with the other parameters. The effect that either of these choices has on the potential is very small and looks like a minute change in the Higgs mass, much less than any possible experimental error. Because we are well within this range we can neglect these corrections.

We therefore calculate the renormalization of the Higgs self-coupling in the minimally coupled standard model and construct an effective potential which fits the renormalization group improved potential around the flat region. The modifications in that fit are very small when the conformal coupling is included. We first consider the implications for the minimally coupled model (where the Jordan and Einstein frames coincide), which had been mentioned in Ref. [29], and then extend the analysis to a conformally coupled model, which is of particular relevance to the noncommutative spectral action approach to the standard model of particle physics.

There is a very good analytic fit to the Higgs potential in the region around this plateau/minimum, which takes the form

$$V_{\rm E} = \lambda_{\rm E}(\phi)\phi^4 = [a \ln^2(b\kappa\phi) + c]\phi^4. \tag{38}$$

The parameters are found to relate to the low energy values of $m_{\rm t}$ in the following way:

$$a(m_{\rm t}) = 4.047\,04 \times 10^{-3} - 4.419\,09 \times 10^{-5}\left(\frac{m_{\rm t}}{\rm GeV}\right) + 1.247\,32 \times 10^{-7}\left(\frac{m_{\rm t}}{\rm GeV}\right)^2,$$
$$b(m_{\rm t}) = \exp\left[-0.979\,261\left(\frac{m_{\rm t}}{\rm GeV} - 172.051\right)\right]. \tag{39}$$

The third parameter, $c = c(m_{\rm t}, m_\phi)$, encodes the appearance of an extremum (see Fig. 1) and depends on the values for $m_{\rm t}$ and m_ϕ. Indeed, $V_{\rm E}(\phi)$ exhibits a sub-Planckian flat region (or local minimum) for suitably tuned parameters. An extremum occurs if and only if $c/a \leq 1/16$, the saturation of the bound corresponding to a perfectly flat region, i.e., $V'_{\rm E}(\phi_0) = V''_{\rm E}(\phi_0) = 0$, where $\phi_0 = e^{-(1/4)}/b(m_{\rm t})$ and e is Euler's constant. The energy at these points is given by

$$V_{\rm E}(\phi_0) = \frac{a(m_{\rm t})}{8eb(m_{\rm t})^4}\kappa^4, \tag{40}$$

which for $169 \leq m_{\rm t} \leq 175$ lies within $10^{-10}\kappa^{-4} \leq V_0 \leq \kappa^{-4}$ [note that $V_{\rm E}(\phi_0)$ increases with $m_{\rm t}$]. This shows that there are regions where the flattening occurs at scales potentially consistent with perturbation amplitudes, given in Eq. (37).

It is convenient to write $c = [(1+\delta)/16]a$, where $\delta = 0$ saturates the bound below which a local minimum is formed. We restrict ourselves to $\delta > 0$, so that the potential contains no metastable vacua. The slow-roll conditions are met only for a narrow region, but for the points in parameter space which are close to $\delta = 0$, both slow-roll parameters vanish simultaneously and we get slow-roll inflation with extremely small ϵ.

From Eq. (37) it follows that for $m_{\rm t} > 171.42$ GeV the two conditions cannot be simultaneously met since the flat region occurs at too high energies. Slow-roll is restricted to the domain where $\max(\epsilon, |\eta|) \leq 1$, and for inflation one should find a point in parameter space which: (i) leads to sufficient e-folds within a region $[\phi_{\rm end}, \phi_*]$, (ii) has an ϵ_* which lies within the bounds imposed by Cosmic Background Explorer normalization, and (iii) satisfies the observational constraints on $n_{\rm s}$ and r. The measured value of perturbation amplitudes serves as a convenient first test of the model. For the scenario to be viable, ϵ at horizon crossing cannot be too small. Since the requirement on a sufficient number of e-folds relies on a potential that has very small ϵ in a small region, the problem is that the valley in ϵ is far narrower than that in η. As a result, within the region $|\eta| \leq 1$, ϵ tends to be very small. The best fit to the observed perturbation amplitude will occur for scenarios in which horizon crossing occurs close to the onset of inflation, i.e., $\eta(\phi_\star) \sim 1$, so that $\epsilon_\star$ takes its largest possible value.

The corrections due to conformal coupling to the potential in the Einstein frame are entirely embodied in the function $f(\phi) \sim 1 + \mathcal{O}(\kappa^2\phi^2)$, since the canonical field

χ feels the potential V_E/f^2. The value of the Higgs field where the plateau occurs in the potential rises with increasing top quark mass, so the greatest effect will be at the highest top quark mass. However, the lower bound on ϵ_* then gets more stringent since V_* is larger. Because of the change in the potential, flatness does not occur at $\delta = 0$ anymore but for fixed values of δ depending on the value of the top quark mass. Sub-Planckian inflation is again reliant on a relationship between the Higgs field and the top quark masses. The values of δ for which the potential has the right flatness are not anymore centered around $\delta = 0$ due to the altered form of the potential. This has an effect on the Higgs masses where flattening occurs: for any $-1 \leq \delta \leq 0$, a given top quark mass fixes the Higgs mass to a value in the range (120–130) GeV with an accuracy of $\Delta m_\phi / m_\phi \sim 10^{-6}$. This means that for inflation to occur via this mechanism, the top quark mass fixes the Higgs mass extremely accurately. As an example, for $m_t = 171.70$ GeV and $\delta = -0.2867$ (corresponding to $m_\phi = 125.735\,368$ GeV), we obtain $\hat{N} = 62$ of e-folds between $\kappa\phi = 0.9570$ and $\kappa\phi_{\rm end} = 0.9417$.

Scanning through parameter space it emerges that sufficient e-folds are indeed generated provided a suitably tuned relationship between m_t and m_ϕ holds. Numerical integration needs to be performed carefully since the slow-roll approximation implies a strongly peaked integrand in the number of e-folds. Using a Runge-Kutta integrator in FORTRAN we identify the curve in parameter space along which sufficient expansion occurs during almost perfect de Sitter inflation, both for minimal and conformal couplings.

The next step is a comparison with astrophysical measurements. To probe the parameter space more finely we use a Monte-Carlo chain. It turns out that in both the minimal and conformal cases, the perturbation amplitudes are too large—the best fit to the ratio $(V_\star/\epsilon_\star)^{(1/4)}$ is still too large by 2 orders of magnitude. Small positive nonminimal couplings such as $\xi = 1/12$ improve the fit only minimally. It should be noted also that when perturbation amplitudes are too large, scenarios where perturbations are generated by a curvaton are in turn ruled out as well, because the quantum fluctuations of the inflaton are already too large.

In Fig. 3 we show the best fit, i.e., the scenario with the largest possible values of ϵ_*, the first slow-roll parameter at horizon crossing, for a given top quark mass along with the potential energy V_*, at horizon crossing. The resulting ratio of perturbation amplitudes is too large for any value of m_t.

On a side note, let us mention that the renormalization of standard model parameters is generally performed in Minkowski space-time, while inflationary perturbations are calculated on a general de Sitter background. The conditions for the generalization of a slow-roll inflationary era should of course be studied in a de Sitter space and the Coleman-Weinberg result should then be recovered as a

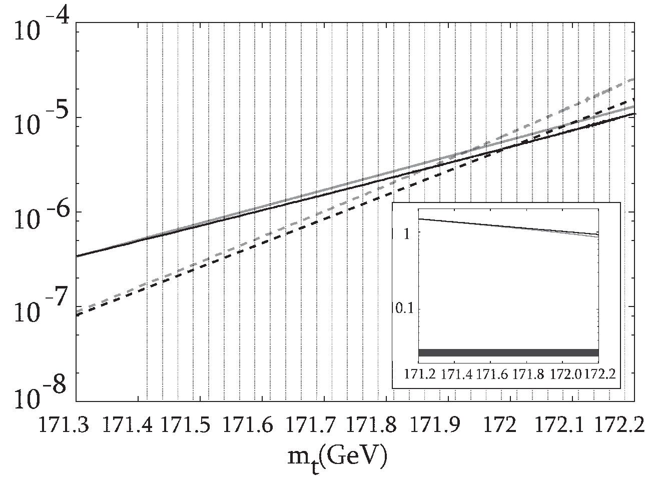

FIG. 3. The value of the potential (solid) in units of κ^{-4} and the maximum value of the first slow-roll parameter (dashed line) at horizon crossing for minimal $\xi = 0$ (black) and conformal $\xi = 1/12$ (grey). The striped area represents the region of the top mass excluded by Eq. (37) from the height of the plateau in the potential. The inset shows the ratio $(V_\star/\epsilon_\star)^{(1/4)}$ in both cases and WMAP7 observations (grey bar). The calculated value of perturbation amplitudes is off by several orders of magnitude and the improvement at conformal coupling minimal.

limit to the flat Minkowski space-time. This analysis has been performed in a recent study [21], where the one-loop improved potential for the nonminimally coupled scalar $\lambda\phi^4$ theory in de Sitter space was calculated. Their analysis poses a stringent constraint on the coupling parameter ξ. The assumption $|\dot{H}| < H^2$ along with the requirement that $f(\phi)$ in the equations of motion remain nonsingular, $f(\phi) < \infty$, implies

$$\frac{1}{16\tilde{N}} \ll |\xi| \ll \frac{1}{48}, \tag{41}$$

where $\tilde{N} \approx N + 1 - \xi$. This rules out most values of ξ used in the literature. However, $|\dot{H}| < H^2$ is in fact a stronger condition than the condition $-\dot{H} < H^2$ for inflationary expansion. The latter implies the former only when $\dot{H}$ is negative. The stronger condition $|\dot{H}| < H^2$ could be circumvented in an inflationary universe where $\dot{H}$ is large and positive. In the minimally coupled case this is clearly not possible, since $\dot{H} = -\dot{\phi}^2/2$. For nonzero ξ we have, however,

$$\dot{H} = \frac{\phi\dot{\phi}}{f(\phi)}\left[-\frac{1}{2}(1 - 4\xi)\frac{\dot{\phi}}{\phi} - 2\xi H + 2\xi\frac{\ddot{\phi}}{\dot{\phi}}\right], \tag{42}$$

which for the slow-roll conditions, Eq. (9), reduces to

$$\dot{H} = -\frac{2\xi\phi\dot{\phi}}{f(\phi)}H. \tag{43}$$

Since the field rolls down the potential, $\text{sign}(\phi) = -\text{sign}(\dot{\phi})$ and $\dot{H}$ is indeed positive when the nonminimal coupling is positive (e.g., conformal) in our notation. This means that the above constraint does not apply in the conformally coupled case. However, it should be mentioned that for negative choices of ξ, popular due to their promise in achieving Higgs driven inflation, $\dot{H} < 0$. The constraint in Eq. (41) is then valid and seems to be in contradiction with large $|\xi|$.

IV. NONCOMMUTATIVE SPECTRAL ACTION AND INFLATION

Using the language of noncommutative geometry and spectral triples, Connes and collaborators have reformulated the standard model in terms of purely geometric data [14]. Based on spectral triples, Connes [32] has developed a new calculus that deals not with the underlying spaces, but with the algebra of functions defined upon them instead. This reformulation allows a natural generalization of the differential calculus on Riemannian manifolds to a wider class of geometric structures, i.e., noncommutative spaces. It is the geometry of these spaces that encodes not only space-time and gravity, but also the matter content of the standard model.

In NCG, the fundamental particles and interactions derive from the spectral data of an action functional defined on noncommutative spaces, the spectral action. The standard model emerges as the asymptotic expansion of this action at an energy Λ below the Planck scale, at which the fundamental noncommutative space is approximated by an almost-commutative space. This space is assumed to be the simplest noncommutative extension of the smooth four-dimensional space-time manifold, and is obtained by taking its tensor product with a finite noncommutative space. Having recovered low energy physics in the framework of NCG, the next step will be to find the true geometry at Planckian energies, for which this product is a low energy limit. We consider here the effective action functional at the scale Λ.

In this section, we will first highlight the main principles of the noncommutative geometry approach and we will then investigate possible inflationary mechanisms driven by one of the available scalar fields.

A. Elements of NCG spectral action

Within general relativity, the group of symmetries of gravity is given by the diffeomorphism of the underlying differentiable manifold of space-time; a key ingredient that one would like to extend to the theory of elementary particles. To achieve such a *geometrization* of the standard model coupled to gravity, one should turn the SM coupled to gravity into pure gravity on a preferred space, whose group of diffeomorphisms is given by the semidirect product of the group of maps from the background manifold to the gauge group of the SM, with the group of diffeomorphisms of the background manifold. Such preferred space cannot be obtained however within ordinary spaces, while noncommutative spaces can easily lead to the desired answer. This is the main reason for extending the framework of geometry to spaces whose algebra of coordinates is noncommutative.

To extend the Riemaniann paradigm of geometry to the notion of metric on a noncommutative space, the latter should contain the Riemaniann manifold with the metric tensor (as a special case), allow for departures from commutativity of coordinates as well as for quantum corrections of geometry, contain spaces of complex dimension, and offer the means of expressing the standard model coupled to Einstein gravity as pure gravity on a suitable geometry. A metric NCG is given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, in the sense that we will discuss below. Thus, within NCG, geometric spaces emerge naturally from purely spectral data. The fermions of the standard model provide the Hilbert space $\mathcal{H}$ of a spectral triple for a suitable algebra $\mathcal{A}$, and the bosons arise naturally as inner fluctuations of the corresponding Dirac operator D. To study the implications of this noncommutative approach coupled to gravity for the cosmological models of the early Universe, we will only consider the bosonic part of the action; the fermionic part is however crucial for the particle physics phenomenology of the model.

More precisely, let us consider a geometric space defined by the product of a continuum compact Riemaniann manifold, $\mathcal{M}$, and a tiny discrete finite noncommutative space, $\mathcal{F}$, composed of only two points. The product geometry $\mathcal{M} \times \mathcal{F}$ has the same dimension as the ordinary space-time manifold, namely, 4. Hence, the noncommutative space $\mathcal{F}$ has zero metric dimension. The space $\mathcal{F}$ represents the geometric origin of the standard model and it is specified in terms of a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a noncommutative $\star$ algebra, $\mathcal{H}$ is a Hilbert space on which $\mathcal{A}$ is realized as an algebra of bounded operators, and D is a suitably defined Dirac operator on $\mathcal{H}$. The Dirac operator can be seen as the inverse of the Euclidean propagator of fermions. Since the action functional only depends on the spectrum of the line element, it is a purely gravitational action. In other words, the physical Lagrangian is entirely determined by the geometric input, which implies that the physical implications are closely dependent on the underlying chosen geometry, see, Ref. [14].

By assuming that the algebra constructed in $\mathcal{M} \times \mathcal{F}$ is *symplectic-unitary*, the algebra $\mathcal{A}$ is restricted to be of the form

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}), \tag{44}$$

where $k = 2a$ and $\mathbb{H}$ is the algebra of quaternions. The choice $k = 4$ is the first value that produces the correct number ($k^2 = 16$) of fermions in each of the three generations [33]. The Dirac operator D connects $\mathcal{M}$ and $\mathcal{F}$ via the spectral action functional on the spectral triple. It is defined as $\mathrm{Tr}(f(D/\Lambda))$, where $f > 0$ is a test function and

Λ is the cutoff energy scale. The asymptotic expression for the spectral action, for large energy Λ, is of the form

$$\mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right) \sim \sum_{k\in\mathrm{DimSp}} f_k \Lambda^k \int\!\!\!\!\!- |D|^{-k} + f(0)\zeta_D(0) + \mathcal{O}(1), \tag{45}$$

where $f_k = \int_0^\infty f(v)v^{k-1}dv$ are the momenta of the function f, the noncommutative integration is defined in terms of residues of zeta functions, and the sum is over points in the *dimension spectrum* of the spectral triple. The test function enters through its momenta f_0, f_2, f_4; these three additional real parameters are physically related to the coupling constants at unification, the gravitational constant, and the cosmological constant. In the four-dimensional case, the term in Λ^4 in the spectral action, Eq. (45), gives a cosmological term, the term in Λ^2 gives the Einstein-Hilbert action functional with the physical sign for the Euclidean functional integral (provided $f_2 > 0$), and the Λ-independent term yields the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the metric. The scale-independent terms in the spectral action have conformal invariance. Note that the arbitrary mass scale Λ can be made dynamical by introducing a scaling dilaton field.

Writing the asymptotic expansion of the spectral action, a number of geometric parameters appear; they describe the possible choices of Dirac operators on the finite noncommutative space. These parameters correspond to the Yukawa parameters of the particle physics model and the Majorana terms for the right-handed neutrinos. The Yukawa parameters run with the renormalization group equations of the particle physics model. Since running toward lower energies implies that nonperturbative effects in the spectral action cannot be any longer safely neglected, any results based on the asymptotic expansion and on renormalization group analysis can only hold for early Universe cosmology. For later times, one should instead consider the full spectral action.

Applying the asymptotic expansion of Eq. (45) to the spectral action of the product geometry $\mathcal{M} \times \mathcal{F}$ gives a bosonic functional S which includes cosmological terms, Riemannian curvature terms, Higgs minimal coupling, Higgs mass terms, Higgs quartic potential, and Yang-Mills terms. Moreover, one can introduce a relation between the parameters of the model, namely, a relation between the coupling constants at unification. More precisely, we impose the relation

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \quad \text{and} \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2, \tag{46}$$

between the coefficient f_0 and the coupling constants g_1, g_2, g_3, which is dictated by the normalization of the kinetic terms. This condition means that the so-obtained spectral action has to be considered as the *bare action* at unification scale Λ, where one supposes the merging of the coupling constants to take place.

The gravitational terms in the spectral action, in Euclidean signature, are of the form

$$\mathcal{S}^{\mathrm{E}}_{\mathrm{grav}} = \int\left(\frac{1}{2\kappa^2}R + \alpha_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tau_0 R^\star R^\star - \xi_0 R|\mathbf{H}|^2\right)\sqrt{g}d^4x. \tag{47}$$

Note that $\mathbf{H}$ is a rescaling $\mathbf{H} = (\sqrt{af_0}/\pi)\phi$ of the Higgs field ϕ to normalize the kinetic energy; the momentum f_0 is physically related to the coupling constants at unification and the coefficient a is related to the fermion and lepton masses and lepton mixing. In the above action, Eq. (47), the first two terms only depend upon the Riemann curvature tensor; the first is the Einstein-Hilbert term with the second one being the Weyl curvature term. The third term

$$R^\star R^\star = \tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}_{\rho\sigma},$$

is the topological term that integrates to the Euler characteristic and hence is nondynamical. Notice the absence of quadratic terms in the curvature; there is only the term quadratic in the Weyl curvature and topological term $R^\star R^\star$. In a cosmological setting, namely, for Friedmann-Lemaître-Robertson-Walker geometries, the Weyl term vanishes. The spectral action contains one more term that couples gravity with the SM, namely, the last term in Eq. (47), which should always be present when one considers gravity coupled to scalar fields.

B. Higgs field inflation

The asymptotic expansion of the spectral action, proposed in Ref. [14], gives rise to the following Gravity-Higgs sector $\mathcal{L}_{\mathrm{GH}} \subset \mathcal{L}_{\mathrm{NCG}}$:

$$S_{\mathrm{GH}} = \int d^4x\sqrt{-g}\left\{\frac{1 - 2\kappa^2\xi H^2}{2\kappa^2}R - \frac{1}{2}(\nabla H)^2 - V(H)\right\}, \tag{48}$$

where $V(H) = \lambda H^4 - \mu^2 H^3$. We work with the real excitation of the Higgs field, $H = |\mathbf{H}|$. In the derivation of the standard model from the spectral action principle, the metric carries Euclidean signature. The discussion of phenomenological aspects of the theory relies on a Wick rotation to imaginary time, into the standard (Lorentzian) signature. While sensible from the phenomenological point of view, there exists as yet no justification on the level of the underlying theory.

To discuss the phenomenology of the aspects of the cutoff scale Λ, the spectral action principle leads to a number of boundary conditions on the parameters of the

Lagrangian. These conditions encode the geometric origin of the standard model parameters. Normalization of the kinetic terms in the action implies the following relations:

$$\kappa^2 = \frac{12\pi^2}{96 f_2 \Lambda^2 - f_0 \mathfrak{c}}, \qquad \xi = \frac{1}{12},$$
$$\lambda = \frac{\pi^2 \mathfrak{b}}{2 f_0 \mathfrak{a}^2}, \qquad \mu = 2\Lambda^2 \frac{f_2}{f_0}. \tag{49}$$

We emphasize that the action, Eq. (48), has to be taken as the bare action at some cutoff scale Λ. The renormalized action will have the same form but with the bare quantities κ, μ, λ and the three gauge couplings g_1, g_2, g_3 replaced with physical quantities.

The factor f_0 is fixed by the canonical normalization of the Yang-Mills terms (not included here) in terms of the common value of the gauge coupling constants g at unification, $f_0 = \pi^2/(2g^2)$. The value of g at the unification scale is determined by standard renormalization group flow, i.e., it is given a value which reproduces the correct observed coupling at low energies. Note that it is not unique since the gauge couplings fail to meet exactly in the nonsupersymmetric standard model (or its extension by right-handed neutrinos). The coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are the Yukawa and Majorana parameters subject to renormalization group flow, see, e.g,. Ref. [14]. The parameter f_2 is *a priori* unconstrained in $\mathbb{R}_+^*$.

Assuming the *big desert* hypothesis, we can connect the physics at low energies with those at $E = \Lambda$ through the standard renormalization procedure. This was carried out at one loop in Ref. [14], and more recently in Ref. [16] where Majorana mass terms for right-handed neutrinos were included and the seesaw mechanism was taken into account. In our renormalization group analysis of the Higgs potential, following Ref. [25], the choice of boundary conditions is the standard one motivated by particle physics considerations. The focus here has of course been on the different boundary conditions at low energies for which a flat section develops in the Higgs potential.

The relations above rely on the validity of the asymptotic expansion at Λ, and are therefore tied intimately to the scale at which the expansion is performed. There is no *a priori* reason for the constraints to hold at scales below Λ—they represent mere boundary conditions. The constraint $\xi(\Lambda) = 1/12$ by itself therefore does not require the coupling to remain conformal all the way down to present energy scales, or even during an inflationary epoch, since it may run with the energy scale. However, we will assume no running in ξ as the arguments laid out (see, discussion in Sec. II A) above still apply.

As we can see from the results presented above, the conformally coupled Higgs field in the spectral action standard model is not a viable candidate for inflaton if the coupling remains conformal at all scales. However, at present it is still unclear whether conformal[4] invariance and $\xi = 1/12$ are a generic feature of models from noncommutative geometry. If it turns out not to be, one can proceed along the line of the analyses presented in Refs. [15,16].

C. Inflation through the massless scalar field

The spectral action gives rise to an additional massless scalar field[5] [17], denoted by σ. Including this field, the cosmologically relevant terms in the Wick rotated action read

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \xi_H R H^2 - \xi_\sigma R \sigma^2 - \frac{1}{2}(\nabla H)^2 - \frac{1}{2}(\nabla\sigma)^2 - V(H, \sigma) \right\}, \tag{50}$$

where

$$V(H, \sigma) = \lambda_H H^4 - \mu_H^2 H^2 + \lambda_\sigma \sigma^4 + \lambda_{H\sigma} |H|^2 \sigma^2. \tag{51}$$

The constants are related to the underlying parameters as follows[6]:

$$\xi_H = \tfrac{1}{12}, \qquad \xi_\sigma = \tfrac{1}{12}, \tag{52}$$

$$\lambda_H = \frac{\pi^2 \mathfrak{b}}{2 f_0 \mathfrak{a}^2}, \qquad \lambda_\sigma = \frac{\pi^2 \mathfrak{d}}{f_0 \mathfrak{c}^2}, \tag{53}$$

$$\mu_H = 2\Lambda^2 \frac{f_2}{f_0}, \qquad \lambda_{H\sigma} = \frac{2\pi^2 \mathfrak{e}}{a \mathfrak{c} f_0}. \tag{54}$$

This action also admits a rescaling of the metric which transforms it to the Einstein frame. The rescaled metric $\hat{g}_{\mu\nu} = f(H, \sigma) g_{\mu\nu}$ with $f(H, \sigma) = 1 - 2\xi_H H^2 - 2\xi_\sigma \sigma^2$ is now accompanied by the new fields χ_H and χ_σ related to the Jordan frame fields by

[4]The coupling term between the Higgs field and the Ricci curvature, appearing in the spectral action functional, is $-f_0/(12\pi^2) a R |\phi|^2$, which, after rescaling $H = (\sqrt{a f_0}/\pi)\phi$, leads to the term $-R|H|^2/12$. This indeed shows the conformal coupling between the background and the Higgs field.

[5]The field σ is unlike all other fields in the theory, such as the Higgs field and gauge fields. Usually one starts with a parameter in the Dirac operator of the discrete space, and then inner fluctuations of the product space would generate the dynamical fields. The only exception being the matrix entry that gives mass to the right-handed neutrinos, where the parameter can either remain as such, or one can use the freedom to make it a dynamical field, which *a priori* may lead to important cosmological consequences [34]. Note that the σ field was not considered in the original noncommutative geometry spectral action analysis presented in Ref. [14], where the authors were mainly interested in recovering the standard model.

[6]Note that a similar action has been studied in Ref [31], but in their analysis the additional scalar field has a nonzero mass and the nonminimal couplings are studied in the previously mentioned large negative ξ regime, which flattens the classical quartic potential in the Einstein frame.

$$\frac{d\chi_H}{dH} = \frac{\sqrt{1 - 2\xi_H(1 - 12\xi_H)H^2}}{f(H, \sigma)} \overset{\text{CC}}{\to} \frac{1}{f(H, \sigma)}, \quad (55)$$

$$\frac{d\chi_\sigma}{d\sigma} = \frac{\sqrt{1 - 2\xi_\sigma(1 - 12\xi_\sigma)\sigma^2}}{f(H, \sigma)} \overset{\text{CC}}{\to} \frac{1}{f(H, \sigma)}. \quad (56)$$

The Einstein frame Lagrangian reads

$$S = \int d^4x\sqrt{-\hat{g}}\Big\{\frac{1}{2\kappa^2}\hat{R} - \frac{1}{2}(\hat{\nabla}\chi_H)^2 - \frac{1}{2}(\hat{\nabla}\chi_\sigma)^2 - P(\chi_H, \chi_\sigma)\hat{\nabla}_\mu\chi_H\hat{\nabla}^\mu\chi_\sigma - V(\chi_H, \chi_\sigma)\Big\}, \quad (57)$$

with

$$\hat{V}(\chi_H, \chi_\sigma) = \frac{V(H, \sigma)}{f(H, \sigma)^2}, \quad (58)$$

and a novel coupling

$$P(\chi_H, \chi_\sigma) = \frac{24\kappa^2\xi_H\xi_\sigma}{f(H, \sigma)^2}\frac{dH}{d\chi_H}\frac{d\sigma}{d\chi_\sigma}H\sigma \overset{\text{CC}}{\to} \frac{\kappa^2}{6}\sigma H. \quad (59)$$

Note that there exists no conformal transformation which gets rid of both the nonminimal coupling to gravity and the cross term $P(\chi_H, \chi_\sigma)$ [35]. However, at conformal coupling $P(\chi_H, \chi_\sigma)$ can be neglected as long as $\sigma H \ll 6\kappa^2$. We are then left with a minimally coupled theory of two scalar fields with potential $\hat{V}(\chi_H, \chi_\sigma)$. When the mass term is negligible the theory is symmetric in the two fields.

Consider the first slow-roll parameter for the σ field, defined as

$$\hat{\epsilon}_\sigma = \frac{1}{2\kappa^2}\frac{1}{\hat{V}^2}\left(\frac{\partial\hat{V}}{\partial\sigma}\right)^2\left(\frac{\partial\chi_\sigma}{\partial\sigma}\right)^{-2}, \quad (60)$$

$$= \frac{1}{2\kappa^2}[\lambda_H H^4 + \lambda_\sigma\sigma^4 + \lambda_{H\sigma}|H|^2\sigma^2]^{-2} \times \sigma^2\Big[(4\lambda_\sigma\sigma^2 + 2\lambda_{H\sigma}H^2)f(H, \sigma) + \frac{2}{3}\kappa^2(\lambda_\sigma\sigma^4 + \lambda_H H^4 + \lambda_{H\sigma}H^2\sigma^2)\Big]^2. \quad (61)$$

For $H = 0$ this reduces to the earlier case and one gets an insufficient number of e-folds below the Planck scale. If we have a nonzero H however, say $H \sim \kappa^{-1}$ close to the Planck mass, then the situation changes somewhat. Because of the additional terms in $\hat{\epsilon}_\sigma$, the coupling constants do not fall out of the expression, and they can therefore influence the magnitude of the integrand in the number of e-folds. For this effect to take place, however, it is necessary that the assisting field maintains a relatively large value throughout the inflationary era driven by the inflaton. This in turn requires the curvature of the potential to be much less in the direction of the constant field. Since only quartic terms arise in the model, the quartic self-coupling of the assisting field is then required to be much lower than that of the inflaton. But in that case, the new terms due to the assisting field are not large enough to enable the inflaton to generate a sufficient number of e-folds. The situation is of course entirely symmetric in the two fields (except for the nonzero H mass which is negligible at high energies), so the roles of the two fields may well be interchanged depending on which constraints lay on the respective coupling constants.

V. RUNNING OF THE GRAVITATIONAL CONSTANT

At the scale Λ, the gravitational constant is related to the geometric parameters of the theory by

$$\kappa^2 = \frac{12\pi^2}{96 f_2\Lambda^2 - f_0\mathfrak{c}}; \quad (62)$$

f_0 is fixed by one of the unification conditions

$$f_0 = \frac{\pi^2}{2g_1^2}, \qquad f_0 = \frac{\pi^2}{2g_2^2}, \qquad f_0 = \pi^2 2g_3^2,$$

f_2 is an unconstrained parameter in $\mathbb{R}_+^*$, and $\mathfrak{c}$ is determined by the renormalization group equations. Note that this value of the gravitational constant does not need to be the same as its present value, $\kappa^2 = [2.43 \times 10^{18}\ \text{GeV}]^{-2}$, since the gravitational constant may run.

Indeed, such a running has been suggested [16] due to the relation between κ^2 and $\mathfrak{c} = \text{Tr}(MM^\dagger)$; M stands for the Majorana mass term. The coefficient $\mathfrak{c}$ is a function of the neutrino mass matrix subject to running with the renormalization group equations dictated by the particle physics content of the model, in this case the standard model with additional right-handed neutrinos with Majorana mass terms. Since the renormalization group flow runs between a unification energy Λ, taken to be of the order of 2×10^{16} GeV, down to the electroweak scale of 100 GeV, the parameter $\mathfrak{c}$ runs as a function of Λ, with assigned initial conditions at the preferential energy scale of unification. One may thus deduce that through the running of κ^2, the number N of e-folds may increase. However, since at conformal coupling N is a logarithmic function of κ^2, the gravitational constant would have to change drastically in order for N to have an interesting change. As previously mentioned, we need a very significant running of the gravitational constant in order to get inflation—a local kind of inflation where the flat region is confined to a small range of energies where the potential is flattened.

In conclusion, unless modification of the spectral triple allows for a nonconformal boundary value of ξ, there seems to be no viable slow-roll scenario for any of the two scalars. Furthermore, if one assumes the validity of the suggestion in Ref. [16] in relating the running of $\mathfrak{c}$ and κ^2, the only situation in which this could trigger inflation (with conformal coupling) would be one in which the running changes drastically, e.g., through the seesaw mechanism. However, the inevitable lack of differentiability of the renormalized couplings at seesaw scales [16,36] makes such a scenario very unlikely and also inaccessible to slow-roll analysis.

VI. CONCLUSIONS

In many realistic cosmological models, the nonminimal coupling of the scalar field to the Ricci curvature cannot be avoided. In particular, there are arguments requiring a conformal coupling between the scalar field and the background curvature. The existence of such a term will generically lead to difficulties in achieving a slow-roll inflationary era. In this paper, we have investigated whether two-loop corrections to the Higgs potential could lead to a slow-roll inflationary period in agreement with the constraints imposed by the CMB measurements. Our findings do not favor the realization of such an era. More precisely, even though slow-roll inflation can be realized, we cannot satisfy the Cosmic Background Explorer normalization constraint for any values of the top quark and Higgs masses allowed from current experimental data.

We have, in particular, investigated Higgs inflation in the context of the noncommutative geometry spectral action, which provides an elegant explanation for the phenomenology of the standard model. Within this context, a conformal coupling arises naturally between the Higgs field and the Ricci curvature. It is also important to note that once conformal coupling is set at the preferential (boundary) energy scale of the spectral action model, then it will remain conformal at all scales. Running of the gravitational constant and corrections by considering the more appropriate de Sitter, instead of a Minkowski, background do not favor the realization of a successful inflationary era. The NCG spectral action provides in addition to the Higgs field, another (massless) scalar field which exhibits no coupling to the matter sector. Our analysis has shown that neither this field can lead to a successful slow-roll inflationary era if the coupling values are conformal. One may be able to improve upon this (negative) conclusion, if important deviations of ξ from its conformal value can be allowed; the value $\xi = 1/12$ may turn out that is not a generic feature of NCG models.

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